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INTEGRAL OPTION*

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(*Translated by D. O. Kramkov*)

Abstract. In the context of diffusion model of the (B, S)-market consisting of two assets: riskless bank account $B = (B_t)_{t \geqq 0}$ and risky stock $S = (S_t)_{t \geqq 0}$ described by (1.1) and (1.2) we consider the option of American type with payment function of "integral type" $f = (f_t)_{t \geqq 0}$:

$$f_t = e^{-\lambda t}\left[\int_0^t S_u\,du + s\psi_0\right],$$

The paper solves the problem of definition of the fair price of the integral option under consideration. The structure of the expiration time is also described.

Key words. Black and Scholes model of (B, S)-market American option, integral option, Asian option, optimal stopping time, Kummer's functions, rational time

1. Introduction. Main results.

1. We consider the diffusion model of the (B, S)-market, consisting of two assets: *bonds* (or bank accounts) and *stocks* (see [2], [11], [12], [13], [16], [17]). The price process of the bonds $B = (B_t)_{t \geqq 0}$ represents the time value of money and appreciates at a constant rate $r \geqq 0$, the *interest rate*:

$$B_t = B_0 e^{rt}, \qquad B_0 > 0. \tag{1.1}$$

The price process of *stocks* $S = (S_t)_{t \geqq 0}$ has random character and is modeled as a *geometric Brownian motion*, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqq 0}, P)$ with constant drift $\mu \in \mathbf{R}$, the *appreciation rate* and constant variance $\sigma > 0$, the *volatility coefficient*:

$$S_t = S_0 e^{\mu t} \cdot \exp\left\{\sigma W_t - \frac{\sigma^2}{2} t\right\}. \tag{1.2}$$

Here $W = (W_t)_{t \geqq 0}$ is a standard Brownian motion and we suppose that the filtration $(\mathcal{F}_t)_{t \geqq 0}$ is generated by W.

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Let $P_t = P|\mathcal{F}_t$ be the restriction of the measure P on $\mathcal{F}_t$, and $P^{\mu-r}$ be the probability on $(\Omega, \mathcal{F})$ such that

$$P_t^{\mu-r}(d\omega) = Z_t^{\mu-r}(\omega)P_t(d\omega), \tag{1.3}$$

where $P_t^{\mu-r} = P^{\mu-r}|\mathcal{F}_t$ and

$$Z_t^{\mu-r}(\omega) = \exp\left\{ -\frac{\mu-r}{\sigma}W_t(\omega) - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t\right\} \tag{1.4}$$

(see [13]).

By Girsanov's theorem the process $W^{\mu-r} = (W_t^{\mu-r})_{t\geqq 0}$ with

$$W_t^{\mu-r} = W_t + \frac{\mu-r}{\sigma}t \tag{1.5}$$

is a Wiener process with respect to the measure $P^{\mu-r}$:

$$\text{Law}\,(W^{\mu-r}|P^{\mu-r}) = \text{Law}\,(W|P). \tag{1.6}$$

Note that from (1.2) and Itô's formula

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{1.7}$$

and by (1.5) we can also rewrite that as

$$dS_t = S_t(r dt + \sigma dW_t^{\mu-r}). \tag{1.8}$$

2. Let $f = (f_t(\omega))_{t\geqq 0}$ be a non-negative progressively measurable process with the function $f_t = f_t(\omega)$ interpreted as a *payment* of an option seller to an option buyer if the option is exercised at time t (see [11], [12], [13], [16], [17]). We consider the American type option which can be exercised at arbitrary stopping time $\tau = \tau(\omega)$. According to the general pricing theory for the American options the *fair* price $\mathbf{C}^*$ or *premium*, which the option buyer pays to the seller at time $t = 0$ is given by the formula

$$\mathbf{C}^* = B_0 \sup \mathbf{E}^{\mu-r}\frac{f_\tau}{B_\tau}, \tag{1.9}$$

where sup is taking over the set of all finite ($P^{\mu-r}$-a.s.) stopping times $\tau = \tau(\omega)$ and $\mathbf{E}^{\mu-r}$ is the expectation with respect to the probability $P^{\mu-r}$.

