Scattering

In the previous lecture, we spent some time discussing the behaviour of two particles that interacted through an attractive two-body central potential. We discussed the nature of the bounded and unbounded orbits which might appear, and said something quantitative in the case of the Kepler problem. However, we can of course also consider the case of a **repulsive** central potential. In this case, it becomes obvious that the two particles will not orbit each other - they will at most approach each other, before the repulsive potential causes them to move away from each other, and never meet again. This type of behaviour is typically referred to as **scattering**, an example of which is illustrated in Figure 1. This type of scenario is incredibly important in a wide range of physics, especially in condensed matter systems (where neutrons being scattered off of a material reveal information about the microscopic details of the material) and in high energy physics (where scattering elementary particles against each other can reveal information about the existence of new fundamental particles). For this reason, we want to understand how to describe the basic physics of such a system.

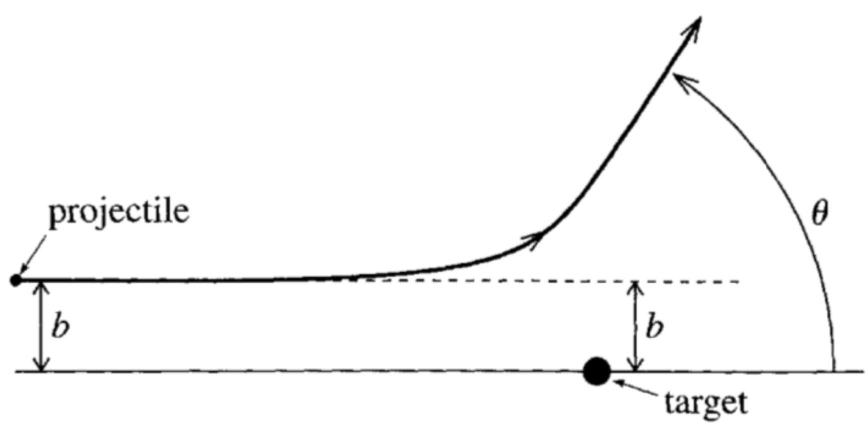

Figure 1: Taylor Figure 14.1: A projectile, with impact parameter b, being scattered by a stationary target. After being scattered, the projectile travels off at some angle θ, never to return to the target.

In what follows, we will mostly be concerned with scattering a **projectile** off of a stationary **target**. Recall that in our two-body problem, when one of the particles is much more massive than the other, we can treat it as essentially being motionless, with its location being the location of the center of mass. In this situation, the larger body is known as the target, while the smaller body is known as the projectile. Many experiments in physics involve firing a small projectile at a very large target, whose motion is essentially unaffected by the collision between the two particles. So we will assume that this is true, although all of what we are about to discuss can be generalized to the case in which both bodies are affected by the collision (which will be the focus of an extra credit problem on the homework).

We will also assume, as before, that the two particles interact through a central potential, which in the case of a stationary target, essentially acts as an external central potential on the smaller body. For this reason, we know that we can still make use of linear momentum, angular momentum, and energy conservation in solving our problem, which means that all of the usual results from the study of the two-body problem should still be applicable. In the case that a collision between two particles conserves total energy, we typically refer to the scattering as being **elastic**.

Before we begin drawing physical conclusions about the motion of our particles, there is some basic terminology we should outline. First, an incoming projectile is typically described in terms of its **impact parameter**. This is the distance that **would** be the radius of closest approach, if there were no interaction between the two particles. This is indicated in Figure 1. Notice that the impact parameter is **not** the actual distance of closest approach - in the presence of the interaction, the particle will begin to scatter away from the target before it has a chance to reach this distance. We will always assume that the projectile starts its motion from a distance very far away from the target, where the forces acting on it are essentially zero, and its motion is described by a straight line.

Second, an outgoing projectile is typically described in terms of its **scattering angle**, which is the angular distance between its incoming and outgoing velocities. Again, this is illustrated in Figure 1. The scattering angle is taken to range anywhere between zero and π radians, with zero radians corresponding to absolutely no scattering, and π radians corresponding to complete back-scattering. A large portion of the problem of scattering is to relate, for a given interaction potential, the scattering angle to the impact parameter. Notice that for a single scattering event, because the two particles interact through a central potential, we can describe the motion using a two-dimensional picture, since the conservation of angular momentum effectively reduces the dimensionality by one.

