Probability and Its Applications

Andrei N. Borodin

Stochastic Processes

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ISSN 2297-0371 ISSN 2297-0398 (electronic)
Probability and Its Applications
ISBN 978-3-319-62309-2 ISBN 978-3-319-62310-8 (eBook)
https://doi.org/10.1007/978-3-319-62310-8

Library of Congress Control Number: 2017956746

Original Russian edition published by LAN Publishing, St. Petersburg, 2013

Printed on acid-free paper

This book is published under the trade name Birkhäuser
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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PREFACE

The aim of the book is to give a rigorous and at the same time accessible presentation of the theory of stochastic processes. This imposed hard restrictions on the selection of material, which reflects of course the preferences of the author.

As befits a theory, this presentation is largely self-contained. The text includes a very limited number of references to assertions, whose proofs should be looked for in other books. As a rule, this does not affect on the main style of the presentation. Accordingly, the different parts of the book can serve a source of ready materials for lecture courses. The text contains many examples showing how to apply theoretical results to solving concrete problems.

We do not dwell on the history of the creation of the theory of stochastic processes. This was covered brilliantly in many monographs that are much closer in time to the milestones of the theory than this book.

A significant part of the book is devoted to the classical theory of stochastic processes. At the same time there are new topics not presented previously in books. As to well-known results, we try to clarify the main ideas of the proofs, sometimes providing them with new approaches.

The first chapter contains basic facts of Probability Theory that will be useful for more detailed treatment in the subsequent chapters. Considerable attention is paid to conditional probabilities and conditional expectations, which are effective tools in the theory of stochastic processes. The foundations of the theory of martingales created by J. L. Doob are explained. The value of this theory is difficult to overestimate. Martingales were effectively applied outside probability theory, for example, in mathematical analysis. We consider only discrete-time martingales. The basic ideas of the theory of martingales are well illustrated for the case of a discrete-time parameter. Furthermore, the results for this case form a basis for the continuous-time theory. For our purposes the discrete time is sufficient and each time we need the continuous case, a corresponding justification will be given. We introduce a fairly detailed description of Markov processes. The Markov property provides a basic way of random changes, which is closest to what happens in reality. The chapter is completed with a consideration of a Brownian motion process. It is no exaggeration to assert that the Brownian motion is the fundamental stochastic process. Far ahead of its time many brilliant conjectures of P. Lévy were confirmed just for this process.

The second chapter is devoted to stochastic calculus, the foundations of which were laid down by K. Itô. At the initial stage of the creation of this theory, it was almost impossible to foresee how fruitful it will become. Its role in the theory of stochastic processes can be compared with the role of differential calculus in mathematical analysis or other disciplines. The theory of stochastic differential equations is the natural development of the theory of ordinary differential equations. Stochastic integrals with respect to Brownian motion, whose sample paths have unbounded variation, differ fundamentally from the classical integrals. This difference leads to the fact that the stochastic differentials of superpositions of smooth functions with the solutions of the stochastic differential equations depend on the second derivatives of functions under differentiation, while this is absolutely impossible in the classical analysis. This difference also shows that descriptions of some physical phenomena, having a constructive nature and including stochastic interactions, are based on second-order differential equations. This gives a possible answer to the following fundamental question: why many phenomena in the real world are described by second-order differential equations? The role of the stochastic analysis in the description of the real world is not completely understood at the present time. At the end of the book, in Appendix 1, the problem of heat transfer is treated by means of the rigorous mathematical arguments. The approach is based on the energy exchange between individual molecules. This description is close to the real physical process.

The third chapter presents the theory of distributions of functionals of Brownian motion. The foundations of this theory were laid by A. N. Kolmogorov and M. Kac. In 1931 the famous forward and backward Kolmogorov's equations were introduced. An impetus to the emergence of probabilistic representations for the solutions of parabolic equations with potential was given in the doctoral thesis of the Nobel prizewinner in physics R. Feynman. He described the solutions of the Schrödinger equation in terms of path integrals. M. Kac saw in this result an analogy with the theory of distributions of integral functionals of Brownian motion and established the basis for this theory. The third chapter is devoted to the investigation of distributions of functionals of Brownian motion stopped at various random times. A sufficiently rich collection of stopping random times is considered. For all of them we derive effective results that enable us to compute distributions of various functionals of Brownian motion stopped at these moments. These results have already been used in Mathematical Statistics, Insurance Theory and Financial Mathematics.

The fourth chapter is devoted to a class of diffusion processes, generalizing the Brownian motion in a natural way. The necessity of studying the diffusion processes was probably realized by physicists earlier than by mathematicians. A striking example of this is the Einstein–Smoluchowski equation, describing the motion of a light particle in a viscous fluid. On the one hand, the random motion of fluid molecules interacting via collisions makes the particle move randomly, and, on the other hand, the viscosity restricts the speed of the movement. These two factors had a significant role in the discovery of the Einstein–Smoluchowski stochastic differential equation. A rigorous mathematical definition of diffusion processes was given by A. N. Kolmogorov. After the appearance of Itô's stochastic calculus, it was shown that under certain assumptions the diffusion processes defined by Kolmogorov are the solutions of the corresponding stochastic differential equations.

The fifth chapter is entirely devoted to a detailed study of the properties of Brownian local time, the cornerstone in the structure of additive functionals of a Brownian motion. The Brownian local time is the simplest positive continuous additive functional of a Brownian motion, in the sense that any such functional can be represented by the Stieltjes integral of the Brownian local time with respect to a non-decreasing function. This shows that the Brownian local time is extremely useful for studying functionals of Brownian motion. Even from a purely mathematical point of view, the Brownian local time is a very interesting object to investigate. This is manifested in a number of deep properties, discovered through careful consideration, and in the beauty of analytical methods used to prove them. The concept of the Brownian local time and its most important properties emerged thanks to the intuition of P. Lévy. A special landmark in the study of the Brownian local time is F. Knight and D. Ray's description of the local time as a Markov process with respect to the space parameter. Methods enabling to compute the distributions of functionals of the Brownian local time play an important role in the study of properties of the Brownian local time. The main progress in this sphere was made possible thanks to the fact that, in contrast to the real potential, which corresponds to the integral functional in the theory of distributions, the Dirac δ-function corresponds to the local time. Involvement of the Dirac δ-function makes differential problems much simpler than those for real potentials.

The subject of the sixth chapter is a class of diffusions with jumps. The appearance of this topic is mainly dictated by needs of Financial Mathematics. In this theory, the continuous variation of the price of some assets is interrupted occasionally by abrupt collapses or, conversely, by growth of their ratings. An important feature here is the presence of random factors affecting the price of the asset. Between moments of jumps the process evolves as a classical diffusion. The most natural way of the appearance of jumps is the following. On disjoint intervals of arbitrarily small length the jumps occur independently with the probability proportional to the interval length. In fact, this probability may also depends on the value of the process at a moment of jump. In this way one comes to the processes having the Markov property. This, in turn, allows us to develop a sufficiently rich theory of such processes, which to some extent is comparable with the theory of classical diffusions.

The final, seventh chapter is devoted to the invariance principle. The Brownian motion and, more generally, diffusions are idealized limiting objects for random walks or more general processes with discrete time that have a recurrent structure. The random walk describes objectively some phenomena occurring in reality. A good example is given by the model of heat transfer (for details, see Appendix 1). This model is a perfect illustration of the following paradigm: the random walk is a simple and intuitively understandable process as regards the structure of sample paths, whereas its finite-dimensional distributions are rather complicated and the distributions of various functionals of the process are in fact extremely complicated. But the sample paths of a Brownian motion process are difficult to imagine at all. They are continuous, but non-differentiable almost everywhere. The level sets of a Brownian motion are Cantor sets (closed, uncountable, with zero topological dimension, and without isolated points), whereas there are only finitely many points in the level sets of a random walk. The finite-dimensional distributions of a Brownian motion are rather simple. Moreover, one can fined explicit formulas for distributions of many functionals of a Brownian motion. In this situation, it is important to know how the Brownian motion process and the correspondingly transformed random walk can be close to each other. It is natural to use the specifics of both the random walk and the Brownian motion, in a certain extent, identifying these processes.

In Appendix 1 the advantages of random walk and Brownian motion modelling are clearly demonstrated in the study of heat transfer. Using the random walk, we visually describe how heat is transferred, whereas using a Brownian motion approximation of it we derive a differential equation describing the change of the temperature in time and in space.

Appendix 2 contains a summary of the main properties of the special functions that are relevant to Probability Theory.

The theory of distributions of nonnegative functionals of processes is largely based on the Laplace transform of the distributions of these functionals. Appendix 3 contains tables of the inverse Laplace transforms that are often used in this theory.

In Appendix 4, we gather certain second-order differential equations and their nonnegative linearly independent solutions that are used to express Laplace transforms of transition functions of various diffusion processes.

Appendix 5 contains examples of transformations of measures generated by some well-known diffusions.

Finally, in Appendix 6 formulas that can be used for computing the moments of the distribution of a functional by its Laplace transform are given.

A. Yu. Zaitsev carefully read the manuscript and made many useful comments, which contributed to a significant improvement of the text. I am truly grateful to him for this great work. I am thankful to I. Ponomarenko for the help with the translation of some parts of the book. My thanks also go to the reviewers for their valuable suggestions.

Finally, I want to thank the staff of Birkhäuser for their excellent work when preparing the book. It is also my pleasure to thank the copy–editor whose careful work made the text more readable.

St. Petersburg, Russia, July 2017 A. N. Borodin

NOTATION

$:=$ stands for "is defined to be equal to."

$[a]$ – largest integer not exceeding a.

$a^+ := \max\{0, a\}$.

$a \vee b := \max\{a, b\}$.

$a \wedge b := \min\{a, b\}$.

$\mathbf{R}$, $\mathbf{R}_+$ – real line and nonnegative axis.

$\mathbb{Z}$ – set of integers.

$\mathbb{N}$ – set of natural numbers.

A^{c} – complement of set A.

$\mathbb{I}_A(\cdot)$ – indicator function.

$(\Omega, \mathcal{F}, \mathbf{P})$ – probability space.

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ – filtered probability space, see § 4 Ch. I.

$\{\mathcal{F}_t\}$ – filtration, see § 4 Ch. I.

$\{\mathcal{G}_0^t\}$ – natural filtration, see § 4 Ch. I.

$\mathcal{B}(E)$ – σ-algebra of Borel subsets of E.

$C(E)$ – space of continuous functions from E to $\mathbf{R}$.

$C_b(E)$ – space of continuous bounded functions from E to $\mathbf{R}$.

$L^2(m)$ – space of functions square integrable with respect to the measure m.

mes – Lebesgue measure.

$\delta_y(x)$ – Dirac δ-function.

$\mathcal{L}_\gamma^{-1}$ – inverse Laplace transform with respect to γ.

$\mathbf{P}_x$ – probability measure associated with a process started at x.

$\mathbf{E}_x$ – expectation associated with a process started at x.

$\mathrm{Cov}(X, Y)$ – covariance of random variables X and Y.

$\mathbf{E}\{X; A\} := \mathbf{E}\{X(\omega)\mathbb{I}_A(\omega)\}$.

$\mathbf{E}\{X \big| \mathcal{Q}\}$ – conditional expectation with respect to the σ-algebra $\mathcal{Q}$.

$W(t)$ – Brownian motion, see § 10 Ch. I.

$W_+(t) := |W(t)|$ – reflected Brownian motion, see § 10 Ch. III.

$U(t)$ – Ornstein–Uhlenbeck process, see § 16 Ch. IV.

$R^{(n)}(t)$ – Bessel process of order $n/2 - 1$, see § 16 Ch. IV.

$Q^{(n)}(t)$ – radial Ornstein–Uhlenbeck process, see § 16 Ch. IV.

$\sigma\big(X(t), t \in \Sigma\big)$ – σ-algebra of events generated by the process X, see § 3 Ch. I.

$m(\cdot)$ – density of the speed measure, see § 11 Ch. IV.

$S(\cdot)$ – scale, see § 11 Ch. IV.

$P(s, x, t, D)$ – transition function of Markov process, see § 6 Ch. I.

$p(s, x, t, y)$ – transition density of Markov process, see § 6 Ch. I.

$p(t, x, y)$ – transition density of homogeneous Markov process.

$G^{\circ}_{\alpha}(x, y)$ – Green's function, see § 11 Ch. IV.

$\psi(x)$ – fundamental increasing solution, see § 12 Ch. II.

$\varphi(x)$ – fundamental decreasing solution, see § 12 Ch. II.

$\mathbb{L}$ – generator of homogeneous diffusion, see § 9 Ch. IV.

$\ell(t, x)$ – local time with respect to the Lebesgue measure, see § 5 Ch. II.

$\varrho(t, z) = \inf\{s : \ \ell(s, z) = t\}$ – inverse local time.

$L(t, x)$ – local time with respect to the speed measure, see § 14 Ch. IV.

$\hat{H}(t) := \inf\{s \le t : X(s) = \inf_{0 \le s \le t} X(s)\}$ – location of the minimum before time t.

$\check{H}(t) := \inf\{s \le t : X(s) = \sup_{0 \le s \le t} X(s)\}$ – location of the maximum before time t.

$H_z := \min\{s : X(s) = z\}$ – first hitting time of z.

$H_{a,b} := \min\{s : X(s) \notin (a, b)\}$ – first exit time from (a, b).

τ – exponentially distributed time independent of Brownian motion.

$\ln(\cdot)$ – natural logarithm

$\mathrm{sh}(\cdot)$, $\mathrm{ch}(\cdot)$, $\mathrm{th}(\cdot)$, $\mathrm{cth}(\cdot)$ – hyperbolic functions, see Appendix 2.

$I_\nu(\cdot)$, $K_\nu(\cdot)$ – modified Bessel functions, see Appendix 2.

$\mathrm{Erfc}(\cdot)$ – error function, see Appendix 2.

$D_\nu(\cdot)$ – parabolic cylinder functions, see Appendix 2.

$M(a, b, \cdot)$, $U(a, b, \cdot)$ – Kummer functions, see Appendix 2.

$M_{n,m}(\cdot)$, $W_{n,m}(\cdot)$ – Whittaker functions, see Appendix 2.

$S_\nu(\cdot,\cdot)$, $C_\nu(\cdot,\cdot)$, $F_\nu(\cdot,\cdot)$ – two-parameter functions associated with Bessel functions, see Appendix 2.

$F(\alpha, \beta, \gamma, \cdot)$, $G(\alpha, \beta, \gamma, \cdot)$, – hypergeometric functions, see Appendix 2.

$P^{\mu}_{\nu}(\cdot)$, $\widetilde{Q}^{\mu}_{\nu}(\cdot)$, – Legendre functions, see Appendix 2.

CHARTER I

BASIC FACTS

§ 1. Random variables

We recall here the classical definition of a probability space.

Let Ω be an abstract set, which is treated as the set of elementary outcomes or sample points of a random experiment.

A collection $\mathcal{F}$ of subsets of Ω is called a *σ-algebra* if it has the following properties:

1) $\Omega \in \mathcal{F}$;

2) if $A \in \mathcal{F}$, then $A^{\rm c} \in \mathcal{F}$, where $A^{\rm c} := \Omega \setminus A$ denotes the complement of A;

3) if $A_l \in \mathcal{F}$, $l = 1, 2, \dots$, then $\bigcup\limits_{l=1}^{\infty} A_l \in \mathcal{F}$.

By the second property, the empty set $\varnothing = \Omega^{\rm c}$, which is the complement of Ω, also belongs to $\mathcal{F}$.

The elements of the σ-algebra $\mathcal{F}$ are called *events*.

The following equalities hold true:

$$\Bigl\{\bigcup_{l=1}^{\infty} A_l\Bigr\}^{\rm c} = \bigcap_{l=1}^{\infty} A_l^{\rm c}, \qquad \Bigl\{\bigcap_{l=1}^{\infty} A_l\Bigr\}^{\rm c} = \bigcup_{l=1}^{\infty} A_l^{\rm c}.$$

The difference of two events is defined as $A \setminus B = A \bigcap B^{\rm c}$.

It follows from the definition that the σ-algebra $\mathcal{F}$ is closed with respect to a countable number of operations on events.

A *probability measure* defined on a σ-algebra of events $A \in \mathcal{F}$ is a function $\mathbf{P} : A \to [0, 1]$ which satisfies the following conditions:

1) $\mathbf{P}(\Omega) = 1$;

2) for any countable sequence of events $\{A_n\}_{n=1}^{\infty}$ that are disjoint ($A_k \bigcap A_l = \varnothing$ if $k \neq l$)

$$\mathbf{P}\Bigl(\bigcup_{n=1}^{\infty} A_n\Bigr) = \sum_{n=1}^{\infty} \mathbf{P}(A_n).$$

The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *probability space*. This space is always assumed to be given initially.

Let $\mathcal{B}(\mathbf{R})$ be the smallest σ-algebra containing all intervals of the form $[a, b)$. This σ-algebra is called the *Borel* σ-algebra on $\mathbf{R}$. Similarly, let $\mathcal{B}(\mathbf{R}^n)$ be the smallest σ-algebra containing all the sets $[a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$.

A measurable mapping $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is called *a random variable*. In other words, it is a mapping $\Omega \to \mathbf{R}$ such that for any Borel set Δ the set $\{\omega : X(\omega) \in \Delta\}$ belongs to $\mathcal{F}$. The reason underlying this definition is that a probability must be assigned to all sets of the form $\{\omega : X(\omega) \in \Delta\}$.

The argument ω in the notation of a random variable $X = X(\omega)$ is usually omitted, because neither the nature of the set Ω, nor its structure are relevant in probability theory.

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_1

A measurable mapping $f : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is called a *Borel function* or a *measurable function.*

With each random variable X we associate its *distribution*

$$\mathcal{P}_X(\Delta) := \mathbf{P}(\{\omega : X(\omega) \in \Delta\}), \qquad \Delta \in \mathcal{B}(\mathbf{R}).$$

It is a probability measure on $\mathcal{B}(\mathbf{R})$. To define a distribution uniquely it is sufficient to consider the sets $(-\infty, x)$, $x \in \mathbf{R}$ instead of arbitrary Borel sets Δ. The function

$$F_X(x) := \mathbf{P}(\{\omega : X(\omega) \in (-\infty, x)\}) =: \mathbf{P}(X < x), \qquad x \in \mathbf{R},$$

is called the *distribution function* of the random variable X.

If there exists a nonnegative measurable function $f_X(x)$, $x \in \mathbf{R}$, such that

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$$

for every $x \in \mathbf{R}$, then $f_X(x)$ is called the *density* of the random variable X.

For a distribution with a continuous density f_X, the mean value theorem for integrals, shows that

$$\mathbf{P}(X \in [x, x+\delta)) \sim f_X(x)\delta \qquad \text{as } \delta \downarrow 0. \tag{1.1}$$

This formula expresses the main meaning of a density: the principal value of the probability that the random variable belongs to an infinitesimally small interval containing a given point is equal to the density at this point multiplied by the length of the interval.

If $g : \mathbf{R} \to \mathbf{R}$ is a measurable function and X is a random variable, then $Y = g(X)$ is also a random variable.

Let $g(x)$, $x \in \mathbf{R}$, be a strictly monotone function, $g(\mathbf{R})$ be the image of $\mathbf{R}$ under the function g, and $g^{(-1)}(y)$, $y \in g(\mathbf{R})$, be the inverse function. Suppose that X has a continuous density f_X. Let g be a continuously differentiable function with nowhere vanishing derivative. Then the random variable $Y = g(X)$ has the density

$$f_Y(y) = \frac{f_X(g^{(-1)}(y))}{|g'(g^{(-1)}(y))|} \mathbb{1}_{g(\mathbf{R})}(y), \tag{1.2}$$

where

$$\mathbb{1}_\Delta(x) = \begin{cases} 1, & \text{if } x \in \Delta, \\ 0, & \text{if } x \notin \Delta, \end{cases}$$

is *the indicator function.* This formula generalizes to piecewise strictly monotone functions g, i.e., functions for which there exists a partition of $\mathbf{R}$ into intervals I_k, $k = 1, 2, \dots, n$, such that g is strictly monotone in each I_k. Assume that g is continuously differentiable on the interiors of these intervals. Let $g_k^{(-1)}$ be the inverse of g restricted to I_k. Then for Y the density f_Y exists and is given by

$$f_Y(y) = \sum_{k=1}^{n} \frac{f_X(g_k^{(-1)}(y))}{|g'(g_k^{(-1)}(y))|} \mathbb{1}_{g(I_k)}(y). \tag{1.3}$$

The *expectation* of a random variable X is defined by the formula

$$\mathbf{E}X := \int_\Omega X(\omega)\mathbf{P}(d\omega) = \int_{-\infty}^{\infty} x\,\mathcal{P}_X(dx) \quad \Big(\text{brief notation} \quad \mathbf{E}X := \int_\Omega X\,d\mathbf{P}\Big)$$

where it is assumed that the *absolute moment* is finite:

$$\mathbf{E}|X| = \int_\Omega |X(\omega)|\mathbf{P}(d\omega) < \infty.$$

In terms of the distribution function F_X, the expectation $\mathbf{E}X$ is expressed by the Stieltjes integral

$$\mathbf{E}X = \int_{-\infty}^{\infty} x\,dF_X(x).$$

The expectation of a measurable function g of X is given by

$$\mathbf{E}g(X) = \int_{-\infty}^{\infty} g(x)\,dF_X(x).$$

The *variance* of X is defined by the formula

$$\operatorname{Var} X := \mathbf{E}(X - \mathbf{E}X)^2 = \mathbf{E}X^2 - \mathbf{E}^2 X.$$

To estimate the expectation of a product of random variables we can apply the *Hölder inequality*

$$|\mathbf{E}XY| \le \mathbf{E}^{1/p}|X|^p\,\mathbf{E}^{1/q}|Y|^q,$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p > 1$, $q > 1$.

The following *Jensen inequality* holds: for any convex function $g(x)$, $x \in \mathbf{R}$, and any random variable X with finite absolute moment,

$$g(\mathbf{E}X) \le \mathbf{E}g(X). \tag{1.4}$$

Indeed, if g is a convex function, then for any number, in particular for $\mathbf{E}X$, there exists a constant C such that

$$g(x) \ge g(\mathbf{E}X) + (x - \mathbf{E}X)C \qquad\qquad \text{for all } x \in \mathbf{R}.$$

Substituting instead of x the random variable X and computing the expectation, we obtain (1.4).

Estimates of probabilities of events generated by a random variable can be expressed via moments of this random variable.

Chebyshev's inequalities hold true: for a nonnegative random variable X and any $\varepsilon > 0$,

$$\mathbf{P}(X \ge \varepsilon) \le \frac{\mathbf{E}X}{\varepsilon}; \tag{1.5}$$

and for any random variable Y with finite variance and any $\delta > 0$,

$$\mathbf{P}(|Y - \mathbf{E}Y| \ge \delta) \le \frac{\operatorname{Var} Y}{\delta^2}. \tag{1.6}$$

The following result is the main tool for the investigation of properties that hold with probability one, or equivalently almost surely (a.s.).

Lemma 1.1 (Borel–Cantelli, part 1). *Let $A_1, A_2, \dots$ be a sequence of events. Then the event*

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

consists of those and only those sample points ω that belong to an infinite number of events A_n, $n = 1, 2, \dots$.

If $\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$, then $\mathbf{P}\Big(\limsup_n A_n\Big) = 0$.

The proof is obvious:

$$\mathbf{P}\Big(\limsup_n A_n\Big) \le \mathbf{P}\Big(\bigcup_{m=n}^{\infty} A_m\Big) \le \sum_{m=n}^{\infty} \mathbf{P}(A_m) \to 0 \qquad \text{as } n \to \infty.$$

Remark 1.1. The first part of the Borel–Cantelli lemma states that if $\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$, then

$$\mathbf{P}\Big(\Big(\limsup_n A_n\Big)^{\mathrm{c}}\Big) = \mathbf{P}\Big(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^{\mathrm{c}}\Big) = 1,$$

i.e., for almost all sample points ω (a.s.) there exists a number $n_0 = n_0(\omega)$, such that $\omega \in A_m^{\mathrm{c}}$ for all $m \ge n_0$.

In probability theory we must deal with different types of convergence of random variables.

A sequence of random variables X_n converges to the variable X *in mean* if $\mathbf{E}|X_n - X| \to 0$, and *in mean square* if $\mathbf{E}(X_n - X)^2 \to 0$.

A sequence of random variables X_n converges to the variable X *in probability* if $\mathbf{P}(|X_n - X| \ge \varepsilon) \to 0$ for any $\varepsilon > 0$.

From (1.5) and (1.6) it follows that the convergence in mean or in mean square implies the convergence in probability.

We say that a sequence of random variables X_n converges to the variable X *with probability one* (a.s.) if the set of all sample points for which convergence of numerical sequences $X_n(\omega) \to X(\omega)$ holds, has probability one, i.e.,

$$\mathbf{P}(\omega : X_n(\omega) \to X(\omega)) = 1.$$

It is convenient to treat the convergence in probability by using the following statement.

Proposition 1.1. *$X_n \to X$ in probability if and only if for any sequence n_m of natural numbers there exists a subsequence n_{m_k} such that $X_{n_{m_k}} \to X$* a.s.

Proof. For any sequences $\varepsilon_k \downarrow 0$ and n_m, set

$$n_{m_k} := \min\Big\{n_m : \mathbf{P}(|X_{n_m} - X| > \varepsilon_k) \le \frac{1}{2^k}\Big\}.$$

Such a sequence n_{m_k} exists by virtue of the convergence $X_{n_m} \to X$ in probability. Thus we have

$$\mathbf{P}(|X_{n_{m_k}} - X| > \varepsilon_k) \le \frac{1}{2^k}.$$

The series of these probabilities converges. Then, by the first part of the Borel–Cantelli lemma (see Remark 1.1), there exists a.s. a number $k_0 = k_0(\omega)$ such that

$$|X_{n_{m_k}} - X| \le \varepsilon_k \qquad \text{for all } k \ge k_0.$$

This implies that $X_{n_{m_k}} \to X$ a.s.

To derive the opposite implication we proceed by contradiction. Suppose that X_n does not converge to X in probability. Then there exist $\varepsilon > 0$, $\delta > 0$, and a sequence n_m, such that

$$\mathbf{P}(|X_{n_m} - X| > \varepsilon) \ge \delta.$$

This is absurd, since we can choose a subsequence n_{m_k} such that $X_{n_{m_k}} \to X$ a.s. □

A family of random variables $\{X_\alpha\}_{\alpha \in A}$ is called *uniformly integrable* if

$$\lim_{c \to \infty} \sup_{\alpha \in A} \int\limits_{\{|X_\alpha| \ge c\}} |X_\alpha| \, d\mathbf{P} = 0.$$

For such families it obviously holds that $\sup\limits_{\alpha \in A} \mathbf{E}|X_\alpha| < \infty$.

Proposition 1.2. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of random variables with finite absolute moments and suppose there exists a nonnegative increasing function $G(x)$, $x \in [0, \infty)$, such that*

$$\lim_{x \to \infty} \frac{G(x)}{x} = \infty \qquad \text{and} \qquad M := \sup_{\alpha \in A} \mathbf{E}\, G(|X_\alpha|) < \infty.$$

Then the family of random variables $\{X_\alpha\}_{\alpha \in A}$ is uniformly integrable.

