Fubini Foiled: Katok's Paradoxical Example in Measure Theory

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Anatole Katok has proved the following (unpublished):

There exists a measurable set E of area one in the unit square $(0, 1) \times [0, 1]$, *together with a family of disjoint smooth real analytic curves* Γ_β *which fill out this square, so that each curve* Γ_β *intersects the set E in at most a single point.*

In other words, we can construct a set of full two-dimensional Lebesgue measure by selecting at most one point from each Γ_β. The construction is completely explicit and natural. (The curves in question depend continuously on the parameter $\beta \in [0, 1]$, and form a topological foliation of the square.) Note however, in any such example, that Γ_β cannot depend smoothly on the parameter β. In the case of a smoothly parameterized family, it follows easily from Fubini's Theorem that E must intersect almost every Γ_β in a set of full one-dimensional Lebesgue measure.)

Figure 1 shows an example of such a family of curves, based on a construction similar but not identical[1] to that of Katok. To begin our construction, we will need Borel's *Strong Law of Large Numbers.* Suppose that we toss a biased coin, which lands with the "zero" side up with probability $p(0) = p$ and lands with the "one" side up with probability $p(1) = 1 - p$. Then, for a sequence of n independent coin tosses, we obtain a sequence of n bits, where the probability of a given sequence $(b_1, \ldots, b_n)$ is given by the product rule

$$\text{probability}(b_1, \ldots, b_n) = p(b_1) \cdots p(b_n). \tag{1}$$

Jakob Bernoulli, 300 years ago, showed that the frequency of ones, that is the ratio $(b_1 + \cdots + b_n)/n$, is likely to be close to $p(1)$ if n is large. Emile Borel sharpened this result as follows. Consider an infinite sequence of coin tosses. Formula (1) gives rise to a probability measure, called the $(p(0), p(1))$-*Bernoulli product measure,* on the space $\{0, 1\}^{\mathbf{N}}$ consisting of all infinite sequences $(b_1, b_2, \ldots)$ of zeros and ones. Borel showed that the limiting frequency

$$\lim_{n \to \infty} (b_1 + \cdots + b_n)/n$$

exists and is precisely equal to $p(1)$, with probability 1. In other words, this limit equals $p(1)$ for all sequences in $\{0, 1\}^{\mathbf{N}}$ outside of a subset of measure zero with respect to the Bernoulli product measure.

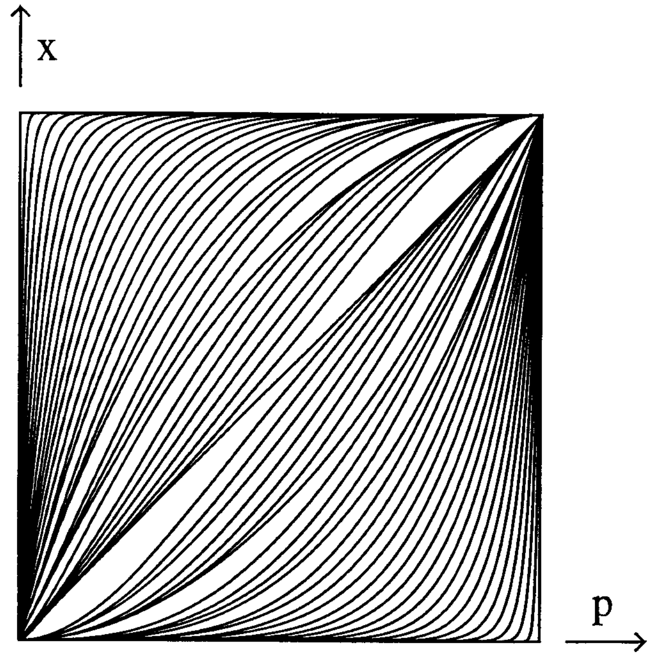
We can restate this result using only real variables (instead of coin tosses) as follows. Let x vary over the circle $\mathbf{R}/\mathbf{Z}$, and let the parameter p vary over the open unit interval $(0, 1)$. For each p, define a piecewise linear map f_p of degree two from the circle $\mathbf{R}/\mathbf{Z}$ to itself by the formula

$$f_p(x) = \begin{cases} x/p & \text{for } x \in I_0(p) = [0, p) \subset \mathbf{R}/\mathbf{Z} \\ (x - p)/(1 - p) & \text{for } x \in I_1(p) = [p, 1) \subset \mathbf{R}/\mathbf{Z}, \end{cases} \tag{2}$$

as plotted in Figure 2.

