Draft version October 8, 2018

Preprint typeset using LaTeX style AASTeX6 v. 1.0

NOTES ON THE RESTRICTED THREE-BODY PROBLEM

Tabare Gallardo

1. JACOBI'S CONSTANT, C

Consider a planet revolving a star with circular orbit of radius a. We redefine units of mass, length and time such that:

$1-m$: mass of star

m: mass planet

the mean motion is $n^2 = \mu/a^3$

the constant for two body problem is: $G(m_1+m_2) = \mu = k'^2((1-m)+m) = k'^2$

taking unit of length equal to a and taking unit of time such as to make $k' = 1$ then

$n = 1 = \frac{2\pi}{P}$ then using these units the orbital period of the planet is $P = 2\pi$. The linear velocity of the planet around the star is $V_p = na = 1$.

We define the system $(\hat{x}, \hat{y}, \hat{z})$ which rotates with the planet around the baricenter of the system with angular velocity $\vec{\omega} = n\hat{z} = 1\hat{z}$.

Consider a particle located in $\vec{r} = (x, y, z)$. We can demonstrate the Jacobi's integral of motion of the particle where v is the particle's velocity in the rotating frame:

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

being r_i the distance to mass i and C is a constant.

Demonstration:

The velocity in the inertial frame $\vec{V}$ and the one in the rotating frame $\dot{\vec{r}}$ are related by

$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r}$

The inertial acceleration is $\vec{\alpha} = -\nabla\mathbb{V}$ where $\mathbb{V}(\vec{r}) = -(1-m)/r_1 - m/r_2$ is the gravitational potential generated by the two masses.

The rotating system rotates with $\vec{\omega} = \hat{z}$ then the relationship between inertial acceleration $\vec{\alpha}$ and the acceleration relative to the rotating system $\ddot{\vec{r}}$ is

$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} + \hat{z} \wedge (\hat{z} \wedge \vec{r})$

but $\vec{r} = z\hat{z} + \vec{\rho}$ being $\vec{\rho} = (x, y, 0)$

then

$\vec{\alpha} = \ddot{\vec{r}} + 2\hat{z} \wedge \dot{\vec{r}} - \vec{\rho}$

multiply by $\dot{\vec{r}}$:

$\vec{\alpha} \cdot \dot{\vec{r}} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right]$

then

$\vec{\alpha} \cdot d\vec{r} = -\nabla \mathbb{V} d\vec{r} = \left[\ddot{\vec{r}} \cdot \dot{\vec{r}} - \vec{\rho} \cdot \dot{\vec{\rho}}\right] dt$

integrating

$$-2\mathbb{V}(\vec{r}) = \dot{\vec{r}}^2 - (x^2 + y^2) + C$$

or

$$v^2 = x^2 + y^2 - 2\mathbb{V}(\vec{r}) - C$$

then, the particle's velocity in the rotating frame becomes

$$v^2 = x^2 + y^2 + \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$$

C is a constant in the R3BP. If planet's eccentricity is different from zero C will oscillate around a mean value.

2. TISSERAND PARAMETER, T

The particle has some orbital elements (a, e, i) and we will make to appear them in Jacobi's integral. We need to express position and velocity in the rotating frame $(\vec{r}, \vec{v})$ as function of position and velocity $\vec{V}$ in the inertial frame.

We have

$\vec{V} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r} = \dot{\vec{r}} + \hat{z} \wedge \vec{\rho}$

Then

$\dot{\vec{r}} = \vec{V} - \hat{z} \wedge \vec{\rho}$

squaring

$v^2 = \vec{V}^2 - 2\vec{V} \cdot (\hat{z} \wedge \vec{\rho}) + \rho^2$

rearranging

$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{\rho} \wedge \vec{V}) + \rho^2$

$v^2 = \vec{V}^2 - 2\hat{z} \cdot (\vec{r} \wedge \vec{V}) + x^2 + y^2$

$\vec{V}^2 - 2\hat{z} \cdot (\vec{r} \wedge \vec{V}) = v^2 - x^2 - y^2 = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$