Note that from (1.6), (1.7), (1.9) it follows that the fair option price does not depend on the appreciation rate μ. So without the loss of generality we can assume that $\mu = r$.

In this paper we suppose that the reward process f has the form

$$f_t = e^{-\lambda t}\left[\int_0^t S_u\,du + s\psi_0\right], \tag{1.10}$$

where $s = S_0$ and $\psi_0 \geqq 0$, $\lambda > 0$ are some constants.

We named this option as "Integral Option" taking into account the formula (1.10) for the reward process. As a matter of fact it is a particular case of Asian options. This option may be looking attractive for "careful" investor because it keeps track of past events in a smooth "integral" way. As a consequence it provides its holder a psychological comfort by reducing regrets. By appearance it is similar to the "Russian Option" introduced by Shepp and Shiryaev ([4], see also [18]); that is also inscribed in the American type and has a reward process

$$f_t = e^{-\lambda t}\max\left(\sup_{u\leqq t} S_u, s\psi_0\right).$$

So the problem of the calculation of $\mathbf{C}^*$ can be formulated as an optimal stopping problem for the process $S = (S_t)_{t \geqq 0}$ satisfying to the stochastic differential equation

$$dS_t = S_t(r\,dt + \sigma\,dW_t), \qquad S_0 = s,$$

with *price*

$$\mathbf{C}^* = B_0 \sup_\tau \mathbf{E}\,\frac{f_\tau}{B_\tau}, \tag{1.11}$$

where $f = (f_t)_{t \geqq 0}$ is given by (1.10) and $\mathbf{E}$ is the expectation with respect to the measure P.

From (1.10) and (1.11) it follows that

$$\mathbf{C}^* = \sup_\tau \mathbf{E} e^{-(\lambda+r)\tau} g\biggl(\int_0^\tau S_u\,du\biggr), \tag{1.12}$$

where

$$g\biggl(\int_0^t S_u\,du\biggr) = \int_0^t S_u\,du + s\psi_0. \tag{1.13}$$

The two-dimensional process $(S_t, \int_0^t S_u\,du)_{t \geqq 0}$ is a Markov process with respect to the measure P. So the "problem (1.12)" belongs to the set of optimal stopping problems for *two-dimensional* Markov processes and in principle can be solved using general methods from [14].

3. It turns out, however, that the problem (1.12) can be reduced to some optimal stopping problem for a *one-dimensional* Markov process. The idea of such reduction is based on the introduction of a *dual* martingale measure (see [13, § 7]) and consists in the following.

Let us introduce the probability measures

$$\widetilde{P}_t(d\omega) = \exp\biggl\{\sigma W_t - \frac{\sigma^2}{2}t\biggr\} P_t(d\omega), \qquad t \geqq 0. \tag{1.14}$$

The family of measures $(\widetilde{P}_t)_{t \geqq 0}$ is consistent and there is a measure $\widetilde{P}$ such that $\widetilde{P}|\mathcal{F}_t = \widetilde{P}_t$, $t \geqq 0$ (compare with [13, § 7]). It can be shown that measures $\widetilde{P}$ and P are locally equivalent (i.e., $\widetilde{P}_t \sim P_t$, $t \geqq 0$), moreover, if τ is a finite ($\widetilde{P}$ and P-a.s.) stopping time then the restrictions $\widetilde{P}_\tau = \widetilde{P}|\mathcal{F}_\tau$ and $P_\tau = P|\mathcal{F}_\tau$ of measures $\widetilde{P}$ and P on σ-field $\mathcal{F}_\tau$ are equivalent and

$$\widetilde{P}_\tau(d\omega) = \exp\biggl\{\sigma W_\tau - \frac{\sigma^2}{2}\tau\biggr\} P_\tau(d\omega) \tag{1.15}$$

(the definition of σ-field $\mathcal{F}_\tau$ can be found for example in [7]).