[1] Katok's example is based on a family of degree-two Blaschke products mapping the unit circle to itself. I am grateful to C. Pugh for describing it to me. A different version of the construction, based on tent maps of the interval, has been given by J. Yorke, also unpublished.

FIGURE 1



(Alternatively, we can think of f_p as a discontinuous map from the unit interval [0, 1] to itself. The argument would be the same in either case.)

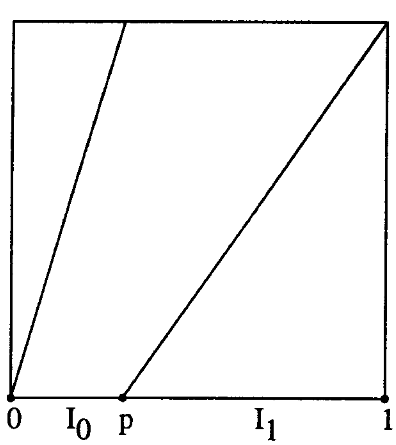
It is easy to see that each f_p is *measure-preserving.* In fact, for any interval $J \subset [0, 1]$ of length $\ell(J)$, the preimage $f_p^{-1}(J)$ consists of an interval in $I_0(p)$ of length $p(0)\ell(J)$ together with an interval in $I_1(p)$ of length $p(1)\ell(J)$, so that the total length is $\ell(f_p^{-1}(J)) = \ell(J)$. [Here again, we set $p(0) = p$ and $p(1) = 1 - p$.] For fixed p, we can code each point $x \in \mathbf{R/Z}$ by an infinite sequence $(b_1, b_2, b_3, \ldots) \in \{0, 1\}^{\mathbf{N}}$ of bits, defined as follows. Let

$$x = x_1 \mapsto x_2 \mapsto x_3 \mapsto \cdots \tag{3}$$

be the orbit of x under f_p, and set each b_n equal to zero or one according to whether x_n belongs to the interval $I_0(p)$ or $I_1(p)$ modulo $\mathbf{Z}$. We will call $(b_1, b_2, \ldots)$ the *symbol sequence* associated with x and f_p.

(*Note*: Almost every symbol sequence in $\{0, 1\}^{\mathbf{N}}$ can occur in this construction. However, sequences ending with infinitely many ones do not occur. Compare the Remarks at the end of this article.)

FIGURE 2



In terms of this coding, note that f_p corresponds to the shift map

$$(b_1, b_2, b_3, \ldots) \mapsto (b_2, b_3, b_4 \ldots).$$

It is not difficult to check that the Lebesgue measure on $\mathbf{R/Z}$ corresponds precisely to the $(p(0), p(1))$–Bernoulli product measure on $\{0, 1\}^{\mathbf{N}}$; that is, the length of the interval consisting of all x for which the associated symbol sequence starts with some specified finite sequence $(b_1, \ldots, b_n)$ of bits is equal to the product $p(b_1) \cdots p(b_n)$.

The Strong Law of Large Numbers can now be restated as follows: Choose some fixed parameter p and consider the map f_p. *For Lebesgue almost every point $x \in \mathbf{R/Z}$, the frequency of ones in the associated symbol sequence $(b_1, b_2, b_3, \ldots)$ is defined and equal to $p(1) = 1 - p$.*

Let $E \subset (0, 1) \times \mathbf{R/Z}$ be the set consisting of all pairs (p, x) such that the frequency of ones, for the symbol sequence of x under the action of f_p, is defined and equal to $1 - p$. It is not difficult to check that E is a measurable set. For each fixed p, let C_p denote the circle $\{p\} \times \mathbf{R/Z} \subset (0, 1) \times \mathbf{R/Z}$. Since the intersection of E with each C_p has one-dimensional Lebesgue measure $\ell_1(E \cap C_p) = 1$, it follows from Fubini's Theorem that E has two-dimensional Lebesgue measure

$$\ell_2(E) = \int_0^1 \ell_1(E \cap C_p)\, dp = 1.$$

Next, define a family of smooth curves Γ_β as follows. Let β be any number in the interval [0, 1). Form the base-two expansion

$$\beta = 0.b_1b_2b_3 \cdots_{\text{(base 2)}} = \sum b_n/2^n,$$

and let Γ_β be the set of all pairs $(p, x) \in (0, 1) \times \mathbf{R/Z}$ such that the symbol sequence of x under the map f_p is equal to $(b_1, b_2, b_3, \ldots)$. Clearly the Γ_β are disjoint sets with union equal to $(0, 1) \times \mathbf{R/Z}$. To prove that each Γ_β is a smooth real analytic curve, we proceed as follows. For each orbit (3), it follows easily from (2) that