(in a numerical integration it is easier to calculate C using the inertial frame than the rotating one)

According to the two body problem baricenter-particle:

$V^2 = 2/r - 1/a$ and $\hat{z} \cdot (\vec{r} \wedge \vec{V}) = \hat{z} \cdot \vec{h} = \sqrt{a(1-e^2)} \cos i$

then

$\frac{2}{r} - \frac{1}{a} - 2\sqrt{a(1-e^2)} \cos i = \frac{2(1-m)}{r_1} + \frac{2m}{r_2} - C$

The orbital elements (a, e, i) are referred to the baricenter of the system Star+planet and the inclination is measured with respect to the orbital plane $\hat{x}\hat{y}$ of the planet. In the case of the solar system $m < 10^{-3}$ so it is possible to assume that (a, e, i) are heliocentric.

If the particle **is not very close to the Sun** we have $r \simeq r_1$ then

$\frac{1}{a} + 2\sqrt{a(1-e^2)} \cos i = 2m[\frac{1}{r_1} - \frac{1}{r_2}] + C$

If the particle is far from the sun and from the planet and taking into account that $m < 10^{-3}$ we obtain

$$C \simeq \frac{1}{a} + 2\sqrt{a(1-e^2)} \cos i = T$$

T is known as the Tisserand parameter. In the R3BP C is constant and T presents some departures if the orbital elements are determined when the conditions above are not satisfied (near the sun or the planet). T should be considered as a simple form of calculating C. It can be easily shown that for $a \geq 0$

$$T_{max}(a) = \frac{1}{a} + 2\sqrt{a} \qquad , \qquad T_{min}(a) = \frac{1}{a} - 2\sqrt{a}$$

For elliptic orbits it is possible to express $T(q, Q, i)$ where q, Q are perihelion and aphelion:

$$T = \frac{2}{q+Q} + 2\sqrt{2qQ/(q+Q)} \cos i$$

This is a useful formula when analyzing regions where encounters are possible ($q < 1, Q > 1$).

3. THE ENCOUNTER VELOCITY, U

Suppose the particle is near the planet ($r_1 \simeq 1$ and $x^2 + y^2 \simeq 1$) but far enough that we can neglect its gravitational attraction ($r \sim R_H$) so the particle is "at infinity" ($m/r_2 \simeq 0$). Then from Jacobi's integral:

$$v_\infty^2 \simeq 1 + 2 + 0 - T$$

then, under the hypothesis above, the planetocentric velocity "at infinity" of the particle is

$$v_\infty \simeq \sqrt{3 - T} = U$$

U is the encounter velocity with the planet **before** the gravitational attraction is felt by the particle (that means "at infinity"). U is determined by T which is constant, so U is also constant. The orbital elements (a, e, i) can evolve but T and U remain constant, only the orientation of $\vec{U}$ is modified (U rotates γ after the encounter).

It follows that when $T > 3$ encounters cannot exist. When $T < 3$ they could exist but they are not guaranteed. For example: $a = 2, e = 0, i = 90^o$ implies $T = 0.5$ but the particle never approaches the planet. This can be showed in a plot of the Minimum Orbit Intersection Distance (MOID) with respect to the Tisserand parameter. Objects with $T > 3$ cannot have low MOID values.

If $U \sim 0$ the planetocentric orbit is quasi-parabolic and a temporary capture by the planet is possible. Then, objects with $T \sim 3$ can experience temporary captures by the planet.

The greatest heliocentric velocity the particle can get after the encounter is $V_p + U = 1 + U$. The escape velocity from the system is $\sqrt{2}$, so if $U \geq \sqrt{2} - 1$ the particle eventually can escape from the solar system and conversely if $U < \sqrt{2} - 1$ the particle will never leave the solar system by this mechanism. Note that only prograde orbits have $U < 1$.

