Ejercicios:

1. Demostrar la 2da Ley de Fick a partir de la 1era Ley de Fick.
2. Demostrar que la función gaussiana dada es solución de la 2da Ley de Fick.
3. Demostrar que existen dos puntos de inflexión en $x = \pm\sqrt{2Dt}$.

Ejercicio 1

$$J = -DA\left(\frac{\partial C}{\partial x}\right)$$

$$(2\ momentos\ temporales):\quad J_1 = -DA\left(\frac{\partial C}{\partial x_1}\right);\quad J_2 = -DA\left(\frac{\partial C}{\partial x_2}\right)$$

$$(Balance\ de\ masas):\quad \Delta J = J_2 - J_1 = -DA\left(\frac{\partial C}{\partial x_2}\right) + DA\left(\frac{\partial C}{\partial x_1}\right) = DA\left(\left(\frac{\partial C}{\partial x_1}\right) - \left(\frac{\partial C}{\partial x_2}\right)\right)$$

$$(intervalo\ infinitesimal):\quad \lim_{\Delta x \to 0} \Delta J = \lim_{\Delta x \to 0} DA\left(\left(\frac{\partial C}{\partial x_1}\right) - \left(\frac{\partial C}{\partial x_2}\right)\right) = DA\left(\frac{\partial^2 C}{\partial x^2}\right)$$

$$si\ j = \frac{J}{A} \Longrightarrow \Delta j = \frac{DA}{A}\left(\frac{\partial^2 C}{\partial x^2}\right) = D\left(\frac{\partial^2 C}{\partial x^2}\right)$$

El flujo de partículas j se define como la cantidad de partículas por unidad de tiempo y área por lo tanto,

$$j \equiv \frac{\partial C}{\partial t}$$

De esta forma, se demuestra que:

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2}\right)$$

Quod erat demonstrandum.

Ejercicio 2

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2}\right)$$

$$C(x,t) = \frac{N_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

$$\begin{aligned}
\frac{\partial C}{\partial t} &= \left(\frac{N_0}{\sqrt{4\pi Dt}}\right)' e^{-\frac{x^2}{4Dt}} + \left(\frac{N_0}{\sqrt{4\pi Dt}}\right)\left(e^{-\frac{x^2}{4Dt}}\right)' = \frac{N_0}{\sqrt{4\pi D}}\left(t^{-\frac{1}{2}}\right)' e^{-\frac{x^2}{4Dt}} + \frac{N_0}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x^2}{4Dt}\right)' \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right) + \frac{N_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\left(-\frac{x^2}{4D}\right)(t^{-1})' \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) + \frac{N_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\left(-\frac{x^2}{4D}\right)(-t^{-2}) \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) + \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(t^{-\frac{1}{2}}\right)\left(-\frac{x^2}{4D}\right)(-t^{-2}) \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(-\frac{1}{2}t^{-\frac{3}{2}} + \frac{x^2}{4D}t^{-\frac{5}{2}}\right) = \frac{1}{2}\frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{\frac{3}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C}{\partial x} &= \frac{N_0}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}}\right)' = \frac{N_0}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x^2}{4Dt}\right)' = \frac{N_0}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{1}{4Dt}\right)(2x) \\
&= \frac{N_0}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x}{2Dt}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 C}{\partial x^2} &= \frac{N_0}{\sqrt{4\pi Dt}}\left(\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x}{2Dt}\right)\right)' = \frac{N_0}{\sqrt{4\pi Dt}}\left(\left(e^{-\frac{x^2}{4Dt}}\right)'\left(-\frac{x}{2Dt}\right) + \left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x}{2Dt}\right)'\right) \\
&= \frac{N_0}{\sqrt{4\pi Dt}}\left(\left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{x}{2Dt}\right)^2 + \left(e^{-\frac{x^2}{4Dt}}\right)\left(-\frac{1}{2Dt}\right)\right) = \frac{N_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{4D^2t^2} - \frac{1}{2Dt}\right) \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(t^{-\frac{1}{2}}\right)\left(\frac{x^2}{4D^2t^2} - \frac{1}{2Dt}\right) = \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{4D^2}\left(t^{-2}t^{-\frac{1}{2}}\right) - \frac{1}{2D}\left(t^{-1}t^{-\frac{1}{2}}\right)\right) \\
&= \frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{4D^2}t^{-\frac{5}{2}} - \frac{1}{2D}t^{-\frac{3}{2}}\right) = \frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{-\frac{3}{2}}\right)
\end{aligned}$$

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2}\right) \Longrightarrow \frac{1}{2}\frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{\frac{3}{2}}\right) = D\left(\frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}} e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{-\frac{3}{2}}\right)\right)$$

Quod erat demonstrandum.

Ejercicio 3

$$\frac{\partial^2 C}{\partial x^2} = 0 \Leftrightarrow x = \pm\sqrt{2Dt}$$

Camino 1: sustituir en la derivada segunda calculada en el ejercicio 2 y verificar que es raíz.

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}}e^{-\frac{x^2}{4Dt}}\left(\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{-\frac{3}{2}}\right) = 0$$

$$\frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}}e^{-\frac{\left(\pm\sqrt{2Dt}\right)^2}{4Dt}}\left(\frac{\left(\pm\sqrt{2Dt}\right)^2}{2D}t^{-\frac{5}{2}} - t^{-\frac{3}{2}}\right) = 0$$

$$\frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}}e^{-\frac{1}{2}}\left(t\left(t^{-\frac{5}{2}}\right) - t^{-\frac{3}{2}}\right) = 0$$

$$\frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}}e^{-\frac{1}{2}}\left(t^{1-\frac{5}{2}} - t^{-\frac{3}{2}}\right) = 0$$

$$\frac{1}{2D}\frac{N_0}{\sqrt{4\pi D}}e^{-\frac{1}{2}}\left(t^{-\frac{3}{2}} - t^{-\frac{3}{2}}\right) = 0$$

Camino 2: encontrar la raíz de la derivada segunda calculada anteriormente y verificar que es el valor dado.

$$\frac{x^2}{2D}t^{-\frac{5}{2}} - t^{-\frac{3}{2}} = 0 \Longrightarrow \frac{x^2}{2D}t^{-\frac{5}{2}} = t^{-\frac{3}{2}} \Longrightarrow \frac{x^2}{2D} = \frac{t^{-\frac{3}{2}}}{t^{-\frac{5}{2}}} \Longrightarrow \frac{x^2}{2D} = t^{-\frac{3}{2}+\frac{5}{2}} \Longrightarrow \frac{x^2}{2D} = t^{-\frac{2}{2}} \Longrightarrow \frac{x^2}{2D} = t \Longrightarrow x^2 = 2Dt$$
$$\Longrightarrow x = \pm\sqrt{2Dt}$$