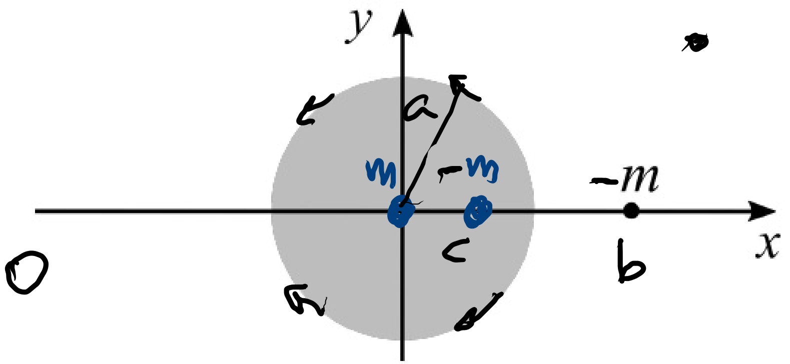
6. Un cilindro de radio a se encuentra centrado en el origen $z = 0$. En $z = b$, hay un sumidero de intensidad m como se muestra en la figura.

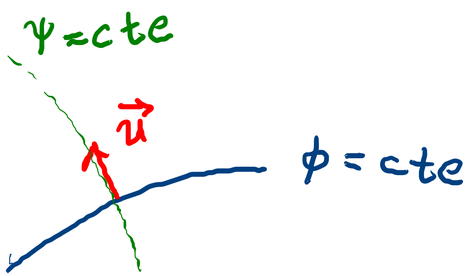


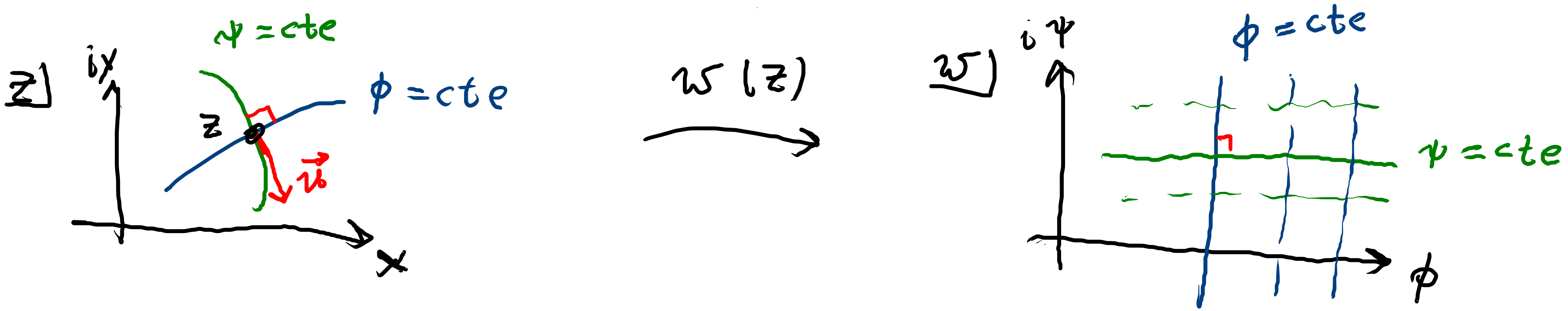
a. Muestre que es posible satisfacer las condiciones de frontera sustituyendo el cilindro por una fuente en el origen y un sumidero sobre el eje x (ambas de intensidad absoluta m). Determine el flujo alrededor del cilindro.

$$w = \phi + i\psi$$

$$\vec{u} = \nabla\phi$$

$$\nabla\phi \cdot \nabla\psi = (\partial_x \phi\,\hat{i} + \partial_x \phi\,\hat{j}) \cdot (\partial_x \psi\,\hat{i} + \partial_y \psi\,\hat{j}) = 0$$





$$z = x + iy$$

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} \quad (\delta z = \delta x) \Rightarrow \frac{dw}{dz} = \partial_x(\phi + i\psi) = u_x - i\,u_y$$

Pared rígida :