The final heliocentric velocity is a vectorial sum:

$\vec{V} = \vec{V_p} + \vec{U'}$ or $V^2 = 1 + U^2 + 2U\cos\theta$

being θ the angle between $\vec{V_p}$ and $\vec{U'}$. If $U > \sqrt{2} - 1$ there exists some θ_∞ so that for $\theta \leq \theta_\infty$ the corresponding V is greater than the ejection velocity. This situation occurs for

$\cos\theta_\infty = \frac{1 - U^2}{2U}$

If we can assume that $\vec{U}'$ is randomized (deflection γ is so great that θ can get all values from 0 to π) then the **probability of ejection per encounter** is equal to the probability $P(\theta \leq \theta_\infty)$ and this is equal to the solid angle subtended by θ_∞ over 4π which is equal to

$$P_\infty = P(\theta \leq \theta_\infty) = \frac{1}{2}(1 - \cos\theta_\infty) = \frac{U^2 + 2U - 1}{4U} \qquad (U > \sqrt{2} - 1, \gamma > 90^o)$$

Conversely, a comet in an hyperbolic heliocentric orbit has a probability of being captured after an encounter and is equal to $1 - P_\infty$. These results are only valid for encounters satisfying the conditions $(U > \sqrt{2} - 1, \gamma > 90^o)$. These are very strong conditions, for example, a particle encountering the Earth never satisfies $\gamma > 90^o$ with $\sigma > \sigma_c$. So, the P_∞ should be weighted with the probability $P(\gamma \geq 90^o)$ which is very low. Weidenschilling (1975) recalculate this issue obtaining more realistic values for the ejection probability.

Finally, it is possible to show that the probability of an encounter of a minor body with the planet with an impact parameter less or equal to σ per orbital revolution of the minor body is given by

$$p(\sigma) = \frac{\sigma^2 U}{\pi \sin i \sqrt{2 - 1/a - a(1 - e^2)}}$$

This is the famous formula given by Öpik (1951), valid for $\sigma < R_H$ where the two body scheme can be applied. An application can be found in www.fisica.edu.uy/∼gallardo/opik/

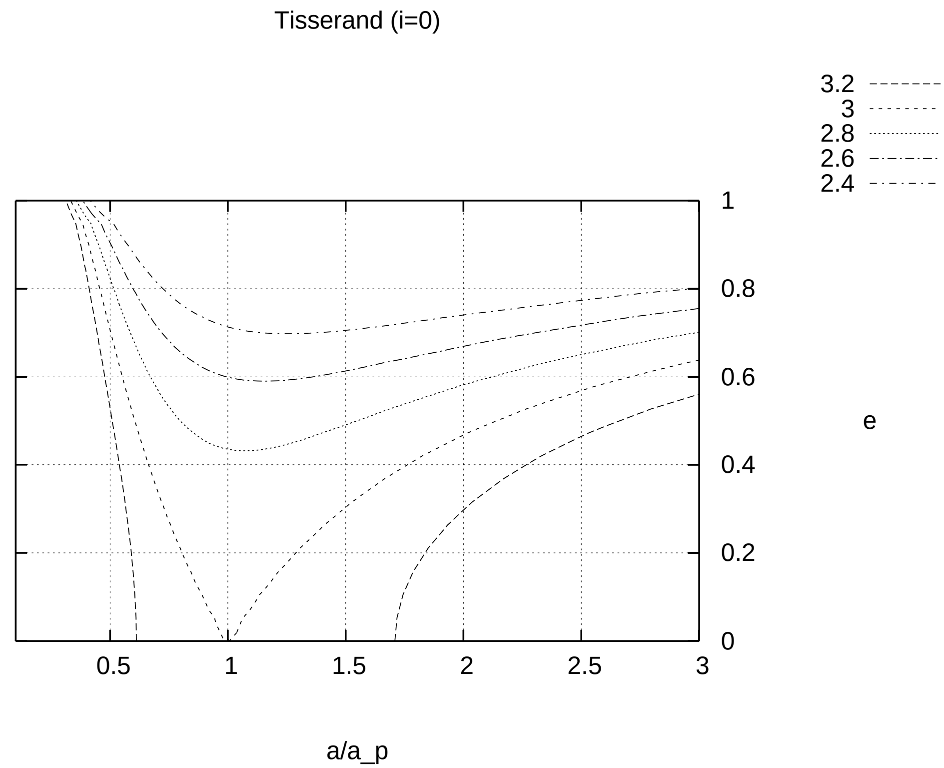

Figure 1. Tisserand parameter $T(a, e, i = 0)$.

