Minimos Cuadrados (cuadrática, polinomica)

Datos:

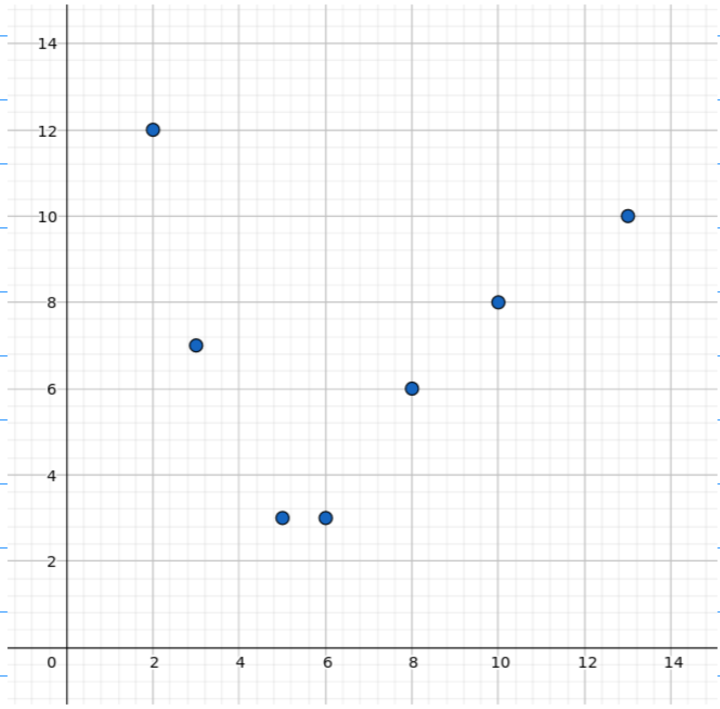
$P_1 = (x_1, y_1)$

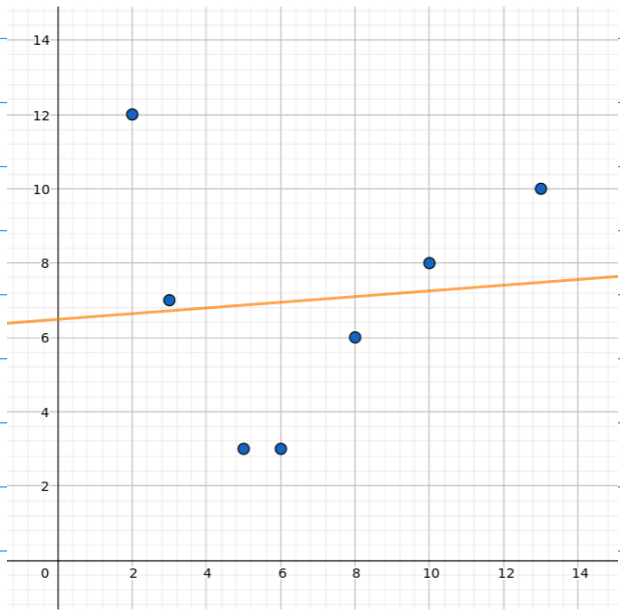
$\vdots$

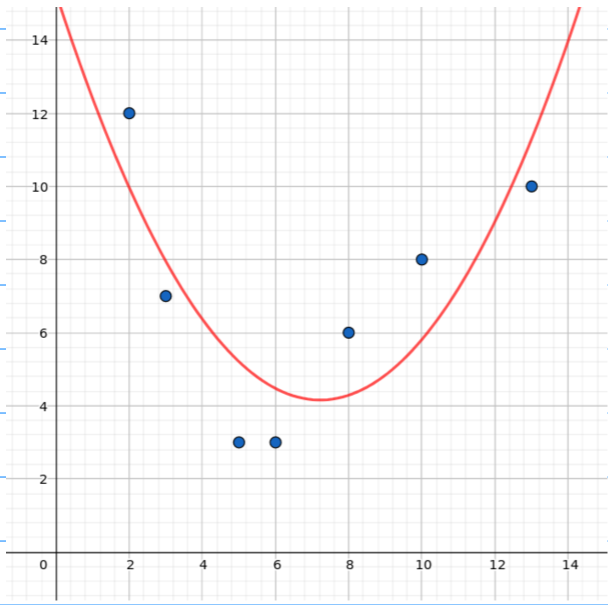
$P_n = (x_n, y_n)$

n puntos

$x_1, \ldots, x_n$ no se repiten.







aprox. lineal

aprox. cuadrática

$y = ax + b$

$y = ax^2 + bx + c$

$a = 0$

$y = bx + c$

(parábola o recta)

$(x_1, y_1), \ldots, (x_n, y_n)$ constantes , a, b, c variables.

