Ejercicio 15 $\quad V$ $\mathbb{K}$-e.v. $\quad (\mathrm{Hom}(V,W))$

$$V^* = \mathrm{Hom}(V, \mathbb{K}) =$$

$$= \{ f : V \to \mathbb{K} \ ; \text{ trans. lineales} \}$$

Operaciones:

$$(f+g)(v) = f(v) + g(v)$$

$$(\alpha . f)(v) = \alpha . f(v)$$

[dimensión finita]

$$V \simeq V^*$$

$$\dim(\mathrm{Hom}(V,W)) = \dim(V) . \dim(W) \longrightarrow \dim V^* = \dim V.$$

- $\mathbb{K}$ como $\mathbb{K}$-e.v. tiene $\dim = 1$.

$\dim V = n < \infty$.

a) $B = \{ e_1, \dots, e_n \}$ base de V.

$\longrightarrow$ B^* base del dual.

$e_1^* \in V^*$

$$e_1^* : V \to \mathbb{K}$$

$$e_1^*(e_1) = 1$$
$$e_1^*(e_2) = 0$$
$$\vdots$$
$$e_1^*(e_n) = 0$$

$e_i^* \in V^*$

$$e_i^* : V \to \mathbb{K}$$

$$e_i^*(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}.$$

Recordar

$t : B \to \mathbb{K}$ $\longrightarrow$ existe $T : V \to \mathbb{K}$

B es base. $\quad$ trans. lineal

con $T|_B = t$.

como $\rightarrow$ se define T a partir de t

V $\quad T$ definida a uma base $\{e_1, \dots, e_n\}$

$v = a_1 e_1 + \dots + a_n e_n$ para únicos a_i

$T(v) = a_1 t(e_1) + \dots + a_n t(e_n)$

c) $V = \mathbb{R}^2$ $\quad B = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

$e_1^* \in (\mathbb{R}^2)^*$

$e_1^* : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$e_1^*(x,y) = x \cdot \overbrace{e_1^*(e_1)}^{=1} + y\, \overbrace{e_1^*(e_2)}^{=0} =$

$= x$

$(x,y) = x e_1 + y e_2$

$e_2^*(x,y) = x\, \cancel{e_2^*(e_1)} + y\, e_2^*(e_2) = y.$

$e_i^*(e_j) = 0 \quad i \neq j$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R} \quad (T \in (\mathbb{R}^2)^*)$

$T = a e_1^* + b e_2^*$

$T(x,y) = ax + by$

$B = \{e_1, \dots, e_n\}$ base de V

<u>$B^* = \{e_1^*, \dots, e_n^*\}$ es base de V^*.</u>

– <u>B^* es l.i.</u> (*) $\underbrace{a_1 e_1^* + \dots + a_n e_n^*}_{= f} = 0$

$$0 = f(e_1) = a_1 e_1^*(e_1) + \overbrace{\dots + a_n e_n^*(e_1)}^{=0} = a_1$$

$$0 = f(e_i) = \underbrace{a_1 e_1^*(e_1) + \dots}_{=0} + a_i e_i^*(e_i) + \underbrace{\dots + a_n e_n^*(e_i)}_{=0} = a_i$$

<u>$a_i = 0 \quad \forall i$.</u>

– <u>B^* es generador:</u> $f \in V^*$

¿ $f \in \langle e_1^*, \dots, e_n^* \rangle$?

$f = \underline{a_1 e_1^* + \dots + a_n e_n^*}$ | $f(e_i) = a_i$

<u>$f = f(e_1) e_1^* + \dots + f(e_n) e_n^*$</u>

Ejercicio 1b $\quad T: V \longrightarrow W$

V y W $\mathbb{K}$-e.v.

$$T^*: W^* \longrightarrow V^*$$

$$T^*(f) = f \circ T$$

$T: V \longrightarrow W$

$f: W \longrightarrow \mathbb{K}$

$f \circ T: V \longrightarrow \mathbb{K}$

y es lineal porque T y f lo son.

T* es lineal $\quad \alpha \in \mathbb{K},\ f, g \in W^*$

$$T^*(\underbrace{\alpha f + g}_{W^*}) \in V^*$$

$$\boxed{T^*(\alpha f + g)}(v) \overset{\text{def } T^*}{=} ((\alpha f + g) \circ T)(v) =$$

$$= \underbrace{(\alpha f + g)}_{\in W^*}(\underset{\in W}{T(v)}) \overset{\text{def. de + y . en } W^*}{=} \alpha . f(T(v)) + g(T(v)) =$$

$$= \alpha T^*(f)(v) + T^*(g)(v).$$

Ejemplo

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad B = \{e_1, e_2\}$ base canónica de $\mathbb{R}^2$

$T(x,y) = (x, x)$

$${}_B[T]_B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$T^*: (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$

$T^*(f) = f \circ T$ $\quad f \in (\mathbb{R}^2)^*$ $\quad f(x,y) = ax + by$

$(T^* f)(x,y) = (f \circ T)(x,y) = f(x,x) = (a+b)x$

$${}_{B^*}[T^*]_{B^*} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = ({}_B[T]_B)^t$$

$B^* = \{e_1^*, e_2^*\}$ $\quad e_1^*(x,y) = x$
$\quad e_2^*(x,y) = y$

$T^*(e_1^*)(x,y) =$

$e_1^*(T(x,y)) = e_1^*(x,x) = x$

$T^*(e_2^*)(x,y) = e_2^*(x,x) = x$

$T^* \quad e_1^* \to e_1^*$

$T^* \quad e_2^* \to e_1^*$

$T : V \longrightarrow W \qquad T^* : W^* \longrightarrow V^*$

$\mathcal{B} = \{ b_1, \ldots, b_n \}$ base de V.

$\mathcal{C} = \{ c_1, \ldots, c_m \}$ base de W.

$_{\mathcal{C}}[T]_{\mathcal{B}} = (a_{ij})_{ij} \qquad ¿\ _{\mathcal{B}^*}[T^*]_{\mathcal{C}^*}$?

$T(b_i) = a_{1i} c_1 + \underbrace{\cdots}_{a_{ji} c_j} + a_{mi} c_m$

$$\begin{pmatrix} a_{j1} & \cdots & a_{jn} \end{pmatrix}$$

$*\downarrow$

$T^*(c_j^*)(b_i) =$

$c_j^*(T(b_i)) = \underline{a_{ji}}$

$_{\mathcal{B}^*}[T^*]_{\mathcal{C}^*}$

$$\begin{pmatrix} & a_{j1} & \\ \cdots & \vdots & \cdots \\ & a_{jn} & \end{pmatrix}$$

$T^*(c_j^*) = a_{j1} b_1^* + \cdots + a_{jn} b_n^*$

$_{\mathcal{B}^*}[T]_{\mathcal{C}^*} = (d_{ij})_{ij} \qquad d_{ij} = a_{ji} \qquad _{\mathcal{C}}[T]_{\mathcal{B}} = (a_{ij})_{ij}.$