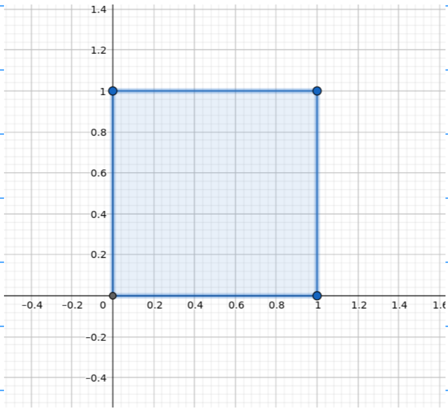
Extremos condicionados (ejemplos)

1) $f(x,y) = x^2y + xy^2 - 2xy$ en $A = [0,1] \times [0,1]$



$A: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

Weierstrass:

f tiene maximo y minimo en A

Interior: puntos criticos de $f(x,y) = x^2y + xy^2 - 2xy$

$$\nabla f(x,y) = (2xy + y^2 - 2y,\ x^2 + 2xy - 2x)$$

$$\begin{cases} 2xy + y^2 - 2y = 0 \\ x^2 + 2xy - 2x = 0 \end{cases} \longleftrightarrow \begin{cases} y(2x + y - 2) = 0 & (1) \\ x(x + 2y - 2) = 0 & (2) \end{cases}$$

(1) $\to$ $y = 0$ 1a; $\to$ $2x + y - 2 = 0$ 1b

(2) $\to$ $x = 0$ 2a; $\to$ $x + 2y - 2 = 0$ 2b

1a 2a: $\begin{cases} x = 0 \\ y = 0 \end{cases}$	1a 2b: $\begin{cases} y = 0 \\ x - 2 = 0 \end{cases}$	1b 2a: $\begin{cases} y - 2 = 0 \\ x = 0 \end{cases}$
$\to (0,0)$	$\to (2,0)$	$\to (0,2)$

1b 2b: $\begin{cases} 2x + y = 2 \\ x + 2y = 2 \end{cases} \to x = y = 2/3 \to (2/3, 2/3)$

Puntos críticos :

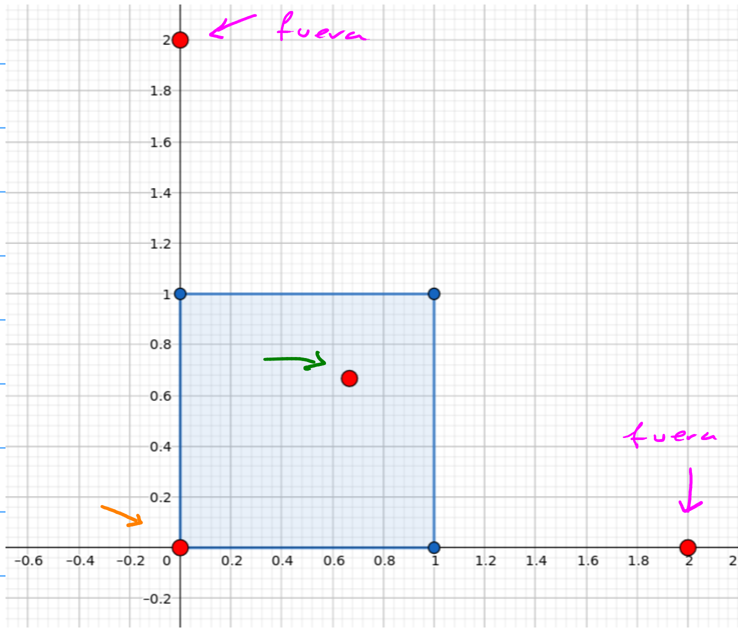
$(0,0)$ ← borde de A

$(2,0)$, $(0,2)$ ← fuera de A

$(2/3, 2/3)$ ← interior ✓

Pues : $0 < x = 2/3 < 1$

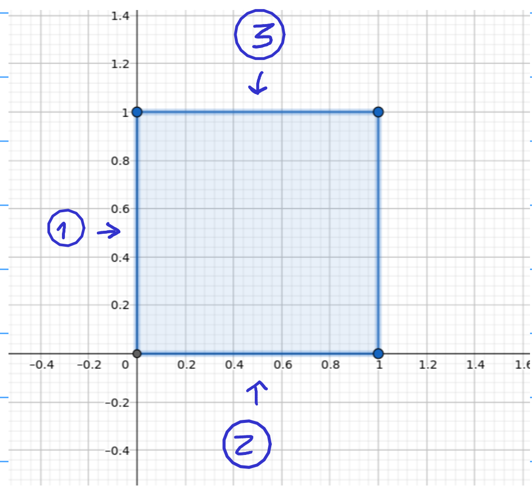
$0 < y = 2/3 < 1$



$$f(x,y) = x^2y + yx^2 - 2xy \longrightarrow \boxed{f(2/3, 2/3) = -8/27}$$

Borde :

$$f(x,y) = x^2y + xy^2 - 2xy$$



1 : $x = 0$, $y \in [0,1]$

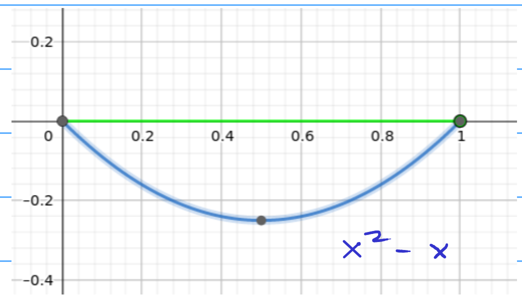
$f(0,y) = 0$ Max/min $= 0$

2 : $y = 0$, $x \in [0,1]$

$f(x,0) = 0$ Max/min $= 0$

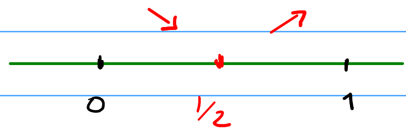
3 : $y = 1$, $x \in [0,1]$

$f(x,1) = \underbrace{x^2 - x}_{g(x)}$ con $x \in [0,1]$



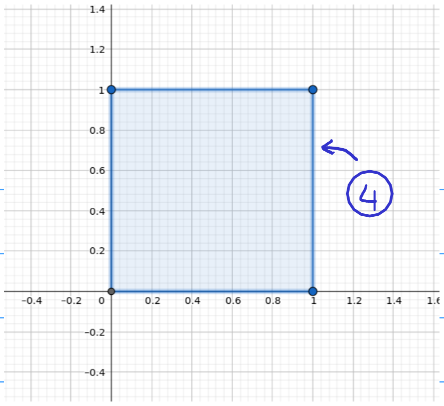
$g'(x) = 2x - 1$

$g'(x) = 0 \Leftrightarrow x = 1/2$



Max/min :