Proof. For any $\varepsilon > 0$ we choose c_ε so large that $G(x)/x \ge M/\varepsilon$ for all $x \ge c_\varepsilon$. Then for any $c \ge c_\varepsilon$ the event $\{|X_\alpha| \ge c\}$ implies $\{|X_\alpha| \le \varepsilon G(|X_\alpha|)/M\}$ and, therefore,

$$\int\limits_{\{|X_\alpha| \ge c\}} |X_\alpha| \, d\mathbf{P} \le \frac{\varepsilon}{M} \int\limits_{\Omega} G(|X_\alpha|) \, d\mathbf{P} \le \varepsilon.$$

This proves the uniform integrability. □

Proposition 1.3. *Let $\{X_n\}_{n \in \mathbb{N}}$ be a uniformly integrable family of random variables and let $X_n \to X$ in probability. Then the random variable X is integrable and $\mathbf{E}|X_n - X| \to 0$.*

Proof. We use Proposition 1.1 and choose a subsequence n_k such that $X_{n_k} \to X$ a.s. Then the integrability of the variable X is a consequence of Fatou's lemma:

$$\mathbf{E}|X| = \mathbf{E} \liminf_{n_k} |X_{n_k}| \le \liminf_{n_k} \mathbf{E}|X_{n_k}| < \infty.$$

Let $A := \{a : \mathbf{P}(X = a) > 0\}$ be the set of *atoms of the distribution* of the random variable X. This set is countable. For positive $b \notin A$ we consider the truncated variables $X_n^b = X_n \mathbb{1}_{\{|X_n|<b\}}$, $X^b = X\mathbb{1}_{\{|X|<b\}}$. Then

$$\mathbf{E}|X_n - X| \le \mathbf{E}|X_n^b - X^b| + \mathbf{E}\{|X_n|\mathbb{1}_{\{|X_n|\ge b\}}\} + \mathbf{E}\{|X|\mathbb{1}_{\{|X|\ge b\}}\}. \qquad (1.7)$$

In view of the integrability of X and the uniform integrability of X_n, for any $\varepsilon > 0$ we can choose $b \notin A$ such that

$$\mathbf{E}\{|X|\mathbb{1}_{\{|X|\ge b\}}\} \le \varepsilon/3 \qquad \text{and} \qquad \sup_n \mathbf{E}\{|X_n|\mathbb{1}_{\{|X_n|\ge b\}}\} \le \varepsilon/3.$$

By the Lebesgue dominated convergence theorem, $\mathbf{E}|X_n^b - X^b| \to 0$. Consequently, there exists a number n_0 such that $\mathbf{E}|X_n^b - X^b| \le \varepsilon/3$ for all $n \ge n_0$. By (1.7), for these n the estimate $\mathbf{E}|X_n - X| \le \varepsilon$ is valid, which completes the proof. □

It is not difficult to understand that one can consider complex-valued random variables. The real and imaginary parts of them are real-valued random variables. The expectation of a complex-valued random variable is defined in a natural way via the expectations of the real and the imaginary parts.

The function

$$\varphi(\alpha) := \int_{-\infty}^{\infty} e^{i\alpha x}\, dF_X(x) = \mathbf{E}e^{i\alpha X}, \qquad \alpha \in \mathbf{R},$$

is called the *characteristic function* of the random variable X.

Proposition 1.4. *Suppose that the random variable X has a finite absolute moment of nth order ($\mathbf{E}|X|^n < \infty$). Then the characteristic function of X is n times differentiable, and for every $0 \le k \le n$,*

$$\mathbf{E}X^k = (-i)^k \frac{d^k}{d\alpha^k}\varphi(\alpha)\Big|_{\alpha=0}.$$

Corollary 1.1. *The expectation and the variance are given by the formulas*

$$\mathbf{E}X = -i\varphi'(0), \qquad \operatorname{Var} X = -\varphi''(0) + (\varphi'(0))^2. \qquad (1.8)$$

In order to distinguish the characteristic function of a given random variable X from other characteristic functions, we sometimes use for it the notation $\varphi_X(\alpha)$.

The next result shows that the distribution function of a random variable is uniquely defined by its characteristic function.

Proposition 1.5 (inversion formula). *The following statements hold*:

1. *For an integer-valued random variable X,*

$$\mathbf{P}(X = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\alpha k}\varphi_X(\alpha)\, d\alpha, \qquad k = 0, \pm 1, \pm 2, \dots .$$

2. *For any points y, z of continuity of the distribution function F_X,*

$$F_X(y) - F_X(z) = \frac{1}{2\pi} \lim_{c\to\infty} \int_{-c}^{c} \frac{e^{-i\alpha y} - e^{-i\alpha z}}{-i\alpha} \varphi_X(\alpha)\, d\alpha. \tag{1.9}$$

3. *If the characteristic function $\varphi_X(\alpha)$ is absolutely integrable, then X has the continuous density*

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \varphi_X(\alpha)\, d\alpha. \tag{1.10}$$

The *covariance* of a two real-valued random variables X and Y is defined to be

$$\mathrm{Cov}(X,Y) := \mathbf{E}\{(X - \mathbf{E}X)(Y - \mathbf{E}Y)\} = \mathbf{E}\{XY\} - \mathbf{E}X\mathbf{E}Y.$$

A collection of random variables $X_1, X_2, \dots, X_n$ is often referred to as a *random vector* $\vec{X} = (X_1, X_2, \dots, X_n)$.

With each random vector $\vec{X}$ we associate its distribution $\mathcal{P}_{\vec{X}}(B)$, $B \in \mathcal{B}(\mathbf{R}^n)$, which is the measure uniquely determined by the *finite-dimensional distributions*

$$\begin{aligned}\mathcal{P}_{\vec{X}}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n) &:= \mathbf{P}\Big(\bigcap_{k=1}^{n} \{\omega : X_k(\omega) \in \Delta_k\}\Big)\\ &=: \mathbf{P}\big(X_1 \in \Delta_1, X_2 \in \Delta_2, \dots, X_n \in \Delta_n\big), \qquad \Delta_k \in \mathcal{B}(\mathbf{R}),\ k = 1, 2, \dots, n.\end{aligned}$$

The function

$$F_{\vec{X}}(\vec{x}) := \mathbf{P}\big(X_1 < x_1, X_2 < x_2, \dots, X_n < x_n\big), \qquad \vec{x} \in \mathbf{R}^n,$$

is called the *finite-dimensional distribution function* of the vector $\vec{X}$.

The random variables X_1, X_2, $\dots$, X_n are said to have a *joint density* if there exists a nonnegative measurable function $f_{\vec{X}}(\vec{x})$, $\vec{x} \in \mathbf{R}^n$, such that

$$F_{\vec{X}}(\vec{x}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f_{\vec{X}}(\vec{y})\, dy_1 dy_2 \dots dy_n$$

for every $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$.

As in the one-dimensional case, for a finite-dimensional distribution with a continuous density $f_{\vec{X}}$ we have

$$\mathbf{P}\big(X_1 \in [x_1, x_1 + \delta_1), \dots, X_n \in [x_n, x_n + \delta_n)\big) \sim f_{\vec{X}}(\vec{x})\, \delta_1 \cdots \delta_n \qquad \text{as } \delta_k \downarrow 0.$$

Suppose that the variables $\vec{X} = (X_1, X_2, \dots, X_n)$ have a joint continuous density $f_{\vec{X}}$. Let $\vec{g} : \mathbf{R}^n \to \mathbf{R}^n$ be a continuously differentiable mapping with non-vanishing Jacobian. Then the random variable $\vec{Y} = \vec{g}(\vec{X})$ has the density

$$f_{\vec{Y}}(\vec{y}) = \begin{cases} f_{\vec{X}}(\vec{g}^{\,(-1)}(\vec{y}))\big|\det J_{\vec{g}^{\,(-1)}}(\vec{y})\big|, & \vec{y} \in G,\\ 0, & \vec{y} \notin G,\end{cases} \tag{1.11}$$

where $G := \{\vec{y} : \vec{y} = \vec{g}(\vec{x}) \text{ for some } \vec{x} \in \mathbf{R}^n\}$ and $J_{\vec{h}}(\vec{x}) := \Big\{\frac{\partial}{\partial x_l} h_k(\vec{x})\Big\}_{k,l=1}^{n}$ is the matrix whose determinant is called the *Jacobian* of the mapping $\vec{h} : R^n \to R^n$.

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a measurable mapping. Then the expectation of the random variable $Y := g(\vec{X})$ is equal to

$$\mathbf{E}Y = \int\limits_{\Omega} Y(\omega)\,\mathbf{P}(d\omega) = \int\limits_{\mathbf{R}^n} g(\vec{x})\,\mathcal{P}_{\vec{X}}(dx_1 \times dx_2 \times \cdots \times dx_n) = \int\limits_{\mathbf{R}^n} g(\vec{x})\,F_{\vec{X}}(d\vec{x}).$$

The *characteristic function of the random vector* $\vec{X} = (X_1, X_2, \dots, X_n)$ is given by the formula $\varphi_{\vec{X}}(\vec{\alpha}) := \mathbf{E}\exp\big(i(\vec{\alpha}, \vec{X})\big)$, $\vec{\alpha} \in \mathbf{R}^n$, where $(\vec{\alpha}, \vec{X}) := \sum\limits_{k=1}^{n} \alpha_k X_k$. The characteristic function $\varphi_{\vec{X}}(\vec{\alpha})$ uniquely determines the distribution of $\vec{X}$.

§ 2. Conditional expectations

If one knows that a event B of positive probability has occurred, one can consider the *conditional probability* of some event A given B. For B such that $\mathbf{P}(B) > 0$ this conditional probability is defined by the formula

$$\mathbf{P}(A|B) := \frac{\mathbf{P}(A \bigcap B)}{\mathbf{P}(B)}. \tag{2.1}$$

Since $\mathbf{P}(\Omega|B) = 1$, the conditional probability $\mathbf{P}(\cdot|B)$ is itself a probability measure defined on the σ-algebra $\mathcal{F}$.

Formula (2.1) becomes clear if one considers a discrete model of sample points with equal probabilities. In this model $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ is the sample space of a finite number of elementary events having equal probabilities

$$\mathbf{P}(\{\omega_1\}) = \mathbf{P}(\{\omega_2\}) = \cdots = \mathbf{P}(\{\omega_n\}) = \frac{1}{n} = \frac{1}{\operatorname{card}\Omega}.$$

By $\operatorname{card} A$ one denotes the cardinality (the number of points) of the event A. Usually the uniform models are motivated by arguments including symmetry and homogeneity properties. For example, in the experiment of tossing a coin it is assumed that the coin is symmetric and is made from a homogeneous material.

For the uniform model, $\mathbf{P}(A) := \dfrac{\operatorname{card} A}{\operatorname{card}\Omega}$. The conditional probability $\mathbf{P}(A|B)$ can be treated as the usual probability for the new sample space $\Omega_1 = B$. We again have the uniform model reduced to the event B. Therefore,

$$\mathbf{P}(A|B) = \frac{\operatorname{card}(A \bigcap B)}{\operatorname{card}\Omega_1} = \frac{\operatorname{card}(A \bigcap B)/\operatorname{card}\Omega}{\operatorname{card} B/\operatorname{card}\Omega} = \frac{\mathbf{P}(A \bigcap B)}{\mathbf{P}(B)}.$$

The event A is said to be *independent* of the event B with $\mathbf{P}(B) > 0$ if

$$\mathbf{P}(A|B) = \mathbf{P}(A).$$

This means that the probability of the event A does not depend on whether B occurs or not. By the definition of the conditional probability, this is equivalent to

$$\mathbf{P}(A\bigcap B) = \mathbf{P}(A)\mathbf{P}(B). \tag{2.2}$$

Since for $\mathbf{P}(A) > 0$

$$\mathbf{P}(B|A) = \frac{\mathbf{P}(B\bigcap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A)\mathbf{P}(B)}{\mathbf{P}(A)} = \mathbf{P}(B),$$

we see that the event B is also independent of the event A and we say that the *events A and B are independent.*

The events $A_1, A_2, \dots, A_n$ are said to be *independent* if for all choices of indices $1 \le l_1 < l_2 < \cdots < l_k \le n$,

$$\mathbf{P}(A_{l_1}\bigcap A_{l_2}\bigcap \dots \bigcap A_{l_k}) = \mathbf{P}(A_{l_1})\mathbf{P}(A_{l_2})\dots\mathbf{P}(A_{l_k}). \tag{2.3}$$

The random variables $X_1, X_2, \dots, X_n$ are said to be *independent* if for any Borel sets Δ_k, $k = 1, 2, \dots, n$,

$$\mathbf{P}(X_1 \in \Delta_1, X_2 \in \Delta_2, \dots, X_n \in \Delta_n) = \prod_{k=1}^{n} \mathbf{P}(X_k \in \Delta_k). \tag{2.4}$$

A necessary and sufficient condition for $X_1, X_2, \dots, X_n$ to be independent is the following: for every $\vec{x} \in \mathbf{R}^n$,

$$F_{\vec{X}}(\vec{x}) = \prod_{k=1}^{n} F_{X_k}(x_k),$$

where $F_{X_k}(x_k) = \mathbf{P}(X_k < x_k)$, $k = 1, 2, \dots, n$, are the *marginal distribution functions.* If the joint density $f_{\vec{X}}$ exists, then this is equivalent to

$$f_{\vec{X}}(\vec{x}) = \prod_{k=1}^{n} f_{X_k}(x_k), \qquad \vec{x} \in \mathbf{R}^n,$$

where $f_{X_k}(x_k)$, $x_k \in \mathbf{R}$, $k = 1, 2, \dots, n$, are the *marginal densities* of the random variables X_k.

One more necessary and sufficient independence condition for random variables is the following. The random variables $X_1, X_2, \dots, X_n$ are independent if and only if (iff) for any bounded Borel functions g_k, $k = 1, 2, \dots, n$,

$$\mathbf{E}\big(g_1(X_1)g_2(X_2)\cdots g_n(X_n)\big) = \prod_{k=1}^{n} \mathbf{E}\, g_k(X_k). \tag{2.5}$$

For the indicator functions, $g_k(x) = \mathbb{1}_{\Delta_k}(x)$, formulas (2.4) and (2.5) coincide. It is clear that (2.4) implies (2.5) for linear combinations of the indicator functions, i.e., for

$$g_m(x) = \sum_{k=1}^{m} c_{m,k} \mathbb{1}_{\Delta_{m,k}}(x), \qquad \Delta_{m,k} \in \mathcal{B}(\mathbf{R}). \tag{2.6}$$

Since every bounded Borel function can be uniformly approximated by functions of the form (2.6), the equivalence of (2.4) and (2.5) is established.

To characterize the independence of random variables it is sufficient to take instead of arbitrary bounded Borel functions the family of complex-valued functions $\{e^{i\alpha x}\}_{\alpha\in\mathbf{R}}$.

The function

$$\varphi_{\vec{X}}(\vec{\alpha}) = \mathbf{E}\exp\Bigl(i\sum_{k=1}^{n}\alpha_k X_k\Bigr), \qquad \vec{\alpha}\in\mathbf{R}^n,$$

is the characteristic function of the random vector $\vec{X}$. Therefore the random variables $X_1, X_2, \dots, X_n$ are independent iff

$$\varphi_{\vec{X}}(\vec{\alpha}) = \prod_{k=1}^{n}\varphi_{X_k}(\alpha_k), \qquad \vec{\alpha}\in\mathbf{R}^n. \tag{2.7}$$

From this formula, as a special case, we have the following result. If $X_1, X_2, \dots, X_n$ are independent random variables and $S_n = \sum_{k=1}^{n} X_k$, then

$$\varphi_{S_n}(\alpha) = \prod_{k=1}^{n}\varphi_{X_k}(\alpha), \qquad \alpha\in\mathbf{R},$$

i.e., the characteristic function of a sum of independent random variables is equal to the product of the characteristic functions of the terms.

The *conditional expectation of a random variable given an event* B is defined for $\mathbf{P}(B)>0$ by the formula

$$\mathbf{E}\{X|B\} := \int_{\Omega} X(\omega)\mathbf{P}(d\omega|B) = \frac{\mathbf{E}\{X\mathbb{1}_B\}}{\mathbf{P}(B)}. \tag{2.8}$$

This equality can be explained in the following way. For the simple case, where $X=\mathbb{1}_A$ this is the consequence of the definition of conditional probability:

$$\mathbf{E}\{\mathbb{1}_A|B\} = \int_A \mathbf{P}(d\omega|B) = \mathbf{P}(A|B) = \frac{\mathbf{P}(A\bigcap B)}{\mathbf{P}(B)} = \frac{\mathbf{E}(\mathbb{1}_A\mathbb{1}_B)}{\mathbf{P}(B)}.$$

By the linearity property, this is true for $X=\sum_{k=1}^{m} c_{m,k}\mathbb{1}_{A_{m,k}}$. Since an arbitrary random variable X with $\mathbf{E}|X|<\infty$ can be approximated by a linear combination of indicator functions, the general case of (2.8) follows.

Further, we define the conditional expectation $\mathbf{E}\{X|\mathcal{Q}\}$ of a random variable X given a σ-algebra $\mathcal{Q}$. If $\mathcal{Q}$ is the algebra generated by a finite number of disjoint sets B_k, $k=1,\dots,m$, then for $\omega\in B_k$,

$$\mathbf{E}\{X|\mathcal{Q}\} := \mathbf{E}\{X|B_k\} = \frac{\mathbf{E}\{X\mathbb{1}_{B_k}\}}{\mathbf{P}(B_k)}. \tag{2.9}$$

Therefore, $\mathbf{E}\{X|\mathcal{Q}\}$ is a random variable that is constant on the sets B_k. For arbitrary $B \in \mathcal{Q}$ we have $B = B_{k_1} \bigcup B_{k_2} \bigcup \dots \bigcup B_{k_m}$ and

$$\int_B \mathbf{E}\{X|\mathcal{Q}\} d\mathbf{P} = \sum_{l=1}^{m} \mathbf{E}\{X|B_{k_l}\}\mathbf{P}(B_{k_l}) = \sum_{l=1}^{m} \mathbf{E}\{X \mathbb{1}_{B_{k_l}}\} = \int_B X\, d\mathbf{P}.$$

As a result, we obtain the equality which underlies the definition of the conditional expectation of a random variable X given an arbitrary σ-algebra $\mathcal{Q}$.

Let X be a random variable with $\mathbf{E}|X| < \infty$. The *conditional expectation* $\mathbf{E}\{X|\mathcal{Q}\}$ of X given a σ-algebra $\mathcal{Q} \subseteq \mathcal{F}$ is the $\mathcal{Q}$-measurable random variable such that

$$\int_B \mathbf{E}\{X|\mathcal{Q}\}\, d\mathbf{P} = \int_B X\, d\mathbf{P} \qquad \Big(\mathbf{E}\{\mathbf{E}\{X|\mathcal{Q}\}\mathbb{1}_B\} = \mathbf{E}\{X\mathbb{1}_B\}\Big) \tag{2.10}$$

for every $B \in \mathcal{Q}$. The a.s. existence of a unique conditional expectation follows from the Radon–Nikodým theorem, because the charge μ defined by $\mu(B) := \int_B X d\mathbf{P}$, $B \in \mathcal{Q}$, is *absolutely continuous* with respect to the measure $\mathbf{P}$ ($\mathbf{P}(\Lambda) = 0$ implies $\mu(\Lambda) = 0$ for $\Lambda \in \mathcal{Q}$). Therefore, the conditional expectation $\mathbf{E}\{X|\mathcal{Q}\}$ is the Radon–Nikodým derivative $\frac{d\mu}{d\mathbf{P}}(\omega)$.

The *conditional probability* $\mathbf{P}(A|\mathcal{Q})$ for a set $A \in \mathcal{F}$ given a σ-algebra $\mathcal{Q} \subseteq \mathcal{F}$ is defined by the formula $\mathbf{P}(A|\mathcal{Q}) := \mathbf{E}\{\mathbb{1}_A|\mathcal{Q}\}$.

According to (2.10), the determining property of conditional probability is

$$\int_B \mathbf{P}(A|\mathcal{Q})\, d\mathbf{P} = \mathbf{P}(A \bigcap B), \qquad \text{for every } B \in \mathcal{Q}. \tag{2.11}$$

We say that a random variable X does not depend on a σ-algebra $\mathcal{Q}$ if for any Borel set Δ and any set $B \in \mathcal{Q}$, the events $\{X \in \Delta\}$ and B are independent.

The *basic properties* of conditional expectations.

1) *Linearity.* For arbitrary constants α, β and random variables X, Y with finite absolute moments,

$$\mathbf{E}\{\alpha X + \beta Y|\mathcal{Q}\} = \alpha\mathbf{E}\{X|\mathcal{Q}\} + \beta\mathbf{E}\{Y|\mathcal{Q}\} \quad \text{a.s.} \tag{2.12}$$

This follows from (2.10).

2) If X does not depend on the σ-algebra $\mathcal{Q}$, then

$$\mathbf{E}\{X|\mathcal{Q}\} = \mathbf{E}X \quad \text{a.s.} \tag{2.13}$$

Indeed, for every $B \in \mathcal{Q}$

$$\int_B \mathbf{E}\{X|\mathcal{Q}\}\, d\mathbf{P} = \int_B X\, d\mathbf{P} = \mathbf{E}(\mathbb{1}_B X) = \mathbf{E}\mathbb{1}_B \mathbf{E}X = \int_B \mathbf{E}X\, d\mathbf{P}.$$

This implies (2.13).

3) If Y is $\mathcal{Q}$-measurable, then

$$\mathbf{E}\{XY|\mathcal{Q}\} = Y\mathbf{E}\{X|\mathcal{Q}\} \quad \text{a.s.} \tag{2.14}$$

It is sufficient to prove this equality for a simple random variable $Y = 1\!\!1_A$, $A \in \mathcal{Q}$, since then linearity and approximation can be applied. For every $B \in \mathcal{Q}$

$$\int_B \mathbf{E}\{X1\!\!1_A|\mathcal{Q}\}\,d\mathbf{P} = \int_B 1\!\!1_A X\,d\mathbf{P} = \int_{B\bigcap A} X\,d\mathbf{P}$$

$$= \int_{B\bigcap A} \mathbf{E}\{X|\mathcal{Q}\}\,d\mathbf{P} = \int_B 1\!\!1_A\mathbf{E}\{X|\mathcal{Q}\}\,d\mathbf{P}.$$

This implies (2.14).

4) For $\mathcal{Q} \subseteq \mathcal{M}$,

$$\mathbf{E}\{X|\mathcal{Q}\} = \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|\mathcal{Q}\} \quad \text{a.s.} \tag{2.15}$$

4′) For $A \in \mathcal{M}$,

$$\mathbf{E}\{X|A\} = \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|A\}.$$

Since for every $B \in \mathcal{Q}$ it holds that $B \in \mathcal{M}$, we have

$$\int_B \mathbf{E}\{X|\mathcal{Q}\}\,d\mathbf{P} = \int_B X\,d\mathbf{P} = \int_B \mathbf{E}\{X|\mathcal{M}\}\,d\mathbf{P} = \int_B \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|\mathcal{Q}\}\,d\mathbf{P}.$$

This implies (2.15).

5) If $X \le Y$ a.s., then $\mathbf{E}\{X|\mathcal{Q}\} \le \mathbf{E}\{Y|\mathcal{Q}\}$ a.s.

This follows from the linearity property and the fact that if $Z \ge 0$ a.s., then, by definition of the conditional expectation, $\mathbf{E}\{Z|\mathcal{Q}\} \ge 0$ a.s.

6) $|\mathbf{E}\{X|\mathcal{Q}\}| \le \mathbf{E}\{|X||\mathcal{Q}\}$ a.s.

This follows from property 5) if one takes into account that $-|X| \le X \le |X|$.

7) If $\mathbf{E}\sup\limits_{n\in\mathbb{N}}|X_n| < \infty$ and $X_n \to X$ a.s., then

$$\mathbf{E}\{X_n|\mathcal{Q}\} \to \mathbf{E}\{X|\mathcal{Q}\} \quad \text{a.s.}$$

Indeed, if $X_n \to X$ a.s., then $\sup\limits_{m\ge n}|X_m - X| \downarrow 0$ a.s. Therefore,

$$|\mathbf{E}\{X_n|\mathcal{Q}\} - \mathbf{E}\{X|\mathcal{Q}\}| \le \mathbf{E}\{|X_n - X|\,|\mathcal{Q}\} \le \mathbf{E}\Big\{\sup_{m\ge n}|X_m - X|\,\Big|\mathcal{Q}\Big\} \quad \text{a.s.}$$

The right-hand side of this inequality decreases monotonically and tends to zero due to the fact that, by the Lebesgue dominated convergence theorem,

$$\mathbf{E}\Big\{\mathbf{E}\Big\{\sup_{m\ge n}|X_m - X|\Big|\mathcal{Q}\Big\}\Big\} = \mathbf{E}\Big\{\sup_{m\ge n}|X_m - X|\Big\} \to 0.$$

7′) If $\{X_n\}_{n\in\mathbb{N}}$ is a uniformly integrable family of random variables and $X_n \to X$ in probability, then $\mathbf{E}\{X_n|\mathcal{Q}\} \to \mathbf{E}\{X|\mathcal{Q}\}$ in mean.

This property is an obvious consequence of Proposition 1.3, because

$$\mathbf{E}|\mathbf{E}\{X_n|\mathcal{Q}\} - \mathbf{E}\{X|\mathcal{Q}\}| \le \mathbf{E}|X_n - X| \to 0.$$

Clearly, being a random variable, the conditional probability $\mathbf{P}(A|\mathcal{Q})$ is defined only up to the set of "exceptionality" Λ_A, which has measure zero ($\mathbf{P}(\Lambda_A) = 0$). For this reason, in the general case the conditional measure $\mathbf{P}(\cdot|\mathcal{Q})$ cannot be defined simultaneously for all sets of the σ-algebra $\mathcal{F}$. Hence, in general the important equality

$$\mathbf{E}\{X|\mathcal{Q}\} := \int\limits_{\Omega} X(\omega)\, \mathbf{P}(d\omega|\mathcal{Q})$$

cannot be obtained. In the case when this is possible, the family of probabilities $\mathbf{P}(A|\mathcal{Q})$, $A \in \mathcal{F}$, is called *regular.*

In general case it is proved (see, for example, Shiryaev (1980) Ch. II § 7) that for a random variable X there exist the *conditional distribution*

$$\mathcal{P}_X(\Delta|\mathcal{Q}) = \mathbf{P}(\{\omega : X(\omega) \in \Delta\}|\mathcal{Q}),$$

defined on the σ-algebra of Borel sets $\mathcal{B}(\mathbf{R})$. This is due to the structure of Borel sets on the real line, in which subintervals with rational endpoints play a decisive role.

For any Borel function $g(x)$, $x \in \mathbf{R}$, satisfying $\mathbf{E}|g(X)| < \infty$, one has the equality

$$\mathbf{E}\{g(X)|\mathcal{Q}\} = \int\limits_{-\infty}^{\infty} g(x)\, \mathcal{P}_X(dx|\mathcal{Q}), \qquad \text{a.s.} \tag{2.16}$$

This formula holds for indicator functions and therefore holds for linear combinations of indicators. Formula (2.16) can be proved for an arbitrary Borel function with the help of approximation of Borel function by indicators.

