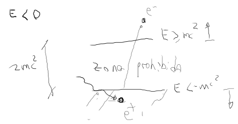
Ecuaciones relativistas

$$i\hbar \frac{\partial \psi}{\partial t} = \left(c\vec{\alpha}\cdot\vec{p} + \beta m c^2 \right)\psi \quad , \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \begin{matrix} \}\ E>0 \\ \\ \}\ E<0 \end{matrix}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$



Interacción campo electromagnético

$$\vec{p} \longrightarrow \vec{p} - \frac{q}{c}\vec{A} \Rightarrow \begin{cases} \text{espectro de H} & \checkmark \\ \text{acopl. espín-órbita} & \checkmark \\ g_e = 2\ ! & \checkmark \end{cases}$$

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi = \left(c\vec{\alpha}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + \beta m c^2 + q\varphi \right)\psi}$$

$$\psi(\vec{r},t) = \psi(\vec{r})\, e^{-iEt/\hbar} \quad , \quad \underbrace{\psi(\vec{r})}_{\text{4-espinor}} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad , \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

bi-espinores

$$i\hbar \begin{pmatrix} \frac{\partial \chi}{\partial t} \\ \frac{\partial \phi}{\partial t} \end{pmatrix} = \left(c \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \cdot \left(\vec{p} - \frac{q}{c}\vec{A}\right) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc^2 + q\varphi \right) \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

$$E\,\psi(\vec{r}) = \left(c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + \beta m c^2 + q\varphi \right)\psi(\vec{r})$$

$$(1) \quad \begin{cases} \left(E - mc^2 - q\varphi\right)\chi = c\,\vec{\sigma}\cdot\vec{\pi}\,\phi \\ \left(E + mc^2 - q\varphi\right)\phi = c\,\vec{\sigma}\cdot\vec{\pi}\,\chi \end{cases} \qquad \vec{\pi} = \vec{p} - \frac{q}{c}\vec{A} \quad , \quad \vec{p} = \frac{\hbar}{i}\vec{\nabla}$$

$$\phi = \frac{c\,\vec{\sigma}\cdot\vec{\pi}}{E + mc^2 - q\varphi}\,\chi \quad \Rightarrow \quad \text{sustituyo}$$

Régimen en que E no es ~~ultrarelativista~~ $\quad E = mc^2 + \overset{\text{energía no relativista}}{E_S} \quad , \quad mc^2 \gg E_S$

$$\phi \sim \frac{c\,\vec{\sigma}\cdot\vec{\pi}}{2mc^2}\,\chi \qquad \pi \sim (v/c)\ , \quad v \ll c \qquad |\phi| \ll |\chi|$$

$$(1) \quad \underset{E - mc^2}{\left(E_S - q\varphi\right)}\chi = \cancel{c^2}\frac{(\vec{\sigma}\cdot\vec{\pi})^2}{2m\cancel{c^2}}\,\chi \quad , \quad (\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\pi}\cdot\vec{\pi} + i\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi})$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \vec{\pi}\cdot\vec{\pi} + i\vec{\sigma}\cdot\left(\left(\vec{p} - \frac{q}{c}\vec{A}\right)\times\left(\vec{p} - \frac{q}{c}\vec{A}\right)\right)$$

$$(\vec{p}\times\vec{p})_x = p_y p_z - p_z p_y = 0 \quad \text{porque conmutan!!}$$

$$= \vec{\pi}\cdot\vec{\pi} + i\vec{\sigma}\cdot\left(-\frac{q}{c}\frac{\hbar}{i}\vec{\nabla}\times\vec{A}\right) = \vec{\pi}\cdot\vec{\pi} - \frac{q\hbar}{c}\vec{\sigma}\cdot\vec{B}$$

$$\boxed{\left(\frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} + q\varphi - \frac{q\hbar}{2cm}\vec{\sigma}\cdot\vec{B} \right)\chi = E_S\,\chi}$$

$$-\frac{q}{2cm}\hbar\vec{\sigma}\cdot\vec{B} = -\frac{q}{2cm}\vec{S}\cdot\vec{B} \quad , \quad \vec{S} = \hbar\vec{\sigma} \qquad \text{Doble!! del acopl. clásico}$$

$$\Rightarrow \boxed{g_e = 2} \quad (g_s = 2)$$

Si seguimos ...

$$H_{S.O} = \frac{e^2}{2m^2c^2}\frac{1}{r^3}\vec{S}\cdot\vec{L}$$

$$E_{nj} = mc^2\left(1 + \left(\frac{\alpha}{n - (j+1/2) + \left((j+1/2)^2 - \alpha^2\right)^{1/2}}\right)^2\right)^{-1/2} \qquad j = \text{mom. ang. total}$$

$2S_{1/2}$ por arriba $2P_{1/2}$ $\Rightarrow$ Corrimiento de Lamb

$(\sim L_S)$

Covarianza relativista de la ec. Dirac

$$(x^\nu)' = \sum_{\mu=0}^{3} a^\nu{}_\mu x^\mu \equiv a^\nu{}_\mu x^\mu \quad , \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \text{ invariante}$$

(S') (S)

A es 4 vector (si transf. como x^μ) $\Rightarrow A\cdot B = A^\mu B^\nu g_{\mu\nu}$ es invariante.
B " " "

