Ejercicio 4:

a) Definicion de $\Bbbk$-coalgebra:

$(C, \Delta, \varepsilon) \in \text{'}Vec_{\Bbbk}\text{'}$

es una categoria monoidal:

$(Vec_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$

$\forall\, X, Y, Z \in Vec_{\Bbbk}$

$$\left.\begin{array}{l} \exists\ a : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z) \\ \lambda : X \otimes \Bbbk \longrightarrow X \\ \rho : \Bbbk \otimes X \longrightarrow X \end{array}\right\} \text{fijos en } Vec_{\Bbbk}$$

$\Delta : C \to C \otimes C$

se llama coproducto

$\varepsilon : C \to \Bbbk$

se llama counidad

cumplen lo siguiente:

```tikzcd
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[d, "id \otimes \Delta"'] & \circlearrowleft & C \otimes C \arrow[d, "\Delta \otimes id"] \\
C \otimes (C \otimes C) \arrow[rr, leftrightarrow, "\cong", "a"'] & & (C \otimes C) \otimes C
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```tikzcd
\Bbbk \otimes C \arrow[r, "\rho", "\cong"'] & C \arrow[d, "\Delta"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & C \otimes \Bbbk \arrow[l, "\lambda"', "\cong"] \\
 & C \otimes C \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] &
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b) $\mathbb{k}[x]$, $\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j$, $\varepsilon(x^n) = \delta_{n,0}$ define una coalgebra.

(hay que ver que $\Delta, \varepsilon$ cumplen los diagramas)

$\Delta : \mathbb{k}[x] \to \mathbb{k}[x] \otimes \mathbb{k}[x]$

$$\begin{tikzcd}
 & x^n \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
\sum_{i+j=n} x^i \otimes x^j \arrow[d, "\Delta \otimes id"'] & & \sum_{i+j=n} x^i \otimes x^j \arrow[d, "id \otimes \Delta"] \\
\sum_{i+j=n} \left(\sum_{l+k} x^l \otimes x^k\right) \otimes x^j \arrow[rr, leftrightarrow, "\simeq"] & & \sum_{i+j=n} x^i \otimes \left(\sum_{l+k=j} x^l \otimes x^k\right)
\end{tikzcd}$$

$$\sum_{i+j=n} \left(\sum_{l+k} x^l \otimes x^k\right) \otimes x^j = \sum_{l+k+j=n} (x^l \otimes x^k) \otimes x^j \qquad \sum_{i+j=n} x^i \otimes \left(\sum_{l+k=j} x^l \otimes x^k\right) = \sum_{i+l+k=n} x^i \otimes (x^l \otimes x^k)$$

$\varepsilon : \mathbb{k}[x] \to \mathbb{k}$

$$\begin{tikzcd}
1 \otimes x^n = \sum_{i+j=n} \delta_{i,0} \otimes x^j \arrow[r, "\simeq"] & x^n \arrow[d] & \sum_{i+j=n} x^i \otimes \underbrace{\delta_{j,0}}_{\substack{0 \text{ si } j \neq 0 \\ 1 \text{ si } \underbrace{j=0}_{\Rightarrow i=n}}} = x^n \otimes 1 \arrow[l, "\simeq"'] \\
 & \sum_{i+j=n} x^i \otimes x^j \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] \arrow[ul, phantom, shift left=6, "G"] \arrow[ur, phantom, shift right=6, "G"] &
\end{tikzcd}$$

<u>Obs</u>:

① $\varepsilon$ es un morfismo de $\mathbb{k}$-alg:

$\varepsilon$ se extiende por linealidad $\Rightarrow \varepsilon\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^n a_i \, \varepsilon(x^i) = a_0$ por lo que es $\mathbb{k}$-lineal. Ademas $\varepsilon(1) = 1$.

Si $p = \sum_{i=0}^n a_i x^i$, $q = \sum_{i=0}^m b_i x^i$ $\Rightarrow \varepsilon(pq) = a_0 b_0 = \varepsilon(p)\,\varepsilon(q)$

② $\Delta$ no es morfismo de $\mathbb{k}$-alg.

$\Delta(x) = x \otimes 1 + 1 \otimes x \Rightarrow \Delta(x) \cdot \Delta(x) = (x \otimes 1 + 1 \otimes x)^2 = x^2 \otimes 1 + 2\, x \otimes x + 1 \otimes x^2$

pero por otro lado $\Delta(x^2) = x^2 \otimes 1 + x \otimes x + 1 \otimes x^2$

c) Definicion de bialgebra:

$(A, m, u, \Delta, \varepsilon)$ es una bialgebra si

- $(A, m, u)$ es un algebra
- $(A, \Delta, \varepsilon)$ una coalgebra.
- $\Delta, \varepsilon$ son morfismos de algebra