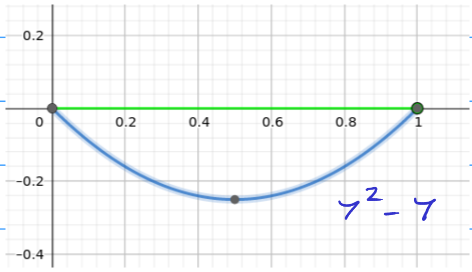
$$\boxed{\begin{array}{l} f(0,1) = 0 \\ f(1,1) = 0 \end{array}} \qquad \boxed{f(1/2, 1) = -1/4}$$



$f(x,y) = x^2y + xy^2 - 2xy$

④ $x = 1 \,,\quad y \in [0,1]$

$f(1,y) = y^2 - y \quad$ con $\quad y \in [0,1]$



Max / min :

$$\boxed{\begin{array}{l} f(1,0) = 0 \\ f(1,1) = 0 \end{array}} \qquad \boxed{f(1, 1/2) = -1/4}$$

Resumen :

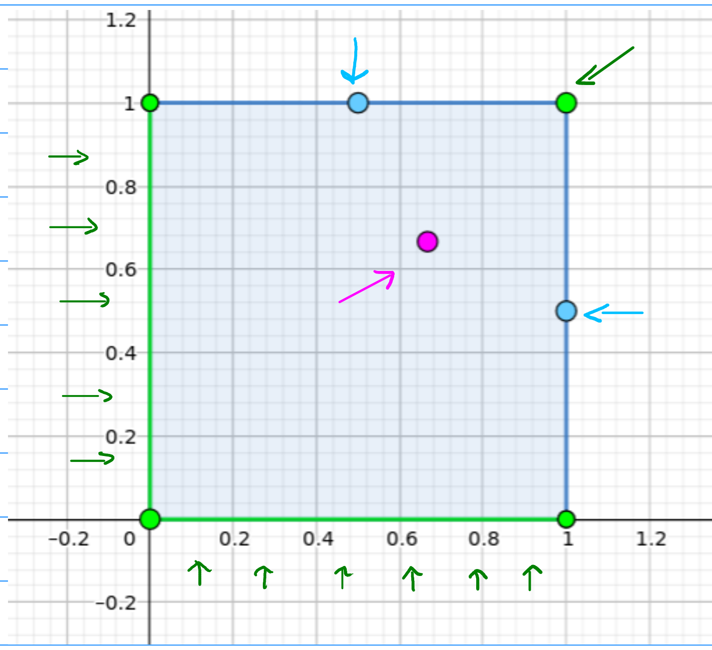
Interior : $f(2/3, 2/3) = -\dfrac{8}{27} \leftarrow$ Min

