THEOREM 6 (Finite propagation speed). *If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone $K(x_0, t_0)$.*

In particular, we see that any "disturbance" originating outside $B(x_0, t_0)$ has no effect on the solution within $K(x_0, t_0)$ and consequently has finite propagation speed. We already know this from the representation formulas (31) and (38), at least assuming $g = u$ and $h = u_t$ on $\mathbb{R}^n \times \{t = 0\}$ are sufficiently smooth. The point is that energy methods provide a *much* simpler proof.

Proof. Define the local energy

$$e(t) := \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t^2(x,t) + |Du(x,t)|^2 \, dx \quad (0 \leq t \leq t_0).$$

Then

$$\begin{aligned} \dot{e}(t) &= \int_{B(x_0,t_0-t)} u_t u_{tt} + Du \cdot Du_t \, dx - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 \, dS \\ &= \int_{B(x_0,t_0-t)} u_t (u_{tt} - \Delta u) \, dx \\ &\quad + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t \, dS - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 \, dS \\ &= \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \, dS. \end{aligned} \tag{46}$$

Now

$$\left| \frac{\partial u}{\partial \nu} u_t \right| \leq |u_t||Du| \leq \frac{1}{2} u_t^2 + \frac{1}{2} |Du|^2, \tag{47}$$

by the Cauchy–Schwarz and Cauchy inequalities (§B.2). Inserting (47) into (46), we find $\dot{e}(t) \leq 0$; and so $e(t) \leq e(0) = 0$ for all $0 \leq t \leq t_0$. Thus u_t, $Du \equiv 0$, and consequently $u \equiv 0$ within the cone $K(x_0, t_0)$. □

A generalization of this proof to more complicated geometry appears later, in §7.2.4. See also §12.1 for a similar calculation for a nonlinear wave equation.

2.5. PROBLEMS

In the following exercises, all given functions are assumed smooth, unless otherwise stated.

1. Write down an explicit formula for a function u solving the initial-value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

2. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

3. Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g\,dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

4. Give a direct proof that if $u \in C^2(U) \cap C(\bar{U})$ is harmonic within a bounded open set U, then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(Hint: Define $u_\varepsilon := u + \varepsilon |x|^2$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}$ at an interior point.)

5. We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v\,dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

6. Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \leq C(\max_{\partial U} |g| + \max_{\bar{U}} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$, for $\lambda := \max_{\bar{U}} |f|$.)

7. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

8. Prove Theorem 15 in §2.2.4. (Hint: Since $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x,y)\, dS(y) = 1$$

for each $x \in B^0(0,1)$.)

9. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show Du is *not* bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$.)

10. (Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1,\ x_n > 0\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U. (Hint: Use Poisson's formula for the ball.)

11. (Kelvin transform for Laplace's equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|)|x|^{2-n} \qquad (x \neq 0),$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.
(Hint: First show that $D_x\bar{x}(D_x\bar{x})^T = |\bar{x}|^4 I$. The mapping $x \to \bar{x}$ is *conformal*, meaning angle preserving.)

12. Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

13. Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$(*) \qquad v'' + \frac{z}{2}v' = 0.$$

Show that the general solution of $(*)$ is

$$v(z) = c\int_0^z e^{-s^2/4}\, ds + d.$$

(b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution. (Hint: What is the initial condition for u?)

14. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

15. Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\, ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = g & \text{on } \{x=0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

16. Give a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(Hint: Define $u_\varepsilon := u - \varepsilon t$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

17. We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \le 0 \quad \text{in } U_T.$$

(a) Prove for a subsolution v that

$$v(x,t) \le \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2}\, dyds$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.

(d) Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

18. (Stokes' rule) Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0,\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Show that $v := u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = h,\ v_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

This is *Stokes' rule*.

19. (a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t, \quad \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

(d) Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

20. Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there exist for *all* profiles ϕ solutions of the wave equation in $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$ having the form

$$u(x,t) = \alpha(r)\phi(t - \beta(r)).$$

Here $r = |x|$ and we assume $\beta(0) = 0$.

Show that this is possible only if $n = 1$ or 3, and compute the form of the functions α, β.