An important consequence of this formula is the *Jensen inequality* for conditional expectations: for any convex function $g(x)$, $x \in \mathbf{R}$, and any random variable X with finite absolute moment,

$$g(\mathbf{E}\{X|\mathcal{Q}\}) \le \mathbf{E}\{g(X)|\mathcal{Q}\}, \quad \text{a.s.}$$

By (2.16), this follows from the Jensen inequality (1.4).

The smallest σ-algebra $\sigma(Y)$ containing the events $\{\omega : Y(\omega) \in \Delta\}$, $\Delta \in \mathcal{B}(\mathbf{R})$, is called the σ-algebra generated by the random variable Y.

For an arbitrary random variable Y there exists a Borel function $\varphi(y)$ such that

$$\mathbf{E}\{X|\sigma(Y)\} = \varphi(Y) \qquad \text{a.s.} \tag{2.17}$$

Indeed, by the definition of the conditional expectation, for any set $\Delta \in \mathcal{B}(\mathbf{R})$

$$\int\limits_{\{Y\in\Delta\}} \mathbf{E}\{X|\sigma(Y)\}\, d\mathbf{P} = \int\limits_{\{Y\in\Delta\}} X\, d\mathbf{P} =: \mu(\Delta).$$

Obviously, μ is a σ-finite charge on the σ-algebra $\mathcal{B}(\mathbf{R})$. In addition, $\mu(\Delta) = 0$ if $\mathbf{P}(Y \in \Delta) = 0$, i.e., μ is absolutely continuous with respect to the distribution of the random variable Y ($\mathcal{P}_Y(\Delta) = \mathbf{P}(Y \in \Delta)$). By the Radon–Nikodým theorem, there exists a Borel function φ such that

$$\mu(\Delta) = \int\limits_{\Delta} \varphi(y)\mathcal{P}_Y(dy).$$

Applying the integration by substitution formula, we obtain

$$\mu(\Delta) = \int\limits_{\{Y\in\Delta\}} \varphi(Y)\, d\mathbf{P}.$$

Comparing this with the definitions of the charge μ and conditional expectation, we have (2.17).

The function φ is defined only on the set $Y(\Omega) := \{y : y=Y(\omega) \text{ for some } \omega\in\Omega\}$. Usually one sets

$$\mathbf{E}\{X|Y = y\} := \varphi(y) \tag{2.18}$$

and call $\varphi(y)$ the *conditional expectation of the random variable* X given $\{Y = y\}$. Here it is important to note that the probability of the event $\{Y = y\}$ can be equal to zero and so the standard definition (see (2.8)) is not applicable.

For this conditional expectation the following analog of the fourth property holds: if $\sigma(Y) \subseteq \mathcal{M}$, then

$$\mathbf{E}\{X|Y = y\} = \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|Y = y\}.$$

A more natural way to define the expectation given $\{Y = y\}$ is the following:

$$\mathbf{E}\{X|Y = y\} := \lim_{\delta\downarrow 0} \mathbf{E}\{X|Y \in [y, y + \delta)\} = \lim_{\delta\downarrow 0} \frac{\mathbf{E}\{X\mathbb{1}_{[y,y+\delta)}(Y)\}}{\mathbf{P}(Y \in [y, y + \delta))}, \tag{2.19}$$

provided the limit exists.

If φ is a right continuous function, then definitions (2.18) and (2.19) are equivalent. Indeed, applying the definition of conditional expectation with respect to a σ-algebra and the mean value theorem for integrals, we obtain

$$\mathbf{E}\{X\mathbb{1}_{[y,y+\delta)}(Y)\} = \int\limits_{Y\in[y,y+\delta)} X\, d\mathbf{P} = \int\limits_{Y\in[y,y+\delta)} \mathbf{E}\{X|\sigma(Y)\}\, d\mathbf{P}$$

$$= \int\limits_{Y\in[y,y+\delta)} \varphi(Y)\, d\mathbf{P} = \int\limits_{[y,y+\delta)} \varphi(z) d_z\mathbf{P}(Y < z) = (\varphi(y) + o(1))\mathbf{P}(Y \in [y, y + \delta)).$$

Here $o(1)$ stands for a variable converging to zero as $\delta \to 0$. From this equality it follows that the limit in (2.19) exists and coincides with (2.18).

Example 2.1. Suppose that the random variables X, Y have a continuous joint density $f(x,y)$. Then the density of the variable Y is given by $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$. Suppose $f_Y(y)$ is strictly positive on the set $Y(\Omega)$. Then for any Borel set Δ

$$\mathbf{P}(X \in \Delta | Y = y) = \frac{\int_\Delta f(x,y)\,dx}{f_Y(y)}, \qquad y \in Y(\Omega). \tag{2.20}$$

Indeed, by (2.19),

$$\mathbf{P}(X \in \Delta | Y = y) = \lim_{\delta \downarrow 0} \frac{\mathbf{P}(X \in \Delta, Y \in [y, y+\delta))}{\mathbf{P}(Y \in [y, y+\delta))}$$

$$= \lim_{\delta \downarrow 0} \frac{\int_y^{y+\delta} \int_\Delta f(x,z)\,dx\,dz}{\int_y^{y+\delta} f_Y(z)\,dz} = \frac{\int_\Delta f(x,y)\,dx}{f_Y(y)}.$$

The function $f(x|y) := \dfrac{f(x,y)}{f_Y(y)}$ is naturally called the *conditional density* of the random variable X given $\{Y = y\}$.

It is easy to see that for any bounded Borel function $g(x)$, $x \in \mathbf{R}$,

$$\mathbf{E}\{g(X)|Y = y\} = \int_{-\infty}^{\infty} g(x) \frac{f(x,y)}{f_Y(y)}\,dx, \qquad y \in Y(\Omega). \tag{2.21}$$

Example 2.2. Suppose that the random variables X, Y have a continuous joint density $f(x,y)$. Let $g(x,y)$ be bounded measurable function continuous with respect to y. Then

$$\mathbf{E}\{g(X,Y)|Y = y\} = \mathbf{E}\{g(X,y)|Y = y\}, \qquad y \in Y(\Omega). \tag{2.22}$$

Indeed, by (2.19),

$$\mathbf{E}\{g(X,Y)|Y = y\} = \lim_{\delta \downarrow 0} \frac{\mathbf{E}\{g(X,Y)\mathbb{1}_{[y,y+\delta)}(Y)\}}{\mathbf{P}(Y \in [y, y+\delta))} = \lim_{\delta \downarrow 0} \frac{\int_y^{y+\delta} \int_{-\infty}^{\infty} g(x,z) f(x,z)\,dx\,dz}{\int_y^{y+\delta} f_Y(z)\,dz}$$

$$= \int_{-\infty}^{\infty} g(x,y) \frac{f(x,y)}{f_Y(y)}\,dx = \lim_{\delta \downarrow 0} \frac{\mathbf{E}\{g(X,y)\mathbb{1}_{[y,y+\delta)}(Y)\}}{\mathbf{P}(Y \in [y, y+\delta))} = \mathbf{E}\{g(X,y)|Y = y\}.$$

For further purposes we need the following statement.

Lemma 2.1. *Let $H(x,\omega)$, $x \in \mathbf{R}$, be a bounded $\mathcal{B}(\mathbf{R}) \times \mathcal{F}$-measurable function independent of the σ-algebra $\mathcal{Q}$. Let X be $\mathcal{Q}$-measurable random variable. Then*

$$\mathbf{E}\big\{H(X,\omega)\big|\mathcal{Q}\big\} = h(X), \tag{2.23}$$

where $h(x) = \mathbf{E}H(x,\omega)$.

Proof. First we consider the special case in which the function $H(x,\omega)$ has the form

$$H(x,\omega) = \sum_{k=1}^{n} b_k(x)\rho_k(\omega), \tag{2.24}$$

where $b_k(x)$ are bounded Borel functions and the random variables $\rho_k(\omega)$, $k = 1,\dots,n$, are independent of the σ-algebra $\mathcal{Q}$. Then

$$h(x) = \sum_{k=1}^{n} b_k(x)\mathbf{E}\rho_k(\omega).$$

Applying the linearity property of the conditional expectation and (2.14), (2.13), we have

$$\mathbf{E}\{H(X,\omega)|\mathcal{Q}\} = \sum_{k=1}^{n} \mathbf{E}\{b_k(X)\rho_k(\omega)|\mathcal{Q}\}$$

$$= \sum_{k=1}^{n} b_k(X)\mathbf{E}\{\rho_k(\omega)|\mathcal{Q}\} = \sum_{k=1}^{n} b_k(X)\mathbf{E}\rho_k(\omega) = h(X).$$

For an arbitrary H one can prove (2.23) with the help of approximation of H by functions taking the form (2.24). □

We conclude this section by proving the following result, which is an addition to Lemma 1.1.

Lemma 2.2 (Borel–Cantelli, part 2). *If the events $A_1, A_2, \dots$ are independent and $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$, then $\mathbf{P}\Big(\limsup_n A_n\Big) = 1$.*

Proof. If A_1, A_2, ... are independent, then the complements A_1^c, A_2^c, ... are also independent. Note that for the complement of the set $\limsup_n A_n$ the following relation holds:

$$\Big(\limsup_n A_n\Big)^c := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c.$$

By independence, for any n

$$\mathbf{P}\Big(\bigcap_{m=n}^{\infty} A_m^c\Big) = \prod_{m=n}^{\infty} \mathbf{P}(A_m^c).$$

In view of the inequality $\ln(1-x) \le -x$, $0 \le x < 1$, we have

$$\ln\Big(\prod_{m=n}^{\infty} \mathbf{P}\big(A_m^{\mathrm{c}}\big)\Big) = \sum_{m=n}^{\infty} \ln\big(1 - \mathbf{P}(A_m)\big) \le -\sum_{m=n}^{\infty} \mathbf{P}(A_m) = -\infty.$$

Consequently, $\mathbf{P}\Big(\bigcap\limits_{m=n}^{\infty} A_m^{\mathrm{c}}\Big) = 0$ for any n. Therefore, $\mathbf{P}\Big(\Big(\limsup\limits_n A_n\Big)^{\mathrm{c}}\Big) = 0$, or $\mathbf{P}\Big(\limsup\limits_n A_n\Big) = 1$. □

Exercises.

2.1. Let $\Omega = \{\omega : \omega \in [-1/2, 1/2]\}$, $\mathcal{F} = \mathcal{B}([-1/2, 1/2])$, $\mathbf{P}(d\omega) = d\omega$. Let $X(\omega) = \omega^2$. Prove that

$$\mathbf{P}(A|\sigma(X)) = \frac{1}{2}\mathbb{1}_A(\omega) + \frac{1}{2}\mathbb{1}_A(-\omega), \quad \mathbf{E}\{Y|\sigma(X)\} = \frac{1}{2}Y(\omega) + \frac{1}{2}Y(-\omega).$$

2.2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the same probability space as in 2.1. Let Y be a random variable with $\mathbf{E}|Y| < \infty$. Compute $\mathbf{E}\{Y|\sigma(X)\}$

1) for $X(\omega) = \omega\, \mathbb{1}_{[0,1/2]}(\omega) - 3\omega\, \mathbb{1}_{[-1/2,0)}(\omega)$;

2) for $X(\omega) = \omega^2\, \mathbb{1}_{[0,1/2]}(\omega) - 2\omega\, \mathbb{1}_{[-1/2,0)}(\omega)$.

2.3. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the same probability space as in 2.1.

1) For $X(\omega) = (\omega - 1/4)^+$ compute $\mathbf{E}\{\omega^2|\sigma(X)\}$.

2) For $X(\omega) = |\omega - 1/4|$ compute $\mathbf{E}\{(\omega + 1/4)^3|\sigma(X)\}$ and $\mathbf{E}\{(\omega + 1/4)^3|X = x\}$.

2.4. Let $\Omega = \{\omega : \omega \in [-1/2, 1/2]\}$, $\mathcal{F} = \mathcal{B}([-1/2, 1/2])$, $\mathbf{P}(d\omega) = (1 - 2\omega)\, d\omega$. Compute $\mathbf{E}\{Y|\sigma(X)\}$ and $\mathbf{E}\{Y|X = x\}$

1) for $X(\omega) = \omega^2$;

2) for $X(\omega) = 2\omega\mathbb{1}_{[0,1/2]}(\omega) - 3\omega\mathbb{1}_{[-1/2,0)}(\omega)$.

2.5. Let $\Omega = \{\omega : \omega \in [-1/2, 1/2]\}$, $\mathcal{F} = \mathcal{B}([-1/2, 1/2])$, $\mathbf{P}(d\omega) = d\omega$. Compute $\mathbf{E}\{(\omega + 1/2)^3|\sigma(X)\}$, where $X(\omega) = (|\omega - 1/4| - 1/8)^+$.

2.6. Suppose that the random variables X, Y, and Z have a strictly positive continuous joint density $f(x, y, z)$, $(x, y, z) \in \mathbf{R}^3$. Compute for a bounded measurable function $g(x, y, z)$ the conditional expectation $\mathbf{E}\{g(X, Y, Z)|Y/Z = t\}$.

2.7. Let Y, X, Z be random variables such that $\mathbf{E}|Y| < \infty$ and the σ-algebras $\sigma(Y, X)$ and $\sigma(Z)$ are independent. Here $\sigma(Y, X)$ is the σ-algebra generated by the two random variables Y, X (the smallest σ-algebra containing the events $\{\omega : Y(\omega) \in \Delta_1, X(\omega) \in \Delta_2, \Delta_k \in \mathcal{B}(\mathbf{R})\}$). Prove that

$$\mathbf{E}\{Y|\sigma(X, Z)\} = \mathbf{E}\{Y|\sigma(X)\} \qquad \text{a.s.}$$

2.8. Let $\mathcal{Q} \subset \mathcal{F}$ and Y be an arbitrary $\mathcal{F}$-measurable random variable with finite second moment. Prove that among all $\mathcal{Q}$-measurable random variables X with finite second moments there is an a.s. unique random variable $X_0 = \mathbf{E}\{Y|\mathcal{Q}\}$ that minimizes the distance $(\mathbf{E}(Y - X)^2)^{1/2}$. The random variable $\mathbf{E}\{Y|\mathcal{Q}\}$ can be treated as a projection of Y on the set of $\mathcal{Q}$-measurable random variables.

2.9. Let X_l, $l = 1, 2, \dots, n$, be independent identically distributed random variables with continuous distribution function F. Set $I := \min\{X_1, X_2, \dots, X_n\}$ and $M := \max\{X_1, X_2, \dots, X_n\}$. Prove that if $x < y$, then

$$\mathbf{P}(x \le I, M < y) = (F(y) - F(x))^n,$$

$$\mathbf{P}(I \ge x|M = y) = \Bigl(1 - \frac{F(x)}{F(y)}\Bigr)^{n-1}.$$

2.10. Let X_l, $l = 1, 2, \dots, n$, be independent identically distributed random variables with continuous distribution function F. Set $M := \max\{X_1, X_2, \dots, X_n\}$. Prove that for every k, $1 \le k \le n$,

$$\mathbf{P}(X_k < x|M = y) = \begin{cases} \dfrac{(n-1)F(x)}{nF(y)}, & \text{if} \quad x < y, \\ 1, & \text{if} \quad x \ge y. \end{cases}$$

§ 3. Stochastic processes. Continuity criterion

Let Σ be an arbitrary set. A *stochastic process* is a family

$$X = \{X(t, \omega),\ t \in \Sigma\}$$

of random variables depending on some parameter t. As in the case of random variables, the argument ω in the notation of a process is, as a rule, omitted.

If Σ is a subinterval of the nonnegative real half-line, then the parameter t is referred to as the *time*, and X is called a *continuous time process.*

If Σ is a subset of $\mathbf{R}^k$, then $X(t)$, $t \in \Sigma$, is called a *multiparameter process.* We assume throughout the presentation that Σ is a subset of $[0, \infty)$. If $\Sigma = \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, then X is called a *stochastic sequence.*

For a given sample point $\omega \in \Omega$, the mapping $t \to X(t, \omega)$ is called a *sample path*, or *trajectory*, or *realization* of the process $X(t)$, $t \in \Sigma$.

The sample space Ω can be very complicated, therefore when considering a particular process it is convenient to interpret Ω as the set of sample paths of the process. To every sample point there corresponds a sample path of the process.

Two stochastic processes $X(t)$ and $Y(t)$, $t \in \Sigma$, defined on the same probability space are said to be *stochastically equivalent* or *modifications* of each other if $\mathbf{P}(X(t) = Y(t)) = 1$ for all $t \in \Sigma$.

Two stochastic processes $X(t)$ and $Y(t)$, $t \in \Sigma$, defined on the same probability space are said to be *indistinguishable* or *equivalent* if there exists a set $\Lambda \in \mathcal{F}$ such that $\mathbf{P}(\Lambda) = 0$ and $X(t, \omega) = Y(t, \omega)$ for all $t \in \Sigma$ and $\omega \in \Omega \setminus \Lambda$.

To every stochastic process X there is associated the family of *finite-dimensional distributions*

$$\mathcal{P}_{t_1,t_2,\dots,t_n}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n)$$

$$:= \mathbf{P}(X(t_1) \in \Delta_1, X(t_2) \in \Delta_2, \dots, X(t_n) \in \Delta_n), \qquad \Delta_k \in \mathcal{B}(\mathbf{R}), \tag{3.1}$$

for all $t_k \in \Sigma$, $k = 1, \dots, n$.

For a fixed $(t_1, \dots, t_n)$, the corresponding element of this family is the finite-dimensional distribution of the random vector $(X(t_1), \dots, X(t_n))$.

A stochastic process X is considered to be defined if the family of its finite-dimensional distributions is given. A process with the given family of finite-dimensional distributions is not unique. Such a process can have different path properties and we should choose the process with the best ones.

Obviously, the family of finite-dimensional distributions satisfies the following conditions of *symmetry and consistency*: for any $n \geq 2$, $t_k \in \Sigma$, $\Delta_k \in \mathcal{B}(\mathbf{R})$, $k = 1, \dots, n$,

1) $\mathcal{P}_{t_1,t_2,\dots,t_n}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n) = \mathcal{P}_{t_{l_1},t_{l_2},\dots,t_{l_n}}(\Delta_{l_1} \times \Delta_{l_2} \times \cdots \times \Delta_{l_n})$, where $(l_1, l_2, \dots, l_n)$ is any permutation of $(1, 2, \dots, n)$;

2) $\mathcal{P}_{t_1,\dots,t_{k-1},t_k,t_{k+1},\dots,t_n}(\Delta_1 \times \cdots \times \Delta_{k-1} \times \mathbf{R} \times \Delta_{k+1} \times \dots \Delta_n) =$
$\mathcal{P}_{t_1,\dots,t_{k-1},t_{k+1},\dots,t_n}(\Delta_1 \times \cdots \times \Delta_{k-1} \times \Delta_{k+1} \times \dots \Delta_n)$ for every $1 \leq k \leq n$.

Theorem 3.1 (Kolmogorov). *Let a family of finite-dimensional distributions $\mathcal{P}_{t_1,t_2,\dots,t_n}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n)$, satisfying the symmetry and consistency conditions be given. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a process $X(t)$, $t \in \Sigma$, defined on this space such that its family of finite-dimensional distributions coincide with the given one.*

The proof of the Kolmogorov theorem can be found, for example, in Bulinskii and Shiryaev (2003).

Two stochastic processes taking values in the same state space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, but not necessarily defined on the same probability space, are said to be *identical in law* if they have the same finite-dimensional distributions.

For a process $X(t)$, $t \in \Sigma$, the sets

$$\{\omega : X(t_1, \omega) \in \Delta_1, X(t_2, \omega) \in \Delta_2, \dots, X(t_n, \omega) \in \Delta_n\},$$

$t_k \in \Sigma$, $\Delta_k \in \mathcal{B}(\mathbf{R})$, are called *cylinder sets*. These sets form an algebra. The smallest σ-algebra containing the algebra of cylinder sets is called the *σ-algebra of events generated by the process* X and is denoted by $\mathcal{G} = \sigma\{X(t), t \in \Sigma\}$. Let $\mathcal{G}_a^b = \sigma\{X(t), t \in \Sigma \cap [a, b]\}$ be the σ-algebra of events generated by the process X when the time is varying from a to b.

For a fixed time $t \in \Sigma$, let $\mathcal{G}_0^t$ be the σ-algebra describing the past of the process, $\mathcal{G}_t^\infty$ be the σ-algebra describing the future of the process, and $\mathcal{G}_t^t = \sigma(X(t))$ be the σ-algebra describing the present state of the process.

A family of finite-dimensional distributions determines a probability measure on the algebra of cylinder sets. According to Kolmogorov's theorem (see Karatzas and Shreve (2000) p. 50) this measure can be extended to a probability measure on $\mathcal{G} = \sigma\{X(t), t \in \Sigma\}$. This measure is usually denoted by $\mathbf{P}_X$ and is called the *measure associated with the process* X.

A stochastic process $X(t)$, $t \in [a, b]$, is called *measurable* if the mapping $(t, \omega) \to X(t, \omega)$ is $\mathcal{B}([a, b]) \times \mathcal{F}$-measurable.

A process $X(t)$, $t \in [a, b]$, is said to be *stochastically continuous* or *continuous in probability* if for any $t \in [a, b]$ and $\varepsilon > 0$,

$$\lim_{s \to t} \mathbf{P}(|X(s) - X(t)| > \varepsilon) = 0, \tag{3.2}$$

and *continuous in mean square* if for every $t \in [a, b]$,

$$\lim_{s \to t} \mathbf{E}|X(s) - X(t)|^2 = 0. \tag{3.3}$$

As we have already mentioned, a process is characterized by its finite-dimensional distributions, however, one can provide many examples of processes that have the same finite-dimensional distributions, but the trajectories of which display essentially different properties.

Example 3.1. Let η be a random variable uniformly distributed on $[0, 1]$ and let $f_1(x) \equiv 0$, $f_2(x) = \mathbb{I}_{\{1\}}(x)$, $x \in [0, \infty)$. Consider the processes $X_1(t) = f_1(t + \eta)$ and $X_2(t) = f_2(t + \eta)$, $t \in [0, 1]$. These processes are stochastically equivalent (from the probabilistic point of view they are modifications of each other), since $\mathbf{P}(X_1(t) \neq X_2(t)) = \mathbf{P}(\eta = 1 - t) = 0$ for any $t \in [0, 1]$. However,

$$\mathbf{P}\Big(\sup_{t \in [0,1]} X_1(t) = 0\Big) = \mathbf{P}\Big(\sup_{t \in [0,1]} X_2(t) = 1\Big) = 1,$$

which is rather unnatural.

In order to avoid unnecessary work, we should introduce some restrictions on processes: a property of weak regularity (separability), thanks to which behavior of trajectories of a process is determined by their values on a countable parameter set.

A process $X(t)$, $t \in [a, b]$, is called *separable* if there exists a countable dense subset $\Delta_0 \subset [a, b]$ such that a.s. for every $t \in [a, b] \setminus \Delta_0$ the value $X(t)$ belongs to the set of all limits of values $X(s_n)$ for different subsequences $s_n \in \Delta_0$, $s_n \to t$.

In other words, a trajectory of the process X must have the following property: there exists an event Λ of zero probability such that the graph $(t, X(t, \omega))$, $t \in [0, 1]$, is contained in the closure of the graph $(s, X(s, \omega))$, $s \in \Delta_0$ for any $\omega \in \Omega \setminus \Lambda$.

Lemma 3.1. *Let $X(t)$, $t \in [a,b]$, be a stochastically continuous process. Then there is a separable process $\widetilde{X}(t)$, $t \in [a,b]$, stochastically equivalent to it, taking values in the extended real line $[-\infty,\infty]$. For the set of separability Δ_0 one can take any dense countable subset of $[a,b]$.*

The proof can be found, for example, in Ventzel (1981).

A process $X(t)$, $t \in [a,b]$, is said to be *continuous* or to have a.s. *continuous paths* if there exists a set $\Lambda \in \mathcal{F}$ such that $\mathbf{P}(\Lambda) = 0$ and the mapping $t \to X(t,\omega)$, $t \in [a,b]$, is continuous for all $\omega \in \Omega \setminus \Lambda$.

The following *continuity criterion for stochastic processes* is due to A. N. Kolmogorov.

Theorem 3.2. *Assume that for a process $X(t)$, $t \in [a,b]$, there exist positive constants α, β, and M such that*

$$\mathbf{E}|X(t) - X(s)|^\alpha \le M|t-s|^{1+\beta} \qquad \text{for any } s,t \in [a,b]. \tag{3.4}$$

Then the process X has a continuous modification $\widetilde{X}$.

For any $0 < \gamma < \beta/\alpha$ the sample paths of the process $\widetilde{X}(t)$, $t \in [a,b]$, satisfy a.s. *the Hölder condition*

$$|\widetilde{X}(t) - \widetilde{X}(s)| \le L_\gamma |t-s|^\gamma, \tag{3.5}$$

where L_γ is a random coefficient independent of s and t.

Proof. The stochastic continuity of X follows from condition (3.4). Indeed, by the Chebyshev inequality, for any $\varepsilon > 0$

$$\mathbf{P}(|X(t) - X(s)| \ge \varepsilon) \le \frac{1}{\varepsilon^\alpha}\mathbf{E}|X(t) - X(s)|^\alpha \le \frac{M}{\varepsilon^\alpha}|t-s|^{1+\beta}. \tag{3.6}$$

This implies the stochastic continuity (3.2).

Choose an arbitrary $\gamma \in (0,\beta/\alpha)$. Consider the dyadic rational (binary rational) points $k/2^n$ from the interval $[a,b]$. Using (3.6), for any neighboring dyadic rational points (whose difference is $1/2^n$) we have

$$\mathbf{P}\Big(\Big|X\Big(\frac{k+1}{2^n}\Big) - X\Big(\frac{k}{2^n}\Big)\Big| \ge 2^{-n\gamma}\Big) \le \frac{M}{2^{-\gamma\alpha n}} 2^{-n(1+\beta)} = M2^{-n}2^{-n(\beta-\alpha\gamma)}.$$

Then for any fixed n

$$\begin{aligned}
&\mathbf{P}\Big(\max_{a2^n \le k \le b2^n - 1}\Big|X\Big(\frac{k+1}{2^n}\Big) - X\Big(\frac{k}{2^n}\Big)\Big| \ge 2^{-n\gamma}\Big) \\
&\quad = \mathbf{P}\Big(\bigcup_{a2^n \le k \le b2^n - 1}\Big\{\Big|X\Big(\frac{k+1}{2^n}\Big) - X\Big(\frac{k}{2^n}\Big)\Big| \ge 2^{-n\gamma}\Big\}\Big) \\
&\quad \le \sum_{a2^n \le k \le b2^n - 1} \mathbf{P}\Big(\Big|X\Big(\frac{k+1}{2^n}\Big) - X\Big(\frac{k}{2^n}\Big)\Big| \ge 2^{-n\gamma}\Big) \\
&\quad \le (b-a)2^n M 2^{-n} 2^{-n(\beta-\alpha\gamma)} = M(b-a)2^{-n(\beta-\alpha\gamma)}.
\end{aligned}$$

Since the series of these probabilities converges, the first part of the Borel–Cantelli lemma, shows that there exists a.s. a number $n_0 = n_0(\omega)$ such that

$$\max_{a \le k/2^n < (k+1)/2^n \le b} \left| X\left(\frac{k+1}{2^n}\right) - X\left(\frac{k}{2^n}\right) \right| < 2^{-n\gamma} \tag{3.7}$$

for all $n \ge n_0$. This estimate can be considered as the analogue of (3.5) for neighboring dyadic points.

