$F(x) = -kx$

1D $\exists\, U /$

$$U(x) \equiv -\int_{x_0}^{x} F(x')\,dx' + \underbrace{cte}_{U(x_0)}$$

$$U(x) = \frac{kx^2}{2} - \frac{kx_0^2}{2} + U(x_0)$$

En 3D no hay garantías que para $\vec{F}(\vec{r})$ genérico $\exists\, U / \vec{F}(\vec{r}) = -\nabla U$

pero, si $\exists\, U / \vec{F}(\vec{r}) = -\nabla U$

$$\dot{\vec{r}}\cdot m\ddot{\vec{r}} = \vec{F}(\vec{r})\cdot\dot{\vec{r}} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = -\left.\frac{\partial U}{\partial x}\right)_{y,z}\frac{dx}{dt} - \left.\frac{\partial U}{\partial z}\right)_{x,y}\frac{dz}{dt} - \left.\frac{\partial U}{\partial y}\right)_{x,z}\frac{dy}{dt} = -\frac{d}{dt}U(x(t),y(t),z(t))$$

$$\| \quad m\frac{d}{dt}\left(\frac{\dot{\vec{r}}^2}{2}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{m\dot{\vec{r}}^2}{2} + U(\vec{r})\right) = 0$$

$$\frac{d}{dt}(\dot{\vec{r}}^2) = \frac{d}{dt}(\dot{\vec{r}}\cdot\dot{\vec{r}}) = \frac{d\dot{\vec{r}}}{dt}\cdot\dot{\vec{r}} + \dot{\vec{r}}\cdot\frac{d\dot{\vec{r}}}{dt} = 2\dot{\vec{r}}\cdot\ddot{\vec{r}}$$

1D $\rightarrow \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + U(x)\right) = 0$

$$\frac{m\dot{x}^2}{2} + U(x) = E_0 = cte \text{ en el tiempo}$$

$$\frac{2}{m}\left[E_0 - U(x)\right] = \dot{x}^2 \rightarrow \int dt' \sqrt{\frac{2}{m}}\sqrt{E_0 - U(x)} = \int dt' \frac{dx}{dt'} = \int \dot{x}\, dt' = x(t) - x(t_0)$$

No conviene (no me lleva a nada)

$$\int_{t_0}^{t} dt' \sqrt{\frac{2}{m}} = \int_{t_0}^{t} \frac{\frac{dx}{dt'}\, dt'}{\sqrt{E_0 - U(x)}} = \int_{x(t_0)}^{x(t)} \frac{dx'}{\sqrt{E_0 - U(x')}}$$

$$x(t)$$

$$dx = \frac{dx}{dt}\, dt$$

La clase pasada discutíamos el caso $F = -kx$

ó $U(x) = \frac{kx^2}{2}$

$$\sqrt{\frac{2}{m}}(t - t_0) = \int_{x_0}^{x} \frac{dx'}{\sqrt{E_0 - \frac{kx'^2}{2}}} = \int \frac{\sqrt{\frac{2E_0}{k}} \cos\theta\, d\theta}{\sqrt{E_0 - \frac{k}{2}\frac{2E_0}{k}\operatorname{sen}^2\theta}} = \sqrt{\frac{2}{k}} \int_{\theta_0}^{\theta(x)} \frac{\cos\theta\, d\theta}{\underbrace{\sqrt{1 - \operatorname{sen}^2\theta}}_{\cos\theta}}$$

$$\operatorname{arcsen}\left(\sqrt{\frac{k}{2E_0}}\, x\right) = \theta(x)$$

$$\operatorname{arcsen}\left(\sqrt{\frac{k}{2E_0}}\, x_0\right) = \theta_0$$

$$= \sqrt{\frac{2}{k}}(\theta - \theta_0)$$

$$x' = \sqrt{\frac{2E_0}{k}} \operatorname{sen}\theta$$

$$dx' = \sqrt{\frac{2E_0}{k}} \cos\theta\, d\theta$$

$$x_0 = \sqrt{\frac{2E_0}{k}} \operatorname{sen}\theta_0$$

$$\sqrt{\frac{k}{2E_0}}\, x_0 = \operatorname{sen}\theta_0$$

$$\operatorname{arcsen}\left(\sqrt{\frac{k}{2E_0}}\, x_0\right) = \theta_0$$

$$\Rightarrow \sqrt{\frac{k}{m}}(t - t_0) + \theta_0 = \theta(x) = \operatorname{arcsen}\left(\sqrt{\frac{k}{2E_0}}\, x\right)$$