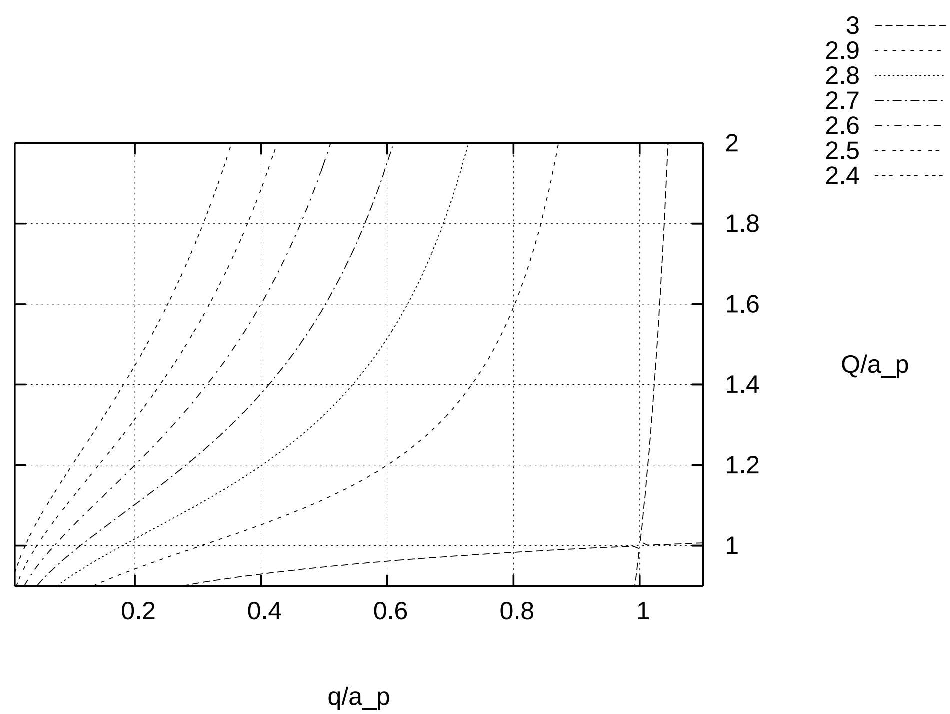

Figure 2. Tisserand parameter $T(q, Q, i = 0)$. The region where encounters are possible.