Remark now that since

$$S_t = S_0 e^{rt} \cdot \exp\biggl\{\sigma W_t - \frac{\sigma^2}{2}t\biggr\},$$

then

$$\frac{S_t}{B_t} = \frac{S_0}{B_0} \exp\biggl\{\sigma W_t - \frac{\sigma^2}{2}t\biggr\}. \tag{1.16}$$

Therefore, from (1.15) we have for finite τ

$$B_0 \mathbf{E} \frac{f_\tau}{B_\tau} = S_0 \mathbf{E}\biggl(\frac{B_0}{B_\tau} \cdot \frac{S_\tau}{S_0} \cdot \frac{f_\tau}{S_\tau}\biggr) = S_0 \mathbf{E}\biggl(\exp\biggl\{\sigma W_\tau - \frac{\sigma^2}{2}\tau\biggr\} \frac{f_\tau}{S_\tau}\biggr) = S_0 \widetilde{\mathbf{E}} \frac{f_\tau}{S_\tau}. \tag{1.17}$$

So from (1.9) it follows that the fair price $\mathbf{C}^*$ can be also calculated as

$$\mathbf{C}^* = S_0 \widetilde{\mathbf{E}} \frac{f_\tau}{S_\tau}, \tag{1.18}$$

where $\widetilde{\mathbf{E}}$ is the expectation with respect to the measure $\widetilde{P}$.

Let $\widetilde{W}_t = W_t - \sigma t$, $t \geqq 0$. It is easy to see that the process $\widetilde{W} = (\widetilde{W}_t)_{t \geqq 0}$ is a Wiener process with respect to $\widetilde{P}$:

$$\mathrm{Law}\left(\widetilde{W}|\widetilde{P}\right) = \mathrm{Law}\,(W|P).$$

So with respect to $\widetilde{P}$ the process $S = (S_t)_{t \geqq 0}$ with $S_t = S_0 e^{rt} \cdot \exp\{\sigma W_t - \frac{\sigma^2}{2} t\}$ can be represented in the form

$$S_t = S_0 \exp\left\{\left(r + \frac{\sigma^2}{2}\right) t + \sigma \widetilde{W}_t\right\}, \tag{1.19}$$

and by Itô's formula

$$dS_t = S_t\left[(r + \sigma^2) dt + \sigma d\widetilde{W}_t\right]. \tag{1.20}$$

Note that according to (1.16) the process $S/B = (S_t/B_t)_{t \geqq 0}$ is a P-martingale. At the same time

$$\frac{B_t}{S_t} = \frac{B_0}{S_0} \exp\left\{-\sigma \widetilde{W}_t - \frac{\sigma^2}{2} t\right\}$$

and so the process $(B/S) = (B_t/S_t)_{t \geqq 0}$ is a $\widetilde{P}$-martingale. That explains the name "dual martingale measure" for $\widetilde{P}$.

4. Let us consider the process $f/S = (f_t/S_t)_{t \geqq 0}$, with respect to $\widetilde{P}$, where f_t is given by (1.10).

If we denote

$$\psi_t = \frac{1}{S_t}\left[\int_0^t S_u\,du + s\psi_0\right], \qquad \psi_0 \geqq 0, \tag{1.21}$$

then

$$\frac{f_t}{S_t} = e^{-\lambda t} \psi_t; \tag{1.22}$$

therefore, the fair price is equal to

$$\mathbf{C}^* = S_0 \sup_\tau \widetilde{\mathbf{E}} e^{-\lambda \tau} \psi_\tau. \tag{1.23}$$

Remark, that with respect to the dual measure $\widetilde{P}$ the process $\psi = (\psi_t)_{t \geqq 0}$ is a Markov one and so (compare with (1.12)) the problem of the calculation of $\mathbf{C}^*$ is the typical optimal stopping problem for the *one-dimensional* Markov process $(\psi_t)_{t \geqq 0}$ with

$$\mathbf{C}^* = S_0 \sup_\tau \widetilde{\mathbf{E}} e^{-\lambda \tau} g(\psi_\tau), \tag{1.24}$$

where $g(\psi) = \psi$ (compare with the corresponding problems for "Russian" options in [4], [10] (continuous time) and [6] (discrete time)). In accordance with the explanations given in [12] and [13] the optimal stopping time τ^*, i.e., the stopping time for which $\widetilde{\mathbf{E}} e^{-\lambda \tau^*} g(\psi_{\tau^*}) = \mathbf{C}^*$, will be referred as *rational.*

The main result of the paper is the following theorem.