$$x = \underbrace{\sqrt{\frac{2E_0}{k}}}_{A} \operatorname{sen}\left(\underbrace{\sqrt{\frac{k}{m}}}_{\omega}\, t + \underbrace{\theta_0 - \sqrt{\frac{k}{m}}\, t}_{\phi}\right)$$

$$x = A \operatorname{sen}(\omega t + \phi)$$

Fuerza depend. de la velocidad

En gral son fuerzas resist. o viscosas y por ello son en gral en la dirección de la velocidad: $\vec{F}(\vec{v}) = -R(v)\,\frac{\vec{v}}{v}$.

función de $v$ que en gral se det. exp.

$\Rightarrow$ Vamos a enfoc. en el caso este.

$$\left.\begin{aligned} m\vec{a} &= \vec{F}(\vec{v}) = -R(v)\,\hat{v} \\ m\vec{a} &= m\frac{d\vec{v}}{dt} = m\frac{dv}{dt}\,\hat{v} \end{aligned}\right\} \Rightarrow \frac{dv}{dt} = -\frac{R(v)}{m}$$

$$\frac{dv}{dt}\cdot\frac{1}{R(v)} = -\frac{1}{m}$$

$$\Downarrow$$

busco $\tilde{F}(v)$ / $\dfrac{d\tilde{F}}{dv} = \dfrac{1}{R(v)} \longrightarrow \dfrac{dv}{dt}\dfrac{d\tilde{F}}{dv} = -\dfrac{1}{m} \Rightarrow \displaystyle\int_{t_0}^{t} dt'\,\frac{dv}{dt'}\frac{d\tilde{F}}{dv} = -\int_{t_0}^{t}\frac{dt'}{m} = -\frac{(t-t_0)}{m}$

Cambio de variable

$$\parallel$$

$\tilde{F}(v(t))$

$\hookrightarrow d\tilde{F} = \dfrac{d\tilde{F}}{dv}\dfrac{dv}{dt}\,dt$

$$\int_{\tilde{F}(v(t_0))}^{\tilde{F}(v(t))} d\tilde{F} = \tilde{F}(v) - \tilde{F}(v_0)$$

luego sólo me falta invertir $\tilde{F}$ y con ello hallar $v(t)$

una vez que tengo $v(t)$ hallo $x(t) = x(t_0) + \displaystyle\int_{t_0}^{t} v(t')\,dt'$

Ejemplo:

$$\vec{F}_A = -\frac{\rho}{2} v^2 C_D A \, \hat{v}$$

son constante

arrastre

$$m\vec{a} = \vec{F}_A \qquad m\frac{d\vec{v}}{dt} = -bv^2\hat{v}$$

Fuerza que exp. un cuerpo al moverse en un fluido

$\rho \to$ densidad del fluido

$A \to$ Área del objeto perp. a la dirección de mov.

$C_D \to$ Coeficiente de arrastre

Identificamos $R(v)$ $\quad \vec{F}(v) = -R(v)\,\hat{v} \Rightarrow R(v) = \frac{\rho v^2 C_D A}{2} = bv^2$

$$b = \frac{\rho C_D A}{2}$$

$$\tilde{F}(v) \Big/ \frac{d\tilde{F}}{dv} = \frac{1}{R(v)} = \frac{1}{bv^2} \Rightarrow \tilde{F}(v) = -\frac{1}{bv}$$