$$x_n = b_n p(0) + x_{n+1}\, p(b_n).$$

A straightforward induction then shows that $x = x_1$ is given by the series

$$\begin{aligned} x &= x(p, \beta) \\ &= p(0)(b_1 + p(b_1)(b_2 + p(b_2)(b_3 + \cdots)\cdots)\cdots) \\ &= p(0)(b_1 + b_2p(b_1) + b_3p(b_1)p(b_2) \\ &\quad + b_4p(b_1)p(b_2)p(b_3) + \cdots). \end{aligned} \tag{4}$$

Set $p(0) = p = (1 + t)/2$ and $p(1) = 1 - p = (1 - t)/2$. If $|t| \le c < 1$, then the nth term in the series (4) has absolute value at most $[(1 + c)/2]^n$. Hence, this series converges uniformly. If fact, this is true even if we allow complex values of t with $|t| \le c < 1$. Hence, by the Weierstrass Uniform Convergence Theorem, for each fixed β the series (4) defines x as an analytic function of t throughout the interval $|t| < 1$, or as an analytic function of p throughout the interval $0 < p < 1$. Evidently, Γ_β is just the graph of this real analytic function $p \mapsto x(p, \beta)$.

Finally, since a given symbol sequence can have at most one limiting frequency $\lim(b_1 + \cdots + b_n)/n = 1 - p$, it follows that each Γ_β can intersect the measurable set E in at most a single point $(p, x(p, \beta))$.

Remarks

The function $\beta \mapsto x(p, \beta)$ is strictly monotone and maps the interval $[0, 1)$ onto itself. Hence, it is a homeomorphism. In fact, it follows easily that the correspondence $(p, \beta) \mapsto (p, x(p, \beta))$ maps the product space $(0, 1) \times \mathbf{R}/\mathbf{Z}$ homeomorphically onto itself.

Here is an alternative argument. It is clear that expression (4) depends continuously on the two variables $p \in (0, 1)$ and $(b_1, b_2, \ldots) \in \{0, 1\}^{\mathbf{N}}$, where we give this space of sequences the cartesian product topology. The correspondence

$$(b_1, b_2, \ldots) \mapsto \sum b_n/2^n \in [0, 1]$$

from symbol sequence to real variable is not quite one-to-one, as every dyadic rational $m/2^n$ has two distinct base-two expansions, ending either with infinitely many zeros or with infinitely many ones. However, a straightforward computation shows that these two expansions give rise to the same sum $x(p, \beta)$, and it follows easily that the correspondence $(p, \beta) \mapsto x(p, \beta)$ is indeed continuous.

We can interpret these constructions dynamically as follows. It is not hard to show that each $f_p : \mathbf{R}/\mathbf{Z} \mapsto \mathbf{R}/\mathbf{Z}$ is uniquely topologically conjugate to the angle-doubling map $f_{1/2}(x) \equiv 2x \pmod{\mathbf{Z}}$; that is, there is a unique homeomorphism $h_p : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ which conjugates $f_{1/2}$ to f_p, so that

$$f_p = h_p \circ f_{1/2} \circ h_p^{-1}.$$

In fact, this conjugating homeomorphism is given by the formula $h_p(\beta) = x(p, \beta)$ and is continuous in both variables. If $\beta \in [0, 1)$ has base-two expansion $\beta = \sum b_n/2^n$, note that the symbol sequence of β under $f_{1/2}$ is $(b_1, b_2, \ldots)$ and that the symbol sequence of $h_p(\beta)$ under f_p is this same sequence $(b_1, b_2, \ldots)$. Details of these arguments will be left to the reader.

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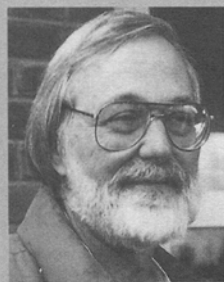

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John Milnor recalled his early years at Princeton in his reminiscence of John Nash in *The Mathematical Intelligencer* 17 (1995), no. 3, 11–17. He has, more recently, been Professor at Princeton University, at the Institute for Advanced Study, and now at SUNY in Stony Brook, where he is Director of the Institute for Mathematical Sciences. He is not usually classified as a measure-theorist.