The heat equation defines a flow in space of functions. Using a similar calculation, we can identify this as a gradient flow for the energy functional $E(u)$:

$$\begin{aligned}\frac{d}{d\sigma}\Big|_{\sigma=0} E(u+\sigma w) &= \frac{d}{d\sigma}\Big|_{\sigma=0} \frac{1}{4}\int |\nabla u + \sigma w|^2\, dx \\ &= \frac{d}{d\sigma}\Big|_{\sigma=0} \Big(\frac{1}{4}\int |\nabla u|^2 + 2\sigma \nabla u\cdot\nabla w + \sigma^2|\nabla w|^2\, dx\Big) \\ &= \frac{1}{2}\int \nabla u\cdot\nabla w\, dx\ .\end{aligned}$$

In order to describe this quantity in terms of the L^2 inner product, one integrates by parts:

$$\frac{d}{d\sigma}\Big|_{\sigma=0} E(u+\sigma w) = -\int \frac{1}{2}\Delta u\, w\, dx\ . \tag{3.24}$$

This identifies $\mathrm{grad}E(u) = -\frac{1}{2}\Delta u$. The (negative) gradient flow is therefore

$$\partial_t u = -\mathrm{grad}E(u) = \frac{1}{2}\Delta u\ ,$$

which is precisely the heat equation (3.15). A classical notation is to write $\delta E(u)$ for the *Gâteau derivative* of $E(u)$; that is, for the directional derivative of $E(u)$ in the direction w, one writes that

$$\frac{d}{d\sigma}\Big|_{\sigma=0} E(u+\sigma w) = \langle \delta E(u), w\rangle\ .$$

Exercises: Chapter 3

Exercise 3.1. This problem concerns the semigroup property of the solution process for the heat equation. Show that the heat kernel satisfies the identity

$$H(t,x) = \int_{-\infty}^{+\infty} H(t-s, x-y)H(y,s)\, dy \quad \forall 0 < s < t\ .$$

Now show that the heat operator satisfies the property that for all $0 < s < t$,

$$\mathbf{H}(t) = \mathbf{H}(t-s)\mathbf{H}(s)\ .$$

Conclude that formula (3.5) holds for the one-parameter family of solution operators $\{\mathbf{H}(t)\}_{t\in\mathbb{R}^1_+}$ of the heat equation.

Exercise 3.2. Justify on a rigorous level of analysis the exchange of integrations in the proof of Proposition 3.4(iii), therefore completing the rigorous proof of the proposition's three parts.

Exercise 3.3. Solve the following initial value problems for the heat equation in explicit terms.

(1) $f(x) = x$

(2) $f(x) = x^2$
(3) $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$
(4) $f(x) = e^{\alpha x}$
(5) $f(x) = \sin(kx)$

What is the asymptotic behavior of $u(t, x)$ as $t \to +\infty$? What is their asymptotic behavior as $t \to +\infty$? On what properties of the initial data does this behavior depend? Does it affect the asymptotic behavior if $f(x) \notin L^1(\mathbb{R}^1)$?

Exercise 3.4 (Moments of the heat kernel)**.** (i) Show that the second moment of the heat kernel

$$H(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = P_t(x)$$

is its variance and that

$$m_2(P_t) = t \ .$$

Hint: Calculate the following integral:

$$G(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha \frac{x^2}{2}} \, dx \ ,$$

and then use the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \, x^2 \, dx = -2\partial_\alpha G(1) \ .$$

(ii) Also calculate all the moments $m_j(P_t)$ for j any odd integer.

(iii) Using the same technique as in (i), give an expression for all of the moments $m_j(P_t)$ for j even.

Exercise 3.5. Suppose that the initial data $f(x) \in L^1(\mathbb{R}^1)$ for the heat equation is piecewise continuous, meaning that it contains a possibly finite number of jump discontinuities

$$\lim_{x \to x_j^-} f(x) = a_j \ , \qquad \lim_{x \to x_j^+} f(x) = b_j$$

at points x_j, $j = 1, N$. Show that the limit of the solution (3.4) of the heat equation satisfies

$$\lim_{t \to 0^+} u(t, x_j) = \frac{1}{2}(a_j + b_j) \ .$$

Exercise 3.6 (Method of images)**.** Give an expression for the solution of the heat equation over the bounded interval $0 \leq x < 2\pi$ for the following cases of boundary conditions.

(1) Dirichlet: $u(t, 0) = 0 = u(t, 2\pi)$. Denote this solution by $u_D(t, x)$.