Borde : ①, ② $\rightarrow 0$ Max

③ $f(1,1) = 0$

$f(1/2, 1) = -1/4$

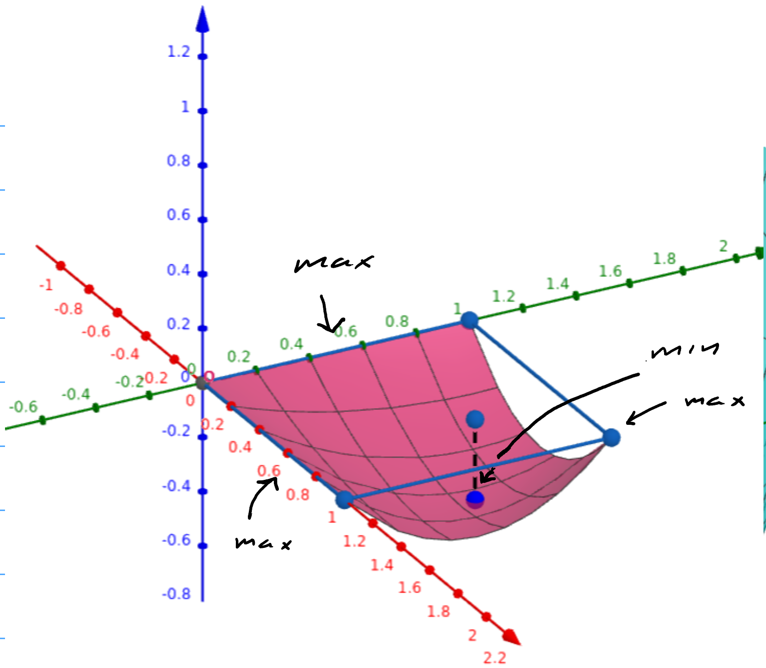
④ $f(1, 1/2) = -1/4$



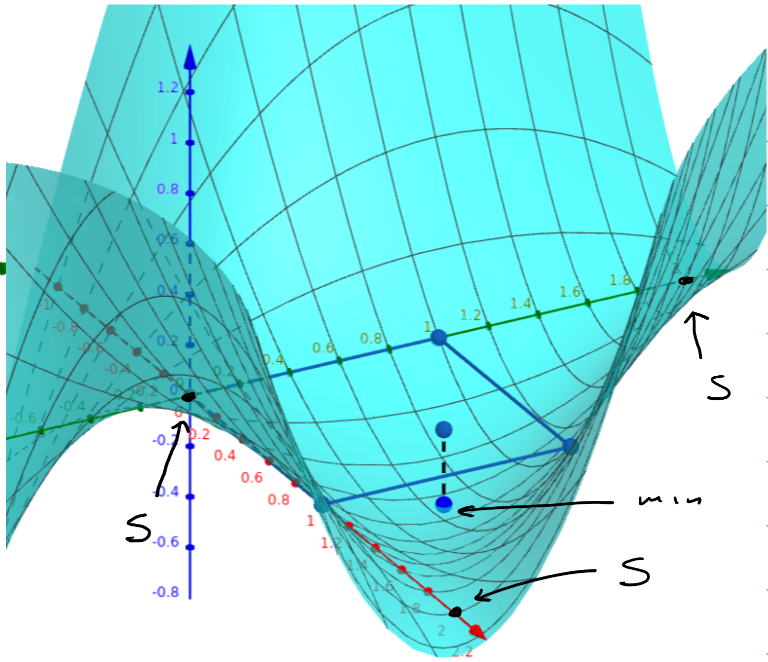
$$-\frac{1}{4} = -\frac{27}{108} > -\frac{8}{27} = -\frac{32}{108}$$

Solución :

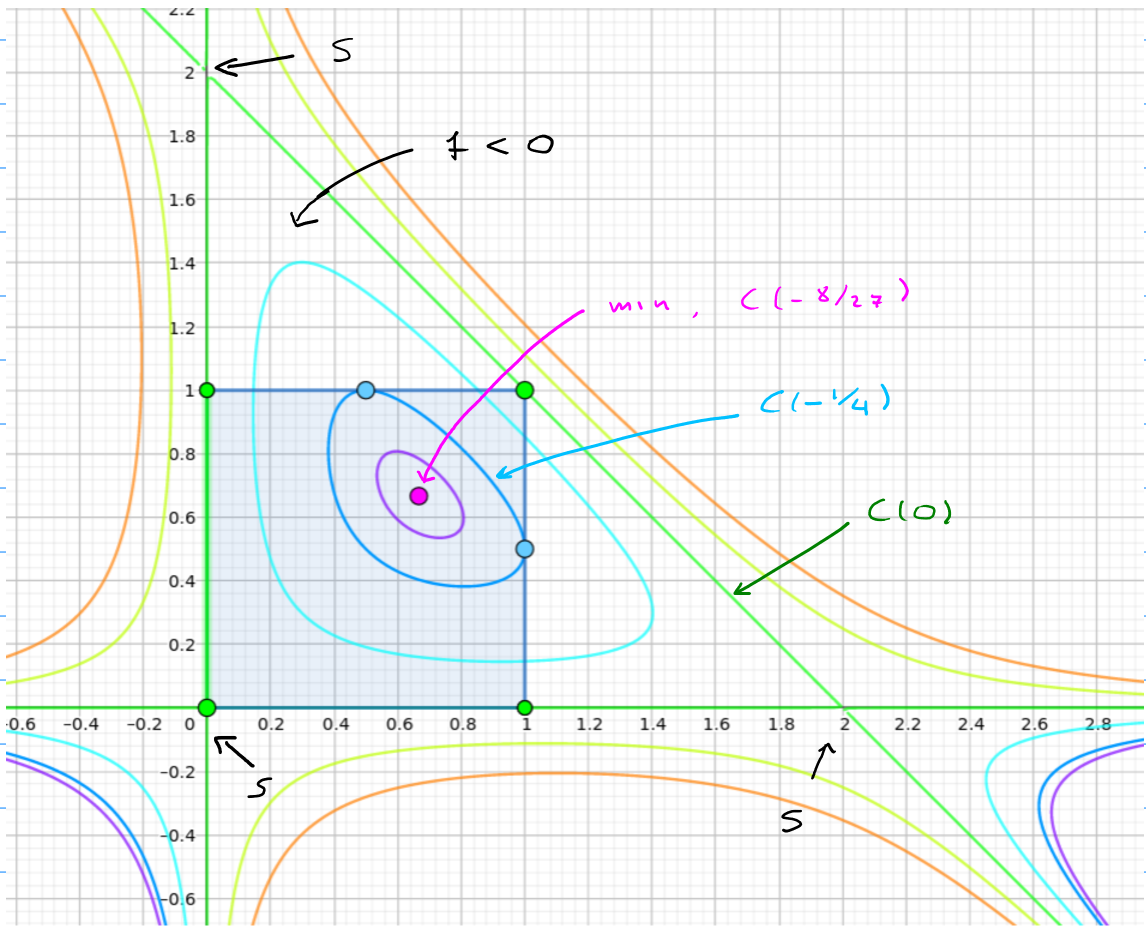
$$\boxed{\begin{array}{llll} \text{Max}: & 0 & \text{en} & (0,y),\ y \in [0,1] \ ,\ (1,1) \\ & & & (x,0),\ x \in [0,1] \\ & & & \\ \text{Min}: & -8/27 & \text{en} & (2/3, 2/3) \end{array}}$$



max
min
max
max



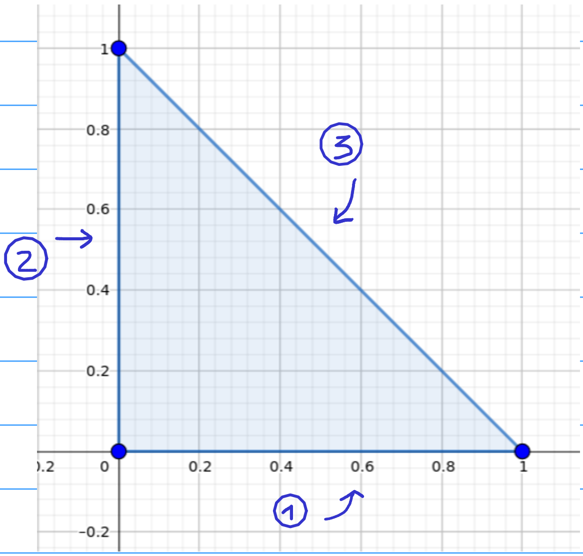
S
S
min
S



S
f < 0
min, C(-8/27)
C(-1/4)
C(0)
S
S

2) $f(x,y) = -4xy + x^2 + y$

en A el triángulo de vértices $(0,0)$, $(1,0)$ y $(0,1)$



Lados:

