CHAPTER I

Introduction

1. THE EQUATIONS OF HYDRODYNAMICS

This book deals with the application of hydrodynamic theory to selected physicochemical problems. Modern hydrodynamical concepts, such as the theories of viscous fluid flow, turbulence, and boundary layers, are used to a considerable extent.

Full treatment of all of these problems is not within the scope of this book. We have therefore assumed that the reader is familiar with the basic principles of hydrodynamics [1]. Certain more specialized aspects of hydrodynamics, however, are presented in appropriate sections of the book. In addition, a brief summary of fundamentals appears in this chapter.

The following discussion is limited to the motion of incompressible liquids, and thus we assume that the fluid density is constant in time and space.

The state of a moving incompressible fluid is fully described if for each point in space and for each instant the following four quantities can be defined: the three components of the fluid velocity **v** and the pressure p. In an incompressible fluid the velocity **v** satisfies the equation of continuity

$$\operatorname{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{1.1}$$

which expresses the law of conservation of matter.

Three further relations required to determine the desired functions are the equations of motion of an element of the fluid. In vectorial form the first is

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \mu \Delta \mathbf{v} + \mathbf{f}. \tag{1.2}$$

This, based upon a unit of fluid volume, is known as the Navier-Stokes equation. The left side comprises the product of the mass of the unit and its acceleration and the right represents the sum of the external forces acting on the unit. The vector **f** represents the volume force exerted on the element of fluid; gravity is an example of a volume force. The negative of the pressure gradient is that volume force which acts on the fluid element when pressure changes from point to point. Thus, if we regard a certain volume of fluid as isolated, the net force acting on it is equal to

$$-\oint p\,d\mathbf{S},$$

where dS is an element of the surface enclosing that volume. Converting the surface integral to a volume integral we obtain

$$-\oint p\,d\mathbf{S} = -\int \operatorname{grad} p\,dv.$$

The latter integral represents the total force acting over the enclosed volume. It follows that - grad p, is the force per unit volume. Since it is not the pressure but only its gradient which is required in the equation of motion, the pressure itself may be reckoned from an arbitrary datum.

The term $\mu\Delta\mathrm{v}$, where μ is the viscosity of the fluid, accounts for the effect of the viscous forces. The viscous nature of the fluid, i.e., its internal friction, is manifested only when one region of the fluid moves relative to another. Faster moving layers of the fluid entrain slower moving ones, and momentum is transferred from the faster to the slower layers. The particular volume force $\mu\Delta\mathrm{v}$ arises in those fluids in which the transfer obeys Newton's law of friction, and the viscous properties are described by a single constant value of viscosity μ. Such fluids are termed normal, or Newtonian.*

Normal fluids include water and aqueous solutions of inorganic and many organic substances. A number of organic liquids, alcohols, hydrocarbons, liquid metals, glycerine, certain resins, glasses, and gases are further examples.

The values of viscosity μ for different fluids show an unusually wide spread. Some examples are given in Table 1.

Despite this wide variation of viscosity, all of these fluids follow Newton's law of viscosity strictly. There is however a broad class of fluids for which Newton's law of friction does not apply. Such fluids are usually termed non-Newtonian, or anomalous (i.e., complex).

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*It should be stressed that the viscous properties of a compressible Newtonian fluid are characterized by two constants: the viscosity ζ and a second viscosity μ. The second enters the equation of motion as a coefficient of the term involving the divergence of velocity. Since div $\mathrm{v} = 0$ is an incompressible fluid, this term does not appear in the equation of motion above, and the second viscosity is not shown.

Table 1

Substance	Viscosity at 20° C	Kinematic viscosity $\nu = \frac{\mu}{\rho}$
Water	0.010	0.010
Air	$1.8 \cdot 10^{-4}$	0.150
Mercury	0.0156	0.0012
Glycerine	8.5	6.8

The properties of non-Newtonian fluids are considered in this book although their study undoubtedly constitutes one of the subjects of physicochemical hydrodynamics. Unfortunately no well-grounded concepts for a theory of flow of non-Newtonian fluids exist at present. Notwithstanding a very large number of theoretical studies in this field, no consistent quantitative theory with which to treat their hydrodynamic properties has been established.

If we break the acceleration into parts and assume the density of the fluid to be constant, we can rewrite equation (1.2) as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\,\mathrm{grad})\,\mathbf{v} = -\mathrm{grad}\,\frac{p}{\rho} + \nu\Delta\mathbf{v} + \frac{\mathbf{f}}{\rho} \qquad (1.3)$$

or in component form as

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 v_i}{\partial x_k^2} + \frac{f_i}{\rho}. \qquad (1.3')$$

In equation (1.3′), as always hereafter, summation is to be taken over subscripts which appear twice. Thus, we sum over the subscript k which has the values 1, 2, 3.

The quantity ν, where

$$\nu = \frac{\mu}{\rho} \quad (\mathrm{cm}^2/\mathrm{sec}),$$

is known as the kinematic viscosity of the fluid.

If f_i is omitted, the Navier-Stokes equation can be considerably simplified. Using equation (1.1), we rewrite equation (1.3′) in the form

$$\frac{\partial \rho v_i}{\partial t} = \frac{\partial}{\partial x_k}\left[-p\delta_{ik} + \rho v_i v_k + \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right]. \qquad (1.4)$$

where $\delta_{ik} = \begin{cases} 1 \text{ when } i = k \\ 0 \text{ when } i \neq k \end{cases}$

The equivalence of expressions (1.4) and (1.3′) is confirmed by the fact that

$$\frac{\partial}{\partial x_k}(v_i v_k) = v_k \frac{\partial v_i}{\partial x_k} + v_i \frac{\partial v_k}{\partial x_k} = v_k \frac{\partial v_i}{\partial x_k}$$

and

$$\frac{\partial^2 v_k}{\partial x_i \, \partial x_k} = \frac{\partial}{\partial x_i} \frac{\partial v_k}{\partial x_k} = 0,$$

since equation (1.1) written in component form is

$$\frac{\partial v_k}{\partial x_k} = 0.$$

If the bracketed expression in equation (1.4) is designated as p_{ik}

$$p_{ik} = -p\delta_{ik} + \rho v_i v_k + \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{1.5}$$

we obtain

$$\frac{\partial \rho v_i}{\partial t} = \frac{\partial p_{ik}}{\partial x_k}. \tag{1.6}$$

The quantity p_{ik} is known as the stress tensor, and since $p_{ik} = -p\delta_{ik} + \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \rho v_i v_k$, it represents the total of the nine quantities p_{xx}, p_{yy}, p_{xy}, p_{xz}, etc.

From its very definition it is clear that the stress tensor is symmetrical in an isotropic medium, i.e.,

$$p_{ik} = p_{ki}.$$

Indeed, for example,

$$p_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \rho v_x v_y = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) + \rho v_y v_x = p_{yx}. \tag{1.7}$$

Thus, only six of the nine quantities of p_{ik} are independent.

To clarify the meaning of the tensor p_{ik}, let us integrate equation (1.6) over an arbitrary volume, and apply the Gauss–Ostrogradsky theorem to the right side of the equation. Because the summation is with respect to the subscript k, the right side of equation (1.6) is actually a divergence

$$\frac{\partial}{\partial t}\int(\rho v_i)\,dv=\int\frac{\partial p_{ik}}{\partial x_k}\,dv=\oint p_{ik}\,ds_k. \tag{1.8}$$

Consequently equation (1.8) describes the momentum change in an arbitrary fluid volume. The change in momentum within the volume is equal to the net momentum transfer across the surface enclosing that volume. Thus, p_{ik} represents momentum transfer. For example, component p_{xy} is the x-component of momentum transfer across a unit surface which is perpendicular to the y-axis.

$$p_{xy}=(\rho v_x)\,v_y+\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right).$$

The first term on the right is the x-component momentum transfer accompanying physical transfer (convection) of fluid volume across a surface perpendicular to the y-axis. The second term represents the momentum transfer caused by the fluid viscosity. The viscous properties of the fluid assure the transfer of a portion of the momentum from regions of greater velocity to regions of lesser velocity.

The system of equations of motion (1.1) and (1.2) must be supplemented by a set of boundary conditions.

Numerous experimental studies of the flow of Newtonian liquids past the surface of a solid body wetted by them have established that the layer of fluid immediately adjacent to the surface remains motionless, or, as often stated, the fluid adheres to the solid surface. Velocity measurements have shown that the thickness of this stationary layer is quite small — limited to several individual molecular layers (see Section 132). Nevertheless, the absence of slip past the surface is highly important to fluid flow in general. An analogous phenomenon takes place in gases, when their densities are sufficiently great.

Thus, we may assume that the boundary condition holds at all solid surfaces in contact with a moving liquid

$$\mathbf{v}=0. \tag{1.9}$$

On this situation the fluid exerts on each unit area of the solid a force which is numerically equal to the rate of momentum transfer across the surface.

At the interface between two flowing phases, e.g., fluids, or a liquid and a gas, the velocity does not vanish. Instead, the following boundary conditions must be satisfied:

1) The tangential components of velocity are equal

$$v_t^{(1)}=v_t^{(2)}; \tag{1.10}$$

2) The normal components of velocity vanish

$$v_n^{(1)}=v_n^{(2)}=0; \tag{1.11}$$

3) The forces acting between the fluids are equal and opposite, i.e.,

$$F_n^{(1)} = F_n^{(2)}, \quad (1.12)$$

$$F_t^{(1)} = F_t^{(2)}, \quad (1.13)$$

where the indices 1 and 2 refer to the different fluids.