$\psi = cte$

$\hat{k} \odot$, $\hat{n}$, $\hat{t}$

$0 = \vec{u} \cdot \hat{n}\big|_s = \nabla\phi \cdot \hat{n}\big|_s$

$0 = (\nabla\psi \times \hat{k}) \cdot \hat{n}\big|_s =$

$$\vec{u} = \nabla\phi = (\partial_x \phi)\hat{i} + (\partial_y \phi)\hat{j} = (\partial_y \psi)\hat{i} - (\partial_x \psi)\hat{j} = \nabla\psi \times \hat{k}$$

$= (\hat{k} \times \hat{n}) \cdot \nabla\psi\big|_s = -\hat{t} \cdot \nabla\psi\big|_s$

— // —

$$w = -\frac{m}{2\pi}\ln(z-b) + \frac{m}{2\pi}\ln z - \frac{m}{2\pi}\ln(z-c)$$

$$z = re^{i\theta} \longrightarrow z - b = (r\cos\theta - b) + i r \operatorname{sen}\theta \quad ; \quad z - c = (r\cos\theta - c) + i r \operatorname{sen}\theta$$

$$= R_b e^{i\beta} \qquad\qquad = R_c e^{i\gamma}$$

$$R_b^2 = b^2 + r^2 - 2br\cos\theta \qquad R_c^2 = c^2 + r^2 - 2cr\cos\theta$$

$$w = \frac{m}{2\pi}\left[\ln r + i\theta - \ln R_b - i\beta - \ln R_c - i\gamma\right] \rightarrow \psi = \frac{m}{2\pi}(\theta - \beta - \gamma)$$

$$w = \frac{m}{2\pi}\left[\ln r + i\theta - \ln R_b - i\beta - \ln R_c - i\gamma\right] \Rightarrow \begin{cases} \psi = \frac{m}{2\pi}(\theta - \beta - \gamma) \\ \phi = \frac{m}{2\pi}(\ln r - \ln R_b - \ln R_c) \end{cases}$$

$$r = a \rightarrow \theta - \beta - \gamma = \overset{\theta_0}{\overset{\shortparallel}{cte}} \quad (=0)$$

$$cte = 0 \Rightarrow \theta = \beta + \gamma \rightarrow tg\theta = tg(\beta+\gamma) = \frac{tg\beta + tg\gamma}{1 - tg\beta\, tg\gamma}$$

$$tg\beta = \frac{a\,sen\theta}{a\cos\theta - b} \qquad tg\gamma = \frac{a\,sen\theta}{a\cos\theta - c}$$

$$\frac{sen\theta}{\cos\theta} = \frac{a\,sen\theta\,(2a\cos\theta - b - c)}{a^2\cos^2\theta - a(b+c)\cos\theta + bc} \Rightarrow a^2 sen\theta\cos^2\theta - a(b+c)\,sen\theta\cos\theta + bc\,sen\theta =$$

$$= 2a^2 sen\theta\cos^2\theta - a(b+c)\,sen\theta\cos\theta$$

$$\Rightarrow a^2 sen\theta\left[\cos^2\theta - bc\right] = 0 \quad \times$$

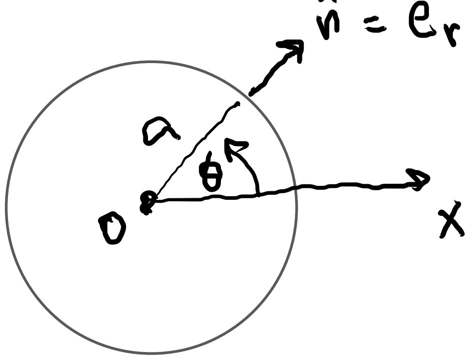
$$\boxed{\theta - \theta_0 = \beta + \gamma} \qquad x^2 + y^2 = a^2$$

$$tg(\theta - \theta_0) = \frac{tg\theta - \overset{k}{tg\theta_0}}{1 + tg\theta_0\, tg\theta} = \frac{sen\theta - k\cos\theta}{\cos\theta + k\,sen\theta}$$

$$w = \frac{m}{2\pi}\left[\ln r + i\theta - \ln R_b - i\beta - \ln R_c - i\gamma\right] \rightarrow \begin{cases} \psi = \frac{m}{2\pi}(\theta - \beta - \gamma) \\ \phi = \frac{m}{2\pi}(\ln r - \ln R_b - \ln R_c) \end{cases}$$

