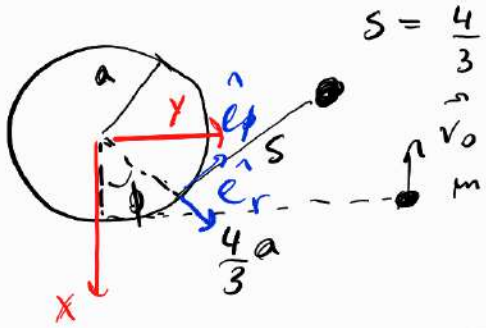
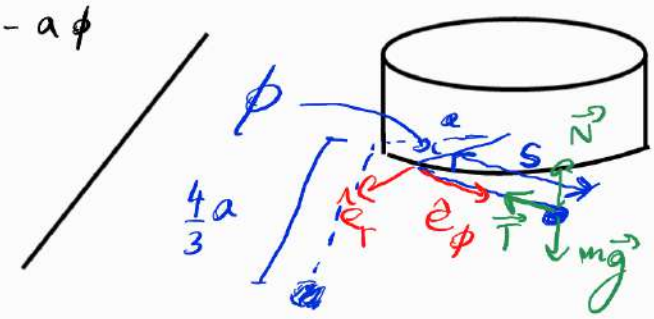
1)

a)



$S = \frac{4}{3}a - a\phi$



$$\vec{r} = a\hat{e}_r + \left(\frac{4}{3} - \phi\right)a\hat{e}_\phi$$

$$\Rightarrow \vec{v} = a\dot{\phi}\hat{e}_\phi - \left(\frac{4}{3}-\phi\right)a\dot{\phi}\hat{e}_r - a\dot{\phi}\hat{e}_\phi = \left(\frac{4}{3}-\phi\right)a\dot{\phi}\hat{e}_r$$

$$\Rightarrow \vec{a} = \left[\left(\frac{4}{3}-\phi\right)a\ddot{\phi} - a\dot{\phi}^2\right]\hat{e}_r - \left(\frac{4}{3}-\phi\right)a\dot{\phi}^2\hat{e}_\phi$$

b) se puede ver que las fuerzas que actúan sobre la partícula no realizan trabajo ya que $m\vec{g}$, $\vec{N}$ y $\vec{T}$ son perpendiculares a la velocidad:

$$\left.\begin{array}{l} m\vec{g} \parallel \hat{k} \\ \vec{N} \parallel \hat{k} \\ \vec{T} \parallel \hat{e}_\phi \end{array}\right\} \text{ y } \vec{v} \parallel \hat{e}_r$$

$$\Rightarrow \begin{cases} W_{m\vec{g}} = \int m\vec{g}\cdot\vec{v}\,dt = 0 \\ W_{\vec{N}} = \int \vec{N}\cdot\vec{v}\,dt = 0 \\ W_{\vec{T}} = \int \vec{T}\cdot\vec{v}\,dt = 0 \end{cases}$$

$$\Rightarrow E = \frac{m\vec{v}^2}{2} = cte$$

$$\Rightarrow E = \frac{m}{2}v_0^2$$

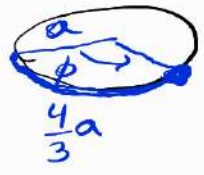
c) $$E = \frac{mv_0^2}{2} = \frac{m}{2}\left(\frac{4}{3}-\phi\right)^2 a^2\dot{\phi}^2 \Rightarrow \dot{\phi}\left(\frac{4}{3}-\phi\right) = \frac{v_0}{a}$$

$$\Rightarrow \int_0^{t_1} \frac{d\phi}{dt}\left(\frac{4}{3}-\phi\right)dt = \int_0^{t_1} \frac{v_0}{a}\,dt = t_1\frac{v_0}{a}$$

$$\Rightarrow \frac{4}{3}\left[\phi(t_1)-\phi(0)\right] - \frac{\left[\phi^2(t_1)-\phi^2(0)\right]}{2} = \frac{v_0 t_1}{a}$$

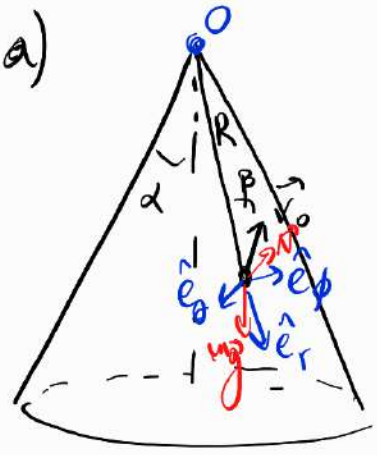
$\phi(0) = 0$

$\phi(t_1) = \frac{4}{3}$



$$\Rightarrow \boxed{t_1 = \frac{8a}{9v_0}}$$

2) a)



$$\Rightarrow m\left[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \operatorname{sen}^2\theta)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \operatorname{sen}\theta\cos\theta)\hat{e}_\theta + (r\ddot{\phi}\operatorname{sen}\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\operatorname{sen}\theta)\hat{e}_\phi\right] = m\vec{g} + \vec{N}$$

Con $m\vec{g} = -mg\cos\theta\,\hat{e}_r + mg\operatorname{sen}\theta\,\hat{e}_\theta$