At the free surface of a fluid the tangential force component vanishes:

$$F_t = 0. \quad (1.14)$$

During the motion of a viscous fluid, energy is dissipated within it. Analysis [2] shows that the energy dissipated in a unit volume is given by

$$\begin{aligned} -\frac{dE}{dt} &= \int \frac{\mu}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 dV = \\ &= \mu\left[-\int (\mathrm{rot}\,\mathbf{v})^2\, dV + \int \frac{\partial \mathbf{v}^2}{\partial n}\, ds - 2\int [\mathbf{v}\ \mathrm{rot}\,\mathbf{v}]\,\mathbf{n}\, ds\right]. \end{aligned} \quad (1.15)$$

From a mathematical viewpoint, solution of the system of hydromechanical equations presents considerable difficulty, because nonlinear partial differential equations are involved. In practice their general solution is possible in but a few special cases. Thus, an effort is generally made to simplify the equations, and then to find approximate solutions for the simplified system.

2. SIMILARITY OF HYDRODYNAMIC PHENOMENA

Dimensional analysis and similarity theory have been applied widely to hydrodynamics and related subjects, especially to heat transfer theory.

Here the exposition is limited to the simplest concepts of the hydrodynamic theory of similarity.

It is to be noted that the methods of similarity theory and dimensional analysis, which represent a scientific basis for the modeling of physical phenomena, are used not only in theoretical studies, but in engineering practice as well. They have found especially wide acceptance in the USSR.

In this book we limit our treatment of similarity theory to a determination of the conditions for similarity in hydrodynamic flow. We make extensive use of this particular aspect of the theory later for background on the theory of similarity. The reader is referred to a number of original papers and monographs [3].

Certain special problems involving application of the theory of similarity to heterogeneous chemical reactions are analyzed in Section 19.

To establish necessary and sufficient conditions for the similarity of two flows, let us consider the flow of a viscous fluid. The equations for flow must be expressed in dimensionless form and therefore all dimensional variables that appear in the hydrodynamic equations must be expressed in terms of factors characteristic of these variables. For instance, let a fluid flow around a body whose characteristic dimension is l, or let the fluid flow inside a tube whose radius is l. Then the dimensions of the body or the radius of the tube are the characteristic length factors for the regions in which fluid motion occurs. All linear dimensions can therefore be dimensionless ratios of the form $X_i = \frac{x_i}{l}$. In like fashion let U_0 represent the velocity of the stream flowing past the body or entering the tube. U_0 represents a characteristic velocity of the motion and may be chosen as the velocity factor. All velocities can then be expressed as dimensionless ratios $V_i = \frac{v_i}{U_0}$. Employing these length and velocity factors, we can write the Navier-Stokes equations for the steady flow of an incompressible fluid as

$$\frac{U_0^2}{l} V_k \frac{\partial V_i}{\partial X_k} = -\frac{1}{\rho} \frac{\partial p}{\partial X_i} \frac{1}{l} + \nu \frac{U_0}{l^2} \frac{\partial^2 V_i}{\partial X_k^2}, \tag{2.1}$$

or

$$V_k \frac{\partial V_i}{\partial X_k} = -\frac{\partial P}{\partial X_i} + \frac{\nu}{U_0 l} \frac{\partial^2 V_i}{\partial X_k^2}, \tag{2.2}$$

where $P = \frac{p}{\rho U_0^2}$ is the dimensionless pressure.

Expressed in dimensionless form equation (1.2) contains a dimensionless parameter known as the Reynolds number:

$$\mathrm{Re} = \frac{U_0 l}{\nu}.$$

Equation (2.2) can now be rewritten

$$V_k \frac{\partial V_i}{\partial X_k} = -\frac{\partial P}{\partial X_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 V_i}{\partial X_k^2}. \tag{2.3}$$

Equation (2.3) must be supplemented by the continuity equation, which in dimensionless form is

$$\frac{\partial V_k}{\partial X_k} = 0. \tag{2.4}$$

Furthermore, if there is to be a single-valued solution of any hydrodynamic problem, boundary conditions must be formulated for the set of surfaces that enclose the space in which fluid motion occurs.

Let us consider two fluid streams moving in geometrically similar regions (i.e., in regions that become interchangeable merely by changing the scale of length).

If the boundary conditions for both flows are identical and if both have identical Reynolds numbers, the dimensionless equations of motion for the two flows will be identical and the flows will be similar both geometrically and dynamically. Thus, geometrical similarity, identity of boundary conditions and equality of Reynolds numbers are made up of sufficient conditions for the similarity of two flows. For example, consider two spheres of radii R_1 and R_2 with streams of the same fluid flowing past them at different velocities U_1 and U_2 such that $\frac{U_1}{U_2} = \frac{R_2}{R_1}$; compare this to two identical spheres in flows of different velocities such that $\frac{U_1}{\nu_1} = \frac{U_2}{\nu_2}$. The Reynolds number contains the arbitrary quantities U_0, l and ν, and viscosity is a physical property of the fluid. The characteristic velocity and dimension may have any values determined by the boundary conditions. Dimensionless parameters, similar to the Reynolds number and made up of arbitrarily chosen values, are known as the controlling dimensionless numbers. All other dimensionless quantities characteristic of the moving fluid are functions of these controlling numbers. Any hydrodynamic variable may, therefore be expressed as a function of the controlling numbers and the dimensionless coordinates. For example, the velocity of the fluid may be expressed as

$$V_i = \frac{v_i}{U_0} = f\left(\mathrm{Re}, \frac{x_i}{l}\right).$$

In steady flow of an incompressible fluid there is only one controlling number — the Reynolds number. Thus, the dimensionless shear force acting on one square centimeter of the surface past which the fluid streams is equal to

$$\tau = f(\mathrm{Re}).$$

In more complex cases, for unsteady flow or for flow in the presence of an external field of volume forces, etc., other controlling parameters enter along with the Reynolds number. In these cases the flows are similar if the geometrical conditions are similar, the initial and boundary conditions are identical, and all the controlling dimensionless numbers have the same respective numerical values.

The more complex cases are not considered in this book. Instead our discussion is confined to the steady flow of an incompressible fluid. For this case the flow regime is determined by the value of the Reynolds number.

3. FLUID FLOW AT HIGH REYNOLDS NUMBERS. BOUNDARY LAYER

In practice the most frequent case encountered is fluid motion at high Reynolds numbers.

When Re $\gg$ 1, the last term in equation (2.3) may be ignored, provided the derivative $\frac{\partial^2 V_i}{\partial X_k^2}$, for one reason or another, does not attain any exceedingly high values.

Omitting the last term in (2.3), we may write

$$V_k \frac{\partial V_i}{\partial X_k} = -\frac{\partial P}{\partial X_i},$$

or upon introducing dimensional quantities

$$v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right).$$

In the general case of unsteady flow in the presence of external forces we have

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\operatorname{grad}\frac{p}{\rho} + \mathbf{f}\cdot\frac{1}{\rho}. \tag{3.1}$$

In equation (3.1) viscosity is disregarded. This indicates that viscous forces at high Reynolds numbers are small and play a secondary role.

A fluid with no viscosity usually is termed ideal. Equation (3.1), expressing the law of motion of an ideal fluid, is known as Euler's equation.

Elimination of the viscosity term, which converts the Navier-Stokes to the Euler equation, constitutes a very important simplification. Euler's equation is first order (unlike the Navier-Stokes equation which is second order) and thus can often be integrated in general form.

In the case of steady motion under the influence of conservative external forces, for which $\mathbf{f} = -\operatorname{grad} U$, Euler's equation may be rewritten in the form

$$(\mathbf{v}\operatorname{grad})\mathbf{v} = -\operatorname{grad}\left(\frac{p}{\rho} + \frac{U}{\rho}\right). \tag{3.2}$$

In order to integrate equation (3.2), we will introduce the concept of streamlines. A streamline is a curve whose tangent at every point has the direction of the velocity vector. In steady flow, the streamlines represent trajectories of the fluid particles. Using the identity

$$\operatorname{grad}\frac{v^2}{2} = (\mathbf{v}\operatorname{grad})\mathbf{v} + [\mathbf{v}\operatorname{rot}\mathbf{v}]$$

and noting that the vector $[\mathbf{v} \operatorname{rot} \mathbf{v}]$ is perpendicular to the velocity vector $\mathbf{v}$, we can express the projection of (3.2) on an arbitrary streamline l,

$$\left(\operatorname{grad}\frac{v^2}{2}\right)_l = -\left[\operatorname{grad}\left(\frac{p}{\rho}+\frac{U}{\rho}\right)\right]_l,$$

or

$$\frac{\partial}{\partial l}\left(\frac{v^2}{2}+\frac{p}{\rho}+\frac{U}{\rho}\right)=0,$$

where (for the given streamline)

$$\frac{v^2}{2}+\frac{p}{\rho}+\frac{U}{\rho}=\text{const.} \tag{3.3}$$

Equation (3.3) is known as Bernoulli's theorem. It represents the general integral of the equations of motion for an ideal incompressible fluid. This theorem is analogous, in some degree, to the energy principle of ordinary mechanics. Bernoulli's theorem shows that as we go from regions of higher flow velocity to regions of lower velocity, the pressure of the fluid changes in the opposite direction.