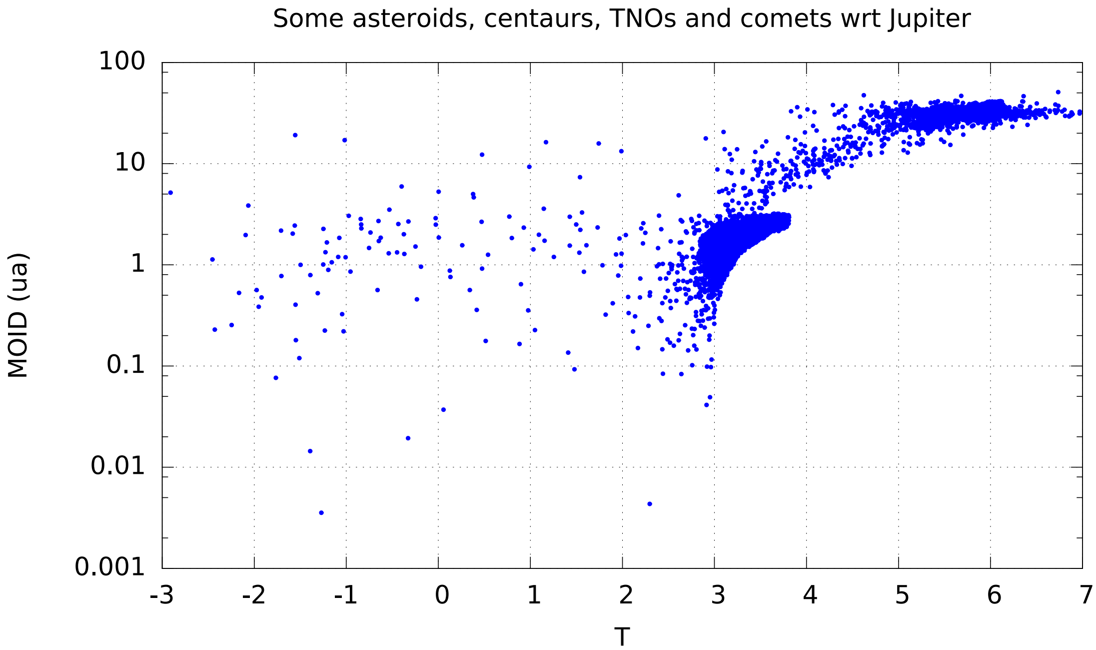

Figure 3. MOID with Jupiter as function of T for some objects. For $T > 3$ there are not close encounters.