① $y = 0$, $x \in [0,1]$

② $x = 0$, $y \in [0,1]$

③ $x + y = 1$, $x \in [0,1]$

$(ax+by = c$; $(1,0) \to a = c$; $(0,1) \to b = c)$

$$A : \begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 1 \end{cases} \qquad \text{Interior} : \begin{cases} x > 0 \\ y > 0 \\ x + y < 0 \end{cases}$$

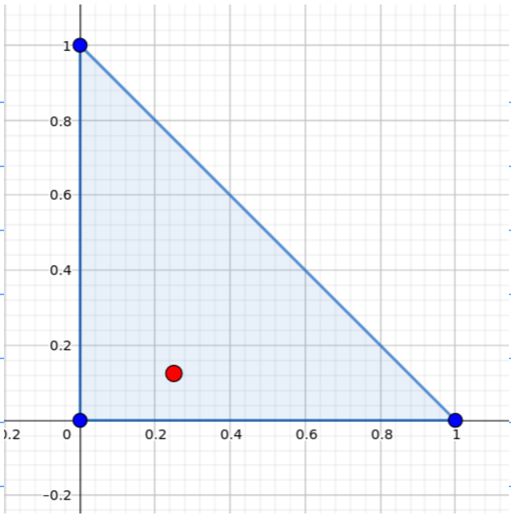
Hallar max/min de $f(x,y) = -4xy + x^2 + y$
en el triángulo A:

Weierstrass: tiene max y min.

Interior: $\nabla f(x,y) = (-4y + 2x, -4x + 1)$

$$\begin{cases} -4y + x = 0 \longrightarrow y = 1/8 \\ -4x + 1 = 0 \to x = 1/4 \end{cases}$$

$\longrightarrow (1/4, 1/8)$



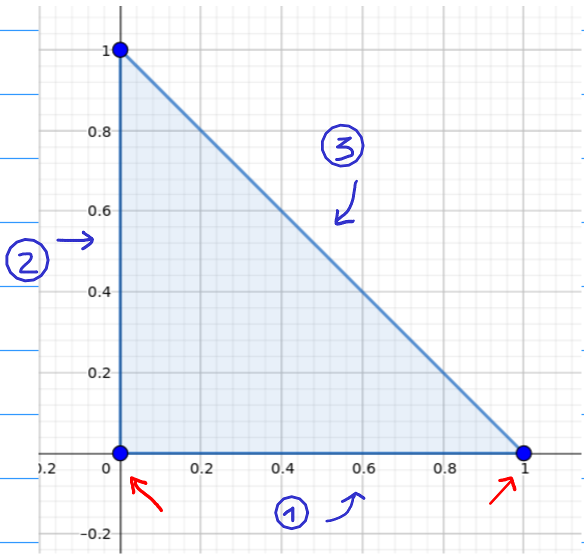
$(1/4, 1/8)$

En el interior: $\begin{cases} 1/4 > 0 \\ 1/8 > 0 \\ 1/4 + 1/8 = 3/8 < 1 \end{cases}$

$$\boxed{f(1/4, 1/8) = 1/16}$$

Borde:

$f(x,y) = -4xy + x^2 - y$

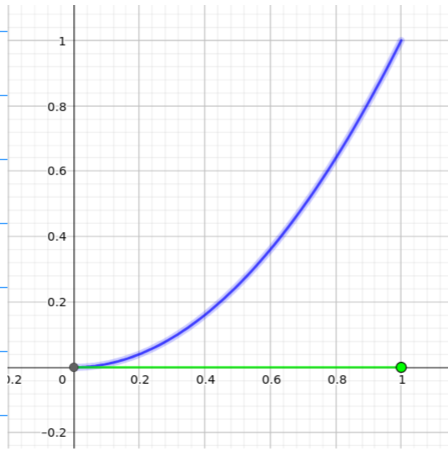


1 $y = 0, \quad x \in [0,1]$

$f(x,0) = x^2$

Max/min:

$$\boxed{\begin{array}{l} f(0,0) = 0 \\ f(1,0) = 1 \end{array}}$$

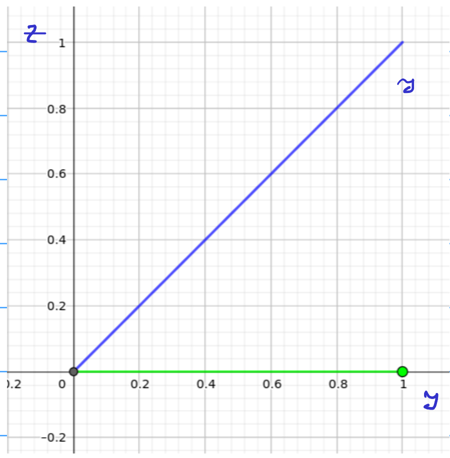


2: $x = 0, \quad y \in [0,1]$

$f(0,y) = y$

Max/min.

$$\boxed{\begin{array}{l} f(0,0) = 0 \\ f(0,1) = 1 \end{array}}$$

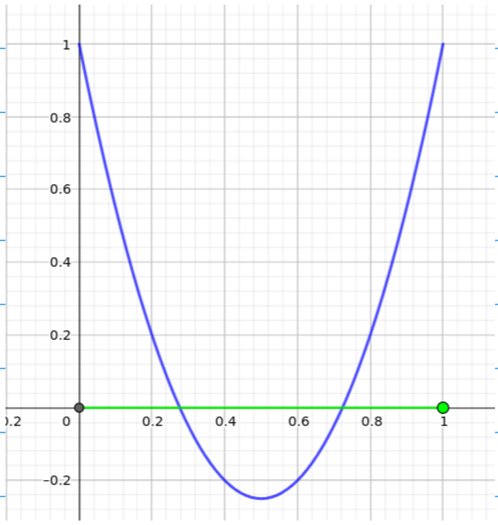


$f(x,y) = -4xy + x^2 + y$

(3) $x + y = 1, \quad x \in [0,1] \quad \to \quad y = 1 - x$

$\to \quad f(x, 1-x) = -4x(1-x) + x^2 + 1 - x$

$= \underbrace{5x^2 - 5x + 1}_{g(x)}$



$g'(x) = 10x - 5 = 0 \iff x = 1/2$

$\searrow \quad \nearrow$

$1/2$

Max/min: $\quad (y = 1 - x)$

$x = 0$	$\to y = 1$	$f(0,1) = 1$
$x = 1$	$\to y = 0$	$f(1,0) = 1$
$x = 1/2$	$\to y = 1/2$	$f(1/2, 1/2) = -1/4$