Since Euler's equations, unlike the Navier-Stokes equations, are first-order differential equations, the boundary conditions for an ideal fluid must be changed. For example, vanishing of all velocity components of a fluid at a solid surface is inconsistent with Euler's equations. In an ideal fluid, interaction with a solid body does not exist, since viscosity is absent. Thus, the tangential velocity component is not restricted, and only the normal velocity component reduces to zero at the surface of the solid body:

$$v_n = 0 \text{ (at the surface of a solid body)}$$

Another important conclusion may be drawn from Euler's equations: in an ideal fluid, velocity circulation is conserved in any closed circuit that moves with the fluid

$$\oint \mathbf{v}\, d\mathbf{l} = \int \operatorname{rot} \mathbf{v}\, d\mathbf{s} = \text{const.}$$

It follows from this that if initially $\operatorname{rot} \mathbf{v} = 0$ for a given streamline, the motion along this streamline will remain irrotational thereafter. In particular, any motion of an ideal fluid, started from rest, is irrotational. (This is valid for fluids for which $\rho = \Phi(p)$.)

The motion of a fluid whose velocity at every point equals zero is termed "potential motion".

In potential motion, the velocity of the fluid may always be expressed as

$$\mathbf{v} = \operatorname{grad} \varphi, \tag{3.4}$$

where φ, a function of time ξ position, is termed the velocity potential. When velocity is so represented, the condition rot $\mathbf{v} = 0$ is automatically satisfied.

If we apply equation (3.4) to the continuity equation (1.1), we see that the velocity potential must satisfy the Laplace equation

$$\Delta\varphi = 0. \tag{3.5}$$

At the surface of solid bodies which confine a region of fluid motion, the velocity potential must satisfy the boundary conditions

$$v_n = \frac{\partial\varphi}{\partial n} = 0$$

(at the surface of the solid body), where n is the normal to the surface.

In an ideal fluid the velocity potential and, consequently, the velocity distribution may be derived from the solution of a boundary value problem well-known in mathematical physics.

The pressure distribution may be determined from Euler's equation. Since in potential flow rot $\mathbf{v} = 0$ everywhere in the fluid, it is possible to write

$$(\mathbf{v}\nabla)\,\mathbf{v} = \operatorname{grad} \frac{v^2}{2}.$$

Then Euler's equation assumes the form

$$\operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \frac{U}{\rho}\right) = 0;$$

hence, it follows that

$$\frac{U}{\rho} + \frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} = \text{const.} \tag{3.6}$$

Another important simplification may be included for the case of a steady flow regime, in which $\frac{\partial\varphi}{\partial t} = 0$ and

$$\frac{p}{\rho} + \frac{v^2}{2} + \frac{U}{\rho} = \text{const.} \tag{3.7}$$

Unlike (3.3), the constant in (3.7) has the same value for all streamlines in the fluid.

It follows from Bernoulli's equation (3.3) that when an ideal fluid flows across a solid body, the greatest pressure is attained at that point where the fluid velocity becomes zero (ignoring the influence of gravity). Such a point is known as the point of incidence of flow, or the stagnation point.

When the potential flow of a fluid is steady, the conditions under which that fluid may be considered incompressible are easily defined. The change in density of an ideal fluid is always both adiabatic and reversible (hence isentropic), since no dissipation of energy occurs:

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_S \delta p,$$

where S is the entropy.

From the Bernoulli equation (3.3) we may conclude that the change in pressure of the fluid is related to its kinetic energy by the approximation

$$\delta p \sim \frac{\rho v^2}{2},$$

since for steady motion $\frac{\partial\varphi}{\partial t} = 0$.

Moreover, it is known that the velocity of sound in a fluid is equal to

$$c = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_S}.$$

Hence,

$$\delta\rho \sim \frac{\rho v^2}{c^2},$$

or

$$\frac{\delta\rho}{\rho} \sim \frac{v^2}{c^2}.$$

A fluid may be considered incompressible when $\frac{\delta\rho}{\rho} \ll 1$. This inequality is satisfied when the fluid velocity v is low in comparison with the speed of sound c in the fluid.

Thus, the steady flow of an ideal fluid may be described by equations of motion whose solution and analysis are relatively simple.

The approximations employed for an ideal fluid are inadequate, however, for the motion of real fluids even at very high Reynolds numbers. This point is well indicated by the so-called d'Alembert paradox. This paradox states that the drag on a solid body in steady motion through an ideal fluid is zero.

This may be seen clearly in the example of a sphere moving steadily through an ideal fluid. Reasoning based on the Bernoulli equation shows that the force acting on the front hemisphere is exactly matched by the force acting on the rear hemisphere.

The absurdity of this conclusion indicates that the laws of motion for an ideal fluid, and Bernoulli's equation in particular, have limited application. It has been found that viscosity exerts a very significant influence in the region immediately adjacent to the surface of a solid body. Furthermore, in this region the law of conservation of velocity circulation is not valid. It has also been shown that the equations of motion of an ideal fluid admit discontinuous solutions. These solutions, moreover, are not single-valued.

The ideal fluid approximation is unsatisfactory for describing fluid motion near a phase interface. The character of flow in the neighborhood of a solid surface is usually treated in classical hydrodynamics. In Section 80 we also deal with the character of flow in the vicinity of a liquid-gas interface.

The flow velocity must become zero in a real fluid at the surface of a solid body. By the same token, it follows from the equations of motion of an ideal fluid — equations inposing no restrictions on the tangential component of fluid velocity near the solid surface — that the fluid in this region moves with a velocity comparable to that of the flow at a significant distance from the solid surface.

Thus, near the surface of the solid body there must be a thin zone in which the tangential velocity component undergoes a very abrupt change from a high value at the outer border of the zone to zero at the solid surface.

The retardation of the fluid in the boundary layer is caused by viscous forces alone. Mathematically speaking, the velocity gradient in the boundary layer is very large in a direction normal to the wall. The viscosity terms in the Navier-Stokes equations, which depend on derivatives in the same direction accordingly are large even if the fluid viscosity is low.

Although the boundary layer occupies an extremely small volume, it exerts a significant influence on the motion of the fluid. The phenomena that take place in the boundary layer are the source of hydrodynamic resistance to the motion of solids through fluids. Thus, the boundary layer is highly important in many of the problems in physicochemical hydrodynamics.

For ease in manipulation the equations of fluid flow in the boundary layer may be simplified considerably. In a thin boundary layer, all quantities change rapidly in the direction perpendicular to the wall while their tangential rate of change is comparatively small.

Moreover, for a sufficiently short length of the body, the flow in the boundary layer may be regarded as laminar (provided the dimensions of the body are large compared to the thickness of the boundary layer).

Let us consider the steady laminar motion of a fluid, choosing the y-axis perpendicular to the surface of the body and the x-axis along the surface in the direction of flow.

The equations of motion, (1.1) and (1.2), for the components in steady motion, take the following forms:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right), \tag{3.8}$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right), \tag{3.9}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{3.10}$$

If we designate the thickness of the boundary layer as δ_0 and the dimensions of the body as l, it may be assumed that the change in velocity along the y-axis takes place over distances of the order of δ_0, and along the x-axis over distances of the order of l.

The entire zone of motion may be roughly subdivided into two regions: a region of inviscid motion and a boundary layer region in which viscosity plays an important part.

In the first region, we can omit the viscosity terms in the Navier-Stokes equation and substitute Euler's equation.

To simplify the Navier-Stokes equations within the boundary layer, we can then utilize the fact that the thickness of this layer is very small compared to its length along the body. Let us first introduce dimensionless coordinates into equations (3.8) to (3.10) and define

$$x = lX, \qquad y = \delta_0 Y. \tag{3.11}$$

These coordinates range between the limits

$$0 \leqslant X \leqslant 1, \qquad 0 \leqslant Y \leqslant 1. \tag{3.12}$$

With the new variables, equations (3.8) to (3.10) take the following forms:

$$\frac{v_x}{l}\frac{\partial v_x}{\partial X} + \frac{v_y}{\delta_0}\frac{\partial v_x}{\partial Y} = -\frac{1}{\rho l}\frac{\partial p}{\partial X} + \frac{\nu}{l^2}\frac{\partial^2 v_x}{\partial X^2} + \frac{\nu}{\delta_0^2}\frac{\partial^2 v_x}{\partial Y^2}, \tag{3.8'}$$

$$\frac{v_x}{l}\frac{\partial v_y}{\partial X} + \frac{v_y}{\delta_0}\frac{\partial v_y}{\partial Y} = -\frac{1}{\rho\delta_0}\frac{\partial p}{\partial Y} + \frac{\nu}{l^2}\frac{\partial^2 v_y}{\partial X^2} + \frac{\nu}{\delta_0^2}\frac{\partial^2 v_y}{\partial Y^2}, \tag{3.9'}$$

$$\frac{1}{l}\frac{\partial v_x}{\partial X} + \frac{1}{\delta_0}\frac{\partial v_y}{\partial Y} = 0. \tag{3.10'}$$

To compare the magnitudes of the velocity components v_x and v_y, we see from (3.10′) that

$$v_y = -\frac{\delta_0}{l}\int_0^1 \frac{\partial v_x}{\partial X}\,dY. \tag{3.13}$$