We establish now the analogue of (3.5) for all dyadic rational points that are close enough to one another. For any dyadic rational points $s < t$ satisfying the condition $t - s < 2^{-n_0}$, there exists $m \ge n_0$ such that $2^{-m-1} \le t - s < 2^{-m}$. Then it is possible to represent the points s and t in the form

$$s = k2^{-m} - 2^{-l_1} - 2^{-l_2} - \dots - 2^{-l_\mu}, \qquad m < l_1 < l_2 < \dots < l_\mu,$$
$$t = k2^{-m} \mp 2^{-v_1} \mp 2^{-v_2} \mp \dots \mp 2^{-v_\eta}, \qquad m < v_1 < v_2 < \dots < v_\eta.$$

There are finitely many terms in these representations. Moreover, in the second formula one should take the minus signs if the points s and t lie in the same interval $((k-1)2^{-m}, k2^{-m}]$, and take the plus signs if they lie in adjoining intervals. For the sake of definiteness we take the plus signs.

The following inequalities hold:

$$|X(t) - X(s)| \le |X(k2^{-m}) - X(s)| + |X(t) - X(k2^{-m})|$$
$$\le |X(k2^{-m}) - X(k2^{-m} - 2^{-l_1})| + |X(k2^{-m} - 2^{-l_1}) - X(s)|$$
$$+ |X(k2^{-m}) - X(k2^{-m} + 2^{-v_1})| + |X(k2^{-m} + 2^{-v_1}) - X(t)|$$
$$\le |X(k2^{-m}) - X(k2^{-m} - 2^{-l_1})| + |X(k2^{-m} - 2^{-l_1}) - X(k2^{-m} - 2^{-l_1} - 2^{-l_2})| + \cdots$$
$$+ |X(k2^{-m}) - X(k2^{-m} + 2^{-v_1})| + |X(k2^{-m} + 2^{-v_1}) - X(k2^{-m} + 2^{-v_1} + 2^{-v_2})| + \cdots .$$

On the right-hand side of these inequalities there are the increments of the process X in the neighboring dyadic rational points, and for them inequality (3.7) holds. Therefore,

$$|X(t) - X(s)| \le \sum_{l_j > m} 2^{-\gamma l_j} + \sum_{v_j > m} 2^{-\gamma v_j}$$

$$\le 2 \sum_{l=m+1}^{\infty} 2^{-\gamma l} = \frac{2^{1-\gamma(m+1)}}{1 - 2^{-\gamma}} \le \frac{2}{1 - 2^{-\gamma}} |t - s|^{\gamma}. \tag{3.8}$$

This means that the process X is uniformly continuous on the set of dyadic rational points of the interval $[a, b]$ and on this set it satisfies the Hölder condition with the parameter γ.

In view of (3.8), due to Cauchy's criterion, the process X can be extended by continuity from the set D of dyadic rational points to the whole interval $[a, b]$. For any $t \in [a, b]$ we set $\widetilde{X}(t) := \lim_{s \to t,\ s \in D} X(s)$. It is clear that a.s. for all $|t - s| < 2^{-n_0}$,

$$|\widetilde{X}(t) - \widetilde{X}(s)| \le \frac{2}{1 - 2^{-\gamma}} |t - s|^{\gamma}. \tag{3.9}$$

Next we verify that the process $\widetilde{X}$ is a modification of X, i.e., $\mathbf{P}(\widetilde{X}(t) = X(t)) = 1$ for every $t \in [a, b]$.

Since the process X is stochastically continuous, we have $X(s) \to X(t)$ as $s \to t$ in probability. According to Proposition 1.1 there exists a subsequence $s_n \in D$, $s_n \to t$, such that $X(t) = \lim_{s_n \to t} X(s_n)$ a.s. However, by the definition of the process $\widetilde{X}$, this limit is equal to $\widetilde{X}(t)$. Hence the processes $\widetilde{X}$ and X are modifications of each other.

To prove (3.5), we apply estimate (3.9). For arbitrary s, t we have

$$|\widetilde{X}(t) - \widetilde{X}(s)| = |\widetilde{X}(t) - \widetilde{X}(s)| \mathbb{1}_{\{|t-s|<2^{-n_0}\}}(\omega) + |\widetilde{X}(t) - \widetilde{X}(s)| \mathbb{1}_{\{|t-s|\geq 2^{-n_0}\}}(\omega)$$

$$\leq \frac{2|t-s|^\gamma}{1-2^{-\gamma}} + |\widetilde{X}(t) - \widetilde{X}(s)| \mathbb{1}_{\{1 \leq (|t-s|2^{n_0})^\gamma\}}(\omega)$$

$$\leq \frac{2|t-s|^\gamma}{1-2^{-\gamma}} + 2^{n_0\gamma}|t-s|^\gamma 2 \max_{t \in [a,b]} |\widetilde{X}(t)| = L_\gamma |t-s|^\gamma,$$

where $L_\gamma = \dfrac{2}{1-2^{-\gamma}} + 2^{1+n_0\gamma} \max_{t \in [a,b]} |\widetilde{X}(t)|$. □

Sometimes we consider a *multiparameter stochastic process*, where $\Sigma = \prod_{k=1}^{r} [a_k, b_k]$. As in the one-parameter case, a process $X(\vec{t})$, $\vec{t} \in \prod_{k=1}^{r} [a_k, b_k]$, is said to be *continuous* if there exists a set $\Lambda \in \mathcal{F}$ such that $\mathbf{P}(\Lambda) = 0$, and the mapping $\vec{t} \to X(\vec{t}, \omega)$, $\vec{t} \in \prod_{k=1}^{r} [a_k, b_k]$, is continuous for all $\omega \in \Omega \setminus \Lambda$.

It is useful to have a similar continuity criterion for multiparameter stochastic processes.

Theorem 3.3. *Let $X(\vec{t})$, $\vec{t} \in \prod_{k=1}^{r} [a_k, b_k]$, be a multiparameter stochastic process. Assume that there exist positive constants α, β, and M such that for any $\vec{v}, \vec{u} \in \prod_{k=1}^{r} [a_k, b_k]$,*

$$\mathbf{E}|X(\vec{u}) - X(\vec{v})|^\alpha \leq M |\vec{u} - \vec{v}|^{r+\beta}. \tag{3.10}$$

Then the process $X(\vec{t})$, $\vec{t} \in \prod_{k=1}^{r} [a_k, b_k]$, has a continuous modification $\widetilde{X}(\vec{t})$.

For any $0 < \gamma < \beta/\alpha$ sample paths of the process $\widetilde{X}(\vec{t})$ a.s. satisfy the Hölder condition

$$|\widetilde{X}(\vec{u}) - \widetilde{X}(\vec{v})| \leq L_\gamma |\vec{u} - \vec{v}|^\gamma, \tag{3.11}$$

where L_γ is a random coefficient independent of $\vec{v}$ and $\vec{u}$.

The proof of this result is analogous to that of Theorem 3.2.

§ 4. Stopping times

A family of σ-algebras $\{\mathcal{F}_t, t \in \Sigma\}$ on $(\Omega, \mathcal{F})$ is called a *filtration* if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \text{ for every } s \le t, \quad s, t \in \Sigma.$$

For $\Sigma = [0, T]$, a filtration is said to be *right continuous* if for every $t \in [0, T)$

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

The collection $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is called a *filtered probability space.* It is said to satisfy the *usual conditions* if

1) $\mathcal{F}$ is $\mathbf{P}$-*complete*,

2) $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$,

3) $\{\mathcal{F}_t\}$ is right continuous.

Condition 1) means that if for set A there exists A_1 and A_2 in $\mathcal{F}$ such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}(A_1) = \mathbf{P}(A_2)$, then A belongs to $\mathcal{F}$.

We say that a process $X(t)$, $t \in \Sigma$, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, is *adapted* to the filtration $\{\mathcal{F}_t, t \in \Sigma\}$ if for every $t \in \Sigma$ the random variable $X(t)$ is $\mathcal{F}_t$-measurable.

Notice that $X(t)$, $t \in \Sigma$, is always adapted to its *natural filtration* $\mathcal{G}_0^t := \sigma\{X(s), s \in \Sigma\}$, which is assumed to satisfy the usual conditions.

A stochastic process $X(t)$, $t \in [0, T]$, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, is said to be *progressively measurable* if for every $t \in [0, T]$ the mapping $(s, \omega) \to X(s, \omega)$ from $[0, t] \times \Omega$ to $\mathbf{R}$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable, i.e., for every $t \in [0, T]$ and any Borel set Δ on the real line, the inclusion $\{(s, \omega) : X(s, \omega) \in \Delta, s \le t\} \in \mathcal{B}([0, t]) \times \mathcal{F}_t$ holds, where $\mathcal{B}([0, t])$ is the Borel σ-algebra on $[0, t]$.

In other words, if the restriction of X to the interval $[0, t]$ is a $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable process for every $t \in [0, T]$, then it is progressively measurable.

A progressively measurable process is clearly adapted to the filtration.

Proposition 4.1. *An adapted process with right or left continuous paths is progressively measurable.*

Proof. Let $X(t)$, $t \in [0, T]$, be a right continuous process. We fix t in the interval $(0, T]$. Set $X_n(s) := X(kt/2^n)$ for $s \in ((k-1)t/2^n, kt/2^n]$, $k = 1, 2, \dots$, $X_n(0) = X(0)$. For every $n \in \mathbb{N}$ the process $X_n(s)$, $s \in [0, t]$, is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable. Since $X(s) = \lim\limits_{n \to \infty} X_n(s)$, $s \in [0, t]$, and the limit of measurable functions is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable, we see that the process X is progressively measurable. □

Progressive measurability is very important. It, for example, guarantees, the validity of the following property: for a progressively measurable integrable process

X the integral $\int_0^t X(s)\,ds$, $t \in [0,T]$, regarded as a process with respect to t, is also progressively measurable.

A stochastic process can be considered not only at deterministic moments, but also at some random times. We consider an important class of random times.

A *stopping time* with respect to a filtration $\{\mathcal{F}_t, t \in \Sigma \subseteq [0,\infty)\}$ is a mapping $\tau : \Omega \to \Sigma \cup \{\infty\}$ such that $\{\tau \le t\} \in \mathcal{F}_t$ for every $t \in \Sigma$.

For $\Sigma = 1, 2, \dots$ and $\Delta \in \mathcal{B}(\mathbf{R})$, the moment $\tau := \min\{k : X(k) \in \Delta\}$ is a stopping time with respect to the natural filtration $\{\mathcal{G}_0^k\}$, $k \in \mathbb{N}$. Indeed,

$$\{\tau \le k\} = \{\tau = 1\}\bigcup\{\tau = 2\}\bigcup \dots \bigcup\{\tau = k\}$$

$$= \{X(1) \in \Delta\} \bigcup_{l=2}^{k} \{X(1) \notin \Delta, \dots, X(l-1) \notin \Delta, X(l) \in \Delta\},$$

and it is clear that this set is $\mathcal{G}_0^k$-measurable.

Given a stopping time τ with respect to the filtration $\{\mathcal{F}_t, t \in \Sigma \subseteq [0,\infty)\}$, the collection of sets

$$\mathcal{F}_\tau := \{A : A\bigcap\{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \in \Sigma\}$$

is a σ-algebra, as one can easily verify.

For the natural filtration $\mathcal{G}_0^t = \sigma\{X(s), s \in \Sigma \cap [0,t]\}$ the σ-algebra $\mathcal{G}_0^\tau$ can be interpreted as σ-algebra of events generated by the process X up to the time τ.

The following *Galmarino's criterion* characterizes stopping times. The proof of this criterion can be found in Itô and McKean (1965).

A measurable with respect to $\mathcal{G}_0^\infty$ functional $\tau(\omega) := \tau(X(s,\omega), s \in \Sigma \subseteq [0,\infty))$, taking values in $\Sigma\bigcup\{\infty\}$, is a stopping time with respect to the filtration $\{\mathcal{G}_0^t\}$ iff for every fixed $t \in \Sigma$ and any sample points ω, $\tilde\omega$, the conditions $X(s,\omega) = X(s,\tilde\omega)$ for all $s \in \Sigma \cap [0,t]$, $\tau(\omega) \le t$ imply $\tau(\omega) = \tau(\tilde\omega)$.

In other words, if for every sample path of X considered up to any fixed time t one can say whether the moment τ occurs or not, then τ is a stopping time.

According to this criterion the *first passage time* $\tau := \min\{s : X(s) \ge b\}$ is a stopping time.

The *last exit time* at zero, i.e., the moment $\rho := \max\{s : X(s) = 0\}$, is not a stopping time. This is due to the fact that the behavior of a sample path up to the time t cannot determine this moment as the last visit to zero point.

Examples of stopping times.

The *first hitting time* of a level z is defined as follows $H_z := \min\{s : X(s) = z\}$.

A significant role for different applications plays the moment

$$H_{a,b} := \min\{s : X(s) \notin (a,b)\},$$

which is the *first exit time* from the interval (a, b).

A process X can be stopped at the *moment inverse of integral functional.* This moment is defined by the formula

$$\nu(t) := \min\Big\{s : \int_0^s g(X(v))\,dv = t\Big\},$$

where g is a nonnegative measurable function.

Let

$$\theta_v = \min\Big\{t : \sup_{0\le s\le t} X(s) - \inf_{0\le s\le t} X(s) \ge v\Big\}$$

be the first moment at which *the range* of X reaches a given value $v > 0$. The moment θ_v is called the *inverse range time* of the process X.

Properties of stopping times.

1. If τ is a stopping time, then $\{\tau < t\} \in \mathcal{F}_t$ and $\{\tau = t\} \in \mathcal{F}_t$.
Indeed,

$$\{\tau < t\} = \bigcup_{k=1}^{\infty} \{\tau \le t - \tfrac{1}{k}\}, \qquad \{\tau \le t - \tfrac{1}{k}\} \in \mathcal{F}_{t-1/k} \subseteq \mathcal{F}_t,$$

and $\{\tau = t\} = \{\tau \le t\} \setminus \{\tau < t\}$.

2. If t_0 is a nonnegative constant, then $\tau = t_0$ is a stopping time.

3. If τ is a stopping time, then $\tau + t_0$ is a stopping time for a nonnegative constant t_0, since $\{\tau + t_0 \le t\} = \{\tau \le t - t_0\} \in \mathcal{F}_{t-t_0} \subseteq \mathcal{F}_t$.

4. If σ and τ are stopping times, then $\sigma \vee \tau := \max\{\sigma, \tau\}$ is a stopping time.
Indeed, $\{\sigma \vee \tau \le t\} = \{\sigma \le t\} \cap \{\tau \le t\} \in \mathcal{F}_t$.

5 If σ and τ are stopping times, then $\sigma \wedge \tau := \min\{\sigma, \tau\}$ is a stopping time.
Indeed, $\{\sigma \wedge \tau \le t\} = \{\sigma \le t\} \cup \{\tau \le t\} \in \mathcal{F}_t$.

6. If τ_n, $n \in \mathbb{N}$, are stopping times, then $\inf_{n\in\mathbb{N}} \tau_n$ and $\sup_{n\in\mathbb{N}} \tau_n$ are stopping times.
Indeed, $\Big\{\sup_{n\in\mathbb{N}} \tau_n \le t\Big\} = \bigcap_{n\in\mathbb{N}} \{\tau_n \le t\} \in \mathcal{F}_t$,

$$\Big\{\inf_{n\in\mathbb{N}} \tau_n \le t\Big\} = \bigcap_{m\in\mathbb{N}} \Big\{\inf_{n\in\mathbb{N}} \tau_n < t + \frac{1}{m}\Big\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \Big\{\tau_n < t + \frac{1}{m}\Big\} \in \bigcap_{m\in\mathbb{N}} \mathcal{F}_{t+1/m} = \mathcal{F}_t.$$

Here we used the fact that the filtration $\{\mathcal{F}_t\}$ is right continuous.

7. If τ_n, $n \in \mathbb{N}$, are stopping times, then $\liminf_n \tau_n$ and $\limsup_n \tau_n$ are stopping times.
Indeed, $\limsup_n \tau_n = \inf_{m\in\mathbb{N}} \sup_{n\ge m} \tau_n$, and $\liminf_n \tau_n = \sup_{m\in\mathbb{N}} \inf_{n\ge m} \tau_n$.

8. A stopping time τ is an $\mathcal{F}_\tau$-measurable random variable.
Set $A := \{\tau \le s\}$. Then for arbitrary $t \in \Sigma$,

$$A \cap \{\tau \le t\} = \{\tau \le s \wedge t\} \in \mathcal{F}_{s\wedge t} \subseteq \mathcal{F}_t.$$

By the definition of $\mathcal{F}_\tau$, we have $A \in \mathcal{F}_\tau$.

9. If σ and τ are stopping times such that $\sigma \le \tau$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
If $A \in \mathcal{F}_\sigma$, we have $A \cap \{\sigma \le t\} \in \mathcal{F}_t$ for every $t \in \Sigma$. Since $\sigma \le \tau$, one has the inclusion $\{\tau \le t\} \subseteq \{\sigma \le t\}$, and

$$A\bigcap\{\tau \le t\} = A\bigcap\{\tau \le t\}\bigcap\{\sigma \le t\} = \big\{A\bigcap\{\sigma \le t\}\big\}\bigcap\{\tau \le t\} \in \mathcal{F}_t.$$

By the definition of $\mathcal{F}_\tau$, we have $A \in \mathcal{F}_\tau$.

Proposition 4.2. *Let $X(t)$, $t \in [0,\infty)$, be a stochastic process progressively measurable with respect to the filtration $\{\mathcal{F}_t, t \in [0,\infty)\}$, and τ be a stopping time with respect to this filtration. Then the random variable $X(\tau)$ is $\mathcal{F}_\tau$-measurable.*

Proof. For any $t \in [0,\infty)$ and an arbitrary Borel sets Δ, consider the event $Q := \{X(\tau) \in \Delta\}\bigcap\{\tau \le t\}$. Obviously, $Q = \{X(\tau \wedge t) \in \Delta\}\bigcap\{\tau \le t\}$. The random variable $X(\tau \wedge t)$ is $\mathcal{F}_t$-measurable as a superposition of measurable mappings: $\omega \to (\tau(\omega) \wedge t, \omega)$ from $(\Omega, \mathcal{F}_t)$ to $([0,t] \times \Omega, \mathcal{B}([0,t]) \times \mathcal{F}_t)$ and $(s,\omega) \to X(s,\omega)$ from $([0,t] \times \Omega, \mathcal{B}([0,t]) \times \mathcal{F}_t)$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. Now, since the events $\{X(\tau \wedge t) \in \Delta\}$ and $\{\tau \le t\}$ are $\mathcal{F}_t$-measurable, the event Q is $\mathcal{F}_t$-measurable. By the definition of the σ-algebra $\mathcal{F}_\tau$, we have $\{X(\tau) \in \Delta\} \in \mathcal{F}_\tau$ and this means that the random variable $X(\tau)$ is $\mathcal{F}_\tau$-measurable. □

Corollary 4.1. *Let $X(t)$, $t \in [0,\infty)$, be a right continuous stochastic process and τ be a stopping time with respect to the natural filtration $\{\mathcal{G}_0^t\}$. Then the random variable $X(\tau)$ is $\mathcal{G}_0^\tau$-measurable.*

Indeed, by Proposition 4.1, a right continuous stochastic process is progressively measurable with respect to the natural filtration.

§ 5. Martingales

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let Σ be a subset of the nonnegative integers or a subset of the nonnegative real line. Let $\{\mathcal{F}_t\}_{t\in\Sigma}$ be a family of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for every $s \le t$, $s,t \in \Sigma$.

A stochastic process $X(t)$, $t \in \Sigma$, is called a *martingale* (respectively, *supermartingale*, *submartingale*) with respect to the filtration $\{\mathcal{F}_t\}_{t\in\Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,

2) for every $t \in \Sigma$ the random variable $X(t)$ is $\mathcal{F}_t$-measurable,

3) $\mathbf{E}\{X(t)|\mathcal{F}_s\} = X(s)$ (respectively, $\mathbf{E}\{X(t)|\mathcal{F}_s\} \le X(s)$, $\mathbf{E}\{X(t)|\mathcal{F}_s\} \ge X(s)$) a.s. for every pair $s,t \in \Sigma$ such that $s < t$.

A martingale, supermartingale or submartingale will be usually denoted by appending the filtration, i.e., $(X(t),\mathcal{F}_t)$, $t\in\Sigma$.

Examples of martingales.

Let η_l, $l=1,2,\dots$, be independent identically distributed random variables and $\mathcal{F}_k=\sigma(\eta_l, 1\le l\le k)$ be the σ-algebra of events generated by these variables.

1) Assume that $\mathbf{E}\eta_1=0$. Then the process $X(k):=\sum\limits_{l=1}^{k}\eta_l$, $k=1,2,\dots$, is a martingale with respect to the σ-algebras $\{\mathcal{F}_k\}_{k=1}^{\infty}$.

Indeed, for $1\le m<k$

$$\mathbf{E}\big\{X(k)\big|\mathcal{F}_m\big\}=\mathbf{E}\Big\{X(m)+\sum_{l=m+1}^{k}\eta_l\Big|\mathcal{F}_m\Big\}=X(m)+\sum_{l=m+1}^{k}\mathbf{E}\eta_l=X(m).$$

Note that for the cases $\mathbf{E}\eta_1<0$ and $\mathbf{E}\eta_1>0$ the process $X(k)$, $k=1,2,\dots$, is a supermartingale and a submartingale, respectively.

2) Let $\mathbf{E}\eta_1=0$, $\mathbf{E}\eta_1^2=\sigma^2<\infty$. Then the process $Y(k):=\big(\sum\limits_{l=1}^{k}\eta_l\big)^2-k\sigma^2$, $k=1,2,\dots$, is a martingale with respect to the σ-algebras $\{\mathcal{F}_k\}_{k=1}^{\infty}$.

Indeed, for arbitrary $m\ge 1$

$$\mathbf{E}\big\{Y(m+1)\big|\mathcal{F}_m\big\}=\mathbf{E}\Big\{Y(m)+2\eta_{m+1}\sum_{l=1}^{m}\eta_l+\eta_{m+1}^2-\sigma^2\Big|\mathcal{F}_m\Big\}$$

$$=Y(m)+2\sum_{l=1}^{m}\eta_l\,\mathbf{E}\eta_{m+1}+\sigma^2-\sigma^2=Y(m).$$

Note that, by the 4th property of conditional expectations, the equalities

$$\mathbf{E}\big\{Y(m+1)\big|\mathcal{F}_m\big\}=Y(m),\qquad m=1,2,\dots,$$

imply $\mathbf{E}\big\{Y(k)\big|\mathcal{F}_m\big\}=Y(m)$ for arbitrary integers $1\le m<k$.

3) Let $\varphi(\alpha)=\mathbf{E}e^{i\alpha\eta_1}$, $\alpha\in\mathbf{R}$, be the characteristic function of the random variable η_1. Then the process $Z(k):=\dfrac{1}{\varphi^k(\alpha)}\exp\Big(i\alpha\sum\limits_{l=1}^{k}\eta_l\Big)$, $k=1,2,\dots$, is a martingale with respect to the σ-algebras $\{\mathcal{F}_k\}_{k=1}^{\infty}$.

Indeed, for arbitrary $m\ge 1$

$$\mathbf{E}\big\{Z(m+1)\big|\mathcal{F}_m\big\}=\frac{1}{\varphi^{m+1}(\alpha)}\exp\Big(i\alpha\sum_{l=1}^{m}\eta_l\Big)\mathbf{E}e^{i\alpha\eta_{m+1}}=Z(m).$$

4) Let η_l, $l=1,2,\dots$, be Bernoulli's random variables such that $\mathbf{P}(\eta_1=1)=p$, $\mathbf{P}(\eta_1=-1)=1-p$. Then the process $U(k):=\Big(\dfrac{1-p}{p}\Big)^{\sum_{l=1}^{k}\eta_l}$, $k=1,2,\dots$, is a martingale with respect to the σ-algebras $\{\mathcal{F}_k\}_{k=1}^{\infty}$.

Indeed, for any $m \geq 1$

$$\mathbf{E}\big\{U(m+1)\big|\mathcal{F}_m\big\} = U(m)\mathbf{E}\Big(\frac{1-p}{p}\Big)^{\eta_1} = U(m).$$

5) Let $f(x)>0$, $x\in\mathbf{R}$, be the density of η_1 and, accordingly, $f(x_1)f(x_2)\cdots f(x_k)$ be the joint density of the i.i.d. random variables η_l, $l = 1, 2, \dots, k$. Let g be some other density. In the theory of hypothesis testing there is the process of likelihood ratio

$$V(k) := \frac{g(\eta_1)g(\eta_2)\cdots g(\eta_k)}{f(\eta_1)f(\eta_2)\cdots f(\eta_k)}, \qquad k = 1, 2, \dots.$$

This process is a martingale with respect to the σ-algebras $\{\mathcal{F}_k\}_{k=1}^{\infty}$.

Indeed, for arbitrary $m \geq 1$

$$\mathbf{E}\big\{V(m+1)\big|\mathcal{F}_m\big\} = V(m)\mathbf{E}\frac{g(\eta_{m+1})}{f(\eta_{m+1})} = V(m)\int\limits_{\mathbf{R}} \frac{g(x)}{f(x)}f(x)\,dx = V(m).$$

6) Let X be a random variable with $\mathbf{E}|X| < \infty$ and $\{\mathcal{F}_t, t \in \Sigma\}$ be an arbitrary filtration. Then the process $X(t) := \mathbf{E}\{X|\mathcal{F}_t\}$, $t \in \Sigma$, is a martingale. This is a simple consequence of the 4th property of conditional expectations: $\mathbf{E}\{\mathbf{E}\{X|\mathcal{F}_t\}|\mathcal{F}_s\} = \mathbf{E}\{X|\mathcal{F}_s\}$ for any $s < t$.

The definition of martingales (supermartingales or submartingales) can be based on corresponding integral relations instead of conditional expectations.

Assume that the random variables μ and η have finite moments $\mathbf{E}|\mu| < \infty$ and $\mathbf{E}|\eta| < \infty$. Let the random variable μ be measurable with respect to a σ-algebra $\mathcal{Q} \subseteq \mathcal{F}$. Then the inequality

$$\mu \geq \mathbf{E}\{\eta|\mathcal{Q}\} \qquad \big(\text{resp. } \mu \leq \mathbf{E}\{\eta|\mathcal{Q}\}\big) \qquad \text{a.s.} \tag{5.1}$$

is equivalent to

$$\int\limits_B \mu\,d\mathbf{P} \geq \int\limits_B \eta\,d\mathbf{P} \qquad \Big(\text{resp. } \int\limits_B \mu\,d\mathbf{P} \leq \int\limits_B \eta\,d\mathbf{P}\Big) \tag{5.2}$$

for every set $B \in \mathcal{Q}$. An analogous statement holds for equality.

This is the consequence of the definition of conditional expectation given the σ-algebra $\mathcal{Q}$ (see (2.10)): $\mathbf{E}\{\eta|\mathcal{Q}\}$ is the $\mathcal{Q}$-measurable random variable such that

$$\int\limits_B \mathbf{E}\{\eta|\mathcal{Q}\}d\mathbf{P} = \int\limits_B \eta d\mathbf{P}$$

for every $B \in \mathcal{Q}$.