Cross Sections

At this point, it would be totally feasible to start listing specific examples of interaction potentials, and then solve for the motion of the projectile in these cases, based on the previous results we derived for the two-body problem. However, in most scattering experiments, there are a variety of experimental limitations which in fact make it almost impossible to know, for a given scattering event, exactly what the initial impact parameter was. For this reason, most scattering experiments involve firing a large number of particles at a collection of targets, measuring the outgoing angles, and then comparing the results against a set of statistical predictions. For this reason, there is some additional terminology we need to develop beyond what we have been discussing for far in the simple two-body problem, terminology related to the concept of a **cross section**.

To understand the basic concept of a cross section, we will start by considering a very simple model of scattering, where the potential experienced by the projectile is given by

$$U(r) = \begin{cases} 0\,, & r > R \\ \infty\,, & r < R \end{cases} \tag{1}$$

This type of scattering potential is known as **hard sphere scattering**, and is illustrated in Figure 2. It describes a light projectile bouncing off of a solid sphere of radius R. While the solid sphere is an extended object, we will assume that the projectile is a point-like mass. In this case, it is clear that if the incoming projectile has an impact parameter larger than R, it will not be deflected at all, while if the impact parameter is less than R, it may suffer a relatively extreme deflection.

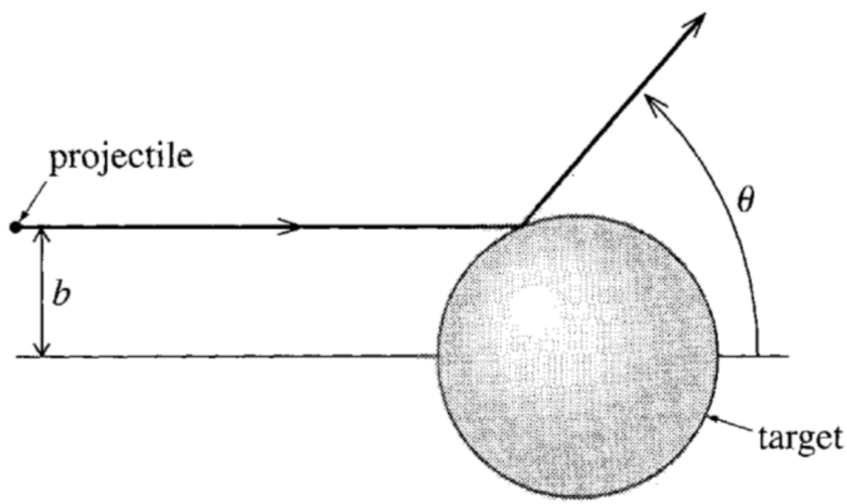

Figure 2: Taylor Figure 14.2: An illustration of hard sphere scattering.

In a moment, we'll make some more quantitative statements about the motion of the projectile, but for now, let's notice something about the overall behaviour of the scattering event in question. Notice that because the solid sphere has an overall cross-sectional area of

$$\sigma = \pi R^2, \tag{2}$$

there is a region of space, with cross-sectional area σ, that projectiles cannot pass through. If the incoming path of a projectile passes through this cross-sectional area, it will be deflected off at some angle. For this reason, we define σ to be the **scattering cross section** for this potential. The scattering cross section gives us an intuitive sense of how much area is blocked out by the target during a scattering event.

To see how this notion is useful to us, let's imagine that instead of one target, we have devised an experiment which in fact contains many targets, as shown in Figure 3. For simplicity, we can assume that the targets are arranged in a sort of thin, two-dimensional sheet, spaced far enough apart from each other that one scattering event will typically involve only one target (we say that the targets are sufficiently **dilute**). This physical picture might represent, for example, a simple toy model for a sheet of Gold atoms (which is a very important example for historical reasons, as we'll mention later). Figure 3 then shows a "head-on view" as seen by an incoming projectile. If we describe the density of targets by the quantity $n_{\rm tar}$, then the total number of targets in the sheet is given by $An_{\rm tar}$, where A is the total area of the target assembly.