The derivative $\frac{\partial v_x}{\partial X}$ has no unusual features, and its integral, within the limits zero and 1, is of the order of v_x. Therefore, in the boundary layer,

$$v_y \sim \frac{\delta_0}{l} v_x \ll v_x. \tag{3.14}$$

Using equation (3.14) the different terms in equation (3.8) may be evaluated. Since Y varies within the limits prescribed by equation (3.12), the derivatives

$$\frac{\partial v_x}{\partial Y} \sim \frac{\partial^2 v_x}{\partial Y^2} \sim v_x.$$

In similar fashion

$$\frac{\partial v_x}{\partial X} \sim \frac{\partial^2 v_x}{\partial X^2} \sim v_x.$$

Therefore, we can disregard the term from the right side of equation (3.8′)

$$\frac{\nu}{l^2}\frac{\partial^2 v_x}{\partial X^2} \sim \frac{v_x}{l^2}\nu$$

as compared to the term

$$\frac{\nu}{\delta_0^2}\frac{\partial^2 v_x}{\partial Y^2} \sim \frac{v_x}{\delta_0^2}\nu.$$

On the left side of (3.8′), however, both terms have the same order of magnitude:

$$\frac{v_x}{l}\frac{\partial v_x}{\partial X} \sim \frac{v_x^2}{l};$$

In view of (3.14),

$$\frac{v_y}{\delta_0}\frac{\partial v_x}{\partial Y} \sim \frac{v_x v_y}{\delta_0} \sim \frac{v_x^2}{l}.$$

Equation (3.8) can therefore be written in its final form:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}. \tag{3.15}$$

Knowing that all terms in equation (3.15) are of the same order of magnitude, we can estimate the thickness of the boundary layer

$$\frac{v_x}{l} \frac{\partial v_x}{\partial X} \simeq \frac{\nu}{\delta_0^2} \frac{\partial^2 v_x}{\partial Y^2}. \tag{3.16}$$

If the velocity v_x at the outer edge of the boundary layer attains the value U_0, then, from equation (3.16), we get

$$\frac{U_0}{l} \sim \frac{\nu}{\delta_0^2},$$

or

$$\delta_0 \sim \sqrt{\frac{\nu l}{U_0}} \sim \sqrt{\frac{l^2}{\mathrm{Re}}} \sim \frac{l}{\sqrt{\mathrm{Re}}}. \tag{3.17}$$

Thus, δ_0 is smaller than the length of the body by a factor approximately equal to the reciprocal of the square root of the Reynolds number. Of course, it is essential here that the inequality $\sqrt{\mathrm{Re}} \gg 1$ be fulfilled. This estimate may be confirmed by more rigorous calculations (see below).

However, it should be emphasized that the concept of a well defined thickness of the boundary layer requires qualification. The transition from viscous flow in the boundary layer to inviscid flow in the main stream is smooth and gradual. The quantity δ_0 represents the thickness of the region across which the principal change in velocity from zero to U_0 takes place.

It also follows from equation (3.15) that the derivative

$$\frac{1}{\rho} \frac{\partial p}{\partial x} \sim \frac{U_0^2}{l}. \tag{3.18}$$

Evaluating the terms of equation (3.9), we find their order of magnitude to be $\frac{U_0^2 \delta_0}{l^2}$, i.e., smaller than that of the terms of equation (3.8) by the factor $\frac{\delta_0}{l}$.

From this it follows that the pressure gradient normal to the surface, which appears in equation (3.9), equals

$$\frac{1}{\rho} \frac{\partial p}{\partial y} \sim \frac{\delta_0 U_0^2}{l^2}.$$

Comparing this with (3.18), we obtain the following approximation

$$\frac{\partial p}{\partial y} \sim \frac{\delta_0}{l} \frac{\partial p}{\partial x}.$$

This means that the pressure gradient in the direction of the normal is small compared to the change of pressure along the surface. Therefore, excluding second-order terms, we can write in place of equation (3.9)

$$\frac{\partial p}{\partial y} = 0. \tag{3.19}$$

Equation (3.19) shows that the pressure does not change in the normal direction but remains equal to the pressure outside the boundary layer. Therefore the pressure variation in the x-direction within the boundary layer is determined by the change in pressure outside. The latter may be determined from Bernoulli's theorem.

The boundary layer equations admit an exact solution for the case of flow past a semi-infinite plate when its leading edge encounters a fluid moving at a velocity U_0 (Figure 1).

In order to find solutions to equations (3.15) and (3.19), for the boundary conditions

$$v_x = v_y = 0 \quad \text{at} \quad y = 0, \tag{3.20}$$

$$v_x \to U_0 \text{ as } y \to \infty \text{ (outside the boundary layer)}, \tag{3.21}$$

we note at the outset that the velocity in the outer region is constant. It follows from the Bernoulli equation that the pressure in the outer region is also constant. As a result, the term containing the pressure gradient in equation (3.15) can be omitted and the equation rewritten

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}. \tag{3.22}$$

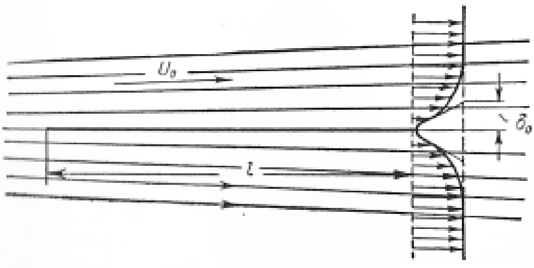

Figure 1. Flow past a plate.

To find the distribution of the boundary layer velocities v_x and v_y satisfying the continuity equation (3.10), we use the stream function ψ, defined by

$$v_x = \frac{\partial \psi}{\partial y}; \tag{3.23}$$

$$v_y = -\frac{\partial\psi}{\partial x}. \tag{3.24}$$

In this case, equation (3.10) becomes an identity.

Equations (3.10) and (3.22) and the boundary conditions contain no characteristic parameter having the dimension of length. It is therefore convenient in equation (3.22) to define a new dimensionless variable

$$\eta = \frac{1}{2}\sqrt{\frac{U_0}{\nu x}}\cdot y. \tag{3.25}$$

Let us put the stream function ψ in the form

$$\psi = \sqrt{\nu U_0 x}\cdot f(\eta). \tag{3.26}$$

Here it is evident that $f(\eta)$ satisfies an ordinary differential equation. Thus, we have

$$v_x = \frac{\partial\psi}{\partial y} = \frac{1}{2}U_0 f'(\eta), \tag{3.27}$$

$$v_y = -\frac{\partial\psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_0}{x}}(\eta f' - f), \tag{3.28}$$

$$\frac{\partial v_x}{\partial x} = -\frac{1}{4}\frac{U_0}{x}\eta f'', \tag{3.29}$$

$$\frac{\partial v_x}{\partial y} = \frac{U_0}{4}\sqrt{\frac{U_0}{\nu x}} f'', \tag{3.30}$$

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{1}{8}U_0\frac{U_0}{\nu x} f'''. \tag{3.31}$$

On substituting the corresponding quantities into equation (3.22), we obtain

$$f''' + f''f = 0. \tag{3.32}$$

Substitution of v_x and v_y into the boundary conditions (3.20) and (3.21) gives

$$f = f' = 0 \qquad \text{at} \quad \eta = 0, \tag{3.33}$$

$$f' = 2 \qquad \text{at} \quad \eta \to \infty. \tag{3.34}$$

If we let $f''(0) = \alpha$, the solution of equation (3.32) can be written in the form suggested in [4]

$$f = \frac{\alpha\eta^2}{2!} - \frac{\alpha^2\eta^5}{5!} + \frac{11\alpha^3\eta^8}{8!} + \ldots \tag{3.35}$$

For large values of η, the series is inconvenient to use. In such cases the limiting value of f that satisfies condition (3.34) is obtained by means of numerical integration.

The following value has been determined in this manner [4]

$$\alpha = 1.33.$$

The drag force F_x acting on one side of the plate is given by

$$F_x = \int_0^b \int_0^l \mu \left(\frac{\partial v_x}{\partial y}\right)_{y=0} dx\, dz = \frac{\alpha \rho U_0^2}{2} bl \sqrt{\frac{\nu}{U_0 l}}, \tag{3.36}$$

where b is the width of the plate.

Instead of the drag force, a so-called drag coefficient is often employed. It is defined as

$$K_f = \frac{F_x}{\rho \frac{U_0^2}{2} \cdot bl} = \frac{\alpha}{\sqrt{\mathrm{Re}}}. \tag{3.37}$$

If the thickness of the boundary layer δ_0 is defined as the distance from the surface of the plate to the point where v_x attains a value equal to 90 percent of the velocity of the main stream, U_0, then, by numerical computation, we obtain the following value for δ_0 (Figure 2):

$$\delta_0 = 5.2 \sqrt{\frac{\nu x}{U_0}}. \tag{3.38}$$

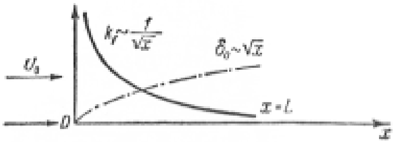

Figure 2. Boundary layer thickness and shear stress as functions of the coordinate x along a plate.

Qualitatively, the formulas derived for a plate are applicable to an arbitrary body with a small curvature.