1. A random time change.

Theorem 5.1. *Let* $(X(k), \mathcal{F}_k)$, $k = 1, 2, \dots,$ *be a supermartingale. Assume that* σ *and* τ *are two integer-valued bounded stopping times with respect to* $\{\mathcal{F}_k\}_{k=1}^{\infty}$ *such that*

$$1 \le \sigma(\omega) \le \tau(\omega) \le n \qquad \text{for almost all } \omega \in \Omega$$

and some integer n*. Then* $X(\sigma)$ *is* $\mathcal{F}_\sigma$*-measurable and*

$$\mathbf{E}\{X(\tau)|\mathcal{F}_\sigma\} \le X(\sigma) \qquad \text{a.s.} \tag{5.3}$$

Proof. We note first that $\mathbf{E}|X(\sigma)| < \infty$ and $\mathbf{E}|X(\tau)| < \infty$. Indeed,

$$\mathbf{E}|X(\tau)| = \sum_{k=1}^{n} \int_{\{\tau=k\}} |X(\tau)|\, d\mathbf{P} = \sum_{k=1}^{n} \int_{\{\tau=k\}} |X(k)|\, d\mathbf{P} \le \sum_{k=1}^{n} \mathbf{E}|X(k)| < \infty.$$

Since for any Borel set B and arbitrary $k = 1, 2, \dots,$

$$\{X(\sigma) \in B\} \bigcap \{\sigma \le k\} = \sum_{l=1}^{k} \big(\{X(l) \in B\} \bigcap \{\sigma = l\}\big) \in \mathcal{F}_k,$$

the variable $X(\sigma)$ is $\mathcal{F}_\sigma$-measurable.

Thanks to the equivalence of (5.1) and (5.2), it suffices to show that for any $B \in \mathcal{F}_\sigma$

$$\int_B X(\sigma)\, d\mathbf{P} \ge \int_B X(\tau)\, d\mathbf{P}. \tag{5.4}$$

We prove first this inequality for the case $\tau - \sigma \le 1$. We have

$$\int_B (X(\sigma) - X(\tau))\, d\mathbf{P} = \sum_{k=1}^{n} \int_{B \cap \{\sigma=k\} \cap \{\tau>\sigma\}} (X(\sigma) - X(\tau))\, d\mathbf{P}$$

$$= \sum_{k=1}^{n} \int_{B \cap \{\sigma=k\} \cap \{\tau>k\}} (X(k) - X(k+1))\, d\mathbf{P}. \tag{5.5}$$

Since τ is a stopping time, $\{\tau > k\} = \Omega \setminus \{\tau \le k\} \in \mathcal{F}_k$ for every $k = 1, 2, \dots, n$. By the definition of the σ-algebra $\mathcal{F}_\sigma$ (see § 3),

$$B \bigcap \{\sigma = k\} = \{B \bigcap \{\sigma \le k\}\} \setminus \{B \bigcap \{\sigma \le k-1\}\} \in \mathcal{F}_k.$$

Thus $B \bigcap \{\sigma = k\} \bigcap \{\tau > k\} \in \mathcal{F}_k$ and from the definition of a supermartingale and (5.2) it follows that

$$\int_{B \cap \{\sigma=k\} \cap \{\tau>k\}} (X(k) - X(k+1))\, d\mathbf{P} \ge 0.$$

As a result, we have that (5.5) implies (5.4) for the special case $\tau - \sigma \le 1$.

Set $\tau_k := \tau \wedge (\sigma + k)$, $k = 0, 1, \dots, n$. It is clear that $\tau_0 = \sigma \le \tau_1 \le \dots \le \tau_n = \tau$ and $\tau_{k+1} - \tau_k \le 1$. From the properties of stopping times it follows that τ_k is again a stopping time. Now we can apply n times the inequality (5.4) for the special case. Finally, we have

$$\int_B X(\sigma)\,d\mathbf{P} \ge \int_B X(\tau_1)\,d\mathbf{P} \ge \int_B X(\tau_2)\,d\mathbf{P} \ge \dots \ge \int_B X(\tau_n)\,d\mathbf{P} = \int_B X(\tau)\,d\mathbf{P}.$$

This proves (5.4) in the general case. □

Theorem 5.1 implies the following result.

Theorem 5.2 (random time change). *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 1, \dots, n$, be a supermartingale and $1 \le \tau_1 \le \tau_2 \le \dots \le \tau_m$ be stopping times with respect to $\{\mathcal{F}_k\}_{k=1}^n$, having the state space $\{1, 2, \dots, n\}$. Then the sequence $X(\tau_l)$, $l = 1, 2, \dots, m$, is a supermartingale with respect to the filtration $\{\mathcal{F}_{\tau_l}\}_{l=1}^m$.*

Indeed, according to the 9th property of stopping times, $\mathcal{F}_{\tau_l} \subseteq \mathcal{F}_{\tau_j}$ for $l < j$. Therefore $\{\mathcal{F}_{\tau_l}\}$, $l = 1, 2, \dots, m$, is a filtration. Moreover, from (5.3) one obtains the supermartingale inequality

$$\mathbf{E}\{X(\tau_j)|\mathcal{F}_{\tau_l}\} \le X(\tau_l) \qquad \text{a.s.}$$

Corollary 5.1. *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 1, \dots, n$, be a supermartingale and $1 \le \rho \le n$ be an integer-valued stopping time with respect to $\{\mathcal{F}_k\}_{k=1}^n$. Then*

$$\mathbf{E}X(1) \ge \mathbf{E}X(\rho) \ge \mathbf{E}X(n). \tag{5.6}$$

This is a consequence of (5.4) for $B = \Omega$ and for the stopping times $\sigma = 1$, $\tau = \rho$ or $\sigma = \rho$, $\tau = n$, respectively.

Remark 5.1. If $\big(X(k), \mathcal{F}_k\big)$, $k = 1, 2, \dots$, is a submartingale, then $\big(-X(k), \mathcal{F}_k\big)$ is a supermartingale. Consequently, for a submartingale there are results analogous to Theorems 5.1, 5.2, and Corollary 5.1 with opposite inequalities in (5.3) and (5.6). For a martingale $\big(X(k), \mathcal{F}_k\big)$ the inequalities (5.3) and (5.6) must be replaced by equalities.

Thus for a martingale $\big(X(k), \mathcal{F}_k\big)$, $k = 1, 2, \dots$, we have

$$\mathbf{E}X(1) = \mathbf{E}X(\rho) = \mathbf{E}X(n).$$

Example 5.1. Set $X(k) := \sum_{j=1}^{k} \eta_j$, where η_j, $j = 1, 2, \dots$, are independent random variables with $\mathbf{E}\eta_1 = 0$. Let $\tau_l := \min\{k : X(k) \ge l\}$, $l = 1, 2, \dots, m$, be the first passage times. It is clear that $1 \le \tau_1 \le \tau_2 \le \dots \le \tau_m$. Then the sequence $X(\tau_l \wedge n)$, $l = 1, 2, \dots, m$, is a martingale with respect to the natural filtration.

2. Martingale inequalities.

Theorem 5.3. *Let* $\big(X(k),\mathcal{F}_k\big)$, $k=1,2,\dots,n$, *be a nonnegative submartingale. Then for any* $y>0$,

$$\mathbf{P}\Big(\max_{1\le k\le n} X(k)\ge y\Big)\le \frac{1}{y}\mathbf{E}\Big\{X(n)\mathrm{1\!I}_{\{\max\limits_{1\le k\le n} X(k)\ge y\}}\Big\}\le \frac{1}{y}\mathbf{E}X(n). \tag{5.7}$$

Proof. Set $A:=\Big\{\max\limits_{1\le k\le n} X(k)\ge y\Big\}$, $A_1:=\{X(1)\ge y\}$,

$$A_k:=\{X(1)<y,\dots,X(k-1)<y,X(k)\ge y\},\quad k=2,\dots,n.$$

Let $\rho:=\min\{k: X(k)\ge y\}$ be the first moment at which the sequence $X(k)$ exceed y. Then $A_k=\{\rho=k\}\in\mathcal{F}_k$. It is clear that $A=\{1\le\rho\le n\}$, the sets A_k are disjoint, and $A=\bigcup\limits_{k=1}^{n} A_k$. Then

$$\mathbf{E}X(n)\ge \mathbf{E}\Big\{X(n)\mathrm{1\!I}_{\{\max\limits_{1\le k\le n} X(k)\ge y\}}\Big\}=\int_A X(n)\,d\mathbf{P}=\sum_{k=1}^{n}\int_{A_k} X(n)\,d\mathbf{P}$$

$$=\sum_{k=1}^{n}\int_{A_k}\mathbf{E}\{X(n)|\mathcal{F}_k\}\,d\mathbf{P}\ge\sum_{k=1}^{n}\int_{A_k} X(k)\,d\mathbf{P}\ge y\sum_{k=1}^{n}\mathbf{P}(A_k)=y\mathbf{P}(A).$$

Thus, the inequalities (5.7) are proved. □

Remark 5.2. The inequality (5.7) is remarkable for the following reason: the probability of the event, depending on the whole sample path of X, is estimated by the expectation of X at the last moment n.

Corollary 5.2 (Doob's inequalities). *Let* $\big(X(k),\mathcal{F}_k\big)$, $k=1,\dots,n$, *be a martingale. Assume that* $\mathbf{E}|X(n)|^p<\infty$ *for some* $1\le p<\infty$. *Then for any* $y>0$

$$\mathbf{P}\Big(\max_{1\le k\le n}|X(k)|\ge y\Big)\le\frac{1}{y^p}\mathbf{E}|X(n)|^p, \tag{5.8}$$

and if $1<p<\infty$, *then*

$$\mathbf{E}\Big\{\max_{1\le k\le n}|X(k)|^p\Big\}\le\Big(\frac{p}{p-1}\Big)^p\mathbf{E}|X(n)|^p. \tag{5.9}$$

Proof. The process $\big(|X(k)|^p,\mathcal{F}_k\big)$, $k=1,\dots,n$, is a submartingale. Indeed, using Jensen's inequality for conditional expectations, we see that for $l<k$

$$\mathbf{E}\{|X(k)|^p|\mathcal{F}_l\}\ge|\mathbf{E}\{X(k)|\mathcal{F}_l\}|^p=|X(l)|^p.$$

For a nonnegative submartingale we can apply (5.7). Then

$$\mathbf{P}\Big(\max_{1\le k\le n}|X(k)|\ge y\Big)=\mathbf{P}\Big(\max_{1\le k\le n}|X(k)|^p\ge y^p\Big)\le\frac{1}{y^p}\mathbf{E}|X(n)|^p.$$

To prove (5.9), we set $M_n := \max_{1\le k\le n} |X(k)|$. Since there is no a priori information that the absolute moment of order p exists for M_n, we must apply a truncation procedure. For an arbitrary constant $C > 0$,

$$(M_n \wedge C)^p = \int_0^{M_n\wedge C} p\, y^{p-1}\, dy = p \int_0^C y^{p-1} 1\!\!1_{\{M_n \ge y\}}\, dy.$$

Applying the inequality on the left-hand side of (5.7), we get

$$\mathbf{E}(M_n \wedge C)^p = p \int_0^C y^{p-1}\mathbf{P}(M_n \ge y)\, dy \le p \int_0^C y^{p-2}\mathbf{E}\{|X(n)| 1\!\!1_{\{M_n \ge y\}}\}\, dy$$

$$= p\,\mathbf{E}\Big\{|X(n)| \int_0^{M_n\wedge C} y^{p-2}\, dy\Big\} = \frac{p}{p-1}\mathbf{E}\Big\{|X(n)|(M_n \wedge C)^{p-1}\Big\}.$$

By Hölder's inequality,

$$\mathbf{E}(M_n \wedge C)^p \le \frac{p}{p-1}\mathbf{E}^{(p-1)/p}(M_n \wedge C)^p \mathbf{E}^{1/p}|X(n)|^p.$$

This implies that

$$\mathbf{E}(M_n \wedge C)^p \le \Big(\frac{p}{p-1}\Big)^p \mathbf{E}|X(n)|^p.$$

Letting $C \to \infty$, we obtain (5.9). □

Example 5.1 (Kolmogorov's inequality). Let η_l, $l = 1, 2, \dots$, be independent identically distributed random variables with $\mathbf{E}\eta_l = 0$, $\mathbf{E}\eta_l^2 = \sigma^2$. The process $X(k) := \sum_{l=1}^{k} \eta_l$, $k = 1, 2, \dots$, is a martingale with respect to the σ-algebras $\mathcal{F}_k = \sigma(\eta_l, 1 \le l \le k)$. Applying (5.8) with $p = 2$, we have

$$\mathbf{P}\Big(\max_{1\le k\le n}\Big|\sum_{l=1}^{k}\eta_l\Big| \ge y\Big) \le \frac{\mathbf{E}X^2(n)}{y^2} = \frac{\sigma^2 n}{y^2}. \tag{5.10}$$

Inequalities analogous to (5.8) and (5.9) hold for a martingale $\big(X(t), \mathcal{F}_t\big)$ with a continuous time parameter from a finite interval.

Corollary 5.3 (Doob's inequalities). *Let $\big(X(t), \mathcal{F}_t\big)$, $t \in [0, T]$, be a right continuous martingale. Let $\mathbf{E}|X(T)|^p < \infty$ for some $1 \le p < \infty$. Then for any $y > 0$*

$$\mathbf{P}\Big(\sup_{0\le t\le T}|X(t)| \ge y\Big) \le \frac{1}{y^p}\mathbf{E}|X(T)|^p, \tag{5.11}$$

and if $1 < p < \infty$, then

$$\mathbf{E}\Big\{\sup_{0\le t\le T}|X(t)|^p\Big\} \le \Big(\frac{p}{p-1}\Big)^p \mathbf{E}|X(T)|^p. \tag{5.12}$$

Proof. Let D be the set of dyadic rational points of the interval $[0,T]$. We can choose an increasing sequence D_n of finite subsets of D such that $\bigcup_n D_n = D$. Applying (5.8) and (5.9), we have

$$\mathbf{P}\Big(\sup_{t\in D_n\cup\{T\}} |X(t)| \ge y\Big) \le \frac{1}{y^p}\mathbf{E}|X(T)|^p,$$

$$\mathbf{E}\Big\{\sup_{t\in D_n\cup\{T\}} |X(t)|^p\Big\} \le \Big(\frac{p}{p-1}\Big)^p \mathbf{E}|X(T)|^p.$$

Since X is a right continuous process,

$$\sup_{t\in D_n\cup\{T\}} |X(t)| \uparrow \sup_{0\le t\le T} |X(t)| \qquad \text{as } n\to\infty \qquad \text{a.s.}$$

and the passage to the limit in the previous inequalities proves the corollary. □

3. Decomposition of submartingales.

Theorem 5.4 (Doob's decomposition). *Any submartingale* $\big(X(k),\mathcal{F}_k\big)$, $k = 0,1,2,\dots$, *can be represented uniquely in the form*

$$X(k) = M(k) + A(k), \tag{5.13}$$

where $\big(M(k),\mathcal{F}_k\big)$ *is a martingale and* $A(k)$, $k = 1,2,\dots$, *is an* $\mathcal{F}_{k-1}$*-measurable nondecreasing process,* $A(0) = 0$.

Proof. Set $M(0) := X(0)$, $A(0) = 0$,

$$M(k) := M(k-1) + (X(k) - \mathbf{E}\{X(k)|\mathcal{F}_{k-1}\}) = X(0) + \sum_{l=1}^{k}(X(l) - \mathbf{E}\{X(l)|\mathcal{F}_{l-1}\}),$$

$$A(k) := A(k-1) + (\mathbf{E}\{X(k)|\mathcal{F}_{k-1}\} - X(k-1)) = \sum_{l=1}^{k}(\mathbf{E}\{X(l)|\mathcal{F}_{l-1}\} - X(l-1)).$$

Since $X(k)$, $k = 0,1,2,\dots$, is a submartingale,

$$\mathbf{E}\{X(k)|\mathcal{F}_{k-1}\} - X(k-1) \ge 0$$

and thus $A(k)$ is nondecreasing process. It is obvious that $A(k)$ is $\mathcal{F}_{k-1}$-measurable.

The process A is usually called *predictable*, since at moment k it is $\mathcal{F}_{k-1}$-measurable.

The process $M(k)$ is $\mathcal{F}_k$-measurable and

$$\mathbf{E}\{M(k)|\mathcal{F}_{k-1}\} = M(k-1) + (\mathbf{E}\{X(k)|\mathcal{F}_{k-1}\} - \mathbf{E}\{X(k)|\mathcal{F}_{k-1}\}) = M(k-1).$$

Therefore $\big(M(k),\mathcal{F}_k\big)$ is a martingale.

It is clear that

$$M(k) + A(k) = X(0) + \sum_{l=1}^{k}(X(l) - X(l-1)) = X(k)$$

and, consequently, (5.13) holds.

Now suppose there is another decomposition: $X(k) = M'(k) + A'(k)$, $k = 0,1,2,\dots$, $A'(0) = 0$. Then $X(k+1) - X(k)$ equals

$$M'(k+1) - M'(k) + A'(k+1) - A'(k) = M(k+1) - M(k) + A(k+1) - A(k).$$

Applying to this equality the conditional expectation given $\mathcal{F}_k$, we have

$$A'(k+1) - A'(k) = A(k+1) - A(k).$$

Since $A'(0) = A(0) = 0$, we have $A'(k) = A(k)$, $k = 0,1,2,\dots$. Thus representation (5.13) is unique. □

From decomposition (5.13) it follows that the submartingale $\big(X(k), \mathcal{F}_k\big)$, $k = 0,1,2,\dots$, transforms into a martingale after the subtraction of the process $A(k)$.

The predictable process $\big(A(k), \mathcal{F}_{k-1}\big)$ involved in Doob's decomposition (5.13) is called a *compensator* of the submartingale $X(k)$.

Doob's decomposition is important for the investigation of the square-integrable martingales $\big(M(k), \mathcal{F}_k\big)$, $k = 0,1,2,\dots$, i.e., the martingales having a finite second moment $\mathbf{E}M^2(k) < \infty$ for every $k \geq 0$.

The process $\big(M^2(k), \mathcal{F}_k\big)$, $k = 0,1,2,\dots$, is a submartingale and, by Doob's decomposition,

$$M^2(k) = m(k) + \langle M\rangle(k),$$

where $\big(m(k), \mathcal{F}_k\big)$, $k = 0,1,2,\dots$, is a martingale and $\langle M\rangle = \big(\langle M\rangle(k), \mathcal{F}_{k-1}\big)$ is a compensator.

The compensator $\langle M\rangle$ is called the *quadratic characteristic* of the martingale M and it essentially determines its structure and properties.

From the proof of Theorem 5.4 it follows that

$$\begin{aligned}\langle M\rangle(k) &= \sum_{l=1}^{k}(\mathbf{E}\{M^2(l)|\mathcal{F}_{l-1}\} - M^2(l-1))\\ &= \sum_{l=1}^{k}\mathbf{E}\{M^2(l) - M^2(l-1)|\mathcal{F}_{l-1}\} = \sum_{l=1}^{k}\mathbf{E}\{(M(l) - M(l-1))^2|\mathcal{F}_{l-1}\}.\end{aligned}$$

The variable on the right-hand side of these equalities is the conditional quadratic variation of the martingale. On the other hand, it is the compensator of the submartingale M^2.

For example, the martingale $M(k) := \sum_{l=1}^{k}\eta_l$, $k = 1,2,\dots$, where η_l are independent random variables with $\mathbf{E}\eta_l = 0$, $\mathbf{E}\eta_l^2 = \sigma_l^2$, has the compensator $\langle M\rangle(k) = \sum_{l=1}^{k}\sigma_l^2$.

4. Convergence of martingales.

Here we present some auxiliary result concerning the number of upcrossings of an interval by a stochastic process.

Let $X(k)$, $k=1,2,\dots$, be a stochastic process and $a<b$. Set $\tau_0:=0$ and $\sigma_1:=\infty$ if $\min\limits_k X(k)>a$. Next we set $\tau_1:=\infty$ if $\sigma_1=\infty$ or $\max\limits_k X(k)<b$. In the remaining cases we set

$$
\begin{aligned}
\sigma_1 &:= \min\{k>0 : X(k)\le a\},\\
\tau_1 &:= \min\{k>\sigma_1 : X(k)\ge b\},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\sigma_m &:= \min\{k>\tau_{m-1} : X(k)\le a\},\\
\tau_m &:= \min\{k>\sigma_m : X(k)\ge b\}.
\end{aligned}
$$

The indices k that lie between σ_m and τ_m correspond to the m-th upcrossing of the interval $[a,b]$ by the process X.

The number of upcrossings of the interval $[a,b]$ before time n by the stochastic process X is defined as follows

$$\beta_n(a,b):=\begin{cases}0, & \text{if } \tau_1>n,\\ \max\{m:\tau_m\le n\}, & \text{if } \tau_1\le n.\end{cases}$$

Theorem 5.5. *Let* $\bigl(X(k),\mathcal{F}_k\bigr)$, $k=1,2,\dots$, *be a submartingale. Then*

$$\mathbf{E}\beta_n(a,b)\le\frac{1}{b-a}\mathbf{E}(X(n)-a)^+, \tag{5.14}$$

where $q^+:=\max\{0,q\}$.

Remark 5.3. An important aspect of the inequality (5.14) is that the expectation of $\beta_n(a,b)$, depending on the whole sample path of the process X up to the time n, is estimated by the expectation of X only at the last moment n.

Proof of Theorem 5.5. The process $Y(k)=(X(k)-a)^+$ is a nonnegative submartingale. Indeed, if Jensen's inequality for the conditional expectation given $\mathcal{F}_l$ is applied to the function $(X(k)-a)^+$ for $1\le l<k$, then

$$\mathbf{E}\{(X(k)-a)^+|\mathcal{F}_l\}\ge\bigl(\mathbf{E}\{X(k)-a|\mathcal{F}_l\}\bigr)^+\ge\max\{0,X(l)-a\}=(X(l)-a)^+.$$

It is clear that the number of upcrossings of the interval $[a,b]$ by the submartingale X coincides with the number of upcrossings of the interval $[0,b-a]$ by the submartingale Y. Therefore, it is sufficient to prove (5.14) only for a nonnegative submartingale X and $a=0$, i.e., to prove the estimate

$$\mathbf{E}\beta_n(0,b)\le\frac{1}{b}\mathbf{E}X(n). \tag{5.15}$$

Let $X(0)=0$. Set $\mathcal{F}_0=\{\varnothing,\Omega\}$, where $\varnothing$ is the empty set. For any $j=1,2,\dots$, set

$$\xi_j:=\begin{cases}1, & \text{if } \sigma_m<j\le\tau_m \text{ for some } m,\\ 0, & \text{if } \tau_{m-1}<j\le\sigma_m \text{ for some } m.\end{cases}$$

Then it is not hard to see that

$$b\,\beta_n(0,b) \le \sum_{j=1}^{n} \xi_j \,\big(X(j) - X(j-1)\big).$$

This holds, because the variable ξ_j is equal to 1 if the index j corresponds to an intersection from downwards to upwards, and it is equal to 0 otherwise. Since

$$\{\xi_j = 1\} = \bigcup_m \big\{\{\sigma_m < j\} \bigcap \{j \le \tau_m\}\big\} = \bigcup_m \big\{\{\sigma_m \le j-1\} \setminus \{\tau_m \le j-1\}\big\}$$

and σ_m, τ_m, $m = 1, 2, \dots$, are stopping times, the event $\{\xi_j = 1\}$ is $\mathcal{F}_{j-1}$-measurable. Using this fact, the definitions of the conditional expectation and of a submartingale, we have

$$b\,\mathbf{E}\beta_n(0,b) \le \mathbf{E}\sum_{j=1}^{n} \xi_j \,\big(X(j) - X(j-1)\big) = \sum_{j=1}^{n} \int\limits_{\{\xi_j=1\}} \big(X(j) - X(j-1)\big)\,d\mathbf{P}$$

$$= \sum_{j=1}^{n} \int\limits_{\{\xi_j=1\}} \big(\mathbf{E}\{X(j)|\mathcal{F}_{j-1}\} - X(j-1)\big)\,d\mathbf{P}$$

$$\le \sum_{j=1}^{n} \int\limits_{\Omega} \big(\mathbf{E}\big\{X(j)\big|\mathcal{F}_{j-1}\big\} - X(j-1)\big)\,d\mathbf{P} = \sum_{j=1}^{n} (\mathbf{E}X(j) - \mathbf{E}X(j-1)) = \mathbf{E}X(n).$$

The estimate (5.15) is valid and, consequently, the theorem is proved. □

Theorem 5.6. *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 1, 2, \dots$, be a submartingale such that*

$$\sup_k \mathbf{E}X^+(k) < \infty. \tag{5.16}$$

Then $X(k)$ converges a.s. *as $k \to \infty$ to a limit X_∞ and $\mathbf{E}|X_\infty| < \infty$.*

Remark 5.4. For a submartingale $X(k)$, $k = 1, 2, \dots$, the inequality (5.16) is equivalent to the inequality $\sup\limits_k \mathbf{E}|X(k)| < \infty$.

Indeed, for a submartingale $X(k)$, $k = 1, 2, \dots$, we have

$$\mathbf{E}X^+(k) \le \mathbf{E}|X(k)| = 2\mathbf{E}X^+(k) - \mathbf{E}X(k) \le 2\mathbf{E}X^+(k) - \mathbf{E}X(1).$$

Proof of Theorem 5.6. We use a proof by contradiction. Assume that

$$\mathbf{P}\Big(\limsup_k X(k) > \liminf_k X(k)\Big) > 0.$$

Since

$$\Big\{\limsup_k X(k) > \liminf_k X(k)\Big\} = \bigcup_{\substack{a<b\\ \text{rational}}} \Big\{\limsup_k X(k) > b > a > \liminf_k X(k)\Big\},$$

there exist a and b such that

$$\mathbf{P}\Big(\limsup_k X(k) > b > a > \liminf_k X(k)\Big) > 0. \tag{5.17}$$

Let $\beta_n(a, b)$ be the number of upcrossings of the interval $[a, b]$ by the submartingale $X(k)$, $k = 1, \dots,$ up to the time n. This number is increasing in n and there exists the limit $\beta_\infty(a, b) := \lim_{n\to\infty} \beta_n(a, b)$, possibly infinite. By (5.14),

$$\mathbf{E}\beta_n(a, b) \le \frac{\mathbf{E}(X(n) - a)^+}{b - a} \le \frac{\mathbf{E}X^+(n) + |a|}{b - a}.$$

Using (5.16) and Fatou's lemma, we obtain

$$\mathbf{E}\beta_\infty(a, b) := \lim_{n\to\infty} \mathbf{E}\beta_n(a, b) \le \frac{\sup_n \mathbf{E}X^+(n) + |a|}{b - a} < \infty.$$

This, however, contradicts (5.17), from which it follows that with positive probability $\beta_\infty(a, b) = \infty$. Thus,

$$\mathbf{P}\Big(\limsup_k X(k) = \liminf_k X(k)\Big) = 1$$

and $X(k)$ converges a.s. as $k \to \infty$ to some limit X_∞. By Fatou's Lemma,

$$\mathbf{E}|X_\infty| = \mathbf{E}\liminf_k |X(k)| \le \liminf_k \mathbf{E}|X(k)| \le \sup_k \mathbf{E}|X(k)| < \infty. \tag{5.18}$$

The theorem is proved. □

Corollary 5.4. *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 1, 2, \dots,$ be a supermartingale such that $\sup_k \mathbf{E}|X(k)| < \infty$. Then $X(k)$ converges* a.s. *to a limit X_∞ and $\mathbf{E}|X_\infty| < \infty$.*

Here we use the first part of Remark 5.1.

