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THE RUSSIAN OPTION: REDUCED REGRET

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We propose a new put option where the option buyer receives the *maximum* price (discounted) that the option has ever traded at during the time period (which may be indefinitely long) between the purchase time and the exercise time, so that the buyer need look at the fluctuations only occasionally and enjoys having little or no regret that he did not exercise the option at an earlier time (except for the discounting). We give an exact simple formula for the optimal expected present value (fair price) that can be derived from the option and the (unique) optimal exercise strategy that achieves the optimum value under the assumption that the asset fluctuations follow the Black–Scholes exponential Brownian motion model, which is widely accepted. It is important to note that the discounting is necessary: If it is omitted or even if it is less than the Black–Scholes drift, then the value to the buyer under optimum performance is infinite. We also solve the same problem under a different model: the original Bachelier linear Brownian market with linear discounting. This model is no longer accepted, but of course the mathematics is consistent.

To our knowledge no such regretless option is currently traded in any existing market despite its evident appeal. We call it the Russian option, partly to distinguish it from the American and European options, where the term of the option is prescribed in advance and where no exact formula for the value has been given.

1. Introduction. Suppose the fluctuations in the price of an asset are given by the geometric Brownian motion model

$$X_t = x \exp\left(\sigma W_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right), \qquad t \geq 0, \tag{1.1}$$

where $x > 0$, $W_0 = 0$ and W_t is a standard Wiener process. The process X, which satisfies the stochastic differential equation $dX = \sigma X\,dW + \mu X\,dt$, forms the basis for the famous option pricing theory of Black and Scholes [4, 5]. The parameters μ, called the drift, and σ, the volatility, are assumed known.

We solve the following mathematical problem, where $r > 0$ and $s \geq x$ are given and we want to find a stopping time $\tau \in [0, \infty)$ (which need not be a fixed time but can depend on the fluctuations observed to date in any way) to

$$\underset{\tau}{\text{maximize}}\, E e^{-r\tau} S_\tau, \tag{1.2}$$

where S is the maximum value, starting at s, for X, that is,

$$S_t = \max\left\{s, \sup_{0 \leq u \leq t} X_u\right\}, \qquad t \geq 0. \tag{1.3}$$

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The motivation for (1.2) is the study of a new financial option that arose first as a consequence of the probability theory developed to solve (1.2) and to our knowledge is not currently traded in any existing market. This (Russian) option allows its owner to choose an exercise date, represented by the stopping time τ, and then pays the owner either s or the maximum stock price achieved up to this exercise date, whichever is larger, discounted by $\exp(-r\tau)$. In problem (1.2) the owner of the option seeks an exercise strategy that will maximize the expected (present) value of his future reward, where r is the interest rate for discounting. Starting with our solution to the mathematical problem (1.2), Duffie and Harrison [9] derive an "arbitrage price" for the Russian option. Their pricing analysis parallels the analysis of European call options by Black and Scholes [4]. Of necessity, this involves a more complete discussion of the interest rate r and drift parameter μ than are appropriate for arbitrage pricing. In the final analysis, for arbitrage pricing it is not necessary that investors agree on the average rate of return earned by the stock underlying a Russian option, or for that matter, on μ and σ; indeed differences may increase the potential for trading.

In this paper the value of the option [i.e., the supremum in (1.2)] will be found exactly. In particular, it will be shown that the maximum in (1.2) is finite if and only if

$$r > \mu. \tag{1.4}$$

Assuming (1.4), an explicit formula is given for both the maximal expected present value and the optimal stopping rule in (2.4), which is not a fixed time rule, but depends heavily on the observed values of X_t and S_t.

We call the financial option described in the preceding text a "Russian option" for two reasons. First, this name serves to (facetiously) differentiate it from American and European options, which have been extensively studied in financial economics, especially with the new interest in market economics in Russia. Second, our solution of the stopping problem (1.2) is derived by the so-called principle of smooth fit, which was first enunciated by the great Russian mathematician A. N. Kolmogorov; cf. [3] and [10]. The Russian option is characterized by "reduced regret" because the owner is paid the *maximum* stock price up to the time of exercise and hence feels less remorse at not having exercised at the maximum.