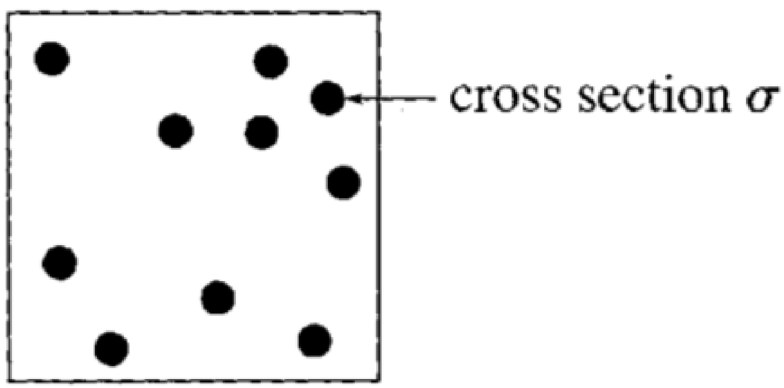

Figure 3: Taylor Figure 14.4: A sheet containing many hard sphere targets, each with a scattering cross section σ.

If we now imagine that our target assembly contains a large number of targets, while still remaining dilute, and that we are interested in describing a large number of scattering events, we can apply statistical considerations when discussing the number of deflections that occur. For any given projectile that is fired at the target assembly, the probability that it will hit one of the spheres and scatter is given by the total cross-sectional area of all of the targets, divided by the total area of the assembly, so that

$$P = \frac{An_{\rm tar}\sigma}{A} = n_{\rm tar}\sigma. \tag{3}$$

If we now imagine that the number of incident projectiles we fire at the sheet is $N_{\rm inc}$, then statistically speaking, the number of scattered particles, $N_{\rm sc}$, should be given by

$$N_{\rm sc} = PN_{\rm inc} = N_{\rm inc}n_{\rm tar}\sigma. \tag{4}$$

Differential Cross Sections

The preceding equation is one of the defining relations of scattering theory. While in most scattering experiments it is essentially impossible to measure the impact parameter of a given projectile, it is possible to measure whether or not a given projectile was scattered. The above formula then allows us to experimentally determine the scattering cross section, by knowing the density of targets, the number of incident particles, and the number of scattered particles.

Of course this formula would have limited utility if it could only describe the scattering of hard spheres. In order to make use of this idea for a realistic system, we need to come up with a suitable generalization of the concept of cross section for situations which involve more general interactions. To do this, we first need to introduce the concept of a **differential cross section**.

To motivate the idea of a differential cross section, let's return to our single hard sphere target, and consider the motion of a single projectile which strikes it with impact parameter b. This is indicated in Figure 4. In this simple model, it is relatively straight-forward to determine the scattering angle as a function of the impact parameter. The trick is to make use of the fact that the angle of incidence on the surface of the sphere must be the same as the angle of reflection, with respect to the surface tangent of the sphere. This certainly seems "obvious" for scattering off of a hard surface, although we could in fact prove this claim if we wanted to, on the basis of angular momentum and energy conservation. If we denote this angle as α (as is done in the figure), then some simple geometric reasoning and some trigonometry lead us to the conclusion that

$$b = R \sin \alpha. \tag{5}$$

This is due to the fact that various theorems in geometry about parallel lines tell us that all of the angles marked as α must in fact be the same. Further study of Figure 4 also indicates that

$$\pi = \theta + 2\alpha, \tag{6}$$

since these angles add up to a full 180 degrees. Combining these two results, we have

$$b = R \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right) = R \cos \frac{\theta}{2} \;\Rightarrow\; \theta = 2 \arccos \left(\frac{b}{R} \right). \tag{7}$$

Notice that this expression becomes undefined when $b > R$, which makes physical sense - there is no scattering in this case.

With our result for the scattering angle in hand, we now in principle have the answer to this dynamical problem. However, in the interest of reformulating our problem into a statistical one, we now want to ask a slightly related, yet different question: If I am interested in particles that scatter only within an infinitesimal range of angles, say $d\theta$, what range of impact parameters, db, is necessary in order to achieve such a scattering? The result we have just derived tells us that if we want our projectile to scatter into an angle $\theta + d\theta$, it must

