Stationary scattering states : $\psi(\vec{r},t) = \varphi(\vec{r})\, e^{-iEt/\hbar}$, $E > 0$

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{r})\right)\varphi(\vec{r}) = E\varphi(\vec{r}) \ , \quad E = \frac{\hbar^2 k^2}{2\mu} \quad V(\vec{r}) = \frac{\hbar^2}{2\mu}U(\vec{r})$$

$$\left(\nabla^2 + k^2 - U(\vec{r})\right)\varphi(\vec{r}) = 0$$

Asymptotic solution $\varphi_k^d(\vec{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f_k(\theta,\varphi)\frac{e^{ikr}}{r}$, $(\nabla^2 + k^2)\frac{e^{ikr}}{r} = 0$ for $r > r_0$ for r_0 sufficiently large

$\sigma(\theta,\varphi) = |f_k(\theta,\varphi)|^2$, f_k amplitude of scattering

Central potential $V(r)$, $r = |\vec{r}|$, partial waves

CCOC $H_0, L^2, L_z \to \frac{\hbar^2 k^2}{2\mu}, \ell(\ell+1)\hbar^2, m\hbar$, angular eigenstates $Y_{\ell m}(\theta,\varphi)$

Stat. states for free particle ($V = 0$)

$E = \frac{\vec{p}^2}{2\mu}$, $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$ $\quad \langle\vec{r}|\vec{p}\rangle = (2\pi\hbar)^{-3/2} e^{i\vec{p}\cdot\vec{r}/\hbar}$, $\vec{k} = \frac{\vec{p}}{\hbar}$

$|\vec{k}\rangle = \hbar^{3/2}|\vec{p}\rangle \Rightarrow \langle\vec{r}|\vec{k}\rangle = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}}$

CCOC $H_0, L^2, L_z \to \frac{\hbar^2 k^2}{2\mu}, \ell(\ell+1)\hbar^2, m\hbar$, angular eigenstates $Y_{\ell m}(\theta,\varphi)$

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$|\vec{k}\rangle = \hbar^{3/2}|\vec{p}\rangle \Rightarrow \langle\vec{r}|\vec{k}\rangle = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}}$

Stat. states with well defined angular momentum: free spherical waves

$|\varphi^0_{k\ell m}\rangle \to \langle\vec{r}|\varphi^0_{k\ell m}\rangle = \varphi^{(0)}_{k\ell m}(\vec{r}) = \left(\frac{2k^2}{\pi}\right)^{1/2} j_\ell(kr)\, Y_{\ell m}(\theta,\varphi)$

$\hookrightarrow$ Bessel 2^{nd} kind functions or spherical

$j_\ell(x) = (-1)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$, $\langle\varphi^0_{k\ell m}|\varphi^0_{k'\ell' m'}\rangle = \delta(k-k')\,\delta_{\ell\ell'}\,\delta_{mm'}$

$\int_0^\infty dk \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} |\varphi^0_{k\ell m}\rangle\langle\varphi^0_{k\ell m}| = 1$

near origin $j_\ell(x) \underset{x\to 0}{\longrightarrow} \frac{x^\ell}{(2\ell+1)!!}$

far away $j_\ell(x) \underset{x\to\infty}{\longrightarrow} \frac{1}{x}\sin\left(x-\ell\frac{\pi}{2}\right)$

phase diff $\ell\pi$

incoming wave

outgoing wave

$$\Rightarrow \varphi^{0}_{k\ell m}(\vec{r}) \underset{r\to\infty}{\longrightarrow} -\left(\frac{2k^2}{\pi}\right)^{1/2} Y_{\ell m}(\theta,\varphi)\,\frac{e^{-ikr}e^{i\ell\pi/2}-e^{ikr}e^{-i\ell\pi/2}}{2ikr}$$