For purposes of comparison and to emphasize the mathematical nature of the contribution here, we conclude the paper by analyzing an optimal stopping problem for the Russian option based on Bachelier's [1] original (1900) linear model of stock price fluctuations:

$$X'_t = x + \sigma W_t + \mu t, \qquad t \geq 0. \tag{1.5}$$

We again introduce the running maximum as in (1.3):

$$S'_t = \max\Big\{s, \sup_{0 \leq u \leq t} X'_u\Big\}, \tag{1.6}$$

where μ, σ, x, and s are as before except x, s can be negative, $x \leq s$, and solve the problem (in Section 3) to

$$\text{(1.7)} \qquad \underset{\tau}{\text{maximize}}\, E(S'_\tau - r\tau),$$

where discounting is now also applied *linearly* (the case of exponential discounting seems to have no simple solution). The simple explicit value in (1.7) is given in (3.8) along with the optimal stopping rule τ, which is rather different. In the geometric case, $S_\tau/X_\tau = \alpha$, for some α, is the form of the stopping time, while in the linear case $S_\tau - X_\tau = \theta$, for some θ.

2. Derivation of the optimal pricing formula for the Russian option. Let x, s, μ, σ, r, X_t, and S_t be as in (1.1) and (1.3) and define

$$\text{(2.1)} \qquad V^*(x, s) = V^*(x, s, \mu, \sigma, r) = \sup_\tau E_{x,s} e^{-r\tau} S_\tau,$$

where the sup is taken over all stopping rules. We will first give a rigorous, but rather *deus ex machina* proof that $V^*(x, s)$ agrees with $V(x, s)$. Then we supply some motivation (or derivation) from the principle of smooth fit as to how V was actually guessed. Because the optimal free boundary here turns out so simply, in a sense this example does not show the full power of the principle, although the form of V suggests it is not so trivial after all. If one intuits that the optimal rule τ is of the form in (2.14), one could try to optimize the choice of α and so derive (2.3) and (2.4). We do not see how to carry this out (even in the case $\mu = \sigma^2/2$), although maybe it can be done, and this might give an alternate derivation of (2.3) and (2.4) as has been suggested by several readers. (*Note added in proof:* See our new papers [19] and [20].)

So assume $r > \max(0, \mu)$ as in (1.5) and let $\gamma = \gamma_1$ and $\gamma = \gamma_2$, $\gamma_1 < 0 < 1 < \gamma_2$, be the two roots of the quadratic equation

$$\text{(2.2)} \qquad \begin{gathered} \frac{1}{2}\sigma^2\gamma^2 + \gamma\left(\mu - \frac{\sigma^2}{2}\right) = r, \\ \gamma_{1,2} = \frac{\sigma^2/2 - \mu \pm \sqrt{(\sigma^2/2 - \mu)^2 + 2\sigma^2 r}}{\sigma^2}, \end{gathered}$$

and set

$$\text{(2.3)} \qquad \alpha = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_2}\right)^{1/(\gamma_2 - \gamma_1)}$$

and

$$\text{(2.4)} \quad V(x, s) = \begin{cases} \dfrac{s}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\dfrac{\alpha x}{s}\right)^{\gamma_1} - \gamma_1\left(\dfrac{\alpha x}{s}\right)^{\gamma_2}\right), & \dfrac{s}{\alpha} \leq x \leq s, \\ s, & 0 < x \leq \dfrac{s}{\alpha}. \end{cases}$$