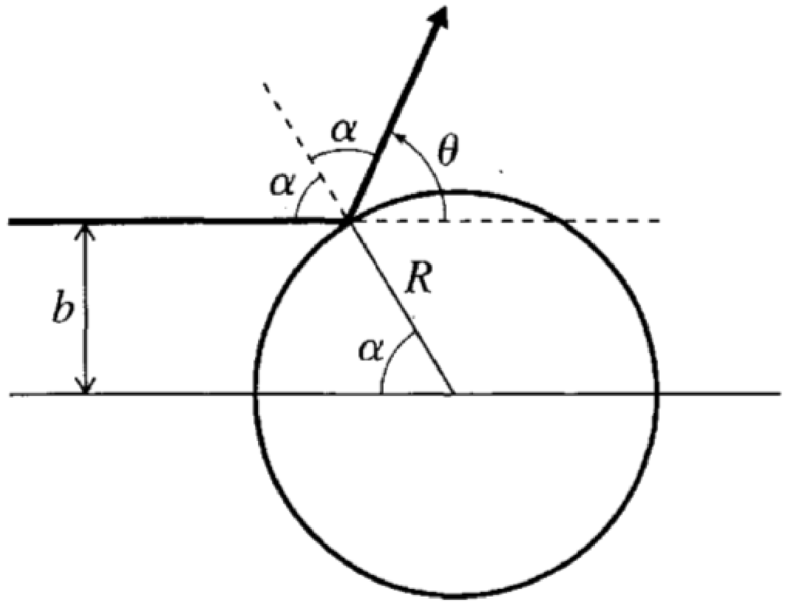

Figure 4: Taylor Figure 14.10: The motion of a projectile which strikes a hard sphere with impact parameter b.

have an impact parameter

$$b+db = R\cos\left(\frac{\theta}{2}+\frac{d\theta}{2}\right) \approx R\cos\left(\frac{\theta}{2}\right)-\frac{R}{2}\sin\left(\frac{\theta}{2}\right)d\theta = b-\frac{R}{2}\sin\left(\frac{\theta}{2}\right)d\theta, \tag{8}$$

where the approximate equality comes from using a Taylor series approximation. Thus, we find that in order to increase the scattering angle by an amount $d\theta$, we must increase the impact parameter by

$$db = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)d\theta. \tag{9}$$

In some sense, we can think of this quantity db as the amount of "infinitesimal cross section" which determines the area (or rather length, in one dimension) within which a particle's path would need to pass in order for it to scatter into a range of angles $d\theta$. What happens when we integrate this quantity over all possible outgoing angles? Before performing this integration, we need to notice that if we admit the possibility of multiple projectiles scattering off of one target, we must now consider a fully three-dimensional problem. This is because while the two-body problem can always be reduced to a two-dimensional one through the use of angular momentum conservation, once we have admitted the possibility of multiple projectiles coming in along multiple paths, we must consider all incoming angles around the central scattering axis. This idea is illustrated in Figure 5. In this case, the infinitesimal amount of scattering area is not db, but rather

$$d\sigma = 2\pi b db. \tag{10}$$

The extra factor of $2\pi b$ comes from the circumference around the central scattering axis. Integrating this expression over all angles, we find

$$\int d\sigma = -\pi R^2 \int_0^\pi \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta = -\pi R^2 \tag{11}$$

Aside from a minus sign, this is simply the total cross section for scattering by the hard sphere.

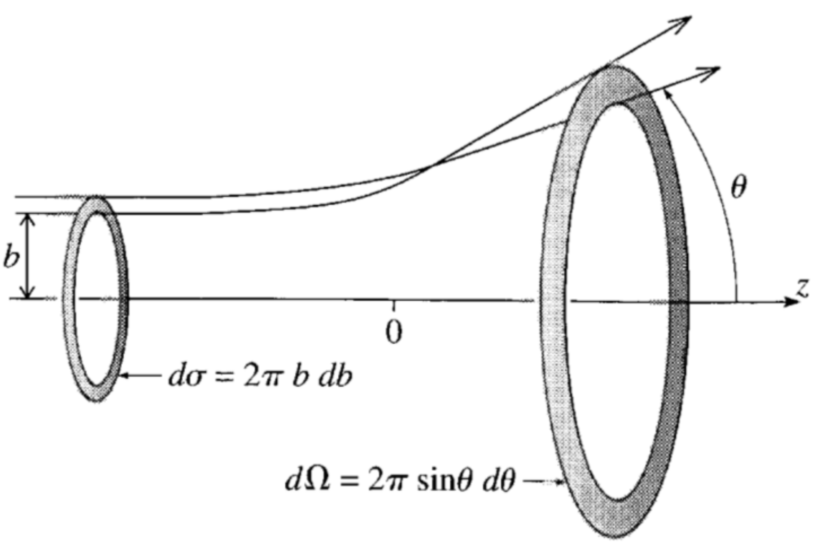

Figure 5: Taylor Figure 14.9: The scattering of multiple projectiles off of a hard sphere, all with an impact parameter which lies between b and $b + db$.