Corollary 5.5. *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 1, 2, \dots,$ be a nonnegative martingale. Then $X(k)$ converges* a.s. *to a limit X_∞.*

Let us discuss what happens with the convergence of submartingales if condition (5.16) fails. Let $H_z := \inf\{k \ge 1 : X(k) > z\}$, $z > 0$, be the *first exceeding moment of the level z*. We set $H_z = \infty$ if $\sup_k X(k) \le z$.

Theorem 5.7. *Let $\big(X(k), \mathcal{F}_k\big)$, $k = 0, 1, 2, \dots,$ be a nonnegative submartingale, $X(0) = 0$, and let $X(k) = M(k) + A(k)$ be its Doob's decomposition. Assume that for every $z > 0$*

$$\mathbf{E}\big\{\Delta X_{H_z} \mathbb{1}_{\{H_z<\infty\}}\big\} < \infty,$$

where $\Delta X_k = X(k) - X(k - 1)$.

Then

$$\{A(\infty) < \infty\} = \{X(k) \quad \text{converges }\} \qquad \text{a.s.} \tag{5.19}$$

Proof. Let $\rho_z := \inf\{k \ge 1 : A(k+1) > z\}$, $z > 0$. Set $\rho_z = \infty$ if $\sup_k A(k) \le z$. Since $M(k)$ is a martingale with zero mean and $A(\rho_z) \le z$, Theorem 5.2 and Remark 5.1 show that

$$\mathbf{E}X(k \wedge \rho_z) = \mathbf{E}A(k \wedge \rho_z) \le z.$$

Set $Y_z(k) := X(k \wedge \rho_z)$. Then $\big(Y_z(k), \mathcal{F}_k\big)$ is a nonnegative submartingale with $\sup_k \mathbf{E}Y_z(k) \le z$. By Theorem 5.6, the process $Y_z(k)$ converges a.s. as $k \to \infty$. Therefore,

$$\{A(\infty) \le z\} = \{\rho_z = \infty\} \subseteq \{X(k) \quad \text{converges }\} \qquad \text{a.s.,}$$

and, consequently,

$$\{A(\infty) < \infty\} = \bigcup_{z>0} \{A_\infty \le z\} \subseteq \{X(k) \quad \text{converges }\} \qquad \text{a.s.}$$

To prove the opposite inclusion, we start with the estimate

$$X(k \wedge H_z) \le X(H_z) \le z + (X(H_z) - X(H_z - 1))\mathbb{1}_{\{H_z<\infty\}}.$$

Thus,

$$\mathbf{E}A(H_z) = \mathbf{E}\liminf_k A(k \wedge H_z) \le \liminf_k \mathbf{E}A(k \wedge H_z) < \infty.$$

Consequently, $A(H_z) < \infty$ a.s. and $\{H_z = \infty\} \subseteq \{A(\infty) < \infty\}$. As a result, we have

$$\{X(k) \quad \text{converges }\} \subseteq \Big\{ \sup_k X(k) < \infty \Big\}$$
$$= \bigcup_{z>0} \Big\{ \sup_k X(k) \le z \Big\} = \bigcup_{z>0} \{H_z = \infty\} \subseteq \{A(\infty) < \infty\}.$$

□

An important application of Theorem 5.7 is the following assertion.

Lemma 5.1 (Borel–Cantelli–Lévy). *Let $\mathcal{F}_l$, $l = 1, 2, \dots$, be an increasing sequence of σ-algebras and $A_l \in \mathcal{F}_l$ for every l. Then the events*

$$\Big\{\omega : \sum_{l=1}^{\infty} \mathbf{P}(A_l|\mathcal{F}_{l-1}) < \infty\Big\} \qquad \text{and} \qquad \Big\{\omega : \sum_{l=1}^{\infty} \mathbb{1}_{A_l}(\omega) < \infty\Big\} \tag{5.20}$$

coincide a.s.

Remark 5.5. The Borel–Cantelli-Lévy lemma states that the number of occurrences of the events A_k is a.s. finite or infinite according to whether the series of their conditional probabilities $\mathbf{P}(A_k|\mathcal{F}_{k-1})$ is a.s. finite or infinite.

Proof of Lemma 5.1. The sequence $X(k) = \sum_{l=1}^{k} \mathbb{1}_{A_l}$, $k = 1, 2, \dots$, is a nonnegative $\mathcal{F}_k$-adapted submartingale. In the proof of Theorem 5.4 we showed that

$$A(k) = \sum_{l=1}^{k} \mathbf{P}(A_l|\mathcal{F}_{l-1})$$

is the corresponding increasing process. Now (5.20) is a direct consequence of (5.19). □

The following results can be useful for proving convergence of sequences.

Lemma 5.2 (Toeplitz). *Let a_k, $k = 1, 2, \dots,$ be a sequence of nonnegative numbers such that $\sum_{k=1}^{\infty} a_k = \infty$. Let $x_n \to x$. Then*

$$\Big(\sum_{k=1}^{n} a_k\Big)^{-1} \sum_{k=1}^{n} a_k x_k \to x.$$

Proof. Let $b_n := \sum_{k=1}^{n} a_k$. For an arbitrary $\varepsilon > 0$ we choose $n_0 = n_0(\varepsilon)$ such that $|x_n - x| \le \varepsilon/2$ for all $n \ge n_0$ and choose $n_1 > n_0$, such that the inequality

$$\frac{1}{b_{n_1}} \sum_{k=1}^{n_0} a_k |x_k - x| \le \frac{\varepsilon}{2}$$

holds. Then for $n > n_1$

$$\Big|\frac{1}{b_n} \sum_{k=1}^{n} a_k x_k - x\Big| \le \frac{1}{b_{n_1}} \sum_{k=1}^{n_0} a_k |x_k - x| + \frac{1}{b_n} \sum_{k=n_0+1}^{n} a_k |x_k - x| \le \varepsilon.$$

□

Lemma 5.3 (Kronecker). *Let x_k, $k = 1, 2, \dots,$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} x_k$ converges. Let b_k, $k = 1, 2, \dots,$ be a monotone sequence of positive numbers tending to infinity ($b_k \uparrow \infty$). Then $\frac{1}{b_n} \sum_{k=1}^{n} b_k x_k \to 0$.*

Proof. Set $s_n := \sum_{k=1}^{n-1} x_k$, $b_0 := 0$. Then $s_n \to s := \sum_{k=1}^{\infty} x_k$. Using this and the Toeplitz lemma, we have

$$\frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = \frac{1}{b_n} \sum_{k=1}^{n} b_k (s_{k+1} - s_k) = s_{n+1} - \frac{1}{b_n} \sum_{k=1}^{n} (b_k - b_{k-1}) s_k \to s - s = 0.$$

□

Theorem 5.8 (Strong law of large numbers). *Let X_k, $k = 1, 2, \dots,$ be independent identically distributed random variables with* $\operatorname{Var} X_1 < \infty$. *Then*

$$\frac{1}{n} \sum_{k=1}^{n} X_k \to \mathbf{E} X_1, \qquad \text{a.s.}$$

Proof. One can assume that $\mathbf{E} X_1 = 0$, since otherwise one should consider the centralized random variables $X_k - \mathbf{E} X_k$. By Kronecker's lemma with $b_k = k$, $x_k = \frac{X_k}{k}$, it suffices to prove that the series $\sum_{k=1}^{\infty} \frac{X_k}{k}$ converges a.s. The process

$Y(n) := \sum\limits_{k=1}^{n} \frac{X_k}{k}$ is a martingale with respect to the natural filtration. Obviously, $\mathbf{E}Y^2(n) = \sum\limits_{k=1}^{n} \frac{\operatorname{Var} X_1}{k^2}$. Therefore,

$$\sup_n \mathbf{E}|Y(n)| \le \sup_n \mathbf{E}^{1/2}|Y(n)|^2 \le \sqrt{\operatorname{Var} X_1}\Bigl(\sum_{k=1}^{\infty} \frac{1}{k^2}\Bigr)^{1/2} < \infty,$$

and by Theorem 5.6, the sequence $Y(n)$ converges a.s as $n \to \infty$. □

To conclude this section we describe the basic construction that leads to martingales with discrete time.

Let $\{\mathcal{F}_n\}$, $n = 1,2,\dots$, be an increasing family of σ-algebras. Let $H(n,x)$, $x \in \mathbf{R}$, be a sequence of bounded stochastic processes such that for every n, the process $H(n,x)$, $x \in \mathbf{R}$, is $\mathcal{B}(\mathbf{R}) \times \mathcal{F}_n$-measurable and independent of the σ-algebra $\mathcal{F}_{n-1}$. We define the process X_n, $n = 1,2,\dots$, by the recurrence formula

$$X_n := X_{n-1} + H(n, X_{n-1}), \qquad X_0 = x. \tag{5.21}$$

Set $h(n,x) := \mathbf{E}H(n,x)$. If $h(n,x) \equiv 0$, then the process X_n is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\ge 1}$.
Indeed, for every n the random variable X_n is $\mathcal{F}_n$-measurable. Applying Lemma 2.1, we get $\mathbf{E}\bigl\{H(n,X_{n-1})\big|\mathcal{F}_{n-1}\bigr\} = h(n,X_{n-1})$. Therefore,

$$\mathbf{E}\bigl\{X_n\big|\mathcal{F}_{n-1}\bigr\} = X_{n-1} + h(n,X_{n-1}) = X_{n-1}.$$

It is clear that if $h(n,x) \ge 0$ for all n, x, then the process X_n is a submartingale. If the opposite inequality holds we get a supermartingale.

The following construction is an important particular case of (5.21). Let ξ_n, $n = 1,2,\dots$, be a sequence of independent random variables with zero mean ($\mathbf{E}\xi_n = 0$). Let $\mathcal{G}_n = \sigma(\xi_k, 1 \le k \le n)$ be the σ-algebra of events generated by these variables up to the time n. We define the process

$$X_n := X_{n-1} + h(n, X_{n-1}) + g(n, X_{n-1})\, \xi_n, \qquad X_0 = x, \tag{5.22}$$

where $h(n,x)$, $g(n,x)$, $n = 1,2,\dots$, are sequences of bounded measurable functions. If $h(n,x) \equiv 0$, then X_n is a martingale with respect to the filtration $\{\mathcal{G}_n\}$.

The analogue of the recurrence relation (5.22) for processes with continuous time leads to the concept of a stochastic differential equation, which will be considered in the Section 7 of the next chapter.

Exercises.

5.1. Let η_k, $k = 1,2,\dots$, be i.i.d. random variables. Let $\mathcal{F}_k = \sigma(\eta_l, 1 \le l \le k)$ be the σ-algebra of events generated by these variables up to the time k. Prove that the process X_n is a martingale

1) if $X_n = \prod_{k=1}^{n} \eta_k$ and $\mathbf{E}\eta_k = 1$;

2) if $X_n = S_n^3 - 3n\sigma^2 S_n - n\mu$, where $S_n = \sum_{k=1}^{n} \eta_k$, $\mathbf{E}\eta_k = 0$, $\sigma^2 = \mathbf{E}\eta_k^2$ and $\mu = \mathbf{E}\eta_k^3$.

5.2. Let η_k, $k = 1, 2, \dots$, be i.i.d. random variables, $\mathcal{F}_k = \sigma(\eta_l, 1 \le l \le k)$ be the σ-algebra of events generated by these variables up to the time k. Suppose that $\mathbf{E}\eta_k^4 < \infty$ and $\mathbf{E}\eta_k = 0$. Compute the compensator of the submartingale $\Big(\sum_{k=1}^{n} \eta_k\Big)^4$.

5.3. Let $X(k)$, $Y(k)$, $k = 0, 1, 2, \dots$, be martingales with respect to a filtration $\mathcal{F}_k$. Suppose that $X(0) = Y(0) = 0$ and $\mathbf{E}X^2(k) < \infty$, $\mathbf{E}Y^2(k) < \infty$ for all k. Prove that

$$\mathbf{E}(X(n)Y(n)) = \sum_{k=1}^{n} \mathbf{E}((X(k) - X(k-1))((Y(k) - Y(k-1))).$$

5.4. Let $(X(k), \mathcal{F}_k)$, $k = 0, 1, 2, \dots$, be a martingale with $\mathbf{E}X^2(k) < \infty$. Prove that for $k < l < m < n$,

$$\operatorname{Cov}((X(n) - X(m))(X(k) - X(l))) = 0.$$

Compute this covariance for $k < m < l < n$ and for $m < k < l < n$.

5.5. Let $(X(k), \mathcal{F}_k)$, $k = 0, 1, 2, \dots$, be a martingale with $\mathbf{E}X^2(k) < \infty$. Assume that σ and τ are two integer-valued bounded stopping times with respect to $\{\mathcal{F}_k\}_{k=1}^{\infty}$ such that $\sigma \le \tau$. Prove that

$$\mathbf{E}\{(X(\tau) - X(\sigma))^2 | \mathcal{F}_\sigma\} = \mathbf{E}\{X^2(\tau) | \mathcal{F}_\sigma\} - X^2(\sigma).$$

5.6. Let $g(x)$, $x \in \mathbf{R}$, be a nondecreasing positive convex function and let $(X(k), \mathcal{F}_k)$, $k = 0, 1, 2, \dots$, be a martingale. Prove that for any $y > 0$

$$\mathbf{P}\Big(\max_{1 \le k \le n} X(k) \ge y\Big) \le \frac{\mathbf{E}g(X(n))}{g(y)}.$$

5.7. Let $(X(k), \mathcal{F}_k)$, $k = 0, 1, 2, \dots$, be a martingale with $\mathbf{E}X(k) = 0$ and $\mathbf{E}X^2(k) < \infty$. Prove that for any $y > 0$

$$\mathbf{P}\Big(\max_{1 \le k \le n} X(k) \ge y\Big) \le \frac{\mathbf{E}X^2(n)}{\mathbf{E}X^2(n) + y^2}.$$

Hint: For every $z \ge 0$, the process $(X(k) + z)^2$ is a submartingale and

$$\mathbf{P}\Big(\max_{1 \le k \le n} X(k) \ge y\Big) \le \mathbf{P}\Big(\max_{1 \le k \le n} (X(k) + z)^2 \ge (y + z)^2\Big).$$

5.8. Let $(M(k), \mathcal{F}_k)$, $k = 0, 1, 2, \dots$, be a martingale with $\mathbf{E}M^2(k) < \infty$. Prove that $\sup_k \mathbf{E}M^2(k) < \infty$ iff

$$\sum_{l=1}^{\infty} \mathbf{E}(M(k) - M(k-1))^2 < \infty.$$

5.9. Let η_k, $k = 1, 2, \dots$, be i.i.d. random variables with $\mathbf{E}\eta_1^2 < \infty$. Let τ be a bounded stopping time with respect to the filtration $\mathcal{G}_k = \sigma(\eta_l, 1 \le l \le k)$. Prove the Wald identities

$$\mathbf{E}\sum_{l=1}^{\tau} \eta_l = \mathbf{E}\tau \mathbf{E}\eta_1, \qquad \mathbf{E}\Big(\sum_{l=1}^{\tau} \eta_l - \tau\mathbf{E}\eta_1\Big)^2 = \mathbf{E}\tau \operatorname{Var}\eta_1,$$

$$\mathbf{E}\Big(\frac{1}{\varphi^{\tau}(\alpha)} \exp\Big(\alpha \sum_{l=1}^{\tau} \eta_l\Big)\Big) = 1,$$

where $\varphi(\alpha) = \mathbf{E}e^{\alpha\eta_1}$, $\alpha \in \mathbf{R}$.

5.10. Write out the analog of the Wald identity for the third moment of $\sum_{l=1}^{\tau} \eta_l$.

§ 6. Markov processes

For a stochastic process $X(t)$, $t \in \Sigma \subseteq [0, \infty)$, we consider

$$\mathcal{G}_u^v = \sigma\{X(t), t \in \Sigma \textstyle\bigcap [u, v]\},$$

i.e., the σ-algebra of events generated by the process X when the time is varying from u to v.

As already mentioned, for a fixed time $t \in \Sigma$, the σ-algebra $\mathcal{G}_0^t$ is called the σ-algebra describing the past of the process, $\mathcal{G}_t^\infty$ is called the σ-algebra describing the future of the process, and $\mathcal{G}_t^t$ is called the σ-algebra describing the present state of the process. Such names are especially appropriate for Markov processes.

A stochastic process $X(t)$, $t \in \Sigma$, is called a *Markov process* if for every $t \in \Sigma$ and any Borel sets $A \in \mathcal{G}_0^t$, $B \in \mathcal{G}_t^\infty$

$$\mathbf{P}(AB|\mathcal{G}_t^t) = \mathbf{P}(A|\mathcal{G}_t^t)\mathbf{P}(B|\mathcal{G}_t^t) \quad \text{a.s.} \tag{6.1}$$

In other words, a process is Markov if for any fixed present state the future of the process does not depend on the past (the *Markov property*).

This definition implies that (6.1) is preserved when one inverts the time, i.e., if $X(t)$, $t \in [0, T]$, is a Markov process, then $Y(t) = X(T - t)$, $t \in [0, T]$, is also a Markov process.

Proposition 6.1. *The Markov property* (6.1) *is equivalent to the following one: for every* $t \in \Sigma$ *and any set* $B \in \mathcal{G}_t^\infty$

$$\mathbf{P}(B|\mathcal{G}_0^t) = \mathbf{P}(B|\mathcal{G}_t^t) \quad \text{a.s.} \tag{6.2}$$

Proof. We prove first that (6.1) implies (6.2). Using the definition of the conditional probability $\mathbf{P}(B|\mathcal{G}_0^t)$ or, equivalently, the conditional expectation $\mathbf{E}\{\mathbb{1}_B|\mathcal{G}_0^t\}$ and applying the properties of conditional expectations, we get

$$\int_A \mathbf{P}(B|\mathcal{G}_0^t)\,d\mathbf{P} = \mathbf{E}\{\mathbb{1}_A\mathbb{1}_B\} = \mathbf{E}\{\mathbf{E}\{\mathbb{1}_A\mathbb{1}_B\big|\mathcal{G}_t^t\}\} = \mathbf{E}\{\mathbf{E}\{\mathbb{1}_B\big|\mathcal{G}_t^t\}\mathbf{E}\{\mathbb{1}_A\big|\mathcal{G}_t^t\}\}$$

$$= \mathbf{E}\{\mathbb{1}_A\mathbf{E}\{\mathbb{1}_B\big|\mathcal{G}_t^t\}\} = \int_A \mathbf{P}(B|\mathcal{G}_t^t)\,d\mathbf{P}$$

for any $A \in \mathcal{G}_0^t$. Since A is an arbitrary set of the σ-algebra $\mathcal{G}_0^t$, this implies (6.2).

We now prove that (6.2) implies (6.1). Let $A \in \mathcal{G}_0^t$ and $B \in \mathcal{G}_t^\infty$ be arbitrary random events. Then, applying the properties of conditional expectations, we have

$$\mathbf{E}\{\mathbb{1}_A\mathbb{1}_B\big|\mathcal{G}_t^t\} = \mathbf{E}\{\mathbf{E}\{\mathbb{1}_A\mathbb{1}_B\big|\mathcal{G}_0^t\}\big|\mathcal{G}_t^t\} = \mathbf{E}\{\mathbb{1}_A\mathbf{E}\{\mathbb{1}_B\big|\mathcal{G}_0^t\}\big|\mathcal{G}_t^t\}$$
$$= \mathbf{E}\{\mathbb{1}_A\mathbf{E}\{\mathbb{1}_B\big|\mathcal{G}_t^t\}\big|\mathcal{G}_t^t\} = \mathbf{E}\{\mathbb{1}_A\big|\mathcal{G}_t^t\}\mathbf{E}\{\mathbb{1}_B\big|\mathcal{G}_t^t\}.$$

This is exactly (6.1). □

If the stochastic process X starts at the time t_0, then in the definition of the Markov property (6.2) the σ-algebra $\mathcal{G}_{t_0}^t$ can be taken instead of $\mathcal{G}_0^t$.

Proposition 6.2. *The process* X *is a Markov process iff for every* $t \in \Sigma$ *and any bounded* $\mathcal{G}_t^\infty$*-measurable function* G

$$\mathbf{E}\{G|\mathcal{G}_0^t\} = \mathbf{E}\{G|\mathcal{G}_t^t\} \quad \text{a.s.} \tag{6.3}$$

Proof. Obviously, (6.2) is a particular case of (6.3) for $G = \mathbb{1}_B$. We deduce (6.3) from (6.2). By the linearity property of the conditional expectation, (6.3) holds for linear combinations of indicators, i.e., for the random variables $G_n(\omega) = \sum_{k=1}^n g_{n,k}\mathbb{1}_{B_{n,k}}(\omega)$, where $g_{n,k} \in \mathbf{R}$, $B_{n,k} \in \mathcal{G}_t^\infty$, $k = 1, 2, \dots, n$.

Any bounded $\mathcal{G}_t^\infty$-measurable function G can be uniformly approximated by functions that are linear combinations of indicators, i.e., for each n there exists a function G_n of the form above such that

$$\sup_{\omega\in\Omega} |G(\omega) - G_n(\omega)| < \frac{1}{n}.$$

Then by the 6th property of conditional expectations, we have

$$|\mathbf{E}\{G(\omega)|\mathcal{G}_0^t\} - \mathbf{E}\{G_n(\omega)|\mathcal{G}_0^t\}| < \frac{1}{n}, \qquad |\mathbf{E}\{G(\omega)|\mathcal{G}_t^t\} - \mathbf{E}\{G_n(\omega)|\mathcal{G}_t^t\}| < \frac{1}{n}.$$

Passing to the limit as $n \to \infty$ and using (6.3) for the functions G_n, we get (6.3) for an arbitrary bounded $\mathcal{G}_t^\infty$-measurable functions G. □

Proposition 6.3. *The process X is a Markov process iff for every $t \in \Sigma$ and any bounded $\mathcal{G}_0^t$-measurable function F, and $\mathcal{G}_t^\infty$-measurable function G*

$$\mathbf{E}\{FG|\mathcal{G}_t^t\} = \mathbf{E}\{F|\mathcal{G}_t^t\}\mathbf{E}\{G|\mathcal{G}_t^t\} \quad \text{a.s.} \tag{6.4}$$

In view of (6.1), the proof of this statement is similar to that for Proposition 6.2.

Proposition 6.4. *The Markov property is equivalent to one of the following*:
1) *for all $t < v$, $t, v \in \Sigma$, and any bounded Borel function g*

$$\mathbf{E}\{g(X(v))|\mathcal{G}_0^t\} = \mathbf{E}\{g(X(v))|\mathcal{G}_t^t\} \quad \text{a.s.,} \tag{6.5}$$

2) *for all $t < v$, $t, v \in \Sigma$ and any Borel set Δ*

$$\mathbf{P}(X(v) \in \Delta|\mathcal{G}_0^t) = \mathbf{P}(X(v) \in \Delta|\mathcal{G}_t^t) \quad \text{a.s.} \tag{6.6}$$

Proof. It is clear that (6.3) implies (6.5), which in turn implies (6.6). Analogously to the proof of Proposition 6.2, we can deduce (6.5) from (6.6). We now prove that (6.5) implies (6.3) for functions of the form $G = \prod_{k=1}^{m} g_k(X(t_k))$, where $t_k \in \Sigma$, $t \le t_1 < t_2 < \cdots < t_m$, and g_k, $k = 1, 2, \dots, m$, are arbitrary bounded Borel functions.

Since $\mathcal{G}_{t_{m-1}}^{t_{m-1}} = \sigma(X(t_{m-1}))$, by (2.17), there exists a Borel function $\varphi_{m-1}(x)$, $x \in \mathbf{R}$, such that

$$\mathbf{E}\big\{g_m(X(t_m))\big|\mathcal{G}_{t_{m-1}}^{t_{m-1}}\big\} = \varphi_{m-1}(X(t_{m-1})).$$

By the properties of the conditional expectation (2.14), (2.15), and (6.5),

$$\mathbf{E}\{G|\mathcal{G}_0^t\} = \mathbf{E}\Big\{\prod_{k=1}^{m-1} g_k(X(t_k))\mathbf{E}\big\{g_m(X(t_m))\big|\mathcal{G}_0^{t_{m-1}}\big\}\Big|\mathcal{G}_0^t\Big\}$$

$$= \mathbf{E}\Big\{\prod_{k=1}^{m-1} g_k(X(t_k))\mathbf{E}\big\{g_m(X(t_m))\big|\mathcal{G}_{t_{m-1}}^{t_{m-1}}\big\}\Big|\mathcal{G}_0^t\Big\}$$

$$= \mathbf{E}\Big\{\prod_{k=1}^{m-1} g_k(X(t_k))\varphi_{m-1}(X(t_{m-1}))\Big|\mathcal{G}_0^t\Big\}.$$

Analogously,

$$\mathbf{E}\{G|\mathcal{G}_t^t\} = \mathbf{E}\Big\{\prod_{k=1}^{m-1} g_k(X(t_k))\varphi_{m-1}(X(t_{m-1}))\Big|\mathcal{G}_t^t\Big\}.$$

Thus we reduced the conditional expectations of the original random function to those of a function that depends on the process X considered in a smaller number of the time moments. Repeating this procedure, we come to a random function that depends on the process X at only one time point. Applying in this case (6.5), we obtain

$$\mathbf{E}\Big\{\prod_{k=1}^{m} g_k(X(t_k))\Big|\mathcal{G}_0^t\Big\} = \mathbf{E}\Big\{\prod_{k=1}^{m} g_k(X(t_k))\Big|\mathcal{G}_t^t\Big\}.$$

The extension of this equality to the similar one for arbitrary bounded $\mathcal{G}_t^\infty$-measurable functions G is realized by means of approximation, a standard procedure in mathematical analysis. □

Remark 6.1. If $\Sigma = \{0, 1, 2, \dots\}$, then a Markov process X is called a *Markov chain* and for every $t \in \Sigma$ in conditions (6.5), (6.6) it suffices to take $v = t + 1$.

Indeed, setting $\varphi_v(X(v)) := g(X(v))$, using (2.17) for $k < v$, and setting

$$\varphi_k(X(k)) := \mathbf{E}\{\varphi_{k+1}(X(k+1))|\mathcal{G}_k^k\} = \mathbf{E}\{\varphi_{k+1}(X(k+1))|\mathcal{G}_0^k\},$$

we get

$$\mathbf{E}\{g(X(v))|\mathcal{G}_0^t\} = \mathbf{E}\{\mathbf{E}\{g(X(v))|\mathcal{G}_0^{v-1}\}|\mathcal{G}_0^t\} = \mathbf{E}\{\mathbf{E}\{\varphi_v(X(v))|\mathcal{G}_{v-1}^{v-1}\}|\mathcal{G}_0^t\}$$
$$= \mathbf{E}\{\varphi_{v-1}(X(v-1))|\mathcal{G}_0^t\} = \dots = \varphi_t(X(t)) = \mathbf{E}\{\varphi_{t+1}(X(t+1))|\mathcal{G}_t^t\}$$
$$= \mathbf{E}\{\mathbf{E}\{\varphi_{t+2}(X(t+2))|\mathcal{G}_0^{t+1}\}|\mathcal{G}_t^t\} = \mathbf{E}\{\varphi_{t+2}(X(t+2))|\mathcal{G}_t^t\} = \dots = \mathbf{E}\{g(X(v))|\mathcal{G}_t^t\}.$$

For the sake of specificity we assume that the *state space* of the Markov process X is $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, although one can consider some other metric space.