Let us write the distribution of velocities for $\eta \ll 1$ in explicit form* (Figure 3):

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*It would be incorrect to verify the inequality $\frac{\partial^2 v_x}{\partial y^2} \gg \frac{\partial^2 v_x}{\partial x^2}$ by substituting the expansion (3.29) in it. In the neighborhood of the point $y = 0$ the behavior of the continuous function v_x is defined by the boundary condition (3.20).

$$v_x = \frac{U_0}{2}\left(\alpha\eta - \frac{\alpha^2\eta^4}{4!} + \dots\right) \approx \frac{1.33U_0}{4}\sqrt{\frac{U_0 y^2}{\nu x}} \approx \frac{U_0 y}{\delta_0}, \tag{3.39}$$

$$v_y \approx \frac{\alpha}{4}\sqrt{\frac{U_0 \nu}{x}}\,\eta^2 \approx \frac{\alpha}{16}\frac{U_0^{3/2} y^2}{\nu^{1/2} x^{3/2}} \approx \frac{\nu y^2}{\delta_0^3}. \tag{3.40}$$

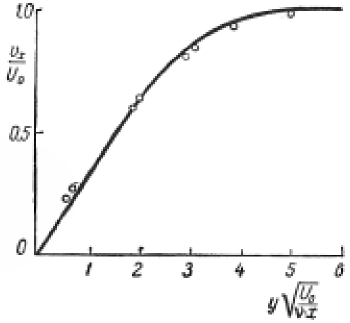

Figure 3. Distribution of the tangential component of velocity in the boundary layer on a plate.

Circles designate experimental values.

4. TURBULENT FLOW

At high Reynolds numbers, the nature of fluid flow usually differs from that examined in the preceding sections.

At a certain value of the Reynolds number, steady laminar flow gives way to distinctly unsteady, chaotic motion in which only on the average is there net flow in a particular direction.

The steady advance of a fluid in separate layers is known as laminar flow. The unsteady, chaotic motion in which the flow velocity fluctuates about some average value is known as turbulent flow.*

Studies of the transition to the turbulent regime have shown that it is related to the inherent instability of laminar flow at Reynolds numbers in excess of some critical value Re_{cr}.

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*Very often in general literature, and especially in physicochemical literature, the discussion of turbulent motion emphasizes its rotational nature. While it is true that turbulent motion is usually rotational, it may sometimes be irrotational, or potential motion (although in such a case the motion is noticeably damped). Furthermore, laminar flow of a viscous fluid is also rotational. The distinguishing feature of the turbulent regime is the chaotic and distinctly unsteady nature of the motion of the fluid particles, and not the rotational character that this motion may have.

The gross over-all motion of a fluid is always subject to infinitesimal disturbances. At $Re < Re_{cr}$, disturbances that occur in the fluid are rapidly damped. At $Re > Re_{cr}$, disturbances are not damped, but rather reinforce each other. If these disturbances are periodic in nature and if their frequencies are incommensurable, the effect of their superposition will be the establishment of a chaotic regime. Random eddies are also superposed on the basic motion. In the region $Re \gg Re_{cr}$ the magnitudes of the eddy velocities are comparable to the average velocity.

If we observe the trajectory of a particular mass of fluid in turbulent flow, we find it highly complicated and involved; its tendency toward a systematic advance can only be described in terms of averages. The trajectory in this case is somewhat similar to the motion of a gas molecule in a stream of gas. This indicates that the theory of turbulent flow must be statistical in nature. A quantitative theory of turbulence has not been perfected. However, thanks to the work of Soviet scientists A. A. Fridman, L. V. Keller, A. N. Kolmogorov, L. D. Landau, A. M. Obukhova, L. G. Loitsyansky and M. D. Millionshchikov, as well as von Karman, Prandtl, Heisenberg, Lin, Taylor and others it has been advanced considerably. We can, therefore, state that a qualitative, or perhaps a semi-quantitative, theory of turbulent motion has evolved [5].

Let us first* examine qualitatively the general nature of turbulent motion at $Re \gg Re_{cr}$. Such motion is known as developed turbulence.

Eddy velocities of extremely varied magnitudes are superposed upon the average motion of a fluid having a velocity U. Turbulence eddies must be characterized by their velocities, and by the distances over which these velocities change significantly. These distances are known as the scale of motion. The most rapid eddy motion also has the largest scale of motion. The velocities v' of the most rapid eddies are approximately

$$v' \approx \Delta U, \tag{4.1}$$

where ΔU is the change in the average velocity over a distance equal to the scale of an eddy l. Thus, for the example of turbulent motion in a tube, the largest scale l of turbulence eddies is equal to the diameter of the tube, and the eddy velocities vary within the range of average velocity over that distance, i.e., they are of the order of the maximum value of the velocity at the center of the tube.

Such large scale eddies contain the main part of the kinetic energy of turbulent motion.

The Reynolds numbers of these motions, defined as $\frac{\Delta U \cdot l}{\nu}$, have values equal to the Reynolds number of the stream taken as a whole.

*What follows is in accordance with the L. D. Landau and M. M. Lifshitz presentation [5].

Together with these large scale eddies, turbulent flow also includes eddies of smaller scale λ, with lesser velocities v_λ. Although the number of such small eddies is very large, they represent only a small portion of the kinetic energy of the stream. Nevertheless, small eddies play an important part in turbulent flow.

To clarify their role, let us examine the Reynolds number that corresponds to an eddy of scale λ, i.e., $Re_\lambda = \frac{v_\lambda \lambda}{v}$. The smaller the value of λ and of the corresponding velocity v_λ, the smaller is the value of Re_λ. For large scale eddies the Reynolds number is very large. Therefore, in fluid motion with a scale $\lambda \approx l$, viscous forces actually have no effect. Such motion takes place without energy dissipation. The superposition of large scale eddies on each other creates small scale eddies, for which the Reynolds numbers rapidly decrease with decreasing λ.

At a certain value of $\lambda = \lambda_0$ the Reynolds number for the corresponding motion $Re_{\lambda_0} = \frac{v_{\lambda_0} \lambda_0}{\nu}$ is approximately unity. This means that in the region of λ_0, viscous forces begin to have a noticeable effect on the motion of the fluid. Eddy motion of a scale λ_0 is accompanied by dissipation of energy.

With a large quantity of small scale motion, there is a considerable dissipation of energy, which is transformed to heat. This energy is continually drawn by small scale motions from the large scale motions, so that one may visualize the existence of a continuous transfer of energy from large scale eddies to eddies progressively smaller in scale, until in eddies with scale on the order of λ_0 it is converted to heat. Small scale motions serve as a "bridge" by means of which the kinetic energy of large scale motions may be converted into thermal energy. For steady state fluid flow the process of energy transfer is also steady in nature. Eddies of a given scale receive as much energy from larger scale eddies as they in turn pass on to smaller scale eddies. Thus, although turbulent motion occurs only at relatively high Reynolds numbers, it is accompanied by considerable dissipation of energy. From this standpoint it is possible to define a certain effective "eddy viscosity" μ_{turb} appropriate to turbulent flow. This "eddy viscosity" expresses energy losses occurring in the flow per second, per unit volume, by an equation analogous to (1.15):

$$\varepsilon = -\frac{dE}{dt} = \mu_{turb}\left(\frac{\Delta U}{l}\right)^2. \tag{4.2}$$

The order of magnitude of the eddy viscosity may be determined from considerations of similarity.

It is clear from the foregoing that the magnitude of $\epsilon = -\frac{dE}{dt}$, is not a function of the scale of the motion but is a characteristic constant for a given flow. In particular, for the largest scale motions, it equals the energy dissipated in the process of creating the smaller scale motions. This process occurs at high Reynolds numbers and can not be a function of the molecular viscosity μ of the fluid. Therefore, ϵ must be determined from quantities characteristic of large scale turbulent motion. These include the following: the velocity ΔU, the scale of motion l and the density of the fluid ρ (no other quantities besides these and μ enter into the hydrodynamic equations). We can combine ΔU, l and ρ into a single dimensional quantity, $[\epsilon]$ in whose units are erg/cm³ sec

$$\varepsilon \approx \rho \frac{(\Delta U)^3}{l}. \tag{4.3}$$

Comparing (4.2) and (4.3), we have

$$\mu_{\text{turb}} = \rho \, \Delta U l. \tag{4.4}$$

The corresponding kinematic viscosity may be expressed as

$$\nu_{\text{turb}} = \frac{\mu_{\text{turb}}}{\rho} = \Delta U l. \tag{4.5}$$

These last equations can also be derived on the basis of analogy already mentioned that exists between turbulent motion and random motion of gas molecules. If the analogy between these two types of motion is closely followed, the scale of motion l may be considered as the analog of the length of the mean free path, and the eddy velocity as the analog of the average velocity of the gas molecules.