THEOREM. *For the Integral option of American type with the reward* $f = (f_t)_{t \geqq 0}$ *given in* (1.10) *the fair price equals*

$$\mathbf{C}^* = S_0 \cdot \begin{cases} c^* u(\psi_0), & 0 \leqq \psi_0 \leqq \psi^*, \\ \psi_0, & \psi_0 > \psi^*, \end{cases} \tag{1.25}$$

where

$$u(\psi) = \int_0^\infty \exp\left(-\frac{2y}{\sigma^2}\right) y^{-(\gamma_1+1)}(1+\psi y)^{\gamma_2}\, dy,$$

constants γ_1 *and* γ_2 *are defined in* (2.5), ψ^* *is the root of the equation*

$$\psi u'(\psi) = u(\psi)$$

and

$$c^* = \frac{\psi^*}{u(\psi^*)}.$$

The rational stopping time in the problem (1.24) *is*

$$\tau^* = \inf\{t \geqq 0 : \psi_t \geqq \psi^*\}. \tag{1.26}$$

The proof of the theorem is given in § 3. In the next section we present some facts about Kummer's functions. These functions are used for another description of the function $u = u(\psi)$ and the constants ψ^*, c^* in (1.25).

The idea to look for the closed form solution for the American option with the integral type reward (1.10) belongs to A. N. Shiryaev whom we would like to express our gratitude for fruitful discussions. We also thank A. A. Afanas'ev for useful remarks.

2. Auxiliary results.

1. For the proof of the main result we need the confluent hypergeometric Kummer's functions $M(a,b,z)$ and $U(a,b,z)$ (see [1, p. 504]):

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots \tag{2.1}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$ and

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right\}, \tag{2.2}$$

where $\Gamma(a)$ is the Gamma function, $a \in \mathbf{R}$.

From the theory of the Kummer's functions (see [1, 13.1.29 and 13.2.5, p. 505]) it is known that if $1+a-b>0$ and $2-b>0$, then $U(a,b,z)$ admits the following representation:

$$U(a,b,z) = \frac{1}{\Gamma(1+a-b)}\int_0^\infty e^{-zt} t^{a-b}(1+t)^{1-a}\, dt. \tag{2.3}$$

Let us denote by γ_1 and γ_2 ($\gamma_1 < \gamma_2$) the roots of the following quadratic equation

$$\frac{\sigma^2}{2}\gamma^2 - \left(\frac{\sigma^2}{2} + r\right)\gamma - \lambda = 0, \tag{2.4}$$

i.e.,

$$\gamma_k = \left\{\left(\frac{\sigma^2}{2} + r\right) + (-1)^k \sqrt{\left(\frac{\sigma^2}{2} + r\right)^2 + 2\lambda\sigma^2}\right\}\Big/\sigma^2, \qquad k = 1, 2. \tag{2.5}$$

For $x \geqq 0$ define the function

$$u(x) = \Gamma(-\gamma_1)x^{\gamma_2}U\left(-\gamma_2, 1-\gamma_2+\gamma_1, \frac{2}{x\sigma^2}\right). \tag{2.6}$$

Notice that since $\lambda > 0$, then $\gamma_1 < 0$ and $\gamma_2 > 1$. That permits to use for the function $U(-\gamma_2, 1-\gamma_2+\gamma_1, 2/(x\sigma^2))$ the *integral representation* (2.3), that after the substitution $zt = 2y/\sigma^2$ gives for $u(x)$ the following expression:

$$u(x) = \int_0^\infty \exp\left(-\frac{2y}{\sigma^2}\right)y^{-(\gamma_1+1)}(1+yx)^{\gamma_2}\,dy. \tag{2.7}$$

Since $\gamma_1 < 0$ and $\gamma_2 > 1$ it follows that $u = u(x)$ is an increasing strictly convex function such that $u(x)/x \to \infty$, $x \to \infty$.

2. In the sequel we will use the fact that the function $u = u(x)$, $x \geqq 0$, satisfies some second order differential equation (see (2.10)).