Para $i = 1, \ldots, n$: $\quad \varepsilon_i = y_i - a x_i^2 - b x_i - c$

Minimizar $\quad E(a,b,c) = \sum \varepsilon_i^2$

$$\boxed{E(a,b,c) = \sum \left(y_i - a x_i^2 - b x_i - c \right)^2}$$

Teorema : ① E tiene mínimo absoluto.

② E tiene un único punto crítico.

$$\frac{\partial E}{\partial a}(a,b,c) = \sum 2 \left(y_i - a x_i^2 - b x_i - c \right)\left(- x_i^2 \right)$$

$$\frac{\partial E}{\partial b}(a,b,c) = \sum 2 \left(y_i - a x_i^2 - b x_i - c \right)\left(- x_i \right)$$

$$\frac{\partial E}{\partial c}(a,b,c) = \sum 2 \left(y_i - a x_i^2 - b x_i - c \right)\left(- 1 \right)$$

$$\frac{\partial E}{\partial a}(a,b,c) = -2 \left(\sum x_i^2 y_i - \left(\sum x_i^4\right) a - \left(\sum x_i^3\right) b - \left(\sum x_i^2\right) c \right)$$

$$\frac{\partial E}{\partial b}(a,b,c) = -2 \left(\sum x_i y_i - \left(\sum x_i^3\right) a - \left(\sum x_i^2\right) b - \left(\sum x_i\right) c \right)$$

$$\frac{\partial E}{\partial c}(a,b,c) = -2 \left(\sum y_i - \left(\sum x_i^2\right) a - \left(\sum x_i\right) b - n c \right)$$

$$\boxed{\left\{ \begin{array}{l} \left(\sum x_i^4\right) a + \left(\sum x_i^3\right) b + \left(\sum x_i^2\right) c = \sum x_i^2 y_i \\ \left(\sum x_i^3\right) a + \left(\sum x_i^2\right) b + \left(\sum x_i\right) c = \sum x_i y_i \\ \left(\sum x_i^2\right) a + \left(\sum x_i\right) b + n c = \sum y_i \end{array} \right.}$$

Obs: es lineal 3×3 de variables a, b, c

Teorema: Si $x_1, \dots, x_n$ no se repiten, es C.D.

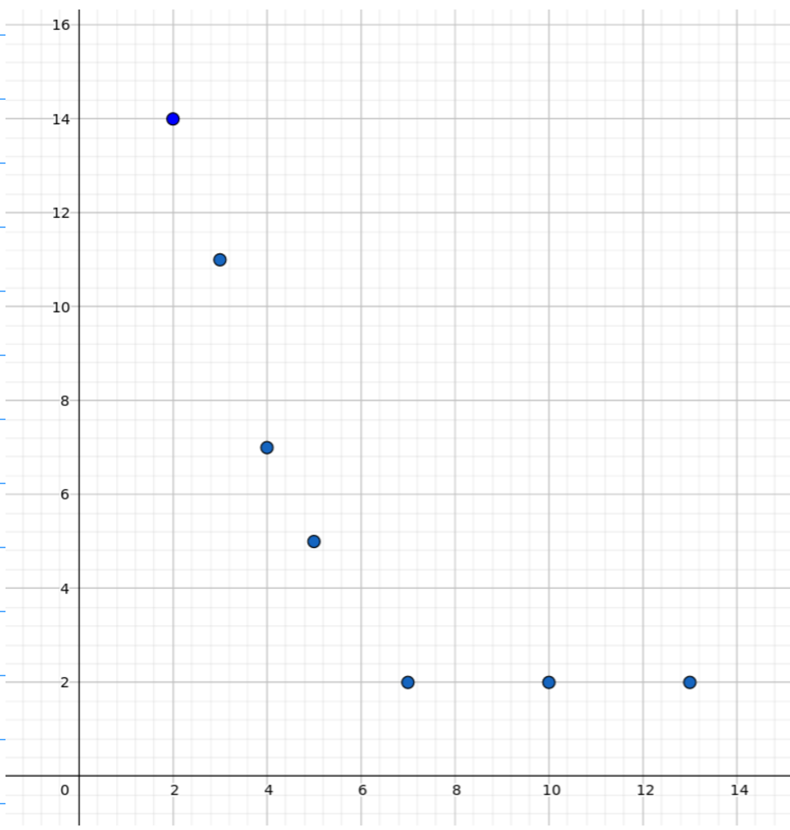
Conclusión: El polinomio de grado $\underline{\underline{2}}$ que más se ajusta a $P_1 = (x_1, y_1), \dots, P_n = (x_n, y_n)$ es