$S \quad \psi(x) = \psi(\vec{r}, t)$

$S' \quad \psi'(x') \overset{?}{\longleftarrow} \psi(x)$

$\beta, \vec{\alpha} \longrightarrow \gamma^0 \equiv \beta \quad \gamma^i \equiv \beta\alpha_i \quad i = 1,2,3 \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$\gamma^\mu = \gamma^0, \gamma^1, \gamma^2, \gamma^3 \qquad \gamma^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$i\hbar \frac{\partial}{\partial t}\psi = (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi \quad , \gamma^0 i\hbar \frac{\partial \psi}{\partial (ct)} = (\gamma^0 \alpha \cdot \vec{p} + \beta mc)\psi$$

$\partial(ct) = \partial x^0$

$\gamma^0\beta = \beta^2 = 1 \qquad \gamma^0\vec{\alpha} = \vec{\gamma} \quad , \vec{p} = \frac{\hbar}{i}\vec{\nabla}$

$$i\hbar\left(\gamma^0 \frac{\partial}{\partial x^0} + \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3}\right)\psi = mc\psi \quad , \left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu}\right)\psi = mc\psi$$

$$\boxed{\left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi = 0} \qquad \not{A} \equiv \gamma^\mu A_\mu \qquad \frac{\partial}{\partial x^\mu} \to \partial_\mu$$

$(i\hbar\not{\partial} - mc)\psi = 0$ unidades naturales $\hbar = c = 1$ $(i\not{\partial} - m)\psi = 0$

$(\not{p} - m)\psi = 0$

en interacción con campo E.M $(\not{p} - \not{A} - m)\psi = 0 \quad (q=1) \quad \left(\bar{p} \to \bar{p} - \frac{q}{c}\bar{A}\right)$

$\not{A} = A^\mu \gamma_\mu = A^\mu \gamma^\nu g_{\mu\nu}$

$A^\mu = (\phi, \vec{A})$

Covarianza: S $\psi(x)$ ec Dirac $\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0$

$S', \psi'(x') \quad , \left(i\hbar\bar{\gamma}^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0 \quad , \bar{\gamma}^\mu$ matrices equivalentes a los γ^μ

$\bar{\gamma}^\mu = U^+\gamma^\mu U \quad , U^+ = U^{-1}$

$\alpha_i^2 = \beta^2 = 1 \qquad \{\alpha_i, \alpha_j\} = 0 \quad i \neq j \qquad \{\alpha_i, \beta\} = 0 \qquad \begin{matrix}\alpha \\ \beta\end{matrix} \to S^+\alpha S \quad , S$ unitaria

γ^μ ? $\qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I \qquad \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$ álgebra de Clifford

$x \to ax$

$\psi'(x') = S(a)\psi(x)$

$\psi(x) = S(a^{-1})\psi'(x' = ax)$

$x' = ax \xrightarrow{a^{-1}} x$

$$\psi'(x') = \psi'(ax) = S(a)\psi(x) = S(a)\psi(a^{-1}x') \quad , S_{4\times4} \qquad \psi' \xrightarrow{S} \psi$$

$x' = ax$ transf Lorentz
- no dep. de $\vec{x}, t$: espacio es homogeneo isótropo
- lineal, ec. lineal

$$\psi(x) = S^{-1}(a)\psi'(x') = S^{-1}(a)\psi'(ax) = S(a^{-1})\psi'(ax) \quad , \quad S(a^{-1}) = S^{-1}(a) \qquad S' \longrightarrow S$$

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0 \Rightarrow \left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)S^{-1}(a)\psi'(x') = 0 \; , \left(i\hbar S(a)\gamma^\mu S^{-1}(a)\frac{\partial}{\partial x^\mu} - mc\right)\psi'(x') = 0$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = a^\nu{}_\mu \frac{\partial}{\partial x'^\nu} \quad , \left(i\hbar \underbrace{S(a)\gamma^\mu S^{-1}(a) a^\nu{}_\mu}\frac{\partial}{\partial x'^\nu} - mc\right)\psi'(x') = 0$$

$x^\mu = x^\mu(x') \qquad x'^\nu = a^\nu{}_\mu x^\mu$

$\Rightarrow S(a)\gamma^\mu S^{-1}(a) a^\nu{}_\mu = \gamma^\nu$ ec. para S, y que permite calcular $S(a)$

$$\boxed{S^{-1}(a)\gamma^\nu S(a) = \gamma^\mu a^\nu{}_\mu}$$

$\Rightarrow \left(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi'(x') = 0$

$\exists\, S$? $\quad S$ se puede calcular, y sera función de la T. Lorentz $a^\nu{}_\mu$, y de los matrices γ

Se puede calc. S para boost
rotación ($s = 1/2$!)

<u>Comentario</u> T.L, boost, eje x: $\quad \operatorname{ch} w = \frac{1}{\sqrt{1 - v^2/c^2}} \quad , \quad \psi'(x') = \underbrace{e^{-\frac{i}{2}w\,\frac{i}{2}[\gamma^0, \gamma^1]}}_{S(a)}\psi(x)$

Sale cons. T.L. infinitesimales $\quad a^\nu{}_\mu = g^\nu{}_\mu + \Delta w^\nu{}_\mu$

$\beta \ll 1, \gamma \simeq 1$

$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\beta & & \\ \beta & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$$a^\nu{}_\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$