In this way equations (4.4) and (4.5) may be derived directly from the well-known equations of the kinetic theory of gases. Developing the analogy further, we can write the approximation for eddy velocity as is usually done in the kinetic theory of gases:

$$\Delta U \approx l \frac{\partial U}{\partial l}. \tag{4.6}$$

The effective eddy viscosity is very large in comparison with the ordinary viscosity. Indeed, having set up the relation $\nu \frac{\nu}{\text{turb}}$, we find

$$\frac{\nu}{\nu_{\text{turb}}} = \frac{\nu}{\Delta U l} \approx \frac{1}{\text{Re}} \ll 1. \tag{4.7}$$

With the aid of μ_{turb}, we can express the drag acting on 1 cm² of a solid surface (shear stress) in the form

$$\tau = \frac{F}{S} \approx \mu_{\text{turb}} \left(\frac{\Delta U}{l}\right) \approx \mu_{\text{turb}} \frac{\partial U}{\partial l} \approx \rho l^2 \left(\frac{\partial U}{\partial l}\right)^2 = \alpha \rho l^2 \left(\frac{\partial U}{\partial l}\right)^2. \tag{4.8}$$

where α is a certain numerical factor.

In the following we consider two special cases of turbulent motion: motion having a scale $\lambda \ll l$, i.e., small scale turbulence at a distance from the solid walls, and turbulent motion close to the solid walls. In these cases it is possible to determine the characteristics of the turbulent flow on the basis of similarity.

Let us first examine small scale motion ($\lambda \ll l$) in a volume of fluid. Let us assume, however, that $\lambda \gg \lambda_0$, so that the type of motion is inviscid. Let us find the velocity v_λ of turbulence eddies in scale λ (or, in other words, the change in the motion velocity over a distance on the order of λ).

The quantity v_λ can be a function only of ρ, λ and the constant ϵ, since these characterize motion on any scale. The motion of the fluid (at $\lambda \gg \lambda_0$) cannot be a function of the fluid viscosity ν. Neither can it be a direct function of the scale l or of the velocity of the flow U (since $\lambda \ll l$).

The only combination of the quantities ρ, λ and ϵ, having the dimensions of the velocity is $\left(\frac{\epsilon\lambda}{\rho}\right)^{1/3}$. Thus,

$$v_\lambda \approx \left(\frac{\epsilon\lambda}{\rho}\right)^{1/3}. \tag{4.9}$$

Using (4.3) to express ϵ in terms of ΔU, we find

$$v_\lambda \approx \Delta U \left(\frac{\lambda}{l}\right)^{1/3}. \tag{4.9'}$$

Thus, the eddy velocities for motion of scale λ are smaller than the velocity of the main flow by the factor $\left(\frac{\lambda}{l}\right)^{1/3}$.

The reductions in velocity and scale are matched by a corresponding reduction in the Reynolds number, according to the relation

$$\mathrm{Re}_\lambda = \frac{v_\lambda \lambda}{\nu} = \frac{\Delta U \lambda^{4/3}}{\nu l^{1/3}} = \mathrm{Re}\left(\frac{\lambda}{l}\right)^{4/3}.$$

At certain scale λ_0, known as the inner scale (microscale) of turbulence, the Reynolds number Re_{λ_0} is found to be approximately unity. Evidently,

$$\lambda_0 \approx \frac{l}{\mathrm{Re}^{3/4}} \approx \left(\frac{\nu^3\rho}{\varepsilon}\right)^{1/4}. \tag{4.10}$$

Starting at this scale level, the motion of the fluid is viscous in nature. Turbulence eddies of a scale $\lambda \leq \lambda_0$ do not suddenly disappear, but are gradually damped due to the effects of viscosity.

Let us now examine the case of turbulent motion near a solid surface by first considering fluid flowing past a flat plate of infinite extent in the plane $y = 0$. Let the mean flow be in the x-direction and the average velocity be $v_x = U$. The average velocity is, in general, a function of the distance of the fluid layer from the surface of the solid body, and thus $U = U(y)$. Upon the average motion of the fluid in the x-direction, there is a superposed eddy motion in all directions.

Let us find the function $U(y)$. To do this we may employ equation (4.8) by rewriting it in the form

$$\frac{\partial U}{\partial y} = \sqrt{\frac{\tau}{\rho\alpha}} \cdot \frac{1}{l}. \tag{4.11}$$

Because all points of the infinite plane $y = 0$ along which there is flow are entirely equivalent, the shear stress τ is constant over all planes. This may be interpreted as follows. The quantity τ represents the momentum transferred from the flowing fluid to the wall. Within the fluid flowing along the wall there is a continuous transfer of momentum (a constant momentum flux equal to τ) from rapidly moving, more distant fluid layers to layers adjacent to the wall.

Since the momentum transfer must satisfy the law of conservation of momentum, and, by assumption, is the same along the entire surface (i.e., it is not a function of the coordinate x), it must be constant and in a direction normal to the wall (i.e., it can not be a function of the coordinate y). This ignores changes in momentum transfer arising from energy dissipation caused by molecular viscosity.

Considering that τ is constant, we can now rewrite equation (4.11)

$$U = \sqrt{\frac{\tau}{\rho\alpha}} \int \frac{dy}{l} + \text{const} = \frac{v_0}{\sqrt{\alpha}} \cdot \int \frac{dy}{l(y)} + C_1, \tag{4.12}$$

where C_1 is a constant of integration and v_0 designates the quantity

$$v_0 = \sqrt{\frac{\tau}{\rho}}. \tag{4.13}$$

In order to integrate the equation (4.12), it is necessary to determine the scale of the motion as a function of the distance $l(y)$ separating the fluid layer from the solid surface. The special feature of the flow situation we are examining is that the conditions determining the flow regime do not include the dimensions of the body or any other linear quantity which could be used to describe a characteristic scale of large turbulence eddies l. It is logical, therefore, to assume that

$$l(y) \approx y. \tag{4.14}$$

The approximation (4.14) shows that the eddy scale increases with distance from the solid wall. And the latter is the only linear quantity which can be used to describe the flow regime. Such an assumption appears to be quite natural: the solid wall retards the fluid moving nearby, so that the velocity gradually diminishes as the wall is approached.

Using the approximation (4.14), we can derive from equation (4.12)

$$U = \frac{v_0}{\sqrt{a}} \ln y + C_1. \qquad (4.15)$$

The physical significance of the quantity v_0 is clarified by noting that the quantity v', according to expression (4.1), is equal to

$$v' \approx \Delta U \approx U(y+l) - U(y) \approx v_0.$$

Thus v_0 is the velocity of the turbulence eddies that are characteristic of the flow. In order to determine the constant C_1 we must remember that the reduction in the scale of turbulence eddies as the wall is approached is matched by a corresponding reduction in the Reynolds number

$$\mathrm{Re} = \frac{v_0 l(y)}{\nu}.$$

At a certain $l = \delta_0$ it is approximately equal to unity.

In the region $y < \delta_0$, known as the viscous sublayer, the nature of the flow is viscous. The thickness of the viscous sublayer is given by the condition

$$\frac{v_0 \delta_0}{\nu} \approx 1,$$

or

$$\delta_0 = a \frac{\nu}{v_0}, \qquad (4.16)$$

where a is a proportionality factor. The constant in (4.15) is chosen in such a manner so that at $y \sim \delta_0$ the average flow velocity becomes a small quantity on the order of the characteristic velocity of the turbulence eddies v_0.

Then, for the average velocity, the so-called logarithmic profile is obtained (Figure 4):

$$U = \frac{v_0}{\sqrt{a}} \ln \frac{v_0 y}{a\nu}. \qquad (4.17)$$

Expressing v_0 in terms of τ by means of equation (4.12), we finally obtain

$$U = \sqrt{\frac{\tau}{\rho\alpha}} \ln \frac{y}{a\nu} \sqrt{\frac{\tau}{\rho}}. \tag{4.18}$$

The eddy viscosity may be written in the form

$$\nu_{turb} \sim v_0 l \sim v_0 y \sim \nu \frac{y}{\delta_0}. \tag{4.19}$$

The logarithmic profile of velocities (4.18) contains two unknown constants: α and a. Their values must be determined from experimental data for the velocity distribution in the vicinity of the solid surface. This may be done most conveniently by introducing the dimensionless factor

$$y_* = \frac{v_0 y}{\nu} \sim \frac{y}{\delta_0}.$$

The dimensionless ratio $\frac{U}{v_0}$ is presented as a function of log y_* in Figure 4. The data for this plot were obtained from numerous measurements of velocity distribution in the vicinity of the solid wall.

It is evident that the velocity distribution can be represented by a simple logarithmic relationship only at $y_* \geq 30$. In that region $\alpha \approx 0.17$. The determination of a directly from the curve in the region $y_* \geq 30$ has no meaning, however, since by definition, a relates to the region in which $Re = y_* \sim 1$, i.e., to the viscous sublayer.

Contemporary hydrodynamics do not offer any singular point of view regarding the velocity distribution in the viscous sublayer. Two hypotheses have been suggested:

1) The Prandtl hypothesis [4], which has been accepted widely, states that in the region $y < \delta_0$, the fluid motion is entirely laminar. Prandtl himself named this region the "laminar sublayer", and based his hypothesis on the fact that at $y < \delta_0$ the Reynolds number is found to be smaller than unity.

The shear stress τ_0 in the laminar sublayer evidently may be expressed by the equation

$$\tau_0 = \rho\nu \cdot \frac{dU}{dy}. \tag{4.20}$$

The velocity distribution can, therefore, be expressed by the linear equation

$$U = \frac{\tau_0}{\rho\nu} y + C.$$

The integration constant must equal zero, since the velocity of the fluid at the solid surface is zero. Therefore, for $y < \delta_0$

$$U = \frac{\tau_0}{\rho\nu} y. \tag{4.21}$$

The linear and logarithmic velocity profiles over the respective ranges up to and beyond their intersection do not provide satisfactory agreement with the experimental distribution data shown in Figure 4.