$y = ax^2 + by + c$ donde a, b, c verifica

$$\boxed{\begin{cases} \left(\sum x_i^4\right) a + \left(\sum x_i^3\right) b + \left(\sum x_i^2\right) c = \sum x_i^2 y_i \\ \left(\sum x_i^3\right) a + \left(\sum x_i^2\right) b + \left(\sum x_i\right) c = \sum x_i y_i \\ \left(\sum x_i^2\right) a + \left(\sum x_i\right) b + n c = \sum y_i \end{cases}}$$

Ejemplo:

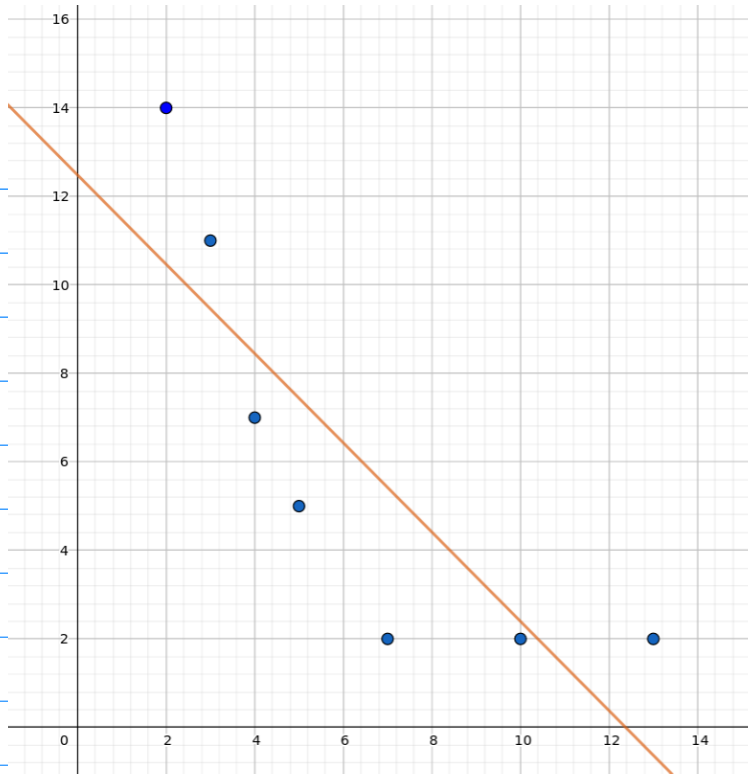
X	Y
2	14
3	11
4	7
5	5
7	2
10	2
13	2



Recordar: aprox. lineal

$$y = -1{,}01\,x + 12{,}48$$

Min $E = 41{,}7$



Mejorar usando aprox. cuadrática ?