<u>Obs.</u> que $\Delta, \varepsilon$ sean morfismos de algebras implica lo siguiente:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] & C \otimes C & & (C \otimes C) \otimes (C \otimes C) \arrow[ll, "m \otimes m"'] \\
C \otimes C \arrow[u, "m"] \arrow[r, "\Delta \otimes \Delta"'] \arrow[ur, phantom, "\circlearrowleft"] & (C \otimes C) \otimes (C \otimes C) \arrow[urr, "\tau"'] & &
\end{tikzcd}
\quad \left.\right\} \quad \begin{array}{c} \Delta(c_1 \cdot c_2) \\ \parallel \\ \Delta(c_1) \cdot \Delta(c_2) \end{array}$$

$$\begin{tikzcd}
C \arrow[r, "\Delta"] & C \otimes C \\
\mathbb{K} \arrow[u, "u"] \arrow[ur, "u \otimes u"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[r, "\varepsilon"] & \mathbb{K} \\
\mathbb{K} \arrow[u, "u"] \arrow[ur, "id"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] &
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\varepsilon"] & \mathbb{K} \\
C \otimes C \arrow[u, "m"] \arrow[r, "\varepsilon \otimes \varepsilon"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{K} \otimes \mathbb{K} \arrow[u]
\end{tikzcd}$$

d) $\mathbb{K}[x]$ con $\Delta(x^n) = \sum_{i+j=n} \binom{n}{i} x^i \otimes x^j$, $\varepsilon(x^n) = \delta_{n,0}$ y la estructura de $\mathbb{K}$-alg usual en $\mathbb{K}[x]$ es una $\mathbb{K}$-bialgebra.

En la parte b) ya vimos que $\varepsilon$ es una counidad y morfismo de $\mathbb{K}$-alg.

Veamos que $\Delta$ es un coprod:

$\Delta : \mathbb{K}[x] \to \mathbb{K}[x] \otimes \mathbb{K}[x]$

$$\begin{tikzcd}[column sep=tiny]
& x^n \arrow[dl, "\Delta"'] \arrow[dr] & \\
\sum\limits_{i+j=n} \binom{n}{i} x^i \otimes x^j \arrow[d, "id\otimes\Delta"] & & \sum\limits_{i+j=n} \binom{n}{i} x^i \otimes x^j \arrow[d, "\Delta\otimes id"] \\
\sum\limits_{i+j=n} \binom{n}{i} x^i \otimes \left( \sum\limits_{k+l=j} \binom{j}{k} x^k \otimes x^l \right) & & \sum\limits_{i+j=n} \binom{n}{i} \left( \sum\limits_{k+l=i} \binom{i}{k} x^k \otimes x^l \right) \otimes x^j
\end{tikzcd}$$

$$\sum_{i+k+l=n} \underbrace{\binom{n}{i}}_{= \binom{n}{n-i} = \binom{n}{k+l}} \binom{k+l}{k} x^i \otimes x^k \otimes x^l \longleftrightarrow \sum_{k+l+j=n} \binom{n}{k+l} \binom{k+l}{k} x^k \otimes x^l \otimes x^j$$

$\Rightarrow (\mathbb{K}[x], \Delta, \varepsilon)$ es una $\mathbb{K}$-coalg.

Para ver que $(\mathbb{K}[x], m, u, \Delta, \varepsilon)$ es una bialgebra nos falta ver que $\Delta$ es morfismo de $\mathbb{K}$-alg.

Como se extiende por linealidad es $\mathbb{K}$-lineal, además $\Delta(1) = 1 \otimes 1$.

Veamos que se porta bien con el producto.

$$\Delta(x^{n+m}) = \sum_{i+j=n+m} \binom{n+m}{i} x^i \otimes x^j$$

$$\Delta(x^n) \cdot \Delta(x^m) = \left( \sum_{i+j=n} \binom{n}{i} x^i \otimes x^j \right) \cdot \left( \sum_{k+l=m} \binom{m}{k} x^k \otimes x^l \right)$$

$$\Rightarrow \Delta(X^n)\Delta(X^m) = \sum_{\substack{i+j=n \\ k+l=m}} \binom{n}{i}\binom{m}{k} X^{i+k} \otimes X^{j+l} = \sum_{t=0}^{n+m} \underbrace{\left(\sum_{s=0}^{t} \binom{n}{s}\binom{m}{t-s}\right)}_{\binom{n+m}{t}} X^t \otimes X^{n+m-t}$$

$$= \sum_{t=0}^{n+m} \binom{n+m}{t} X^t \otimes X^{n+m-t} = \Delta(X^{n+m})$$

e) $\Bbbk M$ es una $\Bbbk$-bialgebra con la estructura trivial de $\Bbbk$-coalgebra, para cualquier monoide $M$.

estructura trivial de coalgebra: $\Delta : \Bbbk M \to \Bbbk M \otimes \Bbbk M$, $m \mapsto m \otimes m$ || $\varepsilon : \Bbbk M \to \Bbbk$, $m \mapsto 1$

veamos que son morfismos de $\Bbbk$-alg.

como los extendemos por linealidad solo hay que ver el producto.

$\Delta(m)\Delta(m') = (m \otimes m)(m' \otimes m') = mm' \otimes mm' = \Delta(mm')$

$\varepsilon(m)\varepsilon(m') = 1 \cdot 1 = 1 = \varepsilon(mm')$