To prove $V^* \equiv V$, verify that $V(x,s)$ in $s/\alpha < x \le s$ satisfies, by direct observation ($V \in C^2$),

$$rV(x,s) = \mu x V_x(x,s) + \tfrac{1}{2}\sigma^2 x^2 V_{xx}(x,s), \tag{2.5}$$

$$V(x,s) \ge s, \tag{2.6a}$$

$$V_s(s,s) = \frac{\partial V}{\partial s}(x,s)\Big|_{x=s} = 0. \tag{2.6b}$$

Now X_t has Itô differential, from (1.1),

$$dX_t = X_t(\mu\, dt + \sigma\, dW_t) \tag{2.7}$$

and so the process

$$Y_t = e^{-rt}V(X_t, S_t) \tag{2.8}$$

is a supermartingale; that is, in the region $0 < X_t \le S_t/\alpha$,

$$dY_t = de^{-rt}S_t = -re^{-rt}S_t\, dt \le 0. \tag{2.9}$$

In the region $S_t/\alpha \le X_t \le S_t$, because S_t grows only when $X_t = S_t$ and $V_s(s,s) = 0$ in (2.6b),

$$\begin{aligned} dY_t &= e^{-rt}\left[V_x(X_t,S_t)\, dX_t + \tfrac{1}{2}V_{xx}(X_t,S_t)(dX_t)^2 - rV(X_t,S_t)\, dt\right] \\ &= e^{-rt}V_x(X_t,S_t)X_t\sigma\, dW_t, \end{aligned} \tag{2.10}$$

using (2.5). So in (2.10), Y_t is a positive local martingale, hence again a supermartingale and

$$E_{x,s}\, dY_t \le 0, \qquad t \ge 0. \tag{2.11}$$

Thus for any stopping time τ (i.e., which does not anticipate the future in the sense that at the time of stopping only information independent of future increments of W is usable), by (2.6a), we can write

$$\begin{aligned} E_{x,s}e^{-r\tau}S_\tau &\le E_{x,s}e^{-r\tau}V(X_\tau, S_\tau) \\ &= E_{x,s}Y_\tau \le E_{x,s}Y_0 \\ &= V(X_0,S_0) = V(x,s), \end{aligned} \tag{2.12}$$

where we used the fact that Y is a supermartingale [(2.9) and (2.11)] to obtain the second inequality. If we sup over all such τ we obtain for all $0 < x \le s$,

$$V^*(x,s) \le V(x,s). \tag{2.13}$$

To prove the reverse inequality, let τ be the first t for which

$$X_t = S_t/\alpha \tag{2.14}$$

starting from $X_0 = x$, $S_0 = s$, $x > s/\alpha$. It is clear that $P(\tau < \infty) = 1$ because

$$\begin{aligned} P\{\tau > T\} = P\Big\{&\text{for } 0 \le u \le t \le T,\ \sigma(W_t - W_u) \\ &+ \left(\mu - \frac{\sigma^2}{2}\right)(t-u) \ge \log\frac{1}{\alpha}\Big\}, \end{aligned} \tag{2.15}$$

which tends to zero as $T \to \infty$ no matter what the sign of $\mu - \sigma^2/2$ is. The first inequality in (2.12) is thus an equality because of (2.15) and (2.4). The second inequality will also be shown to be an equality if we can prove that Y_t, $0 \le t \le \tau$, is a uniformly integrable martingale [15]. We supply a (simpler) direct proof by showing that

$$E \sup_{0 \le t < \infty} Y_t < \infty, \tag{2.16}$$

which will directly prove the equality we need.

To prove (2.16) we note that

$$\begin{aligned} Y_t &= e^{-rt} V(X_t, S_t) \\ &\le e^{-rt} V(S_t, S_t) \\ &= K e^{-rt} S_t, \end{aligned} \tag{2.17}$$

because $V(S, S) = KS$, where the constant K is given from (2.4) by

$$K = \frac{1}{\gamma_2 - \gamma_2} \left(\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2} \right). \tag{2.18}$$

So it is enough to show that $\sup_t \exp(-rt) S_t$ is integrable; that is,

$$\int_0^\infty dy\, P\left\{ \sup_t e^{-rt} S_t > y \right\} < \infty. \tag{2.19}$$