First we note that the functions $M(a,b,z)$ and $U(a,b,z)$ defined in (2.1) and (2.2) with $a = -\gamma_2$ and $b = 1+\gamma_1-\gamma_2$ (with γ_1 and γ_2 from (2.5)) are the *independent solutions* of Kummer's equation

$$z\omega''(z) + (1+\gamma_1-\gamma_2-z)\omega'(z) + \gamma_2\omega(z) = 0, \tag{2.8}$$

that is equivalent (since $\gamma_1+\gamma_2 = 1+2r/\sigma^2$) to the equation

$$z\omega''(z) + \left(\frac{2r}{\sigma^2} - 2 - 2\gamma_2 - z\right)\omega'(z) + \gamma_2\omega(z) = 0. \tag{2.9}$$

It follows that the function $u(x) = x^{\gamma_2}\omega(2/(x\sigma^2))$ with

$$\omega(z) = \Gamma(-\gamma_1)z^{\gamma_2}U(-\gamma_2, 1-\gamma_2+\gamma_1, z)$$

satisfies the following equation

$$\frac{\sigma^2}{2}x^2u''(x) + (1-rx)u'(x) - \lambda u(x) = 0, \qquad x \geqq 0. \tag{2.10}$$

3. Let us define the family of functions $\{u_c(x), c \geqq 0\}$, where $u_c(x) = cu(x)$.

From the properties of the function $u = u(x)$ it follows that there are unique constants $c^* > 0$ and $\psi^* > 0$ such that

$$u_{c^*}(\psi^*) = \psi^*, \qquad u'_{c^*}(\psi^*) = 1, \tag{2.11}$$

i.e., $c^* > 0$ and $\psi^* > 0$ are the solutions of the system

$$c^*u(\psi^*) = \psi^*, \qquad c^*u'(\psi^*) = 1, \tag{2.12}$$

or the equivalent system

$$c^*u(\psi^*) = \psi^*, \qquad c^*u(x) \geqq x, \qquad x \geqq 0. \tag{2.13}$$

From (2.11) it follows that ψ^* is the root of the equation

$$\psi u'(\psi) = u(\psi) \tag{2.14}$$

and the constant c^* is equal to

$$c^* = \frac{\psi^*}{u(\psi^*)}. \tag{2.15}$$

Finally let us denote

$$v(x) = u_{c^*}(x) \ \Big(= c^* u(x)\Big). \tag{2.16}$$

3. The proof of the theorem.

1. From (1.21), (1.20) and by Itô's formula we find that the process $(\psi_t)_{t \geqq 0}$ is the solution of the stochastic differential equation

$$d\psi_t = (1 - r\psi_t)\, dt - \sigma\psi_t d\widetilde{W}_t. \tag{3.1}$$

This equation arises in the change point problem (see [15, (2.1)]) and is sometimes referred as "Shiryaev–Robbins equation"(see [3, p. 168]).

Let us define

$$\widetilde{V}(\psi) = \sup_\tau \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}\psi_\tau, \qquad \psi \geqq 0, \tag{3.2}$$

where $\widetilde{\mathbf{E}}_\psi$ denotes the expectation with respect to the measure $\widetilde{P}$ under the assumption that the process $(\psi_t)_{t \geqq 0}$, defined as the strong solution of the stochastic differential equation (3.1) starts from $\psi_0 = \psi \geqq 0$.

By (1.24)

$$\mathbf{C}^* = S_0 \widetilde{V}(\psi_0) \tag{3.3}$$

with the constant ψ_0 from (1.10).

Along with $\widetilde{V}(\psi)$ we define a function

$$V(\psi) = \begin{cases} v(\psi), & 0 \leqq \psi \leqq \psi^*, \\ \psi, & \psi > \psi^*, \end{cases} \tag{3.4}$$

where the function $v = v(\psi)$, $\psi \geqq 0$, is given in (2.16) and ψ^* is the solution of the equation (2.4).