This result gives us an idea as to how to generalize the notion of cross section. For a given scattering problem, we first compute what impact parameter b is needed to scatter into a given angular range. Then, we find the effective amount of infinitesimal cross section which is needed to scatter into a small range of angles, and finally integrate this expression over all possible outgoing scattering angles. While we've demonstrated this process for the example of hard sphere scattering, this sequence of steps certainly still makes sense for a more general scattering problem.

Notice that in the case of the hard sphere potential, we found that the resulting projectiles experienced **isotropic** scattering. That is to say, the result for b as a function of the scattering angle θ did **not** depend on the angle around the central scattering axis. If we refer to this angle around the axis as ϕ, then we can say that the infinitesimal amount of scattering cross section $d\sigma$ required to scatter into some direction is independent of ϕ.

However, in a more general context, this may not be the case - our result for the required impact parameter as a function of b may depend on both angles θ and ϕ. For this reason, the subject of scattering is typically formulated in a different language, one which requires us to extend our notion of an infinitesimal range of angles to the more general concept of **solid angle**, which is illustrated in Figure 6. In two dimensions, the angular difference between two points on a circle can be defined as

$$\Delta\theta = s/r, \tag{12}$$

where s is the arc length between the two points, and r is the radius of the circle. In three dimensions, we can similarly define the amount of solid angle corresponding to a patch of area on the surface of a sphere. Analogously to the two-dimensional case, the amount of solid angle is defined as

$$\Delta\Omega = A/r^2, \tag{13}$$

The unit of solid angle is the **steradian**, as opposed to the radian which describes regular angles. Notice that since the surface area of a sphere is $4\pi r^2$, the solid angle corresponding to all possible directions in three dimensional space is given by 4π steradians.

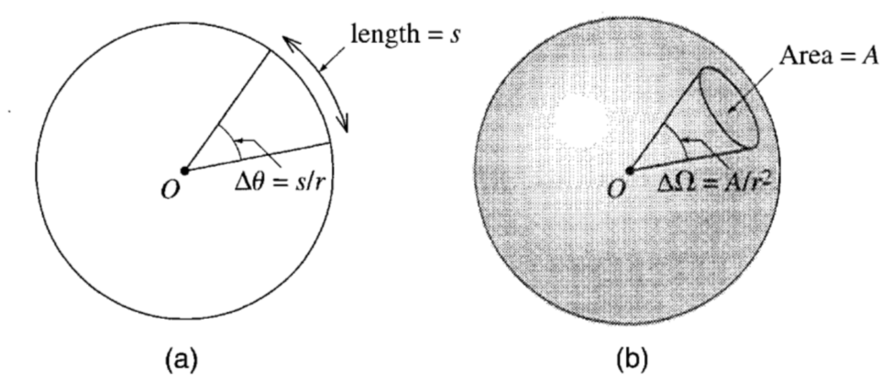

Figure 6: Taylor Figure 14.6: a.) The definition of an angle in two dimensions. b.) The definition of solid angle in three dimensions.

In particular, we will often be interested in knowing the amount of infinitesimal solid angle surrounding a set of angles θ and ϕ which describe a spherical coordinate system. From discussion section last week, we know that in a spherical coordinate system, the volume element is given by

$$dV = r^2 \sin\theta \; dr \; d\theta \; d\phi, \tag{14}$$

so that the infinitesimal amount of area on the surface of a sphere with radius r is given by

$$dA = dV/dr = r^2 \sin\theta \; d\theta \; d\phi. \tag{15}$$

Thus, the infinitesimal amount of solid angle surrounding a given direction in spherical coordinates is given by

$$d\Omega = dA/r^2 = \sin\theta \; d\theta \; d\phi. \tag{16}$$

Having introduced the idea of solid angle, we are now ready to define differential cross section, which is illustrated in Figure 7. We again imagine that we have fired a sequence of projectiles at a large sheet full of many targets, with some density n_{tar}. Instead of asking about the total cross section for scattering, we can ask about the infinitesimal amount of cross section $d\sigma$ that is required for a particle to scatter off at some angle into a small amount of solid angle $d\Omega$. By the same reasoning as before, if we think in statistical terms, then the total number of particles scattered into this small region of solid angle should be

$$N_{\text{sc}}\left(d\Omega\right) = N_{\text{inc}} n_{\text{tar}} d\sigma \left(d\Omega\right). \tag{17}$$

The total cross section can then be found by integrating

$$\sigma = \int d\sigma = \int \frac{d\sigma}{d\Omega} d\Omega. \tag{18}$$

The quantity appearing in the integral,

$$D\left(\theta\right) = \frac{d\sigma}{d\Omega}, \tag{19}$$

is known as the **differential scattering cross section**.