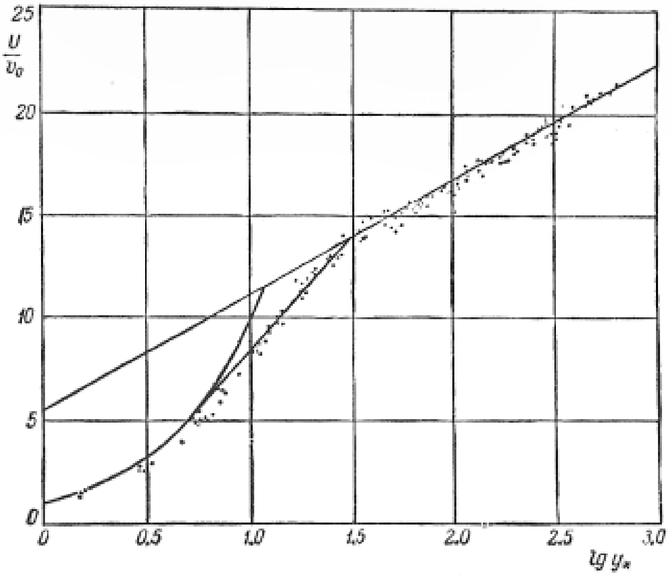

Figure 4. Distribution of average velocity for turbulent flow past a plate.

Von Karman, therefore, proposed a theory for the case of turbulent flow past a solid plane, in which the flow is divided into three regions: (a) a region of turbulent flow, (b) a "buffer" layer, and (c) a laminar sublayer.

According to von Karman [4], the turbulent flow in the buffer layer is damped as it approaches the solid wall, and the same law (4.14) as for the turbulent flow region is also applicable here. In the buffer layer, however, the effect of viscosity becomes significant. Consequently the values of the constants α and a in the logarithmic velocity distribution equation must differ from their values in the main turbulent zone.

2) The other hypothesis, presented jointly by L. D. Landau and the author of this book [1, 6], states that the turbulent motion in the viscous sublayer does not suddenly disappear, but is gradually damped as it approaches the wall. The equations for the damping of turbulence eddies in the viscous sublayer, i.e., the dependence of l on y, can no longer be applied on the basis of dimensional

considerations, as is the case for the region of developed turbulence. All quantities in the viscous sublayer may be functions of viscosity, and the distance from the wall is no longer the sole quantity with a linear dimension. The equations for the damping of the turbulent flow in a viscous sublayer may be derived as follows. The distribution of the average velocity in this layer has the same form as in laminar flow, i.e.,

$$v_x \sim y.$$

Although turbulence eddies do not originate in the viscous sublayer, they enter it from the side $y > \delta_0$. The eddy velocities have the same magnitude as the average velocities in the sublayer. Therefore,

$$v'_x \sim y.$$

In view of the continuity equation

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} = 0$$

the normal component of eddy velocity is

$$v'_y = -\int \frac{\partial v'_x}{\partial x} dy \sim y^2.$$

The proportionality coefficient in the expression for v_y can be evaluated using the condition that, at $y \approx \delta_0$, the eddy velocity v'_y at the boundary of the viscous layer is of the same order of magnitude as the characteristic velocity of the turbulent flow v_0. Therefore,

$$v'_y = v_0 \frac{y^2}{\delta_0^2}. \tag{4.22}$$

Thus, in a viscous sublayer the tangential and normal components of the average velocity and of the eddy velocities vary as a function of distance in the same way as the distribution of velocities in a laminar boundary layer. This, in essence, is the extent of the resemblance between a viscous sublayer and a laminar boundary layer.

In order to evaluate the coefficient of eddy viscosity in the viscous sublayer, the scale of the turbulent motion must be determined. This may be done as follows. In a viscous sublayer Re is less than unity, and the second-order terms in the Navier-Stokes equations are small compared to the first-order terms. The velocity distribution in a viscous sublayer can therefore be determined by linear equations only. If a certain spectrum of eddies penetrates a viscous sublayer, the interaction between separate eddies ceases. The flow then becomes a sum of independent periodic motions, whose periods T remain constant throughout the viscous sublayer.

Thus, it may be assumed that the periods of the turbulence eddies within a viscous sublayer are not a function of the distance y from the wall. The scale of the eddy motion in the y-direction is equal to

$$l \approx v'_y \cdot T, \tag{4.23}$$

or, since T is not a function of y,

$$l \sim y^2.$$

For $y \approx \delta_0$, the scale of the motion must equal that of the turbulent boundary layer, i.e., $l \approx \delta_0$. Therefore, the normalization factor is equal to $\frac{1}{\delta_0}$, and

$$l = \frac{y^2}{\delta_0}. \tag{4.24}$$

The scale of eddy motions in the viscous sublayer decreases with the distance y from the wall more rapidly than in the turbulent boundary layer.

By definition, the momentum transferred by eddies in the direction of the wall is given by the equation

$$\tau_{turb} = \rho \cdot \nu_{turb} \frac{dU}{dy} = \rho \cdot v'_y l \frac{dU}{dy} \sim \frac{\rho v_0 y^4}{\delta_0^3} \frac{dU}{dy}. \tag{4.25}$$

Therefore, the eddy viscosity is

$$\nu_{turb} \sim \frac{v_0 y^4}{\delta_0^3} \sim \nu \left(\frac{y}{\delta_0}\right)^4. \tag{4.26}$$

At $y > \delta_0$, the momentum transferred by eddies is less than the momentum transferred by molecular viscosity, and the eddy viscosity is less than the viscosity ν. In view of this, at $y > \delta_0$, it may be assumed that the shear stress τ coincides approximately with the quantity τ_0, and that the average velocity profile is defined by equation (4.20). Nevertheless, turbulence eddies exist up to the wall itself. It is shown later that the existence of turbulence eddies in the viscous sublayer is of significant inportance in the transfer of heat and mass toward a solid surface. In addition, knowledge of the existence of turbulence eddies in the viscous sublayer provides the basis for a theoretical approach in determining the velocity distribution in the boundary region.

It should be noted that another point of view also exists concerning the nature of damping of turbulence eddies in the viscous sublayer [7]—namely, that the scale of eddy motion is not expressed by formula (4.24) but by the same law (4.14) that applies to the region $y > \delta_0$. This leads to an eddy viscosity given by

$$\nu_{turb} \sim \nu \left(\frac{y}{\delta_0}\right)^3. \tag{4.27}$$

A choice between the two hypotheses, and between the two viewpoints of the second hypothesis concerning the damping of eddies, may be made only on the basis of experimental data. Detection of turbulence eddies and determination of equations describing their damping in the proximity of the wall present a highly complicated problem. Moreover, the existing measurement techniques do not provide the means for a definitive selection of either one of the hypotheses that have been advanced. It has been found that convincing data could be obtained in studying the diffusion of dissolved matter in turbulent flow.

As shown below (see Section 57), facts exist which support the hypothesis of the gradual reduction of turbulence in a viscous sublayer in accordance with equation (4.24). We therefore accept this equation for the purpose of subsequent discussion and utilize expression (4.26) for ν_{turb} in determining the velocity profile in the boundary zone. In deriving the velocity distribution in that zone, we assume that the transfer of momentum is accomplished by turbulence eddies which have already been affected by the viscosity. The significance of this is that the expression of the reduction of eddy viscosity (4.19) is not applicable in the boundary zone. Thus, in the boundary zone, it is convenient to choose an interpolation formula for the coefficient of eddy viscosity (a similar formula was given by G. P. Piterskikh). The formula is intermediate between (4.19) and (4.26)

$$\nu_{turb} = b\nu \left(\frac{y}{\delta_0}\right)^2, \tag{4.28}$$

and for shear stress

$$\tau = \rho(\nu + \nu_{turb})\frac{dU}{dy} = \rho\left[\nu + b\nu\left(\frac{y}{\delta_0}\right)^2\right]\frac{dU}{dy}, \tag{4.29}$$

where b is an undetermined constant.

Integrating (4.29), we obtain the relation describing the distribution of the average velocity in the boundary zone,

$$U = \frac{v_0}{\sqrt{b}} \operatorname{arctg} \sqrt{b}\, \frac{y}{\delta_0} + c. \tag{4.30}$$

The constants b and c are determined by assuming that the velocity distribution corresponds to (4.21) at $y_* \approx 5$, and to (4.17) for $y_* = 30$.

In its final form the relations for the average velocity distribution may be expressed as

$$\frac{U}{v_0} = y_*, \qquad 0 \leqslant y_* \leqslant 5, \tag{4.31}$$

$$\frac{U}{v_0} = 10 \operatorname{arctg}(0.1 y_*) + 1.2, \qquad 5 \leqslant y_* \leqslant 30, \tag{4.32}$$

$$\frac{U}{v_0} = 5.5 + 2.5 \ln y_*, \qquad y_* > 30, \tag{4.33}$$

These show better agreement with measured velocity distributions (See Figure 4) than the previous expressions.

In reality we can never, of course, duplicate the condition of flow past an infinite plane for which these relations were derived. Nevertheless, they can be applied quite effectively to the boundary layer problems of flow pas a plate or within a tube.