$$r = a \rightarrow \theta - \beta - \gamma = \overset{\theta_0}{\overset{\shortparallel}{cte}} \;(= 0)$$

$$\rightarrow \vec{u}\cdot\hat{n}\Big|_a = 0 = \nabla\phi\cdot\hat{e}_r\Big|_a = \frac{\partial\phi}{\partial r}\Big|_a$$



$$\frac{\partial\phi}{\partial r} = \frac{m}{2\pi}\left[\frac{1}{r} - \frac{1}{R_b}\frac{\partial R_b}{\partial r} - \frac{1}{R_c}\frac{\partial R_c}{\partial r}\right]$$

$$R_b^2 = b^2 + r^2 - 2br\cos\theta$$

$$\left.\begin{aligned} \frac{\partial R_b^2}{\partial r} &= 2R_b\frac{\partial R_b}{\partial r} \\ &= 2r - 2b\cos\theta \end{aligned}\right\} \rightarrow \frac{\partial R_b}{\partial r} = \frac{1}{2R_b}(2r - 2b\cos\theta)$$

$\Rightarrow$

$$\frac{1}{R_b}\frac{\partial R_b}{\partial r} = \frac{1}{R_b^2}(r - b\cos\theta) = \frac{r - b\cos\theta}{b^2 + r^2 - 2br\cos\theta}$$

igualmente: $\dfrac{1}{R_c}\dfrac{\partial R_c}{\partial r} = \dfrac{r - c\cos\theta}{c^2 + r^2 - 2cr\cos\theta}$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{m}{2\pi}\left[\frac{1}{r} - \frac{r - b\cos\theta}{b^2 + r^2 - 2br\cos\theta} - \frac{r - c\cos\theta}{c^2 + r^2 - 2cr\cos\theta}\right]$$

$$\left.\frac{\partial \phi}{\partial r}\right|_a = 0 \Longleftrightarrow \frac{1}{a} - \frac{a - b\cos\theta}{b^2 + a^2 - 2ba\cos\theta} - \frac{a - c\cos\theta}{c^2 + a^2 - 2ca\cos\theta} = 0 \quad \forall\, \theta \quad (*)$$

para $\theta = \pi/2$: $(\cos\theta = 0)$

$$\frac{1}{a} - \frac{a}{a^2 + b^2} = \frac{a}{a^2 + c^2} \Rightarrow \frac{a}{a^2 + c^2} = \frac{b^2}{a(a^2 + b^2)} \longrightarrow a^2 + c^2 = \frac{a^2(a^2 + b^2)}{b^2}$$

$$c^2 = \frac{a^2(a^2 + b^2)}{b^2} - a^2 = \frac{a^4}{b^2} \Rightarrow \boxed{c = \frac{a^2}{b}}$$

Se puede ver que al sustituir en $(*)$ este c hace que se verifique.

Flujo

sustituyendo c en $\frac{\partial \phi}{\partial r}$: $u_r = \frac{m}{2\pi}\left[\frac{b^2 - rb\cos\theta}{(b^2+r^2-2br\cos\theta)\,r} - \frac{br - a^2\cos\theta}{b\left(\frac{a^4}{b^2}+r^2-2\frac{a^2}{b}\cos\theta\right)}\right]$