At this point, I should pause to point out that in my personal opinion, one of the largest sources of confusion when learning about scattering is a result of the terrible choice of terminology used in this field. To most reasonable people, the quantity $d\sigma$ should naturally be thought of as the “differential” amount of cross section, because it is indeed a small (differential) amount of cross section. On the other hand, the quantity D is neither a differential, nor a cross section, so it is not clear where this term came from (it is perhaps better thought of as the derivative of the differential cross section). Either way, despite my complaints, it appears that this is the terminology we are stuck with (although it is worth pointing out that many textbook authors, including David Griffiths, seem to agree with me on this point).

Terminology aside, while most scattering experiments cannot determine the initial impact parameter of a given projectile, they can detect the number of particles being scattered into a certain region of solid angle with great accuracy. Given the scattering relation we have derived above, this means that with a knowledge of the number of incident particles, the density of targets in our material, and the number of particles scattered off at some angle, we can learn something about the differential cross section. In many areas of physics, a given

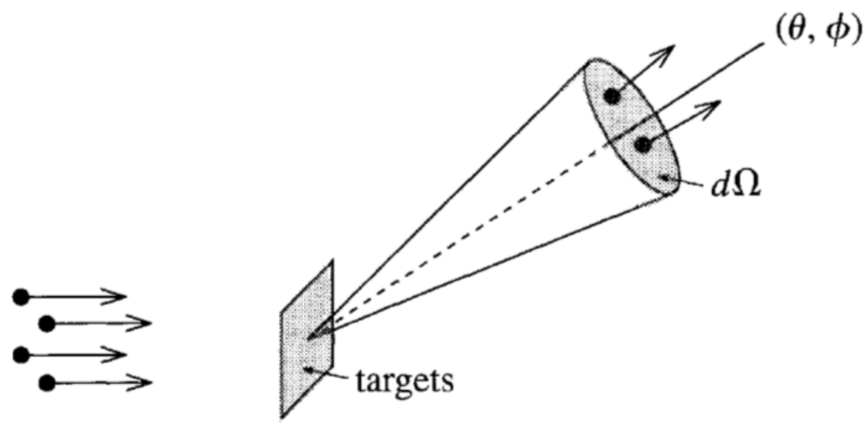

Figure 7: Taylor Figure 14.7: The notion of a differential cross section.

model describing the interactions between particles will result in a theoretical prediction for the differential cross section in a given experiment. This prediction can then be tested against experiment, in order verify (or disprove) the given model.

In order to find the differential cross section in practice, we return to Figure 5. For simplicity, we will again assume that our scattering is isotropic. In this case, assuming that we have found the function $b(\theta)$, the amount of infinitesimal scattering cross section will again be given by

$$d\sigma = 2\pi b db. \tag{20}$$

In this case, the total amount of solid angle, integrated over all values of ϕ, is given by

$$d\Omega = 2\pi \sin\theta d\theta. \tag{21}$$

Dividing these two expressions, we have

$$D(\theta) = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|, \tag{22}$$

where we have added the absolute value signs to ensure that the end result is positive. Notice that for the case of hard sphere scattering, this gives

$$D(\theta) = \frac{R^2}{4}, \tag{23}$$

which is completely independent of angle (as we would expect for a sphere). Integrated over all solid angles, we find

$$\sigma = \frac{R^2}{4}\int d\Omega = \frac{R^2}{4}\int \sin\theta d\theta d\phi = \pi R^2, \tag{24}$$

exactly as we should.

Coulomb Scattering

Now that we've outlined the language of scattering theory, we want to understand how to actually compute a differential cross section for a realistic model. Perhaps the most important model in this context is the Coulomb potential between two electrically charged particles,

$$U(r) = \frac{kqQ}{r}\ ;\ k = 8.988 \times 10^9 \text{ N m}^2 \text{ C}^{-2}. \tag{25}$$

By understanding how positively charged alpha particles would scatter off of the similarly positively charged nuclei of gold atoms, Ernest Rutherford was able to deduce the structure of the atom - a very dense, positively charged nuclei, surrounded by very light electrons.