Let us first examine the flow of a fluid past a semi-infinite plate. As noted in Section 3, the Reynolds number of the boundary zone increases with the distance from the leading edge. If it reaches the critical value $Re_\delta = Re_{cr}$, the flow in the boundary layer becomes turbulent (Figure 5). The critical value of the Reynolds number for a flat plate is approximately 1,500, but this value is substantially reduced by various types of disturbances in the boundary layer, such as are produced by wire probes or other small protuberances. In contrast to the case of the infinite surface examined above, the turbulent boundary layer formed in the stream flowing past a smooth plate not only varies with the coordinate x, but has a finite thickness d as well. (By convention we establish the origin at the leading edge of the plate.) Internally, the boundary layer has a velocity distribution with the logarithmic profile (4.17). At the border of the boundary layer, i.e., at y = d, the flow velocity is equal to the velocity of the main stream U_0:

$$U_0 = \frac{v_0}{\sqrt{\varkappa}} \ln \frac{v_0 d}{10\nu}. \tag{4.34}$$

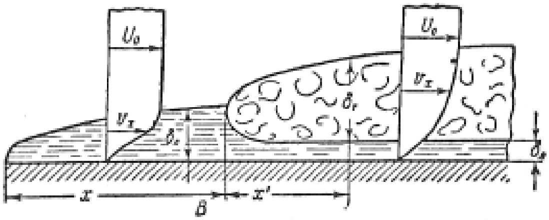

Figure 5. Formation of a turbulent boundary layer on a plate.

Up to point B the flow in the boundary layer layer is laminar, beyond that point it is turbulent.

The thickness of the boundary layer and the characteristic velocity v_0 vary with y in such a manner that this velocity has a constant value U_0 at the border of the boundary layer.

The second equation relating v_0 to d may be derived in the following manner. The slope of the curve d (x) is equal to the ratio of the velocities normal and tangential to the plate

$$\frac{\partial d(x)}{\partial x} \approx \frac{v_y}{v_x}.$$

However, at the boundary of the layer $v_x \approx U$, and the magnitude of v_y is on the order of the velocity v_0. Thus,

$$\frac{\partial d(x)}{\partial x} \approx \frac{v_0}{U_0}.$$

Since v_0 is a weak function of x, the thickness of the layer d can be expressed approximately as

$$d \approx \frac{v_0 x}{U_0}. \qquad (4.35)$$

The thickness of the turbulent boundary layer is proportional to the distance from the leading edge of the plate. Therefore,

$$U_0 = \frac{v_0}{\sqrt{\varkappa}} \ln \frac{v_0^2 x}{10 U_0 \nu}. \qquad (4.36)$$

The drag on the plate is equal to

$$F = \int_0^b \int_0^l \tau \, dx \, dz = K_f \frac{\rho U_0^2}{2} bl,$$

where K_f is the drag coefficient.

Simple computation shows that K_f may be determined from the equation given in [8]

$$\frac{1}{\sqrt{K_f}} = 4.13 \lg (\mathrm{Re}\, K_f). \qquad (4.37)$$

For rough calculation of the drag coefficient in turbulent flow past a plate, we can use the empirical relation of [8] as follows:

$$K_f = \frac{0.074}{\mathrm{Re}^{\frac{1}{5}}}. \qquad (4.37\ )$$

All the other relations derived above for flow past infinite plane may be directly applied to the case of a plate.

The velocity distribution curve shown in Figure 4 applies precisely to the case of flow past a plate. The logarithmic profile of the distribution of the average velocity may be applied to turbulent fluid flow in a tube. Because of the slow change of the logarithmic term in equation (4.17), this formula is applicable in calculating the average fluid velocity in a tube. Thus, for tubes,

$$U_R = \frac{v_0}{\sqrt{a}} \ln \frac{Rv_0}{a\nu},$$

where U_R is the flow velocity in the center of the tube and R is the radius of the tube.

Figure 6 shows the velocity profiles for laminar and turbulent flow regimes in a tube.

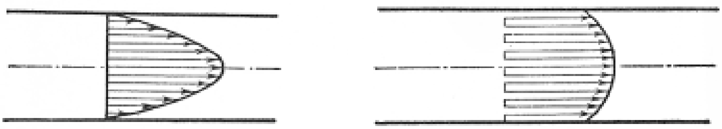

Figure 6. Velocity distribution in laminar (left) and turbulent (right) flow in a tube.

It is often important to relate the eddy velocity v_0 to the velocity U_0.

$$v_0 = \sqrt{\frac{\tau}{\rho}} = \sqrt{\frac{k_f}{2}}\, U_0 \approx \frac{\sqrt{k_f}}{1.41} U_0. \tag{4.38}$$

For flow past a plate and for intermediate values of Reynolds numbers ($\lesssim 10^5$) the approximation derived in [8] is applicable,

$$v_0 \approx \frac{0.27}{0.58} \frac{U_0}{\mathrm{Re}^{4.0}} = 0.47 \frac{U_0}{\mathrm{Re}^{4.0}}. \tag{4.39}$$

Similarly, for a tube the following may be written [8]

$$v_0 \approx \frac{0.16 U_R}{\mathrm{Re}^{\frac{1}{8}}} = \frac{0.2\bar{U}}{\mathrm{Re}^{\frac{1}{8}}}. \tag{4.40}$$

5. FLOW PAST A BODY WITH APPRECIABLE CURVATURE

The flow of a fluid past the surface of a solid plate was discussed above. In practice it is often necessary to study fluid motion past a body having appreciable curvature, such as a cylinder, a sphere, etc. Such bodies are usually described as having a non-streamlined shape.

The configuration of the flow past a non-streamlined surface is more complex than in the case of a plate. By the same token, the transfer of mass toward such a surface is also more complicated.

As a typical example, let us examine flow past a surface of a cylinder whose axis is perpendicular to the direction of the fluid stream.

Direct hydrodynamic measurements have shown that the fluid flows smoothly past the upstream portion of the cylinder forming a boundary layer which differs in nature from the boundary layer formed on a plate (Figure 7).

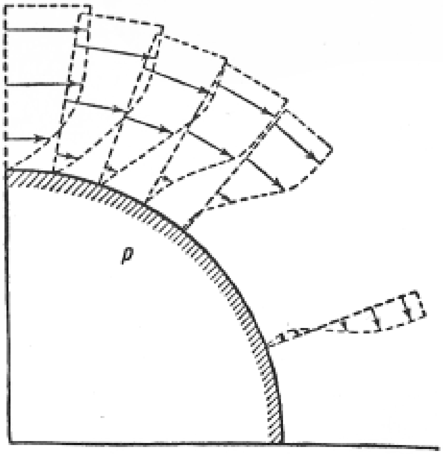

Figure 7. The appearance of separation in flow past a cylinder.

The essential difference between the two boundary layers, i.e., that on the surface of a cylinder and that on a plate, lies in the fact that the velocity and the pressure in the fluid passing around the cylinder vary from one point to the next. Beyond the boundary layer, pressure and flow velocity are determined by the Bernoulli equation (3.3). Therefore, as already noted in Section 3, the velocity of the fluid is least at the stagnation point on the cylinder, and increases uniformly up to the midpoint, beyond which it begins to decline. On the other hand, pressure is greatest at the stagnation point, decreases up to the midpoint, and increases over the downstream portion of the cylinder.

Over the upstream portion of the cylinder the fluid in the boundary layer moves in the direction of the pressure gradient; in the downstream part, against the pressure gradient. It is clear that this gradient initially will retard the slowly moving fluid layers adjacent to the surface of the body. The velocity profile will change as shown in Figure 7. At some point P, pressure in the opposite direction (pressure drag) will completely stop the motion of the fluid layers next to the surface. Beyond that point, there will be flow reversal: fluid will move in the opposite direction at the surface. Here the fluid layers within the boundary layer are forced out into the mainstream flow, so that, effectively, the fluid layers formerly within the boundary layer, are separated from the surface of the solid body. The separated boundary layer forms a jet that flow into the main fluid stream. The surface of the body in Figure 7 is shown by a solid curve; the straight dotted lines indicate the streamlines; and the dotted curved lines show the velocity profile.

It has been experimentally established (in agreement with theory) that at point P eddies begin to separate and are carried away by the stream. The separation of eddies starts at Reynolds numbers around 20.

At higher Reynolds numbers (100 to 200) the motion of the jet is unstable, and the fluid motion becomes turbulent beyond the point of separation (Figure 8). This effect leads to a considerable increase in the resistance of a non-streamlined body immersed in a moving fluid, as compared to a plate. This resistance is known as the drag and is associated with the energy dissipated in the zone of turbulence, which is termed the wake. This resistance is considerably

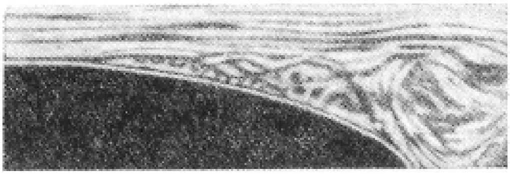

Figure 8. Appearance of turbulence in the downstream region beyond the point of separation.

greater than the usual viscous resistance. The preceding discussion points to the role played by the boundary layer on the surface of a body having appreciable curvature and on the regime of the flow past that body. It has been found that the drag F experienced by the body, may be expressed by the formula

$$F = K_f \frac{\rho U_0^2}{2} S, \tag{5.1}$$

where K_f is a constant coefficient (for a body of a given configuration).

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