By a well-known theorem of Doob [15, 17] for $\alpha > 0$, $\beta > 0$,

$$P\{ W_t \le \alpha t + \beta, 0 \le t < \infty \} = 1 - e^{-2\alpha\beta}. \tag{2.20}$$

If we choose

$$\alpha = \left(r - \mu + \frac{\sigma^2}{2} \right) \Big/ \sigma, \qquad \beta = \frac{1}{\sigma} \log\left(\frac{y}{x} \right), \tag{2.21}$$

then from (1.3) for $y > s$, $y > x$,

$$\begin{aligned} & P\left\{ \sup_t e^{-rt} S_t > y \right\} \\ &\quad = P\left\{ \sup_t \left[\sup_{0 \le u \le t} \left(\sigma W_u + \left(\mu - \frac{\sigma^2}{2} \right) u \right) - \log\left(\left(\frac{y}{x} \right) + rt \right) \right] \right\} > 0. \end{aligned} \tag{2.22}$$

Now if $W_t \le \alpha t + \beta$ for all t, then from (2.21),

$$\begin{aligned} & \sup_{0 \le u \le t} \left(\sigma W_u + \left(\mu - \frac{\sigma^2}{2} \right) u \right) \\ &\qquad \le \sup_{0 \le u \le t} \left(\sigma(\alpha u + \beta) + \left(\mu - \frac{\sigma^2}{2} \right) u \right) \\ &\qquad = \sup_{0 \le u \le t} \left(\log\left(\frac{y}{x} \right) + ru \right), \end{aligned} \tag{2.23}$$

and because $r > 0$, the right side is growing in u and is maximized at $u = t$. It follows from (2.22) and (2.23) using (2.20) that for $y > s$, $y > x$,

$$(2.24) \qquad P\left\{\sup_t e^{-rt} S_t > y\right\} \le e^{-2\alpha\beta} = \left(\frac{y}{x}\right)^{-(1+(2(r-\mu))/\sigma^2)}$$

Thus for $r > \mu$, the integral in (2.19) converges and so (2.16) is proved. Now suppose τ is the stopping time in (2.14). Because Y_t is a martingale (local) for any fixed t,

$$(2.25) \qquad E_{x,s} Y_{t \wedge \tau} = E_{x,s} Y_0.$$

Letting $t \to \infty$, using dominated convergence and (2.16), we see that

$$(2.26) \qquad E_{x,s} Y_\tau = E_{x,s} Y_0.$$

Note that (2.16) also shows that Y_t is a martingale, which implies (2.26) directly. Thus equality holds also in the second inequality in (2.12) for the choice of τ in (2.14). However, this τ is included in the sup in (2.1) so that

$$(2.27) \qquad V^*(x, s) \ge V(x, s)$$

as well. By (2.13) and (2.27) we have proved

$$(2.28) \qquad V^*(x, s) \equiv V(x, s) \quad \text{for } 0 < x \le s$$

as promised.

The proof is unrevealing: How were (2.2)–(2.4) derived? The answer is that we used the "principle of smooth fit." This principle goes back to A. N. Kolmogorov, who discovered it in Russia in the 1950's, and it was later independently found by Chernoff [6] in the United States and also by Lindley in Great Britain. It was used by Grigelionis and Shiryaev [10] and others [2, 18], though even now it is not appreciated widely. A new application to Burkholder–Gundy inequalities is in a paper in preparation [8]. It often enables one to obtain (see especially [2, 3, 6, and 18]) explicit closed form solutions to optimal stopping or optimal control problems in continuous problems where the discrete versions cannot be solved in explicit form. See references 3 and 18 for more details on the technique. In this problem we see that (2.2)–(2.4) were guessed by seeking a C^2 function $V(x, s)$ that satisfies $V(x, s) = s$ for $x \ge g(s)$. Note that the continuation region is intuitively guessed to be of this form; that is, we should not exercise the option if the maximum process, S_t, is just about to take an increase. However, for $0 \le x \le g(s)$ we exercise the option, so $V(x, s) = s$ in this region. The differential equation (2.5) holds in the continuation region, and the principle of smooth fit says, only heuristically of course, that the free boundary g will be determined by $V \in C^2$. This heuristic is only used to guess V as in (2.2)–(2.4); once guessed, the rigorous proof is given in (2.5)–(2.28), in an almost crank-turning way.