$$\Rightarrow \tilde{F}(v) - \tilde{F}(v_0) = -\frac{t-t_0}{m}$$

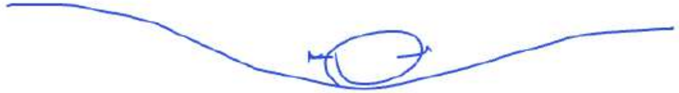
$$\Rightarrow \frac{1}{bv} - \frac{1}{bv_0} = \frac{t-t_0}{m}$$

$$v = \frac{1}{\frac{1}{v_0} + \frac{b(t-t_0)}{m}}$$

$$\Rightarrow \boxed{v(t) = \frac{m v_0}{m + v_0 b (t-t_0)}} \longrightarrow x(t) = x(t_0) + \int_{t_0}^{t} v(t')\,dt'$$

## II . B — Equilibrios y comportamiento en torno a estos

Motivación: En gral encontramos los sistemas en torno a su equilibrio

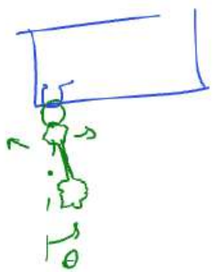


pelota en un pozo

**MÁS COMÚN**

Pelota en un lomo de burro

**RARO**





$$\dot{\theta} = \omega = F_1(\theta,\omega)$$
$$\dot{\omega} = f(\theta,\omega) = F_2(\theta,\omega)$$

Cuando nos enfrentamos a un problema/sistema nuevo lo $1^{ro}$ que tenemos que hacer es modelarlo:

Es decir, describirlo con variables y como se relacionan y cambian estos

En un sist. mecánico

Por ejemplo

$$-m\ddot{x} = f_x(x,\dot{x})$$

$$-m\ddot{y} = f_y(x,y,z,\dot{x},\dot{y},\dot{z})$$

$$\dot{x} = v$$

$$(\ddot{x} =)\ \dot{v} = \frac{f_x(x,v)}{m}$$

En gral.

$$\frac{d}{dt}\eta_1 = F_1(\eta_1, \ldots, \eta_N)$$

$$\vdots$$

$$\frac{d\eta_N}{dt} = F_N(\eta_1, \ldots, \eta_N)$$

donde $\eta_i$ son las variables con los que describo el sistema

Lo $2^{do}$ a hacer (en términos de sencillez) es hallar los equilibrios:

Es decir cuando las cosas no cambian en el tiempo. $f_x = 0$

$F_1 = 0 \qquad F_2 = 0$

$$F_1(\eta_1^*, \dots, \eta_N^*) = 0$$
$$F_2(\eta_1^*, \dots, \eta_N^*) = 0$$
$$\vdots$$
$$F_N(\eta_1^*, \dots, \eta_N^*) = 0$$

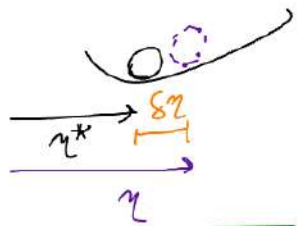
$(\eta_1^*, \dots, \eta_N^*)$ pto fijo o equilibrio

Lo $3^{ro}$ a hacer es hallar el comport. en torno a los equilibrios:

saber como cambian las variables $\delta\eta_1, \dots, \delta\eta_N$ $\boxed{\text{cuando } \delta\eta_i \to 0}$

$$\delta\eta_i = \eta_i - \eta_i^*$$
$$\hookrightarrow \eta_i = \eta_i^* + \delta\eta_i$$
$$\frac{d\eta_i}{dt} = \frac{d(\delta\eta_i)}{dt}$$

$$\frac{d\eta_1}{dt} = \underbrace{F_1(\eta_1^* + \delta\eta_1, \dots, \eta_N^* + \delta\eta_N)}_{} = \frac{d(\delta\eta_1)}{dt}$$
$$\vdots$$





$$\frac{d}{dt}(\delta\eta_1) = F_1(\eta_1^*, \dots, \eta_N^*) + \left.\frac{\partial F_1}{\partial \eta_1}\right|_{\eta_1^*, \dots, \eta_N^*} \delta\eta_1 + \dots + \left.\frac{\partial F_1}{\partial \eta_N}\right|_{\eta_1^*, \dots, \eta_N^*} \delta\eta_N + \boxed{o(\delta\eta_i \delta\eta_j)}$$