We assert that

$$\widetilde{V}(\psi) = V(\psi), \tag{3.5}$$

i.e., that the function $V(\psi)$ defined in (3.4) is the "price" for the problem (3.2), and that the stopping time

$$\tau^* = \inf\{t \geqq 0 : \ \psi_t \geqq \psi^*\} \tag{3.6}$$

is optimal (rational):

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau^*}\psi_{\tau^*} = \widetilde{V}(\psi), \qquad \psi \geqq 0. \tag{3.7}$$

2. To prove these assertions it is sufficient to check the following "verification" properties:

(A_1) $\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau} V(\psi_\tau) \leqq V(\psi)$, $\psi \geqq 0$, for all finite ($\widetilde{P}$-a.s.) stopping times τ

and

(A_2) Stopping time τ^* defined in (3.6) is ($\widetilde{P}$-a.s.) finite and

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau^*} V(\psi_{\tau^*}) = V(\psi), \qquad \psi \geqq 0.$$

Indeed if these conditions are fulfilled then from (3.4) we have $V(\psi) \geqq \psi$ and so (A_1) applied gives

$$\sup_\tau \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}\psi_\tau \leqq \sup_\tau \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau} V(\psi_\tau) \leqq V(\psi). \tag{3.8}$$

This fact together with the definition of τ^* and the property (A_2) gives the optimality of τ^* as well as the equality (3.5).

3. We begin with the proof of the property (A_1).

Since the function $V = V(\psi)$ is two times continuously differentiable we can apply the Itô's formula to $(e^{-\lambda t}V(\psi_t))_{t \geqq 0}$, that gives

$$d\Big(e^{-\lambda t}V(\psi_t)\Big) = e^{-\lambda t}\Big[LV(\psi_t) - \lambda V(\psi_t)\Big]dt - \sigma e^{-\lambda t}V'(\psi_t)\psi_t d\widetilde{W}_t, \tag{3.9}$$

where

$$LV(\psi) = (1 - r\psi)V'(\psi) + \frac{\sigma^2}{2}\psi^2 V''(\psi) \tag{3.10}$$

is the infinitesimal operator of the Markov process $(\psi_t)_{t \geqq 0}$. In the integral form the equation (3.9) becomes

$$e^{-\lambda t}V(\psi_t) = V(\psi_0) + \int_0^t e^{-\lambda s}\Big[LV(\psi_s) - \lambda V(\psi_s)\Big]\,ds - \int_0^t \sigma e^{-\lambda s}V'(\psi_s)\psi_s\,d\widetilde{W}_s. \tag{3.11}$$

Note that if $\psi > \psi^*$ then $V(\psi) = \psi$ (see (3.4)); therefore,

$$LV(\psi) - \lambda V(\psi) = 1 - (r + \lambda)\psi \leqq 1 - (r + \lambda)\psi^* \leqq 0. \tag{3.12}$$

The last inequality $1 - (r+\lambda)\psi^* \leqq 0$ follows from the fact that according to (2.10) and (2.11) at point $x = \psi^*$

$$\frac{\sigma^2}{2}\psi^{*2} r v''(\psi^*) + (1 - r\psi^*) - \lambda\psi^* = 0, \tag{3.13}$$

whereas from (2.7) we have $v''(\psi) \geqq 0$, $\psi \geqq 0$. Hence from (3.13) we obtain that $1 - (r + \lambda)\psi^* \leqq 0$.

In the domain $\psi \leqq \psi^*$ we have $V(\psi) = v(\psi)$; therefore, by (2.10)

$$Lv(\psi) - \lambda v(\psi) = 0. \tag{3.14}$$

So $LV(\psi_s) - \lambda V(\psi_s) \leqq 0$ and thus from (3.11)

$$V(\psi_0) - \int_0^t \sigma e^{-\lambda s}V'(\psi_s)\psi_s\,d\widetilde{W}_s \geqq e^{-\lambda t}V(\psi_t). \tag{3.15}$$

The stochastic integral here (as the process) is a local martingale ([8, Chap. 2, § 2]). Moreover, by (3.15) it is bounded from bellow and, therefore, is a *supermartingale.*

Returning to (3.11) we find that for any finite Markov time τ

$$e^{-\lambda\tau}V(\psi_\tau) \leqq V(\psi_0) - \int_0^\tau \sigma e^{-\lambda s}V'(\psi_s)\,\psi_s d\widetilde{W}_s. \tag{3.16}$$

By the Doob theorem for martingales bounded from below ([5, Chap. VII]), [7, Chap. 2, § 4])

$$\widetilde{\mathbf{E}}_\psi \int_0^\tau \sigma e^{-\lambda s}V'(\psi_s)\psi_s\,d\widetilde{W}_s = 0 \tag{3.17}$$

and, therefore, from (3.15)

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}V(\psi_\tau) \leqq V(\psi_0), \qquad \psi \geqq 0,$$

proving the property (A_1). Similarly it may be shown that the process $(e^{-\lambda t}V(\psi_t))_{t \geqq 0}$ is a $\widetilde{P}_\psi$-supermartingale.