(T. Morley, SIAM Review 27 (1985), 69–71)

21. (a) Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \operatorname{curl} \mathbf{B}, \quad \mathbf{B}_t = -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{E} = 0. \end{cases}$$

Show

$$\mathbf{E}_{tt} - \Delta \mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta \mathbf{B} = 0.$$

(b) Assume that $\mathbf{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\lambda + \mu) D(\operatorname{div} \mathbf{u}) = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \operatorname{div} \mathbf{u}$ and $\mathbf{w} := \operatorname{curl} \mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

22. Let u denote the density of particles moving to the right with speed one along the real line and let v denote the density of particles moving to the left with speed one. If at rate $d > 0$ right-moving particles randomly become left-moving, and vice versa, we have the system of PDE

$$\begin{cases} u_t + u_x = d(v - u) \\ v_t - v_x = d(u - v). \end{cases}$$

Show that both $w := u$ and $w := v$ solve the telegraph equation

$$w_{tt} + 2dw_t - w_{xx} = 0.$$

23. Let S denote the square lying in $\mathbb{R} \times (0, \infty)$ with corners at the points $(0,1), (1,2), (0,3), (-1,2)$. Define

$$f(x,t) := \begin{cases} -1 & \text{for } (x,t) \in S \cap \{t > x+2\} \\ 1 & \text{for } (x,t) \in S \cap \{t < x+2\} \\ 0 & \text{otherwise.} \end{cases}$$

Assume u solves

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0, u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Describe the shape of u for times $t > 3$.

(J. G. Kingston, SIAM Review 30 (1988), 645–649)

24. (Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose g, h have compact support. The *kinetic energy* is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the *potential energy* is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(a) $k(t) + p(t)$ is constant in t,

(b) $k(t) = p(t)$ for all large enough times t.

2.6. REFERENCES

Section 2.2 A good source for more on Laplace's and Poisson's equations is Gilbarg–Trudinger [**G-T,** Chapters 2-4]. The proof of analyticity is from Mikhailov [**M**]. J. Cooper helped me with Green's functions.

Section 2.3 See John [**J2,** Chapter 7] or Friedman [**Fr1**] for further information concerning the heat equation. Theorem 3 is due to N. Watson (Proc. London Math. Society 26 (1973), 385–417), as is the proof of Theorem 4. Theorem 6 is taken from John [**J2**], and Theorem 8 follows Mikhailov [**M**]. Theorem 11 is from Payne [**Pa,** §2.3].

Section 2.4 See Antman (Amer. Math. Monthly 87 (1980), 359–370) for a careful derivation of the one-dimensional wave equation as a model for a vibrating string. The solution of the wave equation presented here follows Folland [**F1**], Strauss [**St2**].

Section 2.5 J. Goldstein contributed Problem 24.

and

$$\|\bar{\mathbf{u}}\|_{L^2(0,T;H^2(V))} \leq C\|\mathbf{u}\|_{L^2(0,T;H^2(U))}, \tag{14}$$

for an appropriate constant C. In addition, $\bar{\mathbf{u}}' \in L^2(0,T;L^2(V))$, with the estimate

$$\|\bar{\mathbf{u}}'\|_{L^2(0,T;L^2(V))} \leq C\|\mathbf{u}'\|_{L^2(0,T;L^2(U))}. \tag{15}$$

This follows if we consider difference quotients in the t-variable, remember the methods in §5.8.2, and observe also that E is a bounded linear operator from $L^2(U)$ into $L^2(V)$.

2. Assume for the moment that $\bar{\mathbf{u}}$ is smooth. We then compute

$$\begin{aligned}\Big|\frac{d}{dt}\Big(\int_V |D\bar{\mathbf{u}}|^2\,dx\Big)\Big| &= 2\Big|\int_V D\bar{\mathbf{u}}\cdot D\bar{\mathbf{u}}'\,dx\Big| = 2\Big|\int_V \Delta\bar{\mathbf{u}}\,\bar{\mathbf{u}}'\,dx\Big|\\ &\leq C\big(\|\bar{\mathbf{u}}\|^2_{H^2(V)} + \|\bar{\mathbf{u}}'\|^2_{L^2(V)}\big).\end{aligned}$$

There is no boundary term when we integrate by parts, since the extension $\bar{\mathbf{u}} = E\mathbf{u}$ has compact support within V. Integrating and recalling (14), (15), it follows that

$$\max_{0\leq t\leq T}\|\mathbf{u}(t)\|_{H^1(U)} \leq C\big(\|\mathbf{u}\|_{L^2(0,T;H^2(U))} + \|\mathbf{u}'\|_{L^2(0,T;L^2(U))}\big). \tag{16}$$

We obtain the same estimate if $\mathbf{u}$ is not smooth, upon approximating by $\mathbf{u}^\varepsilon := \eta_\varepsilon * \mathbf{u}$, as before. As in the previous proofs, it also follows that $\mathbf{u} \in C([0,T];H^1(U))$.