Let us look closer at how to guess, because it has some new features in this problem. The differential equation (2.5) for V given in (2.4) is obtained by observing that if we elect to continue letting the option run for a small time

$h > 0$, then by (2.7), for $g(s) < x < s$,

$$V(x,s) = EV\left(x + x\mu h + \sigma\sqrt{h}\,\eta, s\right)e^{-rh} \tag{2.29}$$

and expanding by Itô calculus gives (heuristically) (2.5). The condition of the principle of smooth fit, that V should be in C^2, makes Itô's rule applicable and, at least heuristically, is sufficient to determine the free boundary $g(s)$. Let us see how it works in the present case.

If $V(x, s)$ satisfies (2.5) in $g(s) < x < s$, then

$$V(x,s) = a(s)x^{\gamma_1} + b(s)x^{\gamma_2}, \tag{2.30}$$

where γ_1, γ_2 are as in (2.2). Because $V(x, s)$ must fit smoothly at $x = g(s)$ with s we must have

$$V(g(s),s) = s, \qquad V_x(g(s),s) = 0. \tag{2.31}$$

Further smoothness conditions [e.g., $V_s(g(s), s) = 1$] now follow automatically and give no further information to determine (guess) the function g. Instead, we must obtain a condition along the known (nonfree) boundary $x = s$. This is apparently a novel feature of this problem and is due to the appearance of the process $S_t = \max_{0 \le u \le t}(X_u)$. Along this boundary we needed the condition

$$V_s(s,s) \equiv 0 \tag{2.32}$$

in (2.10) at $X_t = S_t$ in order to prove that Y_t is a local martingale there. The conditions (2.31) and (2.32) give a differential equation for $g = g(s)$ by eliminating $a(s)$ and $b(s)$:

$$g' = \frac{1/\gamma_1(s/g)^{\gamma_1} - 1/\gamma_2(s/g)^{\gamma_2}}{(s/g)^{\gamma_1+1} - (s/g)^{\gamma_2+1}}. \tag{2.33}$$

It is hard (but possible) to find the general solution to (2.33) for g, but there is one simple solution to (2.33)—the one we need. At this point one can merely guess at this solution:

$$g(s) = s/\alpha \tag{2.34}$$

with α as in (2.3). One might try to develop a further heuristic that will give the extra boundary condition needed:

$$g(0) = 0, \tag{2.35}$$

but the principle of smooth fit is only a heuristic anyway, so we are content to merely "guess" (2.34). However, we remark that (2.33) can be solved explicitly for other values of $g(0)$. If we make the substitution

$$g(s) = \frac{s}{h(s)}, \tag{2.36}$$

then (2.33) separates into

$$\frac{ds}{s} = \frac{h^{\gamma_1-1} - h^{\gamma_2-1}}{(1-1/\gamma_1)h^{\gamma_1} - (1-1/\gamma_2)h^{\gamma_2}}\,dh \tag{2.37}$$

and an explicit integration is possible, although not elementary.