$\vec{N} = -N\hat{e}_\theta \quad (N \geq 0)$

$\theta = \pi - \alpha \Rightarrow \cos\theta = -\cos\alpha$

$\operatorname{sen}\theta = \operatorname{sen}\alpha$

$\dot{\theta} = \ddot{\theta} = 0$

$$\Rightarrow \begin{cases} m\ddot{r} - mr\dot{\phi}^2\operatorname{sen}\alpha = mg\cos\alpha & (\hat{e}_r) \\ mr\dot{\phi}^2\operatorname{sen}\alpha\cos\alpha = mg\operatorname{sen}\alpha - N & (\hat{e}_\theta) \\ r\left[r\ddot{\phi}\operatorname{sen}\alpha + 2\dot{r}\dot{\phi}\operatorname{sen}\alpha = 0\right] & (\hat{e}_\phi) \end{cases}$$

$$\hookrightarrow \frac{d}{dt}(r^2\dot{\phi}\operatorname{sen}\alpha) = 0 \Rightarrow r^2\dot{\phi}\operatorname{sen}\alpha = cte$$

pero $\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\operatorname{sen}\theta\,\hat{e}_\phi$

$\vec{V}_0 = -V_0\cos\beta\,\hat{e}_r + V_0\operatorname{sen}\beta\,\hat{e}_\phi$

$\Rightarrow R\dot{\phi}\operatorname{sen}\alpha = V_0\operatorname{sen}\beta$

$\Rightarrow r^2\dot{\phi}\operatorname{sen}\alpha = RV_0\operatorname{sen}\beta$

b)

$$\Rightarrow \boxed{N = mg\operatorname{sen}\alpha - \frac{m(RV_0\operatorname{sen}\beta)^2\cos\alpha}{r^3\operatorname{sen}\alpha} \geq 0}$$

c)

Si $r = R \Rightarrow N = mg\operatorname{sen}\alpha - m\frac{V_0^2\operatorname{sen}^2\beta}{R\,tg\,\alpha} \geq 0$

$$\Rightarrow \boxed{V_{0\,max} = \frac{\operatorname{sen}\alpha}{\sqrt{\operatorname{sen}\beta\cos\alpha}}\sqrt{gR}}$$

d) como se conserva la energía es

$$mg\Delta y + \frac{m}{2}\frac{r_{min}^2\operatorname{sen}^2\theta\,\dot{\phi}^2}{\frac{(V_0\operatorname{sen}\beta)^2}{r_{min}^2}} = \frac{mV_0^2}{2}$$

, pero $(R - r_{min})\cos\alpha = \Delta y \Rightarrow r_{min} = R - \frac{\Delta y}{\cos\alpha}$

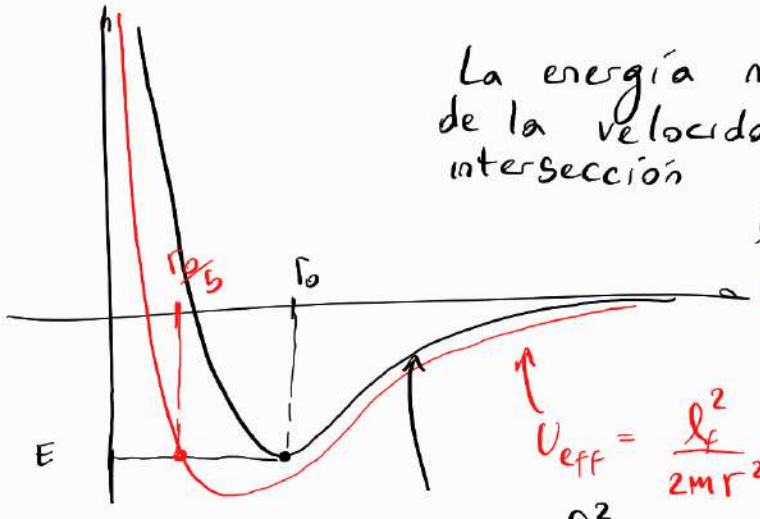
$$\Rightarrow 2g\Delta y = V_0^2\left[1 - \frac{\operatorname{sen}^2\beta}{(R - \frac{\Delta y}{\cos\alpha})^2}\right]$$

3) a) $\frac{mv_0^2}{2} - \frac{mGM}{r_0} = E$

$\frac{mv_0^2}{r_0} = \frac{mGM}{r_0^2} \Rightarrow \frac{mGM}{r_0} = mv_0^2$

$\Rightarrow E = -\frac{mv_0^2}{2}$

b)



La energía no cambia ya que el módulo de la velocidad es el mismo en el punto de intersección

$l_0 = mv_0 r_0$

$l_f = mv_0 \cos\alpha\, r_0$

$U_{eff} = \frac{l_f^2}{2mr^2} - \frac{mGM}{r}$

$U_{ef0} = \frac{l_0^2}{2mr^2} - \frac{mGM}{r}$

c) $l_f\Big|_{\frac{r_0}{5}} = mv_{max}\frac{r_0}{5} = mv_0 \cos\alpha\, r_0$

$E = \frac{mv_{max}^2}{2} - \frac{mGM}{r_0}5 = -\frac{mv_0^2}{2} \Rightarrow \frac{v_{max}^2}{2} = \frac{9}{2}v_0^2 \Rightarrow v_{max} = 3v_0$

$\left(\frac{mGM}{r_0} = mv_0^2\right)$

$\Rightarrow \frac{3}{5}mv_0 r_0 = mv_0 r_0 \cos\alpha \Rightarrow \cos\alpha = \frac{3}{5} \Rightarrow \alpha \approx 53{,}13^\circ$