Resúmen:

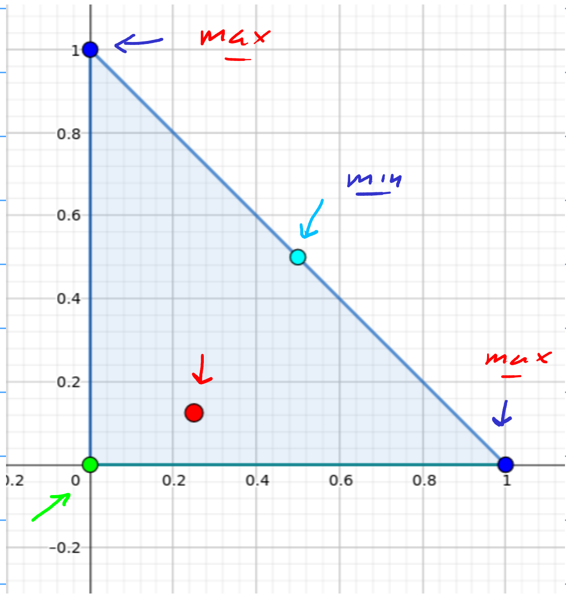
Interior: $\quad f(1/4, 1/8) = 1/16$

Esquinas: $\quad f(0,0) = 0$

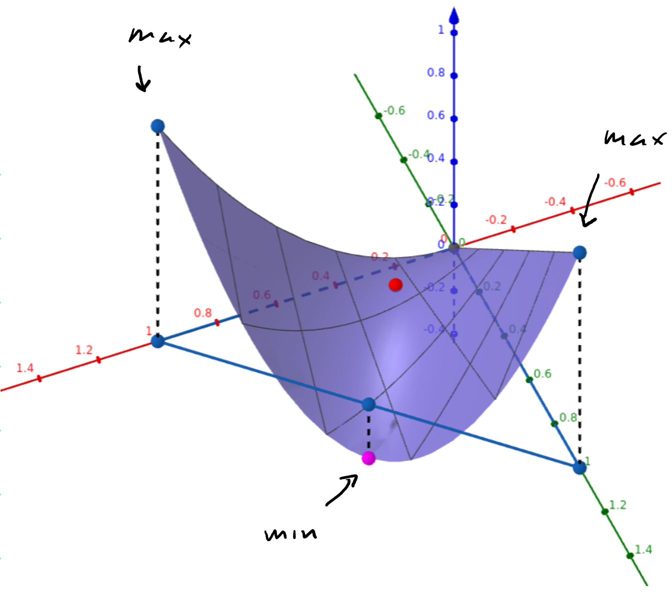
$f(1,0) = 1$

$f(0,1) = 1$ $\quad$ max

Lado (3): $\quad f(1/2, 1/2) = -1/4 \leftarrow$ min



$$\boxed{\begin{array}{l} \text{Máximo } 1 \text{ en } (1,0), (0,1) \\ \text{Mínimo } -1/4 \text{ en } (1/2, 1/2) \end{array}}$$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max
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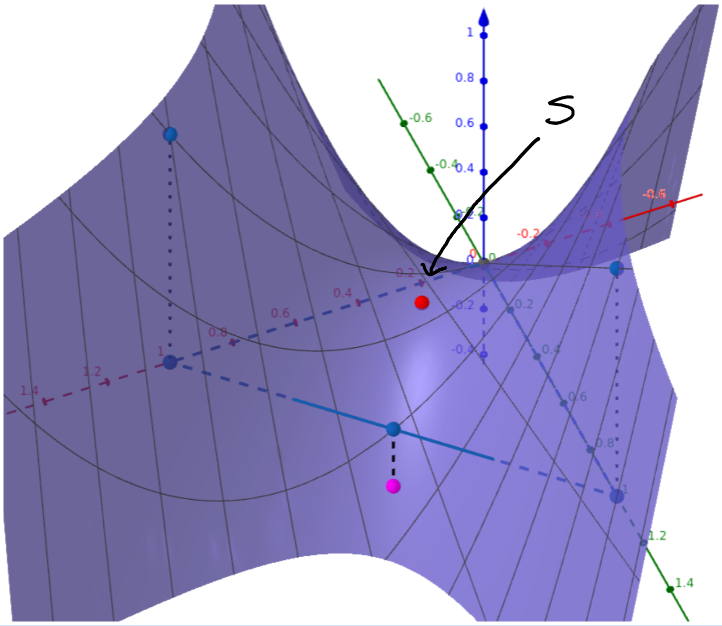
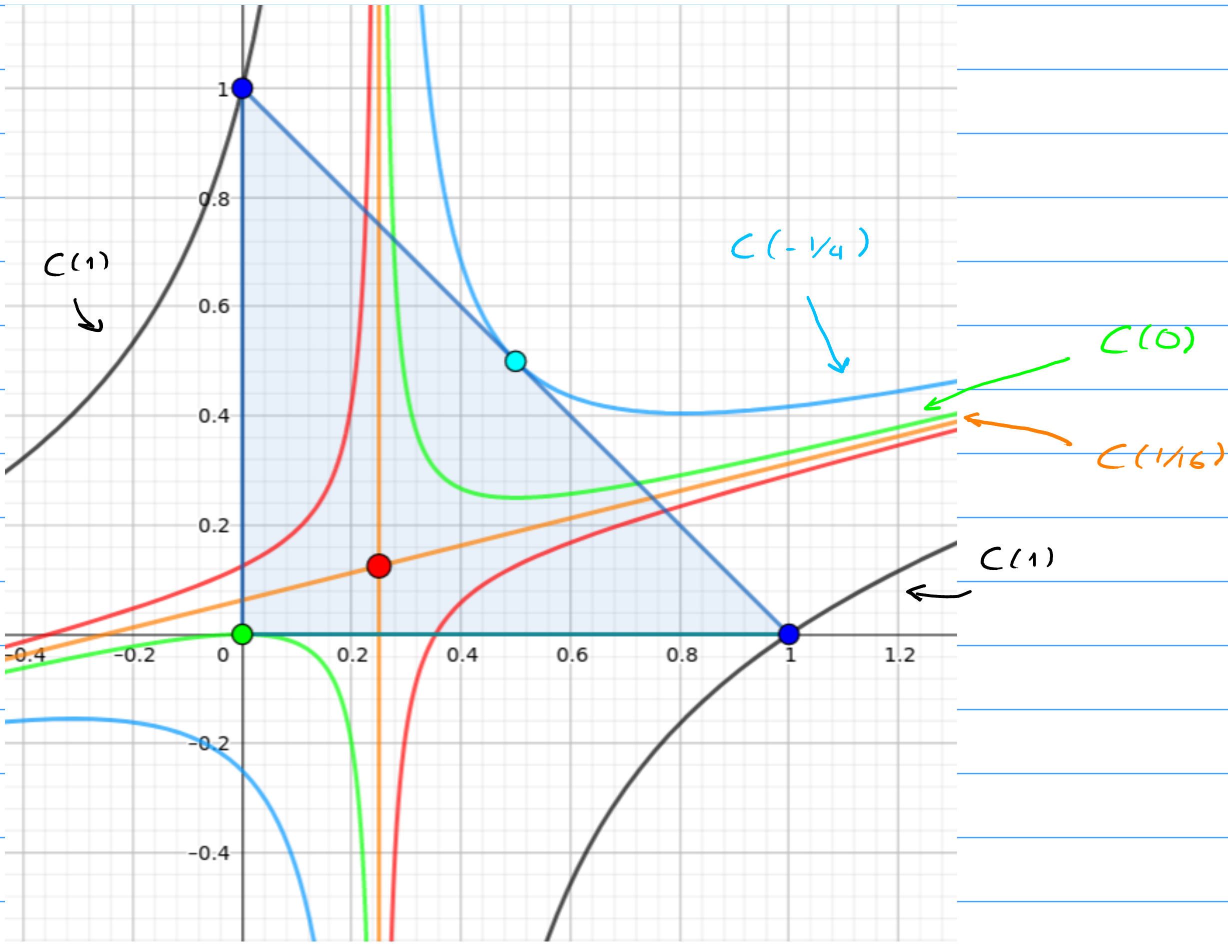