X	Y	X^2	XY		X^4	X^3	YX^2
2	14	4	28	1	16	8	56
3	11	9	33	1	81	27	99
4	7	16	28	1	256	64	112
5	5	25	25	1	625	125	125
7	2	49	14	1	2401	343	98
10	2	100	20	1	10000	1000	200
13	2	169	26	1	28561	2197	338
44	43	372	174	7	41940	3764	1028

$$\begin{cases} 41940\,a + 3764\,b + 372\,c = 1028 \\ 3764\,a + 372\,b + 44\,c = 174 \\ 372\,a + 44\,b + 7\,c = 43 \end{cases}$$

a	0.2083562624
b	-4.121849549
c	20.978978648

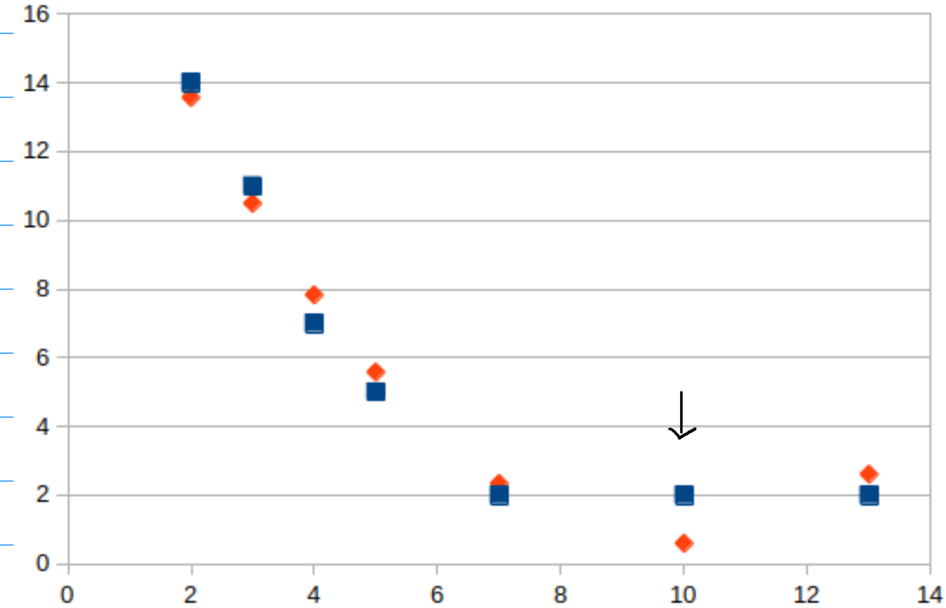
$$\boxed{\begin{aligned} a &\approx 0{,}21 \\ b &\approx -4{,}12 \\ c &\approx 20{,}98 \end{aligned}}$$

aprox. cuadrática:

$$\boxed{y = 0{,}21\,x^2 - 4{,}12\,x + 20{,}98}$$

X	Y	aX^2+bX+c	error	error^2
2	14	13.5687046	0.4312953995	0.186015722
3	11	10.488636364	0.5113636364	0.261492769
4	7	7.8252806516	-0.825280652	0.681088154
5	5	5.5786374642	-0.578637464	0.334821315
7	2	2.3354886639	-0.335488664	0.112552644
10	2	0.5961093991	1.4038906009	1.970908819
13	2	2.6071428571	-0.607142857	0.368622449