## Sist. mecánico de 1 variable

$$\left.\begin{array}{l} m\ddot{x} = F(x,\dot{x}) \\ \dot{x} = v \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x} = F_1(x,v) = v \\ \dot{v} = F_2(x,v) = \dfrac{F(x,\dot{x})}{m} \end{array}$$

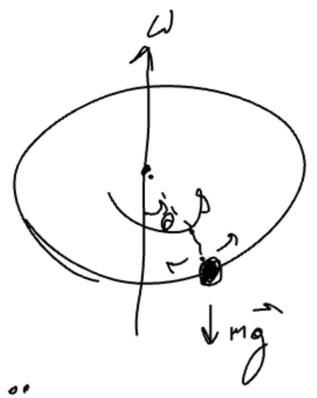
$$F_1(x^*,v^*) = 0 = v^*$$

$$F_2(x^*,v^*) = 0 = \frac{F(x^*,0)}{m}$$

Este es nuestro modelo $\longrightarrow$ Hallamos los equilibrios

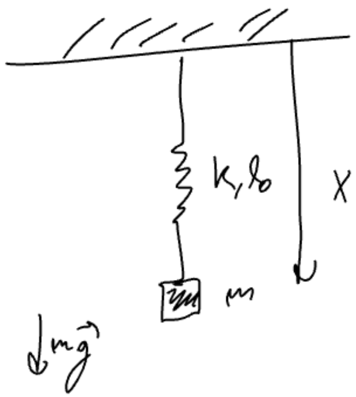
Paso 1 Paso 2

Por ejemplo




$\ddot{\theta} = \ldots$

ó




$$m\ddot{x} = -k(x-\ell_0) + mg$$

$\eta_1 = x$ $\qquad \dot{x} = \dot{\eta}_1 = \eta_2 = F_1(\eta_1,\eta_2)$

$\eta_2 = v$ $\qquad \dot{v} = \dot{\eta}_2 = F_2(\eta_1,\eta_2)$

$$= -\frac{k}{m}(\eta_1 - \ell_0) + g$$

paso 3)

Comportamiento cerca de los equilibrios

$\delta_1 \equiv x - x^*$

$\delta_2 \equiv v - v^*$

$\delta_1, \delta_2 \to 0$

$\dot{x} = \dot{\delta}_1 = F_1(x,v) = F_1(x^*+\delta_1, \underbrace{v^*}_{0}+\delta_2) = \delta_2$

$\dot{v} = \dot{\delta}_2 = F_2(x,v) = F_2(x^*+\delta_1, \underbrace{v^*}_{0}+\delta_2) = \frac{F}{m}(x^*+\delta_1, \delta_2) = \frac{F}{m}(x^*, 0) + \frac{1}{m}\overbrace{\left.\frac{\partial F}{\partial x}\right|_{\substack{x=x^*\\ v=0}}}^{-K}\delta_1 + \frac{1}{m}\overbrace{\left.\frac{\partial F}{\partial v}\right|_{\substack{x=x^*\\ v=0}}}^{-b}\delta_2 + o(\delta_1^2, \delta_2^2, \delta_1\delta_2)$

$$\boxed{\begin{aligned}\dot{\delta}_1 &= \delta_2\\ \dot{\delta}_2 &= -\frac{K}{m}\delta_1 - \frac{b}{m}\delta_2\end{aligned}}$$

cambiamos los nombres

$\dot{y} = V_y$

$\ddot{y} = \dot{V}_y = -\frac{K}{m}y - \frac{b}{m}V_y$

$\Rightarrow m\ddot{y} = -Ky - bV_y$

$b=0$

$\ddot{y} = -\frac{K}{m}y$

$\sqrt{\frac{K}{m}} = \sqrt{\frac{-\left.\frac{\partial F}{\partial x}\right|_{x^*}}{m}}$

$y = A\cos(\omega t + \phi)$