Plane wave in terms of spherical waves:

$$\langle\vec{r}\,|\,0,0,k\rangle = (2\pi)^{-3/2}e^{ikz}$$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell\sqrt{4\pi(2\ell+1)}\; j_\ell(kr)\, Y_{\ell 0}(\theta)$$

al ℓ modulo! $m=0$ (no φ) $L_z|0,0,k\rangle = |0\rangle$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)}\, j_l(kr)\, Y_{l0}(\theta)$$

al l mezclados! $m=0$ ($\propto \varphi$) $L_z|0,0,k\rangle = |0\rangle$

$$Y_{l0}(\theta) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta) \quad , \quad e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

⋮

Partial waves for $V(r) \neq 0$

$$\varphi_{klm}(\vec{r}) = R_{kl}(r)\, Y_{lm}(\theta,\varphi) \quad , \quad R_{kl} = \frac{u_{kl}}{r} \quad , \quad \left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \underbrace{\frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)}_{V_{eff}}\right) u_{kl}(r) = \frac{\hbar^2 k^2}{2\mu} u_{kl}(r)$$

$u_{kl}(0) = 0$

For $r \to \infty$ $\left(\frac{d^2}{dr^2} + k^2\right) u_{kl}(r) \xrightarrow[r\to\infty]{} 0$, $u_{kl}(r) \xrightarrow[r\to\infty]{} A e^{ikr} + B e^{-ikr}$, $|A| = |B|$

$$u_{kl}(r) \xrightarrow[r\to\infty]{} |A| \left(e^{ikr} e^{i\varphi_A} + e^{-ikr} e^{i\varphi_B}\right) \sim C \operatorname{sen}(kr - \beta_l)$$

$\beta_l \to \delta_l$ $\quad u_{kl}(r) \xrightarrow[r\to\infty]{} C \operatorname{sen}\left(kl - \frac{l\pi}{2} + \delta_l\right)$

δ_l → phase shift

$$\varphi_{k\ell m}(\vec{r}) \xrightarrow[r\to\infty]{} C\ \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{r}\ Y_{\ell m}(\theta,\varphi) = -C\,Y_{\ell m}(\theta,\varphi)\ \frac{\overset{\text{incoming}}{e^{-ikr}}\,e^{i(\ell\pi/2-\delta_\ell)} - \overset{\text{outgoing}}{e^{ikr}}\,e^{-i(\ell\pi/2-\delta_\ell)}}{2ir}$$

In order to compare with the free spherical waves, we choose C in such a way that, ~~apart~~ aside from normalization constant $(2k^2/\pi)$: $\varphi \to \tilde{\varphi}$

choosing C $\left(C = \frac{1}{k}\, e^{i\delta_\ell}\right)$

$$\tilde{\varphi}_{k\ell m}(\vec{r}) \xrightarrow[r\to\infty]{} -Y_{\ell m}(\theta,\varphi)\ \frac{e^{-ikr}\,e^{i\ell\pi/2} - e^{ikr}\,e^{-i\ell\pi/2}\,e^{2i\delta_\ell}}{2ikr}$$

Same incoming wave as in free case, but outgoing wave picks a phase $e^{2i\delta_\ell}$! because $V(r) \neq 0$. Only δ_ℓ for ℓ such that $\ell \sim \sqrt{\ell(\ell+1)} > ka$, a range potential are ~~not~~ close to $\underline{0}$ and similar to free case.

Only δ_ℓ for $\ell = 0, 1, \ldots \ell_{max} = ka$ are different from 0 and should be computed.

$$\varphi_k^{+}(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} c_\ell\ \tilde{\varphi}_{k\ell}(r,\theta) \quad , \; c_\ell = i^\ell \sqrt{4\pi(2\ell+1)}$$

only δ_ℓ for $\ell = 0, 1, \dots \ell_{max} = ka$ are different from 0 and should be computed.

$$\varphi_k^i(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} c_\ell \, \tilde{\varphi}_{k\ell 0}(r,\theta) \quad , \; c_\ell = i^\ell \sqrt{4\pi(2\ell+1)}$$

$k\ell 0$!!

