Ecuación de Dirac / ecuaciones relativistas

$E \gg mc^2$

$H = \frac{p^2}{2m}$ $\quad$ $p \to \hat{p}$ $\quad$ $H \to i\hbar\frac{\partial}{\partial t}$ $\quad$ $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \frac{p^2}{2m}|\psi\rangle$

$$\boxed{i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle}$$

Relatividad $E = \sqrt{c^2p^2 + m^2c^4} \longrightarrow H$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (c^2p^2 + m^2c^4)^{1/2}|\psi\rangle$$

$p \to \frac{\hbar}{i}\vec{\nabla}$

$$(c^2p^2 + m^2c^4)^{1/2} = mc^2\left(\frac{p^2}{m^2c^2} + 1\right)^{1/2} = mc^2\left(1 + \frac{p^2}{2m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4} + \dots\right)$$

$\vec{\nabla}^2 \quad \vec{\nabla}^4 \quad \dots$

problemas de causalidad, convergencia

Relatividad especial: x, t transf. linealmente, ec. Schr no relat $\left.\begin{array}{l}\frac{p^2}{2m} \to \vec{\nabla}^2 \\ i\hbar\frac{\partial}{\partial t}\end{array}\right\}$

1) Klein Gordon $E = \sqrt{p^2c^2 + m^2c^4}$, $H^2 = p^2c^2 + m^2c^4$

ec. cuántica relativista

$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \longleftrightarrow H^2$

$$-\hbar^2\frac{\partial^2}{\partial t^2}|\psi\rangle = (p^2c^2 + m^2c^4)|\psi\rangle = \left(-\hbar^2c^2\vec{\nabla}^2 + m^2c^4\right)|\psi\rangle$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \left(\frac{mc}{\hbar}\right)^2\right]|\psi\rangle = 0$$

ec. ~~relativista~~ KG

$\lambda_c = \frac{\hbar}{mc}$

una sola componente? Describe $s = 0$ (piones, ...)

2) Dirac $s = 1/2$

$$p^2c^2 + m^2c^4 = (c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2)^2 = (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)^2$$

$$c^2(p_x^2 + p_y^2 + p_z^2) + m^2c^4 = c^2(\alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2) + \beta^2 m^2c^4 + \left(c^2 p_x p_y(\alpha_x\alpha_y + \alpha_y\alpha_x) + (xz) + (yz)\right)$$
$$+ \left(mc^2 p_x(\alpha_x\beta + \beta\alpha_x)\right) + (y\beta) + (z\beta))$$

$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1 \qquad \beta^2 = 1$

$\alpha_x\alpha_y + \alpha_y\alpha_x = 0$, $\{\alpha_x, \alpha_y\} = 0$, $\{\alpha_i, \alpha_j\} = 0 \quad i \neq j$

$\{\alpha_i, \beta\} = 0$

Soluciones a partir de matrices 4x4 !!

$$\boxed{H = c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2}$$

α, β 4x4, hermíticas, valores propios ± 1

traza cero!

$\alpha_x\alpha_y = -\alpha_y\alpha_x \qquad \alpha_x\alpha_y\alpha_x^{-1} = -\alpha_y\alpha_x^2 = -\alpha_y$

$\mathrm{Tr}(\alpha_x\alpha_y\alpha_x^{-1}) = \mathrm{Tr}(-\alpha_y)$

$\mathrm{Tr}(\alpha_y) = -\mathrm{Tr}(\alpha_y)$

$\vec{\alpha}, \beta$ no es única $\quad \vec{\alpha} \longrightarrow S^{+}\vec{\alpha}S$, S unitaria también válido.

$\beta \longrightarrow S^{+}\beta S$

Representaciones $\vec{\alpha}, \beta$: $\quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ $\quad$ 4x4, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\alpha_x = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \sigma_x \\ \sigma_x & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$ 4x4

$\beta^2 = 1$ ✓ ...... verificar

$\alpha_x^2 = 1$

4x4!

Ecuación cuántica relativista: $\boxed{i\hbar\frac{\partial}{\partial t}|\psi\rangle = (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)|\psi\rangle}$

ec. Dirac $\quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$!!

$E < 0$?

$|\psi\rangle$: espinor, 4 pisos

f.o. $\quad i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = \left(c\frac{\hbar}{i}\,\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\underbrace{\psi(x, t)}_{\text{f.o}}$

4 funciones complejas $\quad \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}$ $\begin{array}{l} \} \to E = mc^2,\ p = 0 \\ \} \to E = -mc^2,\ p = 0 \end{array}$

$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{E = mc^2} \quad \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{E = -mc^2}$

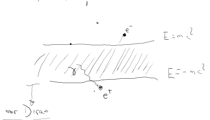
$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$, $\quad \rho = \psi^{+}\psi$, $\quad \vec{J} = c\bar{\psi}\,\vec{\alpha}\,\psi$

$\psi^{+} = (\psi_1^{*}, \psi_2^{*}, \psi_3^{*}, \psi_4^{*})$, $\quad \bar{\psi} = \psi^{+}\gamma^{0}$ $\quad$ matriz relacionada c/ $\vec{\alpha}, \beta$

Ec. relat. Dirac: $\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$ $\begin{array}{l} \to \text{partícula } s = 1/2 \\ \to \text{antipartícula, } s = 1/2 \end{array}$

e^-

$e^+ \longrightarrow$ "$E < 0$"



Interacciones: cEM

$(\vec{A}, \phi)$, $H_{clas} = \left(\left(\vec{p} - \frac{q}{c}\vec{A} \right)^2 c^2 + m^2 c^4 \right)^{1/2} + q\phi$

$\vec{p} \to \vec{p} - \frac{q}{c}\vec{A}$ acoplamiento mínimo

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi = \left(c\vec{\alpha} \cdot \left(\vec{p} - \frac{q}{c}\vec{A} \right) + \beta mc^2 + q\phi \right)\psi}$$

interacción!! $-q\vec{\alpha} \cdot \vec{A}$

$\psi(x,t) = \varphi e^{-iEt/\hbar}$

$$\Rightarrow E\varphi = \left(c\vec{\alpha} \cdot \vec{\pi} + \beta mc^2 + q\phi \right)\varphi \quad , \vec{\pi} = \vec{p} - \frac{q}{c}\vec{A}$$

$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ $\varphi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$, χ, ϕ son biespinores $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

uso expr $\vec{\alpha}$, β: $\begin{pmatrix} E - mc^2 - q\phi & -c\vec{\sigma} \cdot \vec{\pi} \\ -c\vec{\sigma} \cdot \vec{\pi} & E + mc^2 - q\phi \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = 0 \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(E - mc^2 - q\phi)\chi - c\vec{\sigma} \cdot \vec{\pi}\phi = 0$$

$$(E + mc^2 - q\phi)\phi - c\vec{\sigma} \cdot \vec{\pi}\chi = 0 \quad \to \quad \phi = \frac{c\,\vec{\sigma} \cdot \vec{\pi}}{E + mc^2 - q\phi}\chi$$