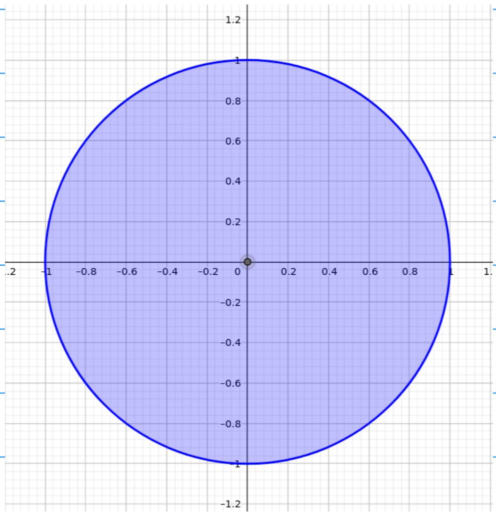

S


C(1)
C(-1/4)
C(0)
C(1/16)
C(1)

3) $f(x,y) = x\, e^{-x^2-y^2}$ en $A = \{(x,y) : x^2+y^2 \le 1\}$



A disco de centro $(0,0)$ y radio 1

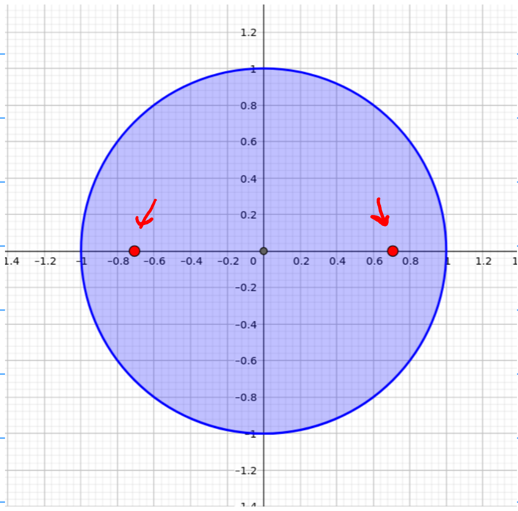
Weierstrass : f tiene max y min en A.

Interior : $x^2+y^2 < 1$

$f(x,y) = x\, e^{-x^2-y^2}$

$$\nabla f(x,y) = \left((1-2x^2)\, e^{-x^2-y^2} \,,\; -2xy\, e^{-x^2-y^2} \right)$$

$$\begin{cases} 1-2x^2 = 0 & \longrightarrow x = \pm 1/\sqrt{2} \\ -2xy = 0 & \longrightarrow x = 0 \quad \times \\ & \longrightarrow y = 0 \quad \checkmark \end{cases}$$