Now let us check the property (A$_2$). To start with we assume that $\widetilde{P}_\psi(\tau^* < \infty) = 1$, $\psi \geqq 0$. This fact will be proved below in item 4.

From (3.11)

$$e^{-\lambda(t\wedge\tau^*)}V(\psi_{t\wedge\tau^*}) = V(\psi_0) + \int_0^{t\wedge\tau^*} e^{-\lambda s}\Big[LV(\psi_s) - \lambda V(\psi_s)\Big]\,ds \tag{3.18}$$
$$- \int_0^{t\wedge\tau^*} \sigma e^{-\lambda s} V'(\psi_s)\psi_s\, d\widetilde{W}_s.$$

The property (A$_2$) is obvious if $\psi \geqq \psi^*$. Therefore, we suppose in (3.18) that $\psi_0 = \psi \leqq \psi^*$. Then for $s \leqq t\wedge\tau^*$ we have $V(\psi_s) = v(\psi_s)$ and hence from (3.14) ($\widetilde{P}_\psi$-a.s., $\psi \leqq \psi^*$)

$$\int_0^{t\wedge\tau^*} e^{-\lambda s}\Big[LV(\psi_s) - \lambda V(\psi_s)\Big]\,ds = 0. \tag{3.19}$$

Let

$$I_t^* = -\int_0^{t\wedge\tau^*} \sigma e^{-\lambda s} V'(\psi_s)\psi_s\, d\widetilde{W}_s.$$

Then from (3.18) and (3.19)

$$I_t^* = e^{-\lambda(t\wedge\tau^*)}V(\psi_{t\wedge\tau^*}) - V(\psi_0). \tag{3.20}$$

It follows that for all $t \geqq 0$

$$-V(\psi_0) \leqq I_t^* \leqq V(\psi^*) - V(\psi_0) \tag{3.21}$$

and, therefore, the local martingale $(I_t^*)_{t\geqq 0}$ is uniformly integrable. Thus by the Doob theorem $\widetilde{\mathbf{E}}_\psi I_\tau^* = 0$ for each ($\widetilde{P}_\psi$-a.s.) finite stopping time τ and in particularly for the stopping time τ^*. Hence the property (A$_2$) (under the assumption that τ^* is finite) is proved.

4. Let us prove that the stopping time τ^* is finite, i.e., that $\widetilde{P}_\psi(\tau^* < \infty) = 1$ for all $\psi \geqq 0$.

With this purpose we consider the harmonic function $K = K(\psi)$ satisfying to the equation

$$LK(\psi) = 0, \qquad \psi > 0, \qquad K(1) = 0, \tag{3.22}$$

where the differential operator L was defined in (3.10). The solution of (3.22) is a strictly increasing function

$$K(x) = \int_1^x y^{\frac{2r}{\sigma^2}} \exp\left(\frac{2}{y\sigma^2}\right) dy, \tag{3.23}$$

where as usually $\int_1^x = -\int_x^1$ for $0 < x \leqq 1$.

From (3.22) and (3.11) (with $\lambda = 0$) we have

$$K(\psi_t) = K(\psi_0) - \int_0^t \sigma\,\psi_s K'(\psi_s)\, d\widetilde{W}_s. \tag{3.24}$$

Since

$$\tau^* = \inf\{t \geqq 0:\ \psi_t \geqq \psi^*\} = \inf\Big\{t \geqq 0:\ K(\psi_t) \geqq K(\psi^*)\Big\},$$

then it is sufficient to show that the process $(K(\psi_t))_{t\geqq 0}$ reaches any level A almost surely:

$$\widetilde{P}_\psi\Big(\sup_t K(\psi_t) < A\Big) = 0, \qquad \psi \geqq 0. \tag{3.25}$$