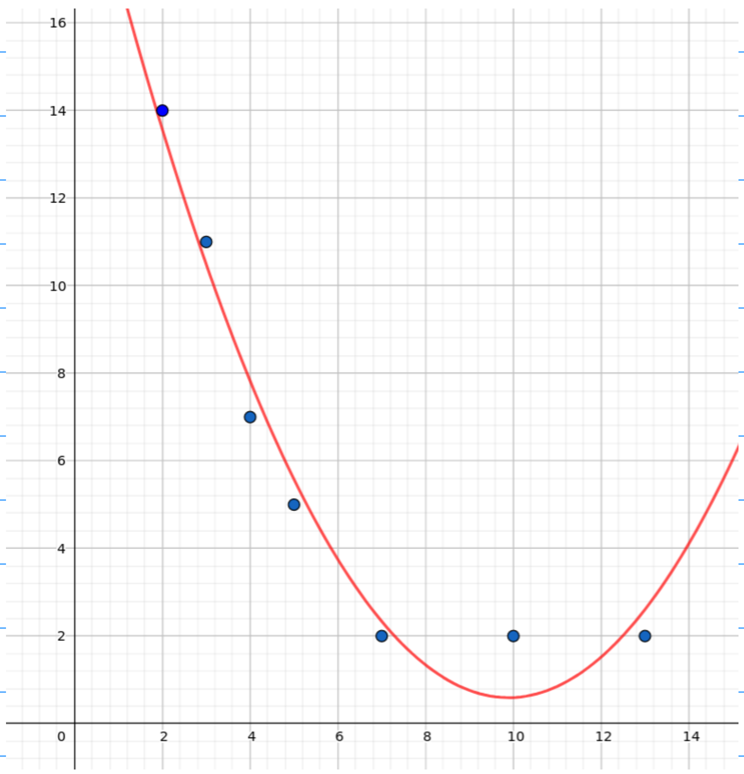
3.915501871

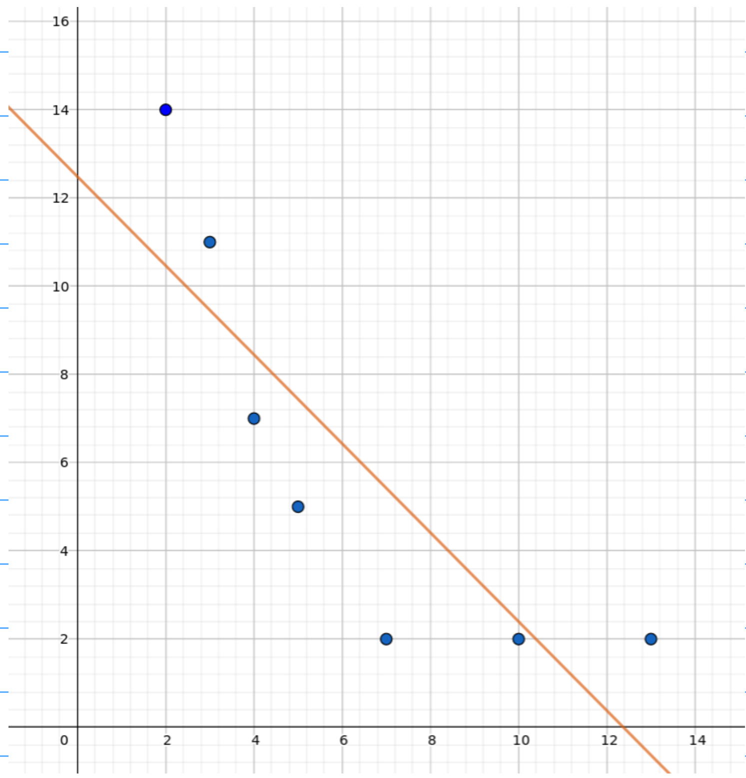


Min $E \approx 3,9$

Aprox. lineal:

Min $E \approx 41,7$





$$\boxed{y = 0,21x^2 - 4,12x + 20,98}$$

$$\boxed{y = -1,01x + 12,48}$$

Obs :

Cuadrática

$$\boxed{y = 0{,}21\,x^2 - 4{,}12\,x + 20{,}98}$$

Lineal

$$\boxed{y = -1{,}01\,x + 12{,}48}$$

No cooinciden

tampoco

(en general)

Para Min. Cuadrados, la aprox. cuadrática NO es agregar un término a la aprox. lineal.

Mínimos Cuadrados con polinomios de grado mayor

3er grado: (cúbica)