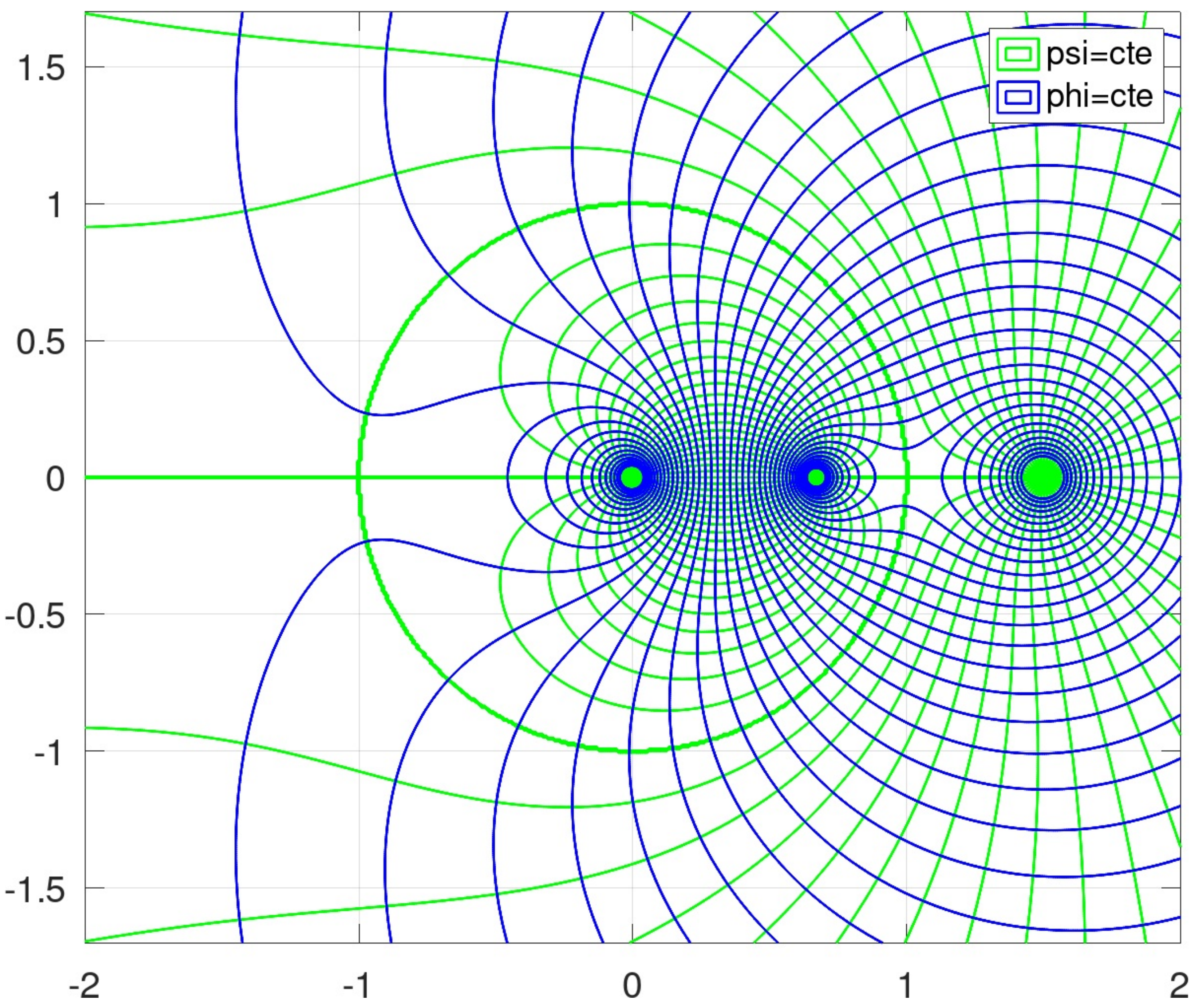
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{m}{2\pi}\left[-\frac{1}{R_b}\frac{\partial R_b}{\partial \theta} - \frac{1}{R_c}\frac{\partial R_c}{\partial \theta}\right]$$

$$R_b^2 = b^2 + r^2 - 2br\cos\theta \longrightarrow 2R_b\frac{\partial R_b}{\partial \theta} = \frac{\partial R_b^2}{\partial \theta} = 2br\,\mathrm{sen}\,\theta \Rightarrow \frac{\partial R_b}{\partial \theta} = \frac{br\,\mathrm{sen}\,\theta}{R_b}$$

igualmente: $\frac{\partial R_c}{\partial \theta} = \frac{cr\,\mathrm{sen}\,\theta}{R_c}$

$$u_\theta = -\frac{m}{2\pi r}\left[\frac{1}{R_b^2}br\,\mathrm{sen}\,\theta + \frac{1}{R_c^2}cr\,\mathrm{sen}\,\theta\right] = -\frac{m}{2\pi}\left[\frac{b}{b^2+r^2-2br\cos\theta} - \frac{c}{c^2+r^2-2cr\cos\theta}\right]\mathrm{sen}\,\theta$$