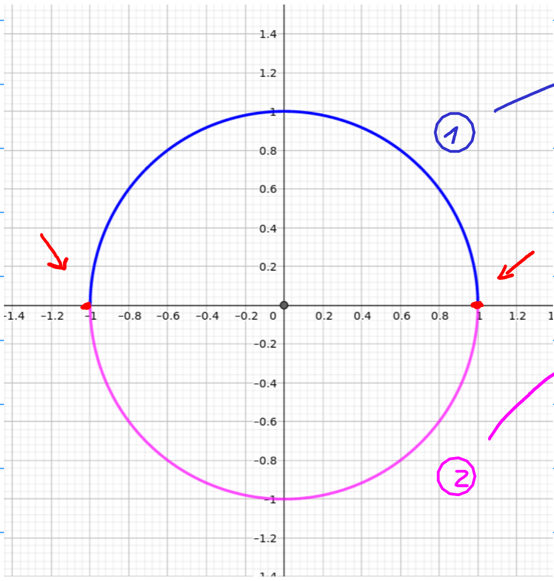


$(1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 0)$

$x^2+y^2 = \frac{1}{2} + 0 = \frac{1}{2} < 1$ ✓

$$\boxed{f(\pm 1/\sqrt{2}, 0) = \pm \frac{1}{\sqrt{2}} e^{-1/2}}$$

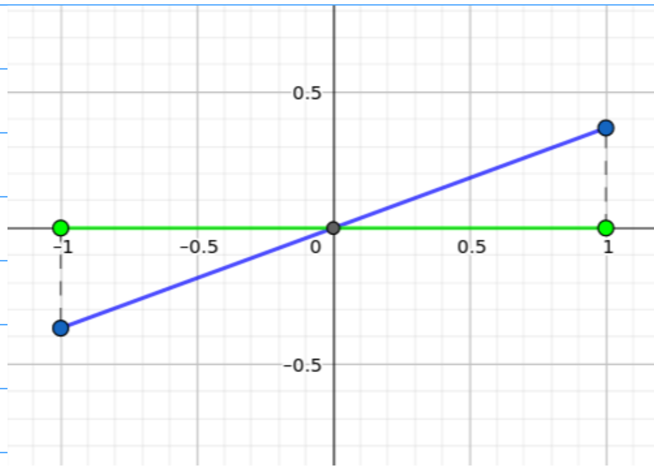
Borde :



① $y = \sqrt{1-x^2}$, $x \in [-1, 1]$

② $y = -\sqrt{1-x^2}$, $x \in [-1, 1]$

① : $f(x,y) = x\, e^{-\underbrace{(x^2+y^2)}_{=1 \text{ en el borde}}}$



$\rightarrow f(x, \sqrt{1-x^2}) = x\, e^{-1}$

Max / min :

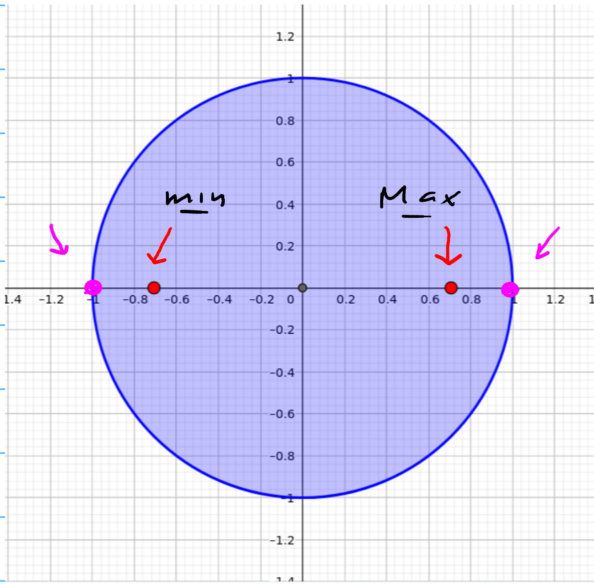
$x = \pm 1 \rightarrow y = 0$ $(\pm 1, 0)$

② $f(x,y) = x\, e^{-\underbrace{(x^2+y^2)}_{=1 \text{ en el borde.}}} \longrightarrow f(x, -\sqrt{1-x^2}) = x\, e^{-1}$

también : $x = \pm 1 \rightarrow y = 0$ $(\pm 1, 0)$

$$\boxed{f(\pm 1, 0) = \pm e^{-1}}$$

Resúmen:



$f(1/\sqrt{2}, 0) = \frac{1}{\sqrt{2}} e^{-1/2}$ ← Max

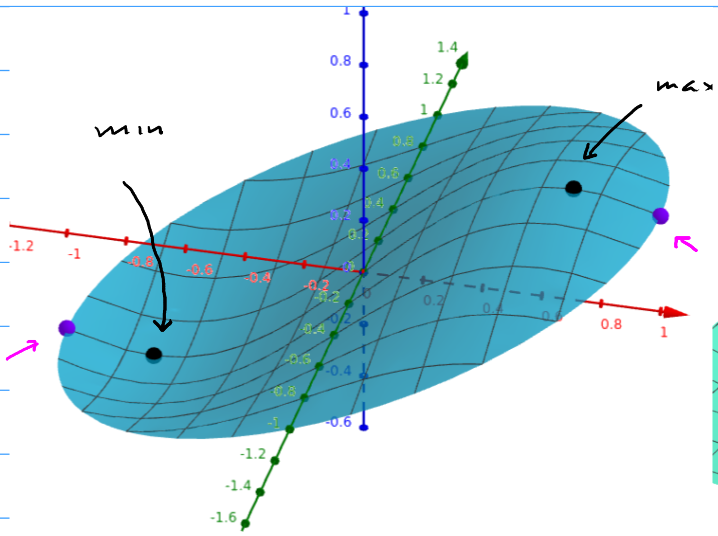
$f(-1/\sqrt{2}, 0) = -\frac{1}{\sqrt{2}} e^{-1/2}$ ← Min