The function $P(s, x, t, \Delta)$ is called a *transition function* (*transition probability*) of the Markov process X if for all $s \le t$, $s, t \in \Sigma$, and every Borel set Δ

$$\mathbf{P}(X(t) \in \Delta|\mathcal{G}_s^s) = P(s, X(s), t, \Delta) \qquad \text{a.s.,} \tag{6.7}$$

and the following conditions hold:

1) for any fixed s, x, t, the mapping $\Delta \to P(s, x, t, \Delta)$ is a measure on the σ-algebra of Borel sets;

2) $P(s, \cdot, t, \Delta)$ is a Borel function for any fixed s, t, Δ;

3) $P(s, x, s, \Delta) = \mathbb{I}_\Delta(x)$ for all $s \in [0, T]$, $x \in \mathbf{R}$ and $\Delta \in \mathcal{B}(\mathbf{R})$;

4) for all $s < v < t$, $s, v, t \in \Sigma$, $x \in \mathbf{R}$, and every $\Delta \in \mathcal{B}(\mathbf{R})$, the *Chapman–Kolmogorov equation* holds:

$$P(s, x, t, \Delta) = \int_{-\infty}^{\infty} P(s, x, v, dy) P(v, y, t, \Delta).$$

If there exists a nonnegative function $p(s, x, t, y)$ that is $\mathcal{B}(\mathbf{R}) \times \mathcal{B}(\mathbf{R})$-measurable as a function of $(x, y) \in \mathbf{R}^2$ for all $s < t$, $s, t \in \Sigma$, and satisfies

$$P(s, x, t, \Delta) = \int_{\Delta} p(s, x, t, y)\, dy$$

for every Borel set Δ, then $p(s,x,t,y)$ is called a *transition density* of the Markov process X.

Note that, according to (2.17), for every Markov process X there exists a Borel function φ such that

$$\mathbf{P}(X(t)\in\Delta|\mathcal{G}_s^s)=\varphi(X(s)) \qquad \text{a.s.}$$

The set of exclusivity, where this equality fails, has probability zero and may depend on s, t and Δ. Therefore, there is no guarantee that the function φ is defined for all s, t, and Δ simultaneously and has good properties with respect to these parameters. This explains the necessity of the conditions 1) and 2).

As to the condition 3), we have $P(s,X(s),s,\Delta)=\mathbb{1}_\Delta(X(s))$ a.s., but this does not guarantee that $P(s,x,s,\Delta)=\mathbb{1}_\Delta(x)$ for all $x\in\mathbf{R}$.

The origin of the condition 4) can be explained as follows. Since $P(s,X(s),t,\cdot)$ is the conditional distribution, using the Markov property and properties (2.15), (2.16) of the conditional expectation, we get for $s<v<t$

$$\begin{aligned}P(s,X(s),t,\Delta)&=\mathbf{P}(X(t)\in\Delta|\mathcal{G}_0^s)=\mathbf{E}\{\mathbf{P}(X(t)\in\Delta|\mathcal{G}_0^v)|\mathcal{G}_0^s\}\\ &=\mathbf{E}\{P(v,X(v),t,\Delta)|\mathcal{G}_s^s\}=\int_{-\infty}^{\infty}P(v,y,t,\Delta)\,P(s,X(s),v,dy) \qquad \text{a.s.} \qquad (6.8)\end{aligned}$$

Thus we have an analogue of 4) with a random variable $X(s)$ instead of a fixed x, and this equality holds a.s. It is necessary additionally to require the validity of such an equality for every fixed x.

From (6.5)–(6.7) it follows that for any bounded measurable function g

$$\mathbf{E}\{g(X(t))|\mathcal{G}_0^s\}=\mathbf{E}\{g(X(t))|\mathcal{G}_s^s\}=\int_{-\infty}^{\infty}g(y)P(s,X(s),t,dy) \qquad \text{a.s.} \qquad (6.9)$$

Suppose that for any fixed s, t, Δ the function $P(s,\cdot,t,\Delta)$ is right continuous. Then, due to (2.18) and (2.19),

$$\mathbf{P}(X(t)\in\Delta|X(s)=x)=P(s,x,t,\Delta). \qquad (6.10)$$

If for any bounded continuous function g the function

$$G(x):=\int_{-\infty}^{\infty}g(y)P(s,x,t,dy)$$

is continuous in x for all $s<t$, then, by (6.9) and (2.17)–(2.19), the equality

$$\mathbf{E}\{g(X(t))|X(s)=x\}=\int_{-\infty}^{\infty}g(y)P(s,x,t,dy) \qquad (6.11)$$

holds.

The finite-dimensional distributions of a Markov process can be expressed in terms of its transition function and the initial distribution.

Proposition 6.5. *Let* $0 = t_0 < t_1 < \cdots < t_n$, $\Delta_k \in \mathcal{B}(\mathbf{R})$, $k = 0, 1, \ldots, n$, *and let* $\mathcal{P}_{t_0}(\Delta) = \mathbf{P}(X(t_0) \in \Delta)$ *be the distribution of the initial value of the process. Then*

$$\mathbf{P}(X(t_0) \in \Delta_0, X(t_1) \in \Delta_1, X(t_2) \in \Delta_2, \ldots, X(t_n) \in \Delta_n)$$

$$= \int_{\Delta_0} \mathcal{P}_{t_0}(dx_0) \int_{\Delta_1} P(t_0, x_0, t_1, dx_1) \int_{\Delta_2} P(t_1, x_1, t_2, dx_2) \cdots$$

$$\cdots \int_{\Delta_{n-1}} P(t_{n-2}, x_{n-2}, t_{n-1}, dx_{n-1}) \int_{\Delta_n} P(t_{n-1}, x_{n-1}, t_n, dx_n). \tag{6.12}$$

Remark 6.2. If one takes for the initial moment a value $t_1 > 0$, then for the initial distribution $\mathcal{P}_{t_1}(\cdot)$ one should choose

$$\mathcal{P}_{t_1}(\Delta) = \int_{-\infty}^{\infty} \mathcal{P}_{t_0}(dx_0)\, P(t_0, x_0, t_1, \Delta), \qquad \Delta \in \mathcal{B}(\mathbf{R}).$$

Remark 6.3. If the moments $t_1, \ldots, t_n$ are not ordered, then the finite-dimensional distributions are determined by the condition of symmetry (permutability of the time moments). The Chapman–Kolmogorov equation implies the consistency condition (see § 3).

Remark 6.4. From (6.12) it follows that a Markov process is determined if its initial distribution and its transition function are given.

Proof of Proposition 6.5. We set

$$g_n(x) := 1, \quad g_{k-1}(x) := \int_{-\infty}^{\infty} g_k(y) \mathbb{1}_{\Delta_k}(y) P(t_{k-1}, x, t_k, dy), \quad k = 1, \ldots, n.$$

Then, taking into account (6.9), we have

$$\mathbf{P}\Big(\prod_{k=0}^{n} \{X(t_k) \in \Delta_k\}\Big) = \mathbf{E}\Big\{\prod_{k=0}^{n} \mathbb{1}_{\Delta_k}(X(t_k))\Big\}$$

$$= \mathbf{E}\Big\{\prod_{k=0}^{n-1} \mathbb{1}_{\Delta_k}(X(t_k)) \mathbf{E}\big\{\mathbb{1}_{\Delta_n}(X(t_n)) \big| \mathcal{G}_0^{t_{n-1}}\big\}\Big\} = \mathbf{E}\Big\{\prod_{k=0}^{n-1} \mathbb{1}_{\Delta_k}(X(t_k))\, g_{n-1}(X(t_{n-1}))\Big\}.$$

Using the iterations, we get

$$\mathbf{P}\Big(\prod_{k=0}^{n} \{X(t_k) \in \Delta_k\}\Big) = \mathbf{E}\Big\{\prod_{k=0}^{n-2} \mathbb{1}_{\Delta_k}(X(t_k))\, g_{n-2}(X(t_{n-2}))\Big\} = \cdots$$

$$=\mathbf{E}\{\mathbb{1}_{\Delta_0}(X(t_0))g_0(X(t_0))\}=\int\limits_{\Delta_0}\mathcal{P}(dx_0)g_0(x_0)=\int\limits_{\Delta_0}\mathcal{P}(dx_0)\int\limits_{\Delta_1}P(t_0,x,t_1,dx_1)g_1(x_1)$$

$$=\cdots=\int\limits_{\Delta_0}\mathcal{P}(dx_0)\int\limits_{\Delta_1}P(t_0,x,t_1,dx_1)\int\limits_{\Delta_2}P(t_1,x_1,t_2,dx_2)\cdots$$

$$\cdots\int\limits_{\Delta_{n-1}}P(t_{n-2},x_{n-2},t_{n-1},dx_{n-1})\int\limits_{\Delta_n}P(t_{n-1},x_{n-1},t_n,dx_n)g_n(x_n).$$

Thus formula (6.12) is proved. □

The following statement is a consequence of Kolmogorov's theorem (see Theorem 3.1 § 3).

Proposition 6.6. *Assume that $P(s,x,t,\Delta)$ satisfies conditions 1)–4). Then there exists a Markov process with the transition function $P(s,x,t,\Delta)$.*

Set $\Sigma = [0,\infty)$ or $\Sigma = \{0,1,2,\dots\}$. The Markov process $X(t)$, $t \in \Sigma$, is called *homogeneous* if its transition function $P(s,x,t,\Delta)$ is invariant under the shift along the time axis: $P(s+h,x,t+h,\Delta) = P(s,x,t,\Delta)$ for any $h \in \Sigma$. In this case we set $P(t,x,\Delta) := P(0,x,t,\Delta)$ and call $P(t,x,\Delta)$ a *transition function* of the homogeneous Markov process X.

In many situations we have to consider the whole family of Markov processes dependent on a nonrandom initial value.

For brevity we denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and the expectation with respect to the process X given $X(0) = x \in \mathbf{R}$, and we consider them as functions of x. This convention enables us, for example, to write $\mathbf{E}_{X(s)}$, where instead of the argument x appears the random variable $X(s)$, which does not mean the expectation with respect to the process, having initial value $X(s)$. With this convention, the equality (6.11) for the homogeneous Markov process X is rewritten as

$$G(x) := \mathbf{E}\{g(X(t))|X(s)=x\} = \int\limits_{-\infty}^{\infty} g(y)P(t-s,x,dy) = \mathbf{E}_x g(X(t-s)).$$

If for any bounded continuous g the function $G(x)$ is continuous in x for every $t-s$, then X is called a *Feller process.*

When studying Markov processes, the following question arises: does the independence of the future from the past hold when instead of a fixed present moment one takes a random time? The answer is affirmative if we restrict our consideration to stopping times as defined in § 4.

Let $\mathcal{G}_0^\tau$ be the σ-algebra of events generated by the process X up to the stopping time τ (see the definition in § 4).

A family of homogeneous Markov processes $X(t)$, $t \in [0,\infty)$, $X(0) = x \in \mathbf{R}$, is called a *strong Markov process* if

1) the process X is progressively measurable with respect to the natural filtration $\{\mathcal{G}_0^t\}$;

2) for a fixed Δ, the transition function $P(t,x,\Delta)$ is $\mathcal{B}([0,\infty))\times\mathcal{B}(\mathbf{R})$-measurable;

3) for any stopping time τ with respect to the natural filtration $\{\mathcal{G}_0^t\}$, any Borel set Δ, and any $t \ge 0$, $x \in \mathbf{R}$

$$\mathbf{P}_x(X(\tau+t) \in \Delta | \mathcal{G}_0^\tau) 1\!\!\mathrm{I}_{\{\tau<\infty\}} = P(t, X(\tau), \Delta) 1\!\!\mathrm{I}_{\{\tau<\infty\}} \qquad \mathbf{P}_x\text{-a.s.} \tag{6.13}$$

Here and in what follows $\mathbf{P}_x$-a.s. means a.s. with respect to the measure $\mathbf{P}_x$.

Proposition 6.7. *Let $X(t)$, $t \in [0,\infty)$, be a homogeneous strong Markov process. Then for any stopping time τ with respect to the filtration $\mathcal{G}_0^t$, for any bounded Borel function $g(\vec{x})$, $\vec{x} \in \mathbf{R}^m$, and any $\vec{t} \in [0,\infty)^m$, $x \in \mathbf{R}$*

$$\mathbf{E}_x\{g(X(\tau+t_1), \dots, X(\tau+t_m)) | \mathcal{G}_0^\tau\} 1\!\!\mathrm{I}_{\{\tau<\infty\}}$$
$$= \mathbf{E}_{X(\tau)} g(X(t_1), \dots, X(t_m)) 1\!\!\mathrm{I}_{\{\tau<\infty\}} \qquad \mathbf{P}_x\text{-a.s.} \tag{6.14}$$

For $m = 1$ this statement is derived from (6.13) by approximating of a bounded Borel function $g(x)$, $x \in \mathbf{R}$, by the functions $\sum_{k=1}^{n} g_{n,k} 1\!\!\mathrm{I}_{\Delta_{n,k}}(x)$.

For functions $g(\vec{x}) = \prod_{k=1}^{m} g_k(x_k)$ the proof is similar to that of Proposition 6.4, because $\mathcal{G}_0^{s+\tau} \subseteq \mathcal{G}_0^{t+\tau}$ for any $s < t$ (see property 9 of § 4). Any bounded Borel function $g(\vec{x})$, $\vec{x} \in \mathbf{R}^m$, is approximated by functions of the described form.

Important examples of separable metric spaces are $C([0,T])$, the space of continuous functions on $[0,T]$ with the uniform norm, and the *Skorohod space* $D([0,T])$. The latter consists of all real-valued right continuous functions on $[0,T]$ that have left-hand limits. We define a metric on $D([0,T])$ by the formula

$$\rho_T(x,y) := \inf_{\phi} \Big\{ \sup_{0 \le t \le T} |x(t) - y(\phi(t))| + \sup_{0 \le t \le T} |t - \phi(t)| \Big\},$$

where the infimum is taken over all monotone continuous mappings $\phi : [0,T] \to [0,T]$, satisfying $\phi(0) = 0$, $\phi(T) = T$.

The metric on $D([0,\infty))$ can be defined by

$$\rho_\infty(x,y) := \sum_{n=1}^{\infty} 2^{-n} \big(\rho_n(x,y) \wedge 1\big).$$

If it is known that the sample paths of a process X belong to the space $D([0,\infty))$, then Proposition 6.7 takes the following form.

Proposition 6.8. *Let $X(t)$, $t \in [0,\infty)$, be a homogeneous strong Markov process with sample paths from the space $D([0,\infty))$. Then for every $x \in \mathbf{R}$, any stopping time τ with respect to the filtration $\{\mathcal{G}_0^t\}$, and any bounded measurable functional $\wp(X(t), 0 \le t < \infty)$, defined on $D([0,\infty))$, we have*

$$\mathbf{E}_x\{\wp(X(\tau+t), 0 \le t < \infty) | \mathcal{G}_0^\tau\} 1\!\!\mathrm{I}_{\{\tau<\infty\}}$$
$$= \mathbf{E}_{X(\tau)} \wp(X(t), 0 \le t < \infty) 1\!\!\mathrm{I}_{\{\tau<\infty\}} \qquad \mathbf{P}_x\text{-a.s.}$$

The strong Markov property means that the process starts anew at each finite stopping time.

Proposition 6.9. *Every right continuous with left-hand limits Feller process is a strong Markov process.*

For the proof, see Dynkin (1960), p. 99, p. 104.

Every homogeneous Markov chain is a strong Markov process (see, for example, Shiryaev (1980)).

Example 6.1. Consider the example of a Markov process with discrete time (Markov chain).

Let ξ_n, $n = 0, 1, 2, \dots$, be a sequence of independent random variables. For $n = 1, 2, \dots$ we set

$$X_n := f_n(X_{n-1}, \xi_n), \qquad X_0 = \xi_0, \tag{6.15}$$

where $f_n(x, y)$, $(x, y) \in \mathbf{R}^2$ are Borel functions. Let $\mathcal{G}_m^k = \sigma(X_n, m \le n \le k)$ be the σ-algebra generated by X_n when $m \le n \le k$. Then X_n, $n = 0, 1, 2, \dots$, is a Markov process.

To prove this, we use Remark 6.1. For an arbitrary bounded Borel function $g(x)$, $x \in \mathbf{R}$, denote $h_n(x) := \mathbf{E} g(f_n(x, \xi_n))$. For every n the variable X_n is $\mathcal{G}_0^n$-measurable and the variable ξ_{n+1} is independent of the σ-algebra $\mathcal{G}_0^n$. Applying Lemma 2.1, we get

$$\mathbf{E}\big\{g(X_{n+1})\big|\mathcal{G}_0^n\big\} = \mathbf{E}\big\{g(f_{n+1}(X_n, \xi_{n+1}))\big|\mathcal{G}_0^n\big\} = h_{n+1}(X_n).$$

Applying Lemma 2.1 again, we similarly obtain

$$\mathbf{E}\big\{g(X_{n+1})\big|\mathcal{G}_n^n\big\} = \mathbf{E}\big\{g(f_{n+1}(X_n, \xi_{n+1}))\big|\mathcal{G}_n^n\big\} = h_{n+1}(X_n).$$

Therefore, by (6.5) with $v = t + 1$, the Markov property holds.

§ 7. Processes with independent increments

Let $\Sigma = [0, \infty)$ or $\Sigma = \{0, 1, 2, \dots\}$.

A process $X(t)$, $t \in \Sigma$, is called a *process with independent increments* if for any collection of increasing times $0 = t_0 < t_1 < t_2 < \cdots < t_n$ the variables $X(t_0)$, $X(t_1) - X(t_0)$, $\dots$,$X(t_n) - X(t_{n-1})$ are independent.

The differences $X(t) - X(s)$, $s < t$, are called the *increments of the process.*

Remark 7.1. In order to determine a process with independent increments it suffices to specify its initial distribution and the distributions of its increments.

Indeed, using the independence of increments, the characteristic function of the finite-dimensional distribution can be transformed as follows:

$$\mathbf{E}\exp\Big(i\sum_{k=0}^{n}\alpha_k X(t_k)\Big) = \mathbf{E}\exp\Big(i\sum_{k=0}^{n}\Big(\sum_{l=k}^{n}\alpha_l\Big)(X(t_k) - X(t_{k-1}))\Big)$$

$$= \prod_{k=0}^{n}\mathbf{E}\exp\Big(i\Big(\sum_{l=k}^{n}\alpha_l\Big)(X(t_k) - X(t_{k-1}))\Big),$$

where we set $X(t_{-1}){:=}0$. As a result, the distributions of the increments uniquely determine the characteristic function of the finite-dimensional distribution and therefore the finite-dimensional distribution itself.

A process with independent increments, $X(t)$, $t \in [0,\infty)$, is called *homogeneous* if $X(0) = x$, $x \in \mathbf{R}$, and the distributions of increments $X(u) - X(v)$, $v < u$, depend on v, u only via the difference $u - v$.

For the characteristic function of a homogeneous process with independent increments there holds the *Lévy–Khintchine formula*

$$\mathbf{E}e^{i\alpha(X(t)-X(0))} \tag{7.1}$$

$$= \exp\Big(it\alpha\gamma - \frac{1}{2}t\sigma^2\alpha^2 + t \int\limits_{0<|y|\le 1} \big(e^{i\alpha y} - 1 - i\alpha y\big)\varPi(dy) + t\int\limits_{|y|>1}\big(e^{i\alpha y} - 1\big)\varPi(dy)\Big),$$

where $\gamma \in \mathbf{R}$, $\sigma \in \mathbf{R}$, and $\varPi$ is a measure defined on $\big(\mathbf{R}\setminus[-\varepsilon,\varepsilon], \mathcal{B}\big(\mathbf{R}\setminus[-\varepsilon,\varepsilon]\big)\big)$ for every $\varepsilon > 0$ such that $\lim\limits_{\delta\downarrow 0} \int\limits_{\delta<|y|<\infty} \frac{y^2}{1+y^2}\,\varPi(dy) < \infty$.

The proof of this and of more general formulas for nonhomogeneous processes with independent increments can be found in Skorohod (1991).

Example 7.1. Let ξ_n, $n = 0, 1, 2, \dots$, be a sequence of independent random variables. Then the sequence of partial sums

$$X(n) = \sum_{k=1}^{n} \xi_k, \qquad X(0) = 0,$$

is a process with independent increments.

Example 7.2. A *Poisson process* with intensity $\lambda > 0$ is a process $N(t)$, $t \in [0,\infty)$, $N(0) = 0$, with independent increments having the Poisson distribution

$$\mathbf{P}(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k = 0, 1, 2, \dots.$$

Proposition 7.1. *The process $N(t)$ can be represented as follows:*

$$N(t) = \begin{cases} \max\Big\{l : \sum\limits_{k=1}^{l} \tau_k \le t\Big\}, & \text{for} \quad \tau_1 \le t, \\ 0, & \text{for} \quad \tau_1 > t, \end{cases}$$

where τ_k, $k = 1, 2, \dots$, are independent exponentially distributed with the parameter $\lambda > 0$ random variables, $\mathbf{P}(\tau_k \ge t) = e^{-\lambda t}$, $t \ge 0$.

The proof of Proposition 7.1 can be found, for example, in Karlin and Taylor (1981).

This proposition states that the sample paths of the Poisson process are arranged in the following way: they take zero value during the time τ_1, then the process increases by one and remains the same during the time τ_2, after that the process increases again by one and remains the same during the time τ_3, and so on.

In other words, the process $N(t)$, $t \geq 0$, describes the number of events occurring up to the time t if the events occur through independent random intervals, distributed exponentially with the parameter λ.

The Poisson process has an important practical application due to the following interpretation that underlies its structure.

We divide the time interval $[0, t]$ into small subintervals of the length Δ and then let their length go to zero. Suppose that on each interval $[k\Delta, (k+1)\Delta)$ the moment of the unit jump may occur independently with probability $\lambda\Delta$. Then in the limit as $\Delta \to 0$ the time of the first occurrence of a jump (the moment τ_1) has the exponential distribution with the parameter λ. Indeed,

$$\mathbf{P}(\tau_1 \geq t) \approx \bigl(1 - \lambda\Delta\bigr)^{[t/\Delta]} \approx e^{-\lambda t}.$$

Due to this interpretation of the occurrence of jumps, the consecutive moments of jumps occur through the independent time intervals distributed according to the exponential law. From here it follows also that if for a given nonrandom moment the process has some particular meaning, then regardless of when the previous jump occurred, the time until the subsequent moment of jump is again distributed exponentially with the parameter λ. This property is called *no aftereffects* or the *Markov property*, considered in the previous section.

Example 7.3. The *compound Poisson process* is the process

$$N_c(t) := \sum_{k=1}^{N(t)} Y_k, \qquad t \geq 0,$$

where Y_k, $k = 1, 2, \ldots$, are i.i.d. random variables independent of the process N.

In contrast to the Poisson process, here the sizes of the jumps are identically distributed random variables. The interpretation of the occurrence of moments of jumps is the same as for the Poisson process.

Let $F(y)$, $y \in \mathbf{R}$, be the distribution function of the variables Y_k, and $\varphi(\alpha)$, $\alpha \in \mathbf{R}$, be their characteristic function. Then the characteristic function of the increments of the process $N_c(t)$ can be computed as follows:

$$\mathbf{E}e^{i\alpha(N_c(t)-N_c(s))} = \mathbf{E}\exp\Biggl(i\alpha \sum_{k=1}^{N(t-s)} Y_k\Biggr) = e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{(\lambda(t-s))^k}{k!} \varphi^k(\alpha)$$

$$= e^{\lambda(t-s)(\varphi(\alpha)-1)} = \exp\Biggl(\lambda(t-s)\int_{-\infty}^{\infty} (e^{i\alpha y} - 1)\, dF(y)\Biggr). \qquad (7.2)$$

This is a particular case of the Lévy–Khintchine formula (7.1).

The most important example of a continuous process with independent increments is Brownian motion, which is considered in § 10.

Proposition 7.2. *Every process with independent increments is a Markov process.*

Proof. Let X be a process with independent increments. We use Proposition 6.4. For any bounded Borel function $g(x)$, $x \in \mathbf{R}$, and any fixed $t < v$ we denote $h(x) := \mathbf{E}g(X(v) - X(t) + x)$. Set $\mathcal{G}_s^t := \sigma(X(v), s \le v \le t)$. The variable $X(t)$ is $\mathcal{G}_0^t$-measurable and the variable $X(v) - X(t)$ is independent of the σ-algebra $\mathcal{G}_0^t$. Applying Lemma 2.1, we get

$$\mathbf{E}\{g(X(v))\big|\mathcal{G}_0^t\} = \mathbf{E}\{g(X(v) - X(t) + X(t))\big|\mathcal{G}_0^t\} = h(X(t)).$$

Applying Lemma 2.1 again, we analogously obtain

$$\mathbf{E}\{g(X(v))\big|\mathcal{G}_t^t\} = \mathbf{E}\{g(X(v) - X(t) + X(t))\big|\mathcal{G}_t^t\} = h(X(t)).$$

This proves the Markov property (6.5).

For $g(z) = \mathbb{1}_\Delta(z)$ the last equality implies that the transition probability is given by

$$P(t, x, v, \Delta) = \mathbf{P}(X(v) - X(t) + x \in \Delta). \tag{7.3}$$

□

Proposition 7.3. *Let $X(t)$, $t \ge 0$, be a right continuous homogeneous process with independent increments. Then X is a strong Markov process.*

Proof. A right continuous process is progressively measurable with respect to the natural filtration σ-algebras $\{\mathcal{G}_0^t\}$. We prove (6.14), from which (6.13) obviously follows. For this proof it suffices to consider an arbitrary continuous bounded function $g(\vec{y})$, $\vec{y} \in \mathbf{R}^m$.

Note that $\{\tau < \infty\} \in \mathcal{G}_0^\tau$. Therefore, instead of (6.14) it suffices to prove that for all $\vec{t} \in [0, \infty)^m$ and any $A \in \mathcal{G}_0^\tau$,

$$\begin{aligned}&\mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} g(X(\tau + t_1), \dots, X(\tau + t_m))\}\\ &= \mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} \mathbf{E}_{X(\tau)} g(X(t_1), \dots, X(t_m))\}.\end{aligned} \tag{7.4}$$

For every $s \ge 0$, we set

$$h(y) := \mathbf{E}_x g(X(s + t_1) - X(s) + y, \dots, X(s + t_m) - X(s) + y).$$

By the homogeneity and independence of increments, we have

$$\mathbf{E}_y g(X(t_1), \dots, X(t_m)) = \mathbf{E}_x g(X(t_1) - X(0) + y, \dots, X(t_m) - X(0) + y) = h(y).$$

Therefore (7.4), is rewritten in the form

$$\mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} g(X(\tau + t_1), \dots, X(\tau + t_m))\} = \mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} h(X(\tau)). \tag{7.5}$$

Applying Lemma 2.1, we obtain

$$\mathbf{E}_x\{g(X(s + t_1), \dots, X(s + t_m))\big|\mathcal{G}_0^s\}$$

$$= \mathbf{E}_x\big\{g(X(s+t_1) - X(s) + X(s), \dots, X(s+t_m) - X(s) + X(s))\big|\mathcal{G}_0^s\big\} = h(X(s))$$

a.s. with respect to the measure $\mathbf{P}_x$.

