Supongamos: $\Psi(r,\theta,\phi) = f(r) \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\theta,\phi)$

Veamos porqué $P\left(L^2 = \ell(\ell+1)\hbar^2, L_z = m\hbar\right) = |a_{\ell m}|^2$

- $\left\{ |\ell m\rangle \otimes |r\rangle \right\}_{\substack{\ell \in \mathbb{N} \\ -\ell \leq m \leq \ell \\ r \in \mathbb{R}^+}}$ es una base (impropia) de los estados de una partícula en el espacio

- $|\Psi\rangle = \sum_{\ell,m} a_{\ell m} |\ell m\rangle \otimes |\Psi_r\rangle$ y entonces

$$\langle r\theta\phi|\Psi\rangle = \sum_{\ell m} a_{\ell m} \underbrace{\langle\theta\phi|\ell m\rangle}_{= Y_{\ell m}(\theta,\phi)} \underbrace{\langle r|\Psi_r\rangle}_{= f(r)}$$

- $$P\left(L^2 = (\ell+1)\ell\hbar^2, L_z = m\hbar\right) = \left\| \frac{\overbrace{\int dr\, |\ell m\rangle|r\rangle\langle r|\langle \ell m|}^{\mathbb{P}}\, \Psi\rangle}{\langle\Psi|\Psi\rangle} \right\|^2$$

Proyector a ℓ,m

$$\mathbb{P}|\Psi\rangle = a_{\ell m} |\ell m\rangle \otimes |\Psi_r\rangle$$

$$\| \mathbb{P}|\Psi\rangle\|^2 = |a_{\ell m}|^2 \langle\Psi_r|\Psi_r\rangle = |a_{\ell m}|^2 \int_0^\infty dr\, r^2 f(r)^2$$

$$\langle\Psi|\Psi\rangle = \sum_{\ell m} |a_{\ell m}|^2 \int_0^\infty dr\, r^2 f(r)^2 \Rightarrow$$

$$P\left(L^2 = \ell(\ell+1)\hbar^2, L_z = m\hbar\right) = \frac{|a_{\ell m}|^2 \int_0^\infty dr\, r^2 f(r)^2}{\sum_{\ell m} |a_{\ell m}|^2 \int_0^{+\infty} dr\, r^2 f(r)^2} = \frac{|a_{\ell m}|^2}{\sum_{\ell m} |a_{\ell m}|^2}$$

- extra: $\langle r|r'\rangle = \dfrac{\delta(r-r')}{r^2}$