The process $(K(\psi_t))_{t\geqq 0}$ is the diffusion Markov process such that

$$dK(\psi_t) = -\sigma(\psi_t)d\widetilde{W}_t, \tag{3.26}$$

where

$$\sigma(\psi) = \sigma\psi K'(\psi).$$

Since

$$c = \inf_{\psi>0} \sigma(\psi) = \inf_{\psi>0} \left(\sigma\psi^{(2r)/\sigma^2+1} \exp\left(\frac{2}{\psi\sigma^2}\right)\right)$$
$$= \sigma\left(\frac{2}{2r+\sigma^2}\right)^{(2r)/\sigma^2+1} \exp\left(\frac{2r}{\sigma^2}\right) > 0,$$

then the quadratic variation $(\langle K(\psi)\rangle_t)_{t\geqq 0}$ (see [8, Chap. 1, § 8]) of the continuous martingale $(K(\psi_t))_{t\geqq 0}$ is equal to

$$\left\langle K(\psi)\right\rangle_t = \int_0^t \sigma^2(\psi_s)\, ds \geqq c^2 t.$$

Hence $\widetilde{P}_\psi(\langle K(\psi)\rangle_t \to \infty) = 1$ and the stopping time ("random time change")

$$\varkappa(t) = \inf\left\{s \geqq 0 : \left\langle K(\psi)\right\rangle_s \geqq t\right\}$$

is finite with $\widetilde{P}_\psi$-probability one.

Remark now that the process $\overline{W} = (\overline{W_t})_{t\geq 0}$ with $\overline{W_t} = K(\psi_{\varkappa(t)})$ is a continuous local martingale with quadratic variation $\langle \overline{W}\rangle_t = t$, $t \geqq 0$. Therefore, by Levy's theorem [7, Theorem 4.1] this process is a Wiener process. Hence with probability one it reaches any level. It follows that the process $(K(\psi_t))_{t\geqq 0}$ also possesses such property, proving (3.25).

The theorem is proved.

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MEAN-VARIANCE HEDGING OF OPTIONS ON STOCKS WITH MARKOV VOLATILITIES *

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(*Translated by the authors*)

Abstract. We consider the problem of hedging an European call option for a diffusion model where drift and volatility are functions of a Markov jump process. The market is thus incomplete implying that perfect hedging is not possible. To derive a hedging strategy, we follow the approach based on the idea of hedging under a mean-variance criterion as suggested by Föllmer, Sondermann, and Schweizer. This also leads to a generalization of the Black–Scholes formula for the corresponding option price which, for the simplest case when the jump process has only two states, is given by an explicit expression involving the distribution of the integrated telegraph signal (known also as the Kac process). In the Appendix we derive this distribution by simple considerations based on properties of the order statistics.

Key words. Black–Scholes formula, call option, stochastic volatility, incomplete market, mean-variance hedging, Kac process

1. Introduction. In the famous Black–Scholes model of option pricing it is assumed that the dynamics of stocks is given by a linear stochastic differential equation

$$dS_t = aS_t dt + \sigma S_t dW_t, \tag{1.1}$$

where a and σ are deterministic functions (in the simplest case they are constants). Nevertheless, it has been observed that many financial assets do not have a deterministic volatility σ and the basic assumption of the Black–Scholes model fails. There are attempts to extend the model by describing the evolution of a and σ by stochastic differential equations ([4], [5], [8], [15], [17]) and find more or less explicit formulae for the hedging strategies.

We present here a model where the coefficients a and σ are "modulated" by a Markov jump process Y_t which is independent of W. The setting is similar to that considered in [1] by Di Masi, Platen, and Runggaldier for the discrete time. We consider the problem of hedging an European call option with contingent claim $H = f(S_T)$. Since there is an additional source of randomness, the market is incomplete and perfect hedging is not possible. Following ideas proposed by Föllmer, Sondermann, and Schweizer we derive a hedging strategy that is locally risk-minimizing. In the particular case when $f(S_T) = (S_T - K)^+$ we also obtain for the corresponding option price a generalization of the Black–Scholes formula which, for the

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