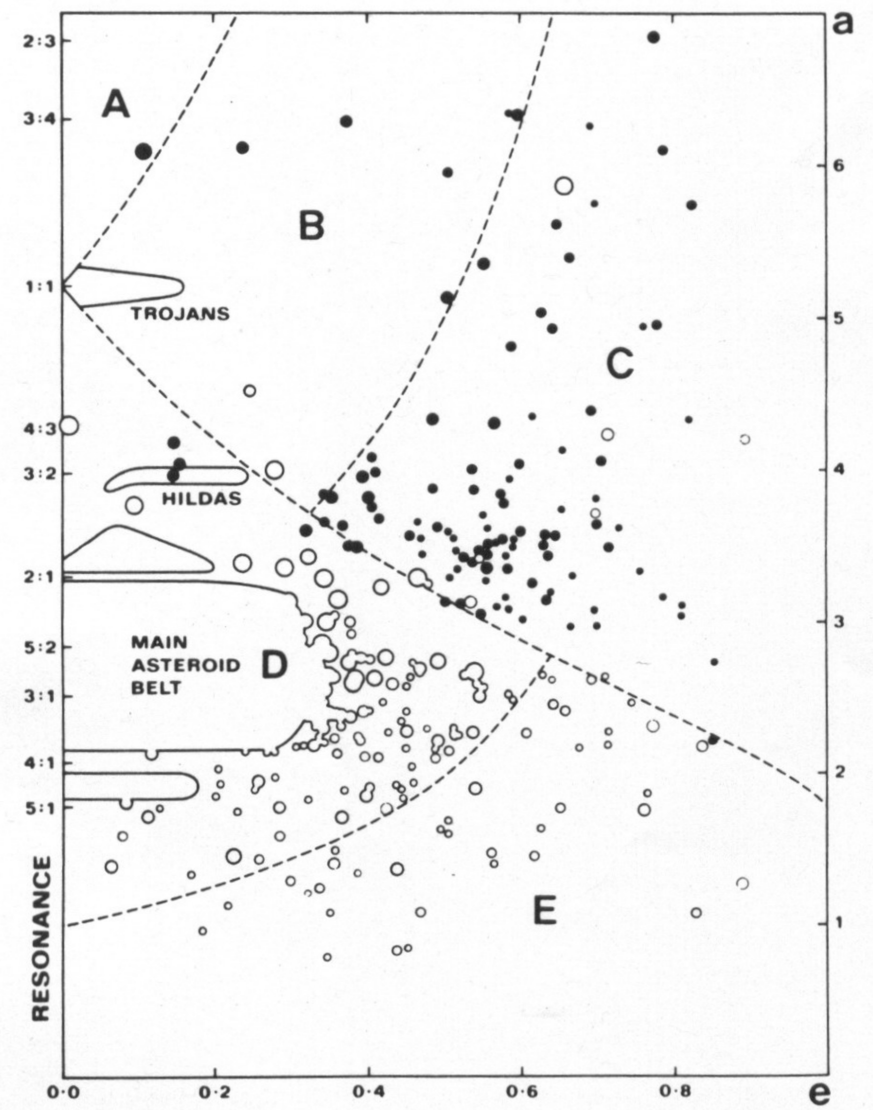

Fig. 1. Short-period comets (solid circles) and asteroids (open circles) plotted on a scatter diagram of semimajor axis vs eccentricity (Kresák 1985). Increasing circle size indicates estimated size of the objects: diameter < 1 km or lost, 1 to 3 km, 3 to 10 km, 10 to 30 km and > 30 km. Different regions identified within the diagram are: (A) transjovian region, (B) Jupiter domain of weak cometary activity, (C) Jupiter domain of strong cometary activity, (D) minor planets region, and (E) Apollo-Aten region. The dashed line going from upper left to lower right corresponds to a Tisserand invariant of 3.0, the usual dividing line between comets and asteroids. However, note the several asteroids above the line in the cometary region C; the figure has been modified to include seven new asteroids in or near region C discovered since Kresák's (1985) work was published.

Figure 4. Kresak's diagram. Regions B and C corresponds to $T < 3$ and regions A, D and E to $T > 3$. Asteroids II.

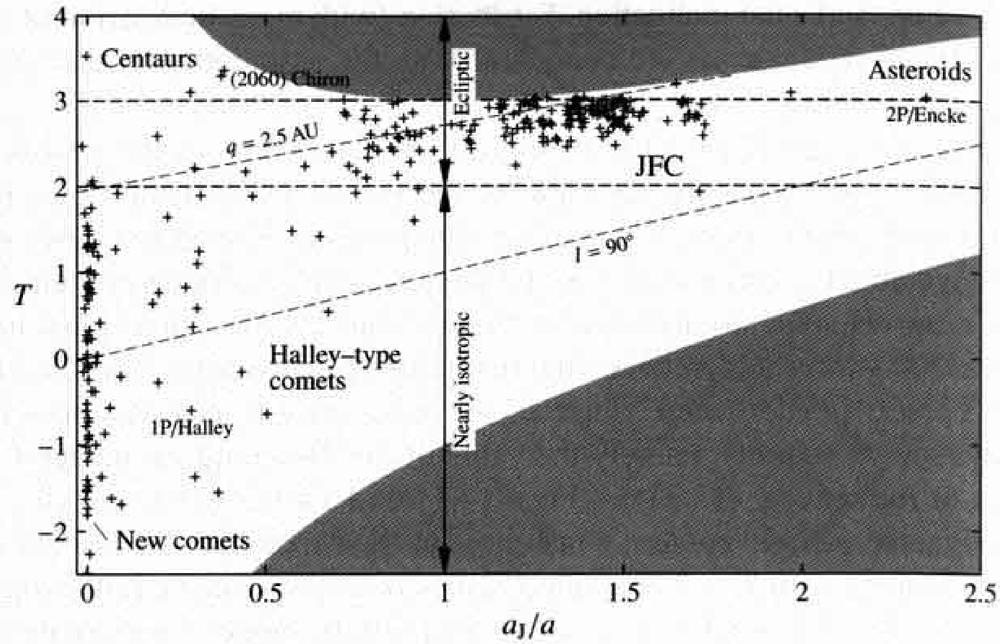

Figure 14.12. Position of known comets in the plane of the Tisserand parameter T and the ratio a_J/a between the semimajor axes of Jupiter and the comet; the unphysical regions are in dark grey. Dashed lines indicate the values $T = 2$ and 3 (see Fig. 14.11); the thin dashed line labelled $q = 2.5$ shows T for a 2.5 AU perihelion comet in the ecliptic (objects above and to the left of this curve are very difficult to detect because they never get close to the Sun). Adapted from H. Levison, Comet taxonomy, in: *Completing the Inventory of the Solar System*, eds. T.W. Rettig and J.M. Hahn (Astronomical Society of the Pacific), p. 173 (1996).

Figure 5. Populations of minor bodies in space $(1/a, T)$. Bertotti et al. 2003.