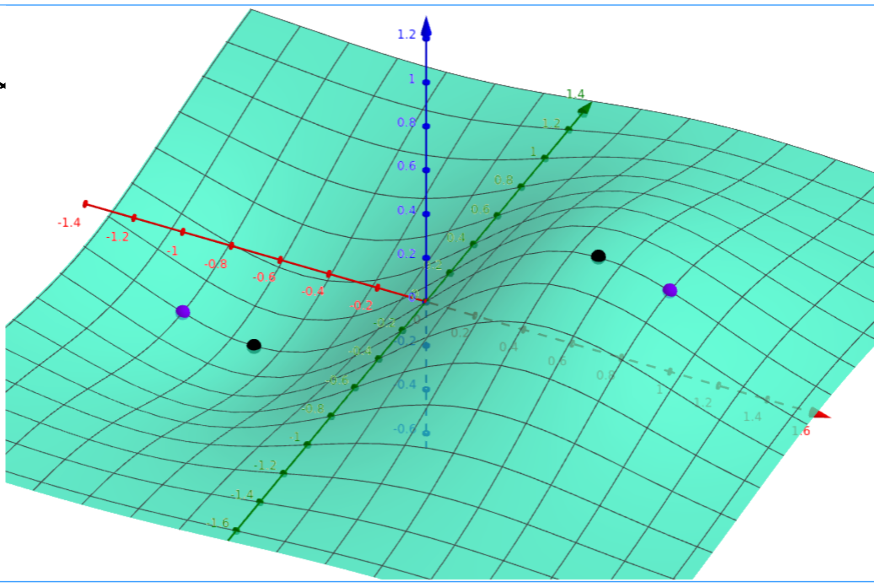
$f(1, 0) = e^{-1}$

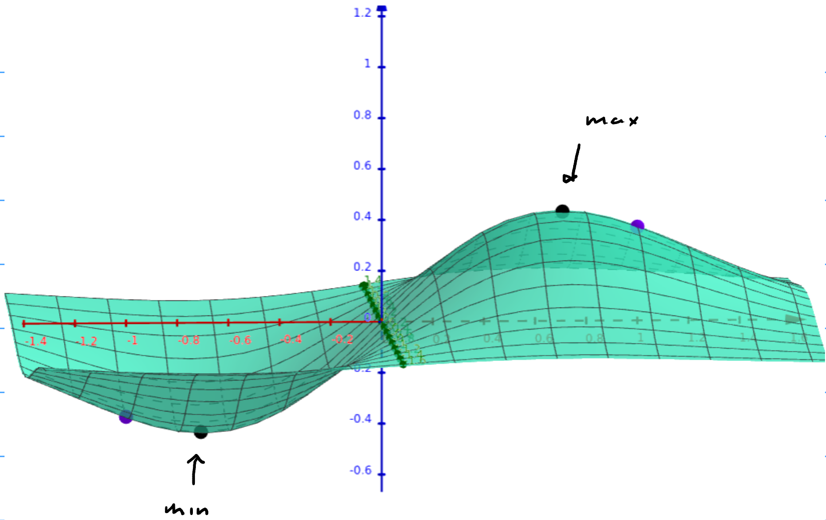
$f(-1, 0) = -e^{-1}$

$e^{-1} \approx 0,367\ldots \quad < \quad \frac{1}{\sqrt{2}} e^{-1/2} \approx 1,165\ldots$

Máximo	$\frac{1}{\sqrt{2}} e^{-1/2}$	en	$(1/\sqrt{2}, 0)$
Mínimo	$-\frac{1}{\sqrt{2}} e^{-1/2}$	en	$(-1/\sqrt{2}, 0)$






max
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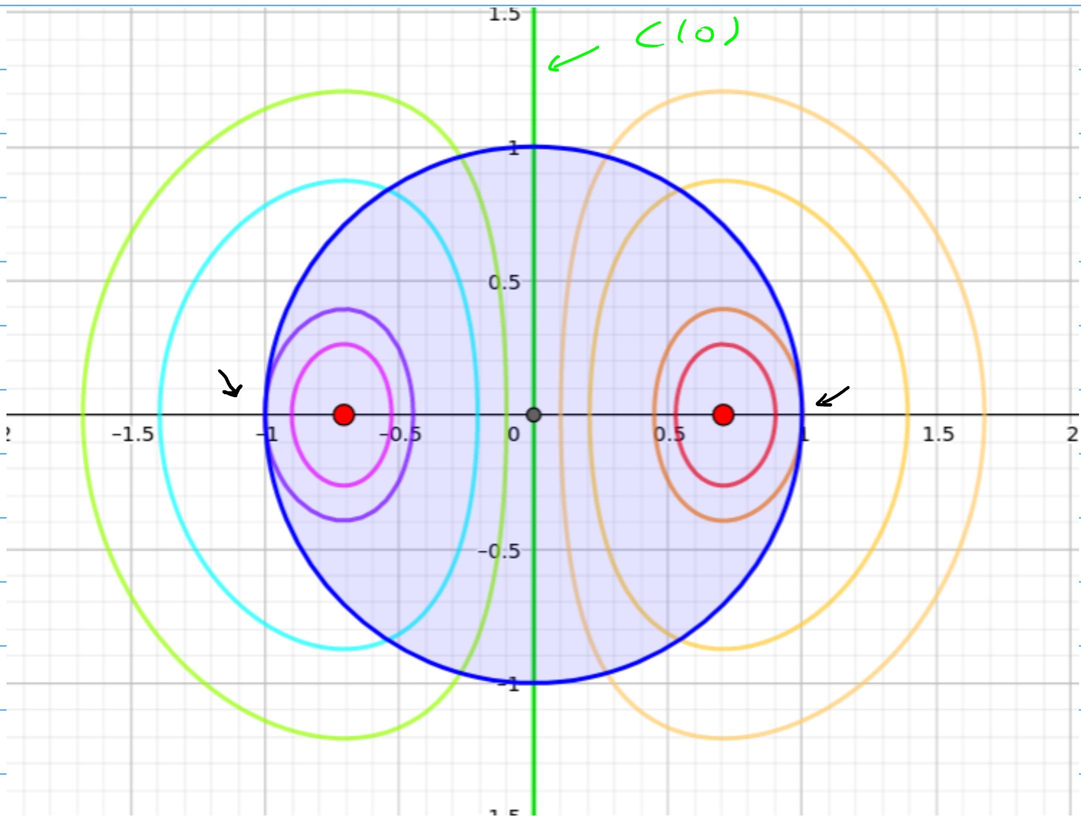
C(0)