3. In the general case that $m \geq 1$, we let α be a multiindex of order $|\alpha| \leq m$ and set $\mathbf{v} := D^\alpha\mathbf{u}$. Then

$$\mathbf{v} \in L^2(0,T;H^2(U)),\ \mathbf{v}' \in L^2(0,T;L^2(U)).$$

We apply estimate (16), with $\mathbf{v}$ replacing $\mathbf{u}$, and sum over all indices $|\alpha| \leq m$, to derive (13). □

5.10. PROBLEMS

In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

1. Suppose $k \in \{0,1,\dots\}$, $0 < \gamma \leq 1$. Prove $C^{k,\gamma}(\bar{U})$ is a Banach space.

2. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

3. Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, \;\; |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, \;\; |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, \;\; |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, \;\; |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$?

4. Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.
   (a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$.
   (b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p dt \right)^{1/p}$$

   for a.e. $x, y \in [0,1]$.

5. Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W.)

6. Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i $(i = 1, \dots, N)$ such that

$$\begin{cases} 0 \leq \zeta_i \leq 1, \; \operatorname{spt} \zeta_i \subset V_i \;\; (i = 1, \dots, N) \\ \sum_{i=1}^N \zeta_i = 1 \quad \text{on } U. \end{cases}$$

The functions $\{\zeta_i\}_{i=1}^N$ form a *partition of unity.*

7. Assume that U is bounded and there exists a smooth vector field $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \geq 1$ along ∂U, where $\boldsymbol{\nu}$ as usual denotes the outward unit normal. Assume $1 \leq p < \infty$.

Apply the Gauss–Green Theorem to $\int_{\partial U} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{\nu} \, dS$, to derive a new proof of the trace inequality

$$\int_{\partial U} |u|^p \, dS \leq C \int_U |Du|^p + |u|^p \, dx$$

for all $u \in C^1(\bar{U})$.

8. Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

9. Integrate by parts to prove the interpolation inequality:

$$||Du||_{L^2} \leq C||u||_{L^2}^{1/2}||D^2u||_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.
(Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$.)

10. (a) Integrate by parts to prove

$$||Du||_{L^p} \leq C||u||_{L^p}^{1/2}||D^2u||_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and all $u \in C_c^\infty(U)$.
(Hint: $\int_U |Du|^p\,dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2}\,dx$.)

(b) Prove

$$||Du||_{L^{2p}} \leq C||u||_{L^\infty}^{1/2}||D^2u||_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(U)$.

11. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U.$$

Prove u is constant a.e. in U.

12. Show by example that if we have $\|D^h u\|_{L^1(V)} \leq C$ for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(V)$.

13. Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$, such that u is *not* Lipschitz continuous on U. (Hint: Take U to be the open unit disk in $\mathbb{R}^2$, with a slit removed.)

14. Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

15. Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C, depending only on n and α, such that

$$\int_U u^2\,dx \leq C \int_U |Du|^2\,dx,$$

provided

$$|\{x \in U \mid u(x) = 0\}| \geq \alpha\,, \quad u \in H^1(U).$$

16. (Variant of Hardy's inequality) Show that for each $n \geq 3$ there exists a constant C so that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \leq C \int_{\mathbb{R}^n} |Du|^2\, dx$$

for all $u \in H^1(\mathbb{R}^n)$.
(Hint: $|Du + \lambda \frac{x}{|x|^2} u|^2 \geq 0$ for each $\lambda \in \mathbb{R}$.)

17. (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u)u_{x_i} \quad (i = 1, \dots, n).$$

18. Assume $1 \leq p \leq \infty$ and U is bounded.
    (a) Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$.
    (b) Prove $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\}, \end{cases}$$
$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

    (Hint: $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, for

$$F_\varepsilon(z) := \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \geq 0 \\ 0 & \text{if } z < 0.) \end{cases}$$

    (c) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

19. Provide details for the following alternative proof that if $u \in H^1(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

Let ϕ be a smooth, bounded and nondecreasing function, such that ϕ' is bounded and $\phi(z) = z$ if $|z| \leq 1$. Set

$$u^\epsilon(x) := \epsilon\phi(u/\epsilon).$$

Show that $u^\epsilon \rightharpoonup 0$ weakly in $H^1(U)$ and therefore

$$\int_U Du^\epsilon \cdot Du \, dx = \int_U \phi'(u/\epsilon)|Du|^2 \, dx \to 0.$$

Employ this observation to finish the proof.

20. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n.

21. Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)},$$

the constant C depending only on s and n.

5.11. REFERENCES

Sections 5.2–8 See Gilbarg–Trudinger [**G-T,** Chapter 7], Lieb–Loss [**L-L**], Ziemer [**Z**] and [**E-G**] for more on Sobolev spaces.

Section 5.5 W. Schlag showed me the proof of Theorem 2.

Section 5.6 J. Ralston suggested an improvement in the proof of Theorem 4.

Section 5.9 See Temam [**Te,** pp. 248–273].

Section 5.10 Problem 16: see Tartar [**Tr,** Chapter 17]. H. Brezis taught me the trick in Problem 19.