$$u_\theta = -\frac{m}{2\pi b}\left[\frac{b^2}{b^2+r^2-2br\cos\theta} - \frac{a^2}{\frac{a^4}{b^2}+r^2-2\frac{a^2}{b}r\cos\theta}\right]\mathrm{sen}\,\theta$$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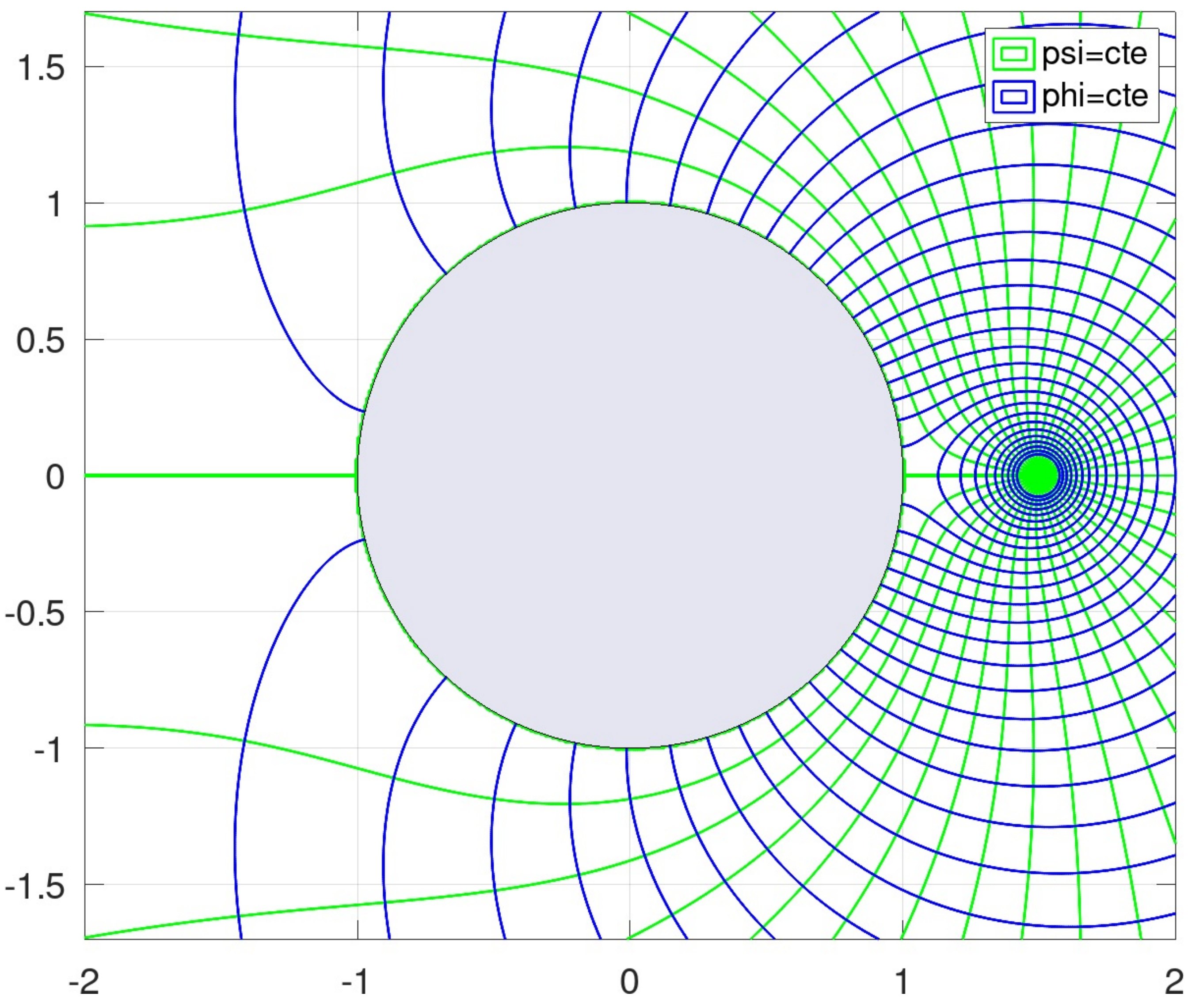
phi=cte
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2


psi=cte
phi=cte
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2