$$y = a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0$$

Minimizar

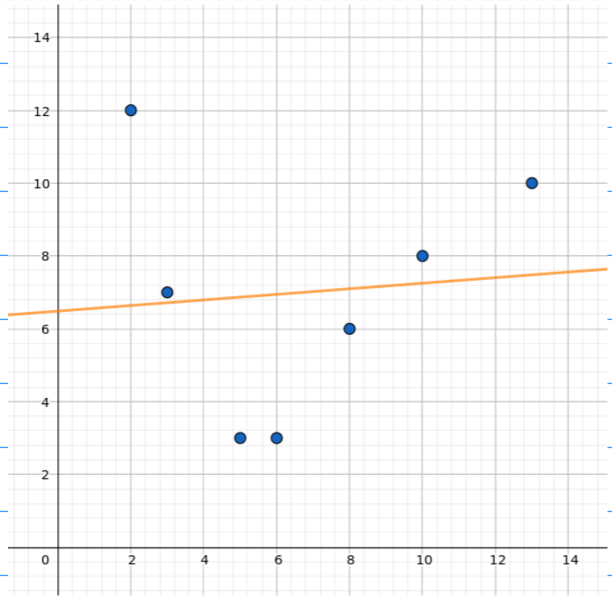
$$E(a_3, a_2, a_1, a_0) = \sum \varepsilon_i^2$$

⇝ Sistema 4 x 4 en a_3, a_2, a_1, a_0

⇝ Solución : coef. de la aprox. cúbica.

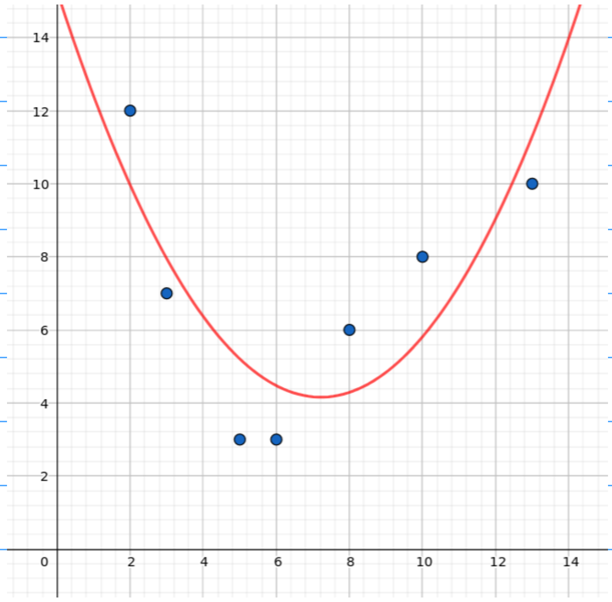
Ejemplo :

X	2	3	5	6	8	10	13
Y	12	7	3	3	6	8	10



Aprox. Lineal

$$y = 0{,}07\,x + 6{,}48$$

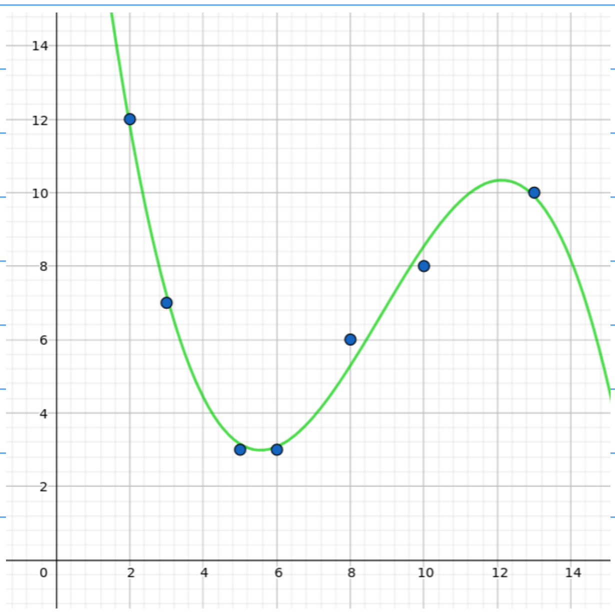


Aprox. cuadrática

$$y = 0{,}21\,x^2 - 3{,}07\,x + 15{,}27$$

Aprox. cúbica

$$y = -0{,}05\,x^3 + 1{,}38\,x^2 - 10{,}53\,x + 27{,}77$$



Grado k: $y = a_k x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0$

$\rightsquigarrow$ Sistema $(k+1) \times (k+1)$

A mayor k: $\longrightarrow$ $E = \sum \varepsilon_i^2$ más chico

$\searrow$ Cálculo más complejo.

$k = n-1 \longrightarrow E = 0$ (pasa por todos los puntos)

$\searrow$ si n es grande, cálculo muy difícil.