In fact, explicit computation

$$\sum_{\ell=0}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \, \tilde{\varphi}_{k\ell 0}(r,\theta) \xrightarrow[r\to\infty]{} -\sum_{\ell=0}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \, Y_{\ell 0}(\theta) \frac{1}{2ikr}\left(e^{-ikr} e^{i\ell\pi/2} - e^{ikr} e^{-i\ell\pi/2} e^{2i\delta_\ell}\right)$$

$$e^{2i\delta_\ell} = 1 + 2i e^{i\delta_\ell} \sin\delta_\ell \, .$$

$$\Rightarrow \sum_{\ell=0}^{\infty} i^\ell (4\pi(2\ell+1))^{1/2} \tilde{\varphi}_{k\ell 0}(r,\theta) \xrightarrow[r\to\infty]{} -\sum_{\ell=0}^{\infty} i^\ell (4\pi(2\ell+1))^{1/2} Y_{\ell 0}(\theta) \left(\frac{e^{-ikr} e^{i\ell\pi/2} - e^{ikr} e^{-i\ell\pi/2}}{2ikr} - \frac{e^{ikr}}{r} \frac{1}{k} e^{-i\ell\pi/2} e^{i\delta_\ell} \sin\delta_\ell \right)$$

$$\downarrow$$

$$e^{ikz} + f_k(\theta) \frac{e^{ikr}}{r}$$

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\, e^{i\delta_l} \operatorname{sen}\delta_l\, Y_{l0}(\theta)$$

$$\sigma(\theta) = |f_k(\theta)|^2 \quad , \sigma = \int d\Omega\, \sigma(\theta) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\operatorname{sen}^2\delta_l$$

δ_l : resolve radial equation for $V(r)$!! $\delta_l = \delta_l(k)$ depends on energy

s-wave ($l=0$) isotropic $\sigma(\theta)$, other have $l \neq 0$ contributions!

Scatt resonance: for some energy $\delta_l = \pi/2$ and $\sigma_l(\theta)$ is maximal

Example: low energy scatt by hard sphere

$$V(r) = \begin{cases} 0 & r > r_0 \\ \infty & r < r_0 \end{cases}$$

low energy: only $l=0$, $kr_0 \ll 1$

$$f_k(\theta) = \frac{1}{k} e^{i\delta_0(k)} \operatorname{sen}\delta_0(k) \quad , \sigma(\theta) = \frac{\operatorname{sen}^2\delta_0(k)}{k^2} \quad , \sigma = \frac{4\pi}{k^2}\operatorname{sen}^2\delta_0(k)$$

$V(r) = 0 \quad r > r_0$

$\quad\quad\; \infty \quad r < r_0$

low energy: only $l = 0$, $kr_0 \ll 1$

$f_k(\theta) = \frac{1}{k} e^{i\delta_0(k)} \operatorname{sen} \delta_0(k)$, $\sigma(\theta) = \frac{\operatorname{sen}^2 \delta_0(k)}{k^2}$, $\sigma = \frac{4\pi}{k^2} \operatorname{sen}^2 \delta(k)$

radial eq: $\left(\frac{d^2}{dr^2} + k^2\right) u_{k0}(r) = 0 \quad r > r_0$, $u_{k0}(r_0) = 0 \quad r = r_0$

$$\Rightarrow u_{k0}(r) = \begin{cases} C \operatorname{sen}\{k(r - r_0)\} & r > r_0 \\ 0 & r < r_0 \end{cases}$$

$\delta_0(k)$ from asymptotic behaviour $u_{k0}(r) \xrightarrow[r \to \infty]{} \operatorname{sen}(kr + \delta_0) \Rightarrow \delta_0(k) = -kr_0$

$\Rightarrow \sigma = \frac{4\pi}{k^2} \operatorname{sen}^2 kr_0 \underset{kr_0 \ll 1}{\sim} 4\pi r_0^2$!! 4 times geometrical x section!

(kind of diffraction for hard border!)

Obs: for large k one finds $\sigma \sim 2\pi r_0^2$!!

This is due to the fact that the potential always varies appreciably within a distance that is always smaller (in fact, zero!) that the wavelength of the particle ($a < \lambda_{dB}$, a distance potential varies appreciable $\Rightarrow$ wave effects)