To determine the differential cross section for this model, we need to determine what angle a given projectile will scatter at, as a function of its impact parameter. Again, since our potential is a two-body central potential, we can make use of all of the conclusions we derived when studying the two-body problem. In particular, the infinitesimal change in angle which occurs during an infinitesimal change in radius is given by

$$d\theta = \pm \frac{l}{\sqrt{2m_*}} \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}}. \tag{26}$$

In this case, the reduced mass of the system is simply m, the mass of the projectile. As the particle approaches from the left and its radius decreases, its angle will also be decreasing, from π down to some smaller angle. Thus, as the projectile advances towards its point of closest approach, we must choose the positive sign in the above expression. Since the particle initially approaches from infinitely far to the left, its initial angle is π, and its initial radius is infinity. If we denote its radius of closest approach as r_*, and corresponding angle at that point by θ_*, we can integrate both sides, to get

$$\int_\pi^{\theta_*} d\theta = \frac{l}{\sqrt{2m}} \int_\infty^{r_*} \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}}, \tag{27}$$

or

$$\theta_* = \pi + \frac{l}{\sqrt{2m}} \int_\infty^{r_*} \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}}. \tag{28}$$

If we swap the order of integration, we get a minus sign, and can write

$$\theta_* = \pi - \frac{l}{\sqrt{2m}} \int_{r_*}^{\infty} \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}}. \tag{29}$$

When the particle is outgoing, it will have an increasing radius, while its angle is still decreasing (remember that conservation of angular momentum shows that the sign of $\dot{\theta}$ is always the same, no matter how hard the particle might scatter back). Thus, we now have

$$d\theta = -\frac{l}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E-U_{\text{eff}}(r)}}. \tag{30}$$

The particle will begin from its closest approach angle and radius, and go off to infinite radius, with some final angle. We can thus integrate both sides, to get

$$\int_{\theta_*}^{\theta_f} d\theta = -\frac{l}{\sqrt{2m}}\int_{r_*}^{\infty}\frac{dr}{r^2\sqrt{E-U_{\text{eff}}(r)}}, \tag{31}$$

or,

$$\theta_f = \theta_* - \frac{l}{\sqrt{2m}}\int_{r_*}^{\infty}\frac{dr}{r^2\sqrt{E-U_{\text{eff}}(r)}}. \tag{32}$$

If we use the value we previously found for θ_*, then we can write

$$\theta_f = \pi - l\sqrt{\frac{2}{m}}\int_{r_*}^{\infty}\frac{dr}{r^2\sqrt{E-U_{\text{eff}}(r)}}. \tag{33}$$

We now have an expression for the final scattering angle in terms of an integral expression. As you will show on the homework, for an incoming projectile with impact parameter b and initial speed v_0, its angular momentum is given by

$$l = mbv_0. \tag{34}$$

Using this expression for the angular momentum, and using the explicit form of the Coulomb potential in the integral, a relatively straight-forward (though slightly tedious) calculation reveals that

$$\theta_f = 2\arcsin\left[\left\{1+\left(\frac{mbv_0^2}{kqQ}\right)^2\right\}^{-1/2}\right]. \tag{35}$$

If we invert this expression to find b in terms of the outgoing angle, we find

$$b(\theta) = \frac{kqQ}{mv_0^2}\sqrt{\frac{1}{\sin^2(\theta/2)}-1}. \tag{36}$$

Using this in our expression for the differential cross section, we find, after a bit of algebra,

$$\frac{d\sigma}{d\Omega} = \left(\frac{kqQ}{2mv_0^2\sin^2(\theta/2)}\right)^2. \tag{37}$$

This famous result is the **Rutherford scattering formula**.

While in this case we were able to find an exact expression for the differential cross section, this is typically not the case in most physics problems. An enormous amount of condensed matter and high energy physics is dedicated to developing perturbation theory methods for computing cross sections. In many cases, a combination of perturbative and numerical techniques are needed to make any sort of progress. Some of these calculations can become quite involved, and some physicists have essentially devoted their entire careers to computing scattering cross sections. However, the amount of work involved is worth it - scattering cross sections are one of the most important ways that physicists learn things about the microscopic world around us.