3. The Bachelier version. For purposes of comparison we also obtain the pricing formula that the option seller should use to find his break-even point should he believe in the Bachelier rather than the Black–Scholes model of asset fluctuation. We will use primes to denote this model, where the price or value of the asset follows the nonexponential evolution

$$X'_t = x + \mu t + \sigma W_t, \qquad t \geq 0, \tag{3.1}$$

where $\sigma > 0$, x and μ are given parameters. Again analogous to (1.3), we set

$$S'_t = \max\Big(s, \max_{0 \leq u \leq t} X'_u\Big). \tag{3.2}$$

The buyer is allowed to exercise his option at any time $t > 0$ and obtains the payoff [analogous to (1.4)]

$$S'_t - rt, \tag{3.3}$$

where $r > 0$ is the cost of retaining the option for time t. Analogously to (1.5), we assume $r > 0$ and $s \geq x$ and

$$r > \mu. \tag{3.4}$$

The problem of determining the price of option (3.3) is simpler than that of (1.4), but has apparently not been solved before despite its simplicity, although very similar problems have been discussed [7]. If we overlooked a prior solution, perhaps it has not been solved with the smooth fit principle, but the elementary solution could have been simply guessed in some other way. Again the full power of smooth fit is perhaps better shown in examples where the free boundary is more difficult to guess than this one.

So we want to determine

$$V^{*\prime}(x, s) = V^{*\prime}(x, s, \mu, \sigma, r) = \sup_{\tau} E_{x,s}[S_\tau - r\tau], \tag{3.5}$$

where the sup is taken over all stopping rules τ with the important proviso that

$$E\tau < \infty, \tag{3.6}$$

to avoid $\infty - \infty$ in (3.5).

The answer is shown to be $V^{*\prime} = V'$, where we first define $\theta > 0$ by

$$\theta = \frac{\sigma^2}{2\mu} \log \frac{1}{1 - \mu/r} \tag{3.7}$$

and then take

$$V'(x, s) = \begin{cases} s + \dfrac{r}{\mu}(x - s + \theta) \\ \quad - \dfrac{r\sigma^2}{2\mu^2}\left(1 - \exp\left(-\dfrac{2\mu}{\sigma^2}(x - s + \theta)\right)\right), & s - \theta \leq x \leq s, \\ s, & -\infty \leq x \leq s - \theta. \end{cases} \tag{3.8}$$

It is easy to check that $V' \in C^2$ in $-\infty < x \leq s$, that

$$\frac{\partial V'}{\partial s}(s, s) = 0 \tag{3.9}$$

and that V' satisfies the O.D.E.

$$r = \mu V_x'(x, s) + \tfrac{1}{2}\sigma^2 V_{xx}'(x, s), \qquad x - \theta \leq x \leq s. \tag{3.10}$$

Because $\exp(-y) \geq 1 - y$, it is easy to see that

$$V'(x, s) \geq s. \tag{3.11}$$

Thus if we define the process

$$Y_t' = V'(X_t', S_t') - rt, \qquad t \geq 0, \tag{3.12}$$

and note that X' has the Itô differential

$$dX_t' = \mu\, dt + \sigma\, dW_t, \tag{3.13}$$

we see that

$$dY_t' = \frac{r\sigma}{\mu}\, dW_t \quad \text{in } S_t - \theta \leq X_t; \qquad dY_t' = -r\, dt \quad \text{in } X_t \leq S_t - \theta, \tag{3.14}$$

so that Y_t' is a supermartingale [note (3.9) is needed at $X_t = S_t$]. Thus we can write for any stopping rule τ, by (3.11) and (3.12),

$$E_{x,s}(S_\tau' - r\tau) \leq E_{x,s} Y_\tau' \tag{3.15}$$

and because Y' is a supermartingale and $(x_0, s_0) = (x, s)$, we have

$$E_{x,s} Y_\tau' \leq EY_0' = V'(x, s) \tag{3.16}$$

so that

$$V^{*\prime}(x, s) \leq V'(x, s). \tag{3.17}$$

Now letting τ be the first t for which

$$X_t = S_t - \theta \tag{3.18}$$

gives equality in (3.15) and (3.16) by arguments analogous to those in Section 2 for W and so $V^{*\prime}(x, s) = V^{*\prime}(x, s)$. The choice of V' in (3.8) can be (and was) "derived" or guessed by using the principle of smooth fit in a similar way as V in (2.4).

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brought to several other papers with related methodology and direction [11–14]. However, these papers do not contain any of our results.

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