Consider the random variables

$$\tau_n := \sum_{k=1}^{\infty} k2^{-n} 1\!\!1_{\Omega_{k,n}}, \qquad n \in \mathbb{N},$$

where $\Omega_{1,n} = \{\tau \le 2^{-n}\}$, $\Omega_{k,n} = \{(k-1)2^{-n} < \tau \le k2^{-n}\}$ for $k = 2, 3, \dots$. It is obvious that $\tau_n \downarrow \tau$ as $n \to \infty$ for all $\omega \in \{\tau < \infty\}$. In addition, for every $n \in \mathbb{N}$ the variable τ_n is a stopping time with respect to the filtration $\{\mathcal{G}_0^t\}$, since $\{\tau_n \le t\} = \{\tau \le [t2^n]2^{-n}\} \in \mathcal{G}_0^{[t2^n]2^{-n}} \subseteq \mathcal{G}_0^t$, where $[a]$ denotes the integer part of a.

By the definition of the σ-algebra $\mathcal{G}_0^\tau$, we have

$$A \bigcap \{\tau_n = k2^{-n}\} = A \bigcap \Omega_{k,n} \in \mathcal{G}_0^{k2^{-n}}.$$

Since $\{\tau < \infty\} = \bigcup_{k=1}^{\infty} \Omega_{k,n}$,

$$\mathbf{E}_x\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} g(X(\tau_n + t_1), \dots, X(\tau_n + t_m))\}$$

$$= \sum_{k=1}^{\infty} \mathbf{E}_x\{1\!\!1_A 1\!\!1_{\{\tau_n = k2^{-n}\}}\, g(X(k2^{-n} + t_1), \dots, X(k2^{-n} + t_m))\}$$

$$= \sum_{k=1}^{\infty} \mathbf{E}_x\{1\!\!1_{A\cap\{\tau_n = k2^{-n}\}}\, \mathbf{E}_x\{g(X(k2^{-n} + t_1), \dots, X(k2^{-n} + t_m))|\mathcal{G}_0^{k2^{-n}}\}\}$$

$$= \sum_{k=1}^{\infty} \mathbf{E}_x\{1\!\!1_{A\cap\{\tau_n = k2^{-n}\}}\, h(X(k2^{-n})\} = \mathbf{E}\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} h(X(\tau_n))\}. \tag{7.6}$$

The process X is right continuous, therefore $X(\tau_n + t) \to X(\tau + t)$ a.s. for every $t \ge 0$. Since the function $h(y)$ is continuous and bounded together with $g(\vec{y})$, by the Lebesgue dominated convergence theorem, we have

$$\mathbf{E}\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} g(X(t_1 + \tau), \dots, X(t_m + \tau))\}$$

$$= \lim_n \mathbf{E}\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} g(X(t_1 + \tau_n) \dots, X(t_m + \tau_n))\}$$

and

$$\mathbf{E}\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} h(X(\tau))\} = \lim_n \mathbf{E}\{1\!\!1_A 1\!\!1_{\{\tau<\infty\}} h(X(\tau_n))\}.$$

Now, passing in (7.6) to the limit as $\tau_n \downarrow \tau$, we get (7.5). □

We have actually proved the following statement.

Proposition 7.4. *Let τ be an a.s. a finite stopping time with respect to the natural filtration $\{\mathcal{G}_0^t\}$ of the right continuous homogeneous process with independent increments $X(t)$, $t \ge 0$, $X(0) = 0$. Then $X(t+\tau) - X(\tau)$, $t \ge 0$, is a process independent of the σ-algebra $\mathcal{G}_0^\tau$ and identical in law to X.*

Indeed, regarding a function g as a function of increments of the process, we only need to prove the following analogue of (7.4):

$$\mathbf{E}\{\mathbb{1}_A g(X(t_1+\tau) - X(\tau), \dots, X(t_m+\tau) - X(\tau))\}$$

$$= \mathbf{E}\{\mathbb{1}_A\}\mathbf{E}g(X(t_1) - X(0), \dots, X(t_m) - X(0))\}.$$

Obviously, the analogue of (7.6) is

$$\mathbf{E}\{\mathbb{1}_A g(X(t_1+\tau_n) - X(\tau_n), \dots, X(t_m+\tau_n) - X(\tau_n))\}$$

$$= \mathbf{E}\{\mathbb{1}_A\}\mathbf{E}\{g(X(t_1) - X(0), \dots, X(t_m) - X(0))\},$$

and this, after passage to the limit, gives the required result.

Proposition 7.5. *A separable stochastically continuous process with independent increments* a.s. *has no discontinuities of the second kind.*

For the proof see Skorohod (1991).

§ 8. Gaussian processes

A random variable X is called *Gaussian or normally distributed* with an average m and a variance σ^2 if its density is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}, \qquad x \in \mathbf{R}. \tag{8.1}$$

Its distribution is also uniquely determined by the characteristic function

$$\varphi_X(\alpha) = \mathbf{E}e^{i\alpha X} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\alpha x} e^{-(x-m)^2/2\sigma^2}\, dx = e^{i\alpha m - \alpha^2\sigma^2/2}, \quad \alpha \in \mathbf{R}. \tag{8.2}$$

A stochastic process $X(t)$, $t \in [0,T]$, is called *Gaussian* if its finite-dimensional distributions are Gaussian, i.e., for any $t_1 < t_2 < \dots < t_n$ the characteristic function of the random vector $\vec{X} := (X(t_1), X(t_2), \dots, X(t_n))$ has the form

$$\varphi_{\vec{X}}(\vec{\alpha}) = \mathbf{E}\exp\big(i(\vec{\alpha}, \vec{X})\big) = \exp\Big(i(\vec{\alpha}, \vec{m}) - \frac{1}{2}(R\vec{\alpha}, \vec{\alpha})\Big)$$

$$= \exp\Big(i\sum_{k=1}^n \alpha_k m_k - \frac{1}{2}\sum_{k=1}^n\sum_{l=1}^n \alpha_k\alpha_l r_{k,l}\Big), \qquad \vec{\alpha} \in \mathbf{R}^n,$$

where $\vec{m} = (m_1, m_2, \dots, m_n)$ is the vector of expectations, $m_k := \mathbf{E}X(t_k)$, and $R = \{r_{k,l}\}_{k,l=1}^n$ is the covariance matrix,

$$r_{k,l} := \operatorname{Cov}(X(t_k), X(t_l)) = \mathbf{E}\big\{(X(t_k) - m_k)(X(t_l) - m_l)\big\}.$$

The covariance matrix is a symmetric positive semi-definite matrix, because for any real vector $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$

$$(R\vec{\alpha}, \vec{\alpha}) = \sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l r_{k,l}$$

$$= \mathbf{E} \sum_{k=1}^n \sum_{l=1}^n \alpha_k \alpha_l (X(t_k) - m_k)(X(t_l) - m_l) = \mathbf{E}\bigg(\sum_{k=1}^n \alpha_k (X(t_k) - m_k)\bigg)^2 \geq 0.$$

The characteristic function uniquely determines the finite-dimensional distributions of the vector $\vec{X}$. Thus the finite-dimensional distributions of the Gaussian process X are uniquely determined by the expectations $\mathbf{E}X(t)$, $t \in [0, T]$, and the covariance function $\operatorname{Cov}(X(s), X(t))$, $(s, t) \in [0, T] \times [0, T]$.

In the nondegenerate case, in which the covariance matrix is strongly positive definite $\big((R\vec{\alpha}, \vec{\alpha}) = 0$ only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0\big)$, the Gaussian distribution of the vector $\vec{X}$ has the density

$$f(x_1, x_2, \dots, x_n) = \frac{|A|^{1/2}}{(2\pi)^{n/2}} \exp\bigg(-\frac{1}{2} \sum_{k,l=1}^n a_{k,l}(x_k - m_k)(x_l - m_l)\bigg), \qquad \vec{x} \in \mathbf{R}^n,$$

where $|A| := \det A$ and $A = \{a_{k,l}\}_{k,l=1}^n$ is the inverse of the covariance matrix R.

The general form of the two-dimensional density of the Gaussian vector (X_1, X_2) is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}} \exp\Big(-\frac{(x_1 - m_1)^2}{2(1 - r^2)\sigma_1^2} + \frac{r(x_1 - m_1)(x_2 - m_2)}{(1 - r^2)\sigma_1\sigma_2} - \frac{(x_2 - m_2)^2}{2(1 - r^2)\sigma_2^2}\Big), \tag{8.3}$$

where $m_1 = \mathbf{E}(X_1)$, $m_2 = \mathbf{E}(X_2)$, $\sigma_1^2 = \operatorname{Var}(X_1)$, $\sigma_2^2 = \operatorname{Var}(X_2)$, $r = \dfrac{\operatorname{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$.

Proposition 8.1. *For a Gaussian random vector $\vec{X} = (X(t_1), \dots, X(t_n))$ the noncorrelatedness of the coordinates* $(\operatorname{Cov}(X(t_k), X(t_l)) = 0,\ k \neq l)$ *is equivalent to their independence.*

Proof. If the coordinates of the Gaussian random vector $\vec{X}$ are uncorrelated, then the matrix R is diagonal and the characteristic function $\varphi_{\vec{X}}(\vec{\alpha})$ has the form

$$\varphi_{\vec{X}}(\vec{\alpha}) = \prod_{k=1}^n \varphi_{X(t_k)}(\alpha_k),$$

i.e., it is equal to the product of the characteristic functions of the coordinates. Therefore, the coordinates are independent. The converse is obvious. □

Proposition 8.2. *A stochastic process X is Gaussian iff for any $t_k \in [0,T]$ and any $\alpha_k \in \mathbf{R}$, $k = 1,2,\dots,n$, the linear combination*

$$L := \alpha_1 X(t_1) + \alpha_2 X(t_2) + \cdots + \alpha_n X(t_n)$$

is a Gaussian random variable.

Proof. Let R and $\vec{m}$ be the covariance matrix and the expectation of the random vector $\vec{X} = (X(t_1), X(t_2), \dots, X(t_n))$. Set $\vec{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $L = (\vec{\alpha}, \vec{X})$. It is clear that $m := \mathbf{E}L = (\vec{\alpha}, \vec{m})$ and $\sigma^2 := \mathbf{E}(L - \mathbf{E}L)^2 = (R\vec{\alpha}, \vec{\alpha})$.

If X is a Gaussian process, then for arbitrary real γ

$$\mathbf{E}e^{i\gamma L} = \mathbf{E}\exp\big(i\gamma(\vec{\alpha}, \vec{X})\big) = \exp\Big(i\gamma(\vec{\alpha}, \vec{m}) - \frac{\gamma^2}{2}(R\vec{\alpha}, \vec{\alpha})\Big) = \mathbf{E}\exp\Big(i\gamma m - \frac{\gamma^2\sigma^2}{2}\Big).$$

This implies that L is a Gaussian random variable.

The opposite implication follows easily from the equalities

$$\mathbf{E}\exp\big(i(\vec{\alpha}, \vec{X})\big) = \mathbf{E}e^{iL} = \mathbf{E}\exp\Big(im - \frac{\sigma^2}{2}\Big) = \exp\Big(i(\vec{\alpha}, \vec{m}) - \frac{1}{2}(R\vec{\alpha}, \vec{\alpha})\Big).$$

□

Proposition 8.3. *Let $X(s)$, $s \in [0,T]$, $X(0) = x$, be a Gaussian process. Then its finite-dimensional distributions are uniquely determined by the family of one-dimensional distributions of the increments $X(u) - X(v)$ for all $0 \le v < u \le T$.*

Proof. The proof of this statement can be based on the fact that for such a process the covariances are expressed in terms of the expectations and the variances of the increments. Since $X(0) = x$, we have $\mathbf{E}X(u) = x + \mathbf{E}(X(u) - X(0))$ and

$$\mathbf{E}X^2(u) = x^2 + 2x\mathbf{E}(X(u) - X(0)) + \mathbf{E}(X(u) - X(0))^2.$$

The covariance function can be expressed as

$$\begin{aligned}\mathrm{Cov}(X(v), X(u)) &= \mathbf{E}(X(v)X(u)) - \mathbf{E}X(v)\mathbf{E}X(u)\\ &= \frac{1}{2}\big(\mathbf{E}X^2(v) + \mathbf{E}X^2(u) - \mathbf{E}(X(u) - X(v))^2\big) - \mathbf{E}X(v)\mathbf{E}X(u).\end{aligned}$$

Thus the covariance function is expressed via the expectations of increments and the expectation of the square of increments. □

Proposition 8.4. *Let X_n, $n = 1,2,\dots$, be a sequence of Gaussian random variables that converges in probability to a variable X. Then X is also a Gaussian random variable.*

Proof. Let $m_n = \mathbf{E}X_n$, $\sigma_n^2 = \mathbf{D}X_n$. For an arbitrary $\alpha \in \mathbf{R}$, by the Lebesgue dominated convergence theorem

$$\mathbf{E}e^{i\alpha X} = \lim_{n\to\infty} \mathbf{E}e^{i\alpha X_n} = \lim_{n\to\infty} e^{i\alpha m_n - \alpha^2\sigma_n^2/2}, \qquad \alpha \in \mathbf{R}.$$

This implies the existence of the limits $m := \lim_{n\to\infty} m_n$, $\sigma^2 := \lim_{n\to\infty} \sigma_n^2$. Consequently, $\mathbf{E}e^{i\alpha X} = e^{i\alpha m - \alpha^2\sigma^2/2}$. □

Corollary 8.1. *Let $X_n(t)$, $t \in \Sigma$, $n = 1, 2, \dots$, be a sequence of Gaussian processes. Suppose that $X_n(t) \to X(t)$ in probability for every $t \in \Sigma$. Then the process X is Gaussian.*

This statement follows from Propositions 8.2 and 8.4, since the limit of an arbitrary linear combination $\sum_{k=1}^{l} \alpha_k X_n(t_k)$, $\alpha_k \in \mathbf{R}$, is the linear combination $\sum_{k=1}^{l} \alpha_k X(t_k)$, which has a Gaussian distribution.

§ 9. Stationary processes

Set $\Sigma := [0, \infty)$ or $\Sigma = (-\infty, \infty)$ in the case of continuous time, and $\Sigma := \{0, 1, 2, \dots\}$ or $\Sigma = \{0, \pm 1, \pm 2, \dots\}$ in the case of discrete time.

A process $X(t)$, $t \in \Sigma$, is said to be *strictly stationary* if for arbitrary $t_k \in \Sigma$, $k = 1, \dots, n$, and any $t \in \Sigma$ the finite-dimensional distribution of the random vector $\big(X(t_1 + t), X(t_2 + t), \dots, X(t_n + t)\big)$ is independent of t.

In other words, a strictly stationary process is a process whose finite-dimensional distributions are invariant under any shift of the parameter belonging to the parameter set.

Among the moment characteristics of distributions of a stochastic process the first two moments: $m(t) := \mathbf{E}X(t)$, the expectation (mean), and

$$R(s,t) := \mathrm{Cov}(X(s), X(t)) = \mathbf{E}\big((X(s) - m(s))(X(t) - m(t))\big),$$

the *covariance function* (*correlation function*), are of particular importance.

For a strictly stationary processes it is obvious that $m(t)$ is constant as a function of $t \in \Sigma$ and $R(s,t) = R(0, t-s)$ depends only on the difference of the arguments for all $s, t \in \Sigma$. This is due to the fact that the shift of the parameter does not change the one-dimensional and the two-dimensional distributions. As a rule, one sets $R(t) := R(0, t - s)$ and calls $R(t)$, $t \in \Sigma$, the covariance function of the stationary process.

Often, only such conditions on the moments are realized, although the process is not a strictly stationary.

A process $X(t)$, $t \in \Sigma$, is said to be a *wide sense stationary* if $\mathbf{E}|X(t)|^2 < \infty$, $m(t) = m$ for all $t \in \Sigma$, and $R(s,t) = R(0, t - s)$ for all $s, t \in \Sigma$.

Remark 9.1. A Gaussian wide sense stationary process is a strictly stationary.

Indeed, the finite-dimensional distributions of a Gaussian process are uniquely determined by the mean and the covariance function.

Example 9.1. Let $A > 0$, $\alpha > 0$, and φ be independent random variables and φ be uniformly distributed in $[0, 2\pi]$. Then the process $X(t) := A\sin(\alpha t + \varphi)$, $t \in \mathbf{R}$, (a random sinusoid) is strictly stationary.

The concepts of stationary process in a strict and in a wide sense can be extended to the complex-valued processes X if the covariance function is defined by

$$R(s,t) := \mathbf{E}\big((X(s)-m(s))\overline{(X(t)-m(t))}\big),$$

where the bar denotes complex conjugation. The determining conditions for a wide sense stationary complex-valued process are $m(t) = m$ for all $t \in \Sigma$ and $R(s,t) = R(t-s)$ for all $s,t \in \Sigma$.

In order to describe the structure of a wide sense stationary complex-valued process it is necessary to introduce the integral of a nonrandom measurable function with respect to the orthogonal stochastic measure defined below.

Let $\mathfrak{M}$ be the collection of sets consisting of all intervals of the form $\Delta = [a,b)$.

Suppose that $G(\cdot)$ is a σ-additive measure defined on $\mathfrak{M}$. Such measure can be extended to a measure on the σ-algebra of Borel sets.

Let a complex random variable $Z(\Delta)$ be associated to each $\Delta \in \mathfrak{M}$ so that the following conditions are satisfied:

1) $\mathbf{E}|Z(\Delta)|^2 < \infty$, $Z(\varnothing) = 0$, where $\varnothing$ is the empty set;

2) for any disjoint sets Δ_1 and Δ_2 from $\mathfrak{M}$,

$$Z(\Delta_1 \bigcup \Delta_2) = Z(\Delta_1) + Z(\Delta_2) \qquad \text{a.s.};$$

3) $\mathbf{E}\{Z(\Delta_1)\overline{Z(\Delta_2)}\} = G(\Delta_1 \bigcap \Delta_2)$ (orthogonality property).

The family of random variables $\{Z(\Delta)\}$, $\Delta \in \mathfrak{M}$ is called an *orthogonal stochastic measure* and $G(\Delta)$ is called its *structure function.*

Consider a class $S(\mathfrak{M})$ of simple complex-valued functions of the form

$$f(y) = \sum_{k=1}^{m} b_k \mathrm{I\!I}_{\Delta_k}(y), \qquad \Delta_k \in \mathfrak{M}, \tag{9.1}$$

where b_k, $k = 1,2,\dots,m$, are some complex constants.

The stochastic integral of $f \in S(\mathfrak{M})$ with respect to the orthogonal stochastic measure Z is defined by the formula

$$I(f,Z) := \int_{-\infty}^{\infty} f(y)\, Z(dy) := \sum_{k=1}^{m} b_k Z(\Delta_k). \tag{9.2}$$

Any two functions f and g from $S(\mathfrak{M})$ can be written as a linear combination of indicator functions of the same disjoint sets. One can assume that

$$g(y) = \sum_{k=1}^{m} c_k \mathrm{I\!I}_{\Delta_k}(y), \qquad \Delta_k \in \mathfrak{M},\ k = 1,2.\dots,m, \tag{9.3}$$

where $\Delta_k \bigcap \Delta_l = \varnothing$ for $k \neq l$.

By the orthogonality property of the stochastic measure $Z(\cdot)$,

$$\mathbf{E}\Bigl(\int_{\mathbf{R}} f(y)\,Z(dy)\overline{\int_{\mathbf{R}} g(y)\,Z(dy)}\Bigr)=\sum_{k=1}^{m} b_k\bar{c}_k G(\Delta_k)=\int_{-\infty}^{\infty} f(y)\overline{g(y)}\,G(dy). \tag{9.4}$$

Let $L^2(G(dy))$ be the space of complex-valued functions whose modulus squared is integrable with respect to G. The complex-valued random variables $I(f, Z)$ for $f \in S(\mathfrak{M})$ belong to the space $L^2(\mathbf{P}(d\omega))$ of complex-valued random variables with finite absolute second moment. This space is equipped with the scalar product $(\mu, \bar{\eta}) = \mathbf{E}\{\mu\bar{\eta}\}$. The class of simple functions $S(\mathfrak{M})$ is dense in $L^2(G(dy))$. By (9.4), the mapping (9.2) is a linear isometric mapping from $S(\mathfrak{M})$ into $L^2(\mathbf{P}(d\omega))$, therefore it can be extended uniquely to a linear isometric mapping from the whole space $L^2(G(dy))$ into $L^2(\mathbf{P}(d\omega))$. Thus for any function f in $L^2(G(dy))$ the integral $\int_{\mathbf{R}} f(y)\,Z(dy)$ is well defined. For more information on the extension by isometry see § 1 Ch. II, where the construction of a stochastic integral with respect to a Brownian motion is considered. According to this extension, for any $f \in L^2(G(dy))$ and any sequence of simple functions $f_n \in S(\mathfrak{M})$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(y) - f_n(y)|^2\,G(dy) = 0,$$

the relation

$$\int_{-\infty}^{\infty} f(y)\,Z(dy) = \mathrm{l.\,i.\,m.} \int_{-\infty}^{\infty} f_n(y)\,Z(dy) \tag{9.5}$$

holds, where l. i. m. denotes the limit in mean square.

The integral (9.5) defines an orthogonal stochastic measure

$$Z(\Delta) := \int_{-\infty}^{\infty} \mathbb{1}_{\Delta}(y)\,Z(dy), \qquad \Delta \in \mathcal{B}(\mathbf{R}),$$

because, by isometry, we have the equality

$$\mathbf{E}\{Z(\Delta_1)\overline{Z(\Delta_2)}\} = \int_{-\infty}^{\infty} \mathbb{1}_{\Delta_1}(y)\mathbb{1}_{\Delta_2}(y)\,G(dy) = G\bigl(\Delta_1 \bigcap \Delta_2\bigr). \tag{9.6}$$

Let $\Sigma = \mathbf{R}$. We provide a description of covariance functions as well as a wide sense complex-valued stationary processes themselves that are continuous in mean square.

Let $R(t)$, $t \in \mathbf{R}$, be the covariance function of X. If $\mathbf{E}X(t) = 0$, then for every $t \in \mathbf{R}$

$$|R(t+h)-R(t)|=|\mathbf{R}((X(t+h)-X(t))\overline{X(0)})| \le \mathbf{E}^{1/2}|X(t+h)-X(t)|^2\mathbf{E}^{1/2}|X(0)|^2.$$

Therefore, if the process X is continuous in mean square, then the covariance function is continuous.

Theorem 9.1 (Bochner–Khintchine). *Let $R(t)$, $t \in \mathbf{R}$, be a covariance function continuous at zero. Then*

$$R(t) = \int_{-\infty}^{\infty} e^{ity}\, dG(y), \qquad t \in \mathbf{R}, \tag{9.7}$$

where $G(y)$, $y \in \mathbf{R}$, is a bounded right continuous nondecreasing function.

The function $G(y)$, $y \in \mathbf{R}$, is called the *spectral function* of the stationary process X. With the function G one can associate the measure defined on the sets $[a, b)$ by the formula $G([a, b)) := G(b) - G(a)$, and in (9.7) the Stieljes integral can be replaced by the integral with respect to the measure $G(dy)$.

Theorem 9.2. *Let $X(t)$, $t \in \mathbf{R}$, be a centered ($\mathbf{E}X(t) = 0$) stationary process continuous in mean square. Then*

$$X(t) = \int_{-\infty}^{\infty} e^{ity}\, Z(dy), \qquad t \in \mathbf{R}, \qquad \text{a.s.}, \tag{9.8}$$

where $Z(dy)$ is an orthogonal stochastic measure such that $\mathbf{E}|Z(dy)|^2 = G(dy)$.

The measure Z is called the *spectral random measure* of the stationary process X.

Remark 9.2. Formula (9.8) informally represents the wide sense stationary process X as a "continuous" sum of mutually uncorrelated harmonic oscillations of different frequencies with random amplitudes. The spectral function determines the average of the squares of amplitudes.

Proof of Theorem 9.2. The family of functions $\mathcal{F}(\mathbf{R})$ that are finite linear combinations for different $t \in \mathbf{R}$ of the functions e^{ity}, $y \in \mathbf{R}$, is dense in $L^2(G(dy))$, since G is a finite measure. For any function $f(y) = \sum_{k=1}^{n} \alpha_k e^{it_k y}$, $y \in \mathbf{R}$, $\alpha_k \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, we define the mapping $\Psi : \mathcal{F}(\mathbf{R}) \to L^2(\mathbf{P}(d\omega))$ by the formula $\Psi(f) := \sum_{k=1}^{n} \alpha_k X(t_k)$. Let $\Psi(g) = \sum_{k=1}^{n} \beta_k X(t_k)$. By choosing the coefficients, we may assume that the linear combinations of values of the process X are taken at the same points $\{t_k\}$. By (9.7), the equalities

$$\mathbf{E}\big\{\Psi(f)\overline{\Psi(g)}\big\} = \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_k \bar{\beta}_l \mathbf{E}\big\{X(t_k)\overline{X(t_l)}\big\}$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_k \bar{\beta}_l \int_{-\infty}^{\infty} e^{i(t_k - t_l)y}\, dG(y) = \int_{-\infty}^{\infty} f(y)\overline{g(y)}\, dG(y)$$

hold. On the right-hand side there is the scalar product of the functions f and g in the space $L^2(G(dy))$, and on the left-hand side there is the scalar product of the random variables $\Psi(f)$ and $\Psi(g)$ in the space $L^2(\mathbf{P}(d\omega))$. Thus the mapping Ψ is isometric. We extend it by isometry to the whole space $L^2(G(dy))$. As a result, we get a linear isometric mapping $\Psi : L^2(G(dy)) \to L^2(\mathbf{P}(d\omega))$. If $\Delta \in \mathcal{B}(\mathbf{R})$, then $1\!\!\mathrm{I}_\Delta(y) \in L^2(G(dy))$. Therefore, we can set $Z(\Delta) := \Psi(1\!\!\mathrm{I}_\Delta)$. By the isometry, (9.6) holds. By linearity, for any disjoint sets Δ_1 and Δ_2 from $\mathcal{B}(\mathbf{R})$ one has that $Z(\Delta_1 \bigcup \Delta_2) = Z(\Delta_1) + Z(\Delta_2)$ holds a.s., since $1\!\!\mathrm{I}_{\Delta_1 \cup \Delta_2} = 1\!\!\mathrm{I}_{\Delta_1} + 1\!\!\mathrm{I}_{\Delta_2}$. Therefore, $Z(\cdot)$ is an orthogonal stochastic measure.

Set $Y(t) := \int\limits_{-\infty}^{\infty} e^{ity}\, Z(dy)$ and prove that the processes X and Y are modifications of each other, i.e., $\mathbf{P}(X(t) = Y(t)) = 1$ for every $t \in \mathbf{R}$. Indeed, in view of the isometric mapping Ψ, we have

$$\mathbf{E}\big\{X(t)\overline{Z(\Delta)}\big\} = \mathbf{E}\big\{\Psi(e^{it\cdot})\overline{\Psi(1\!\!\mathrm{I}_\Delta(\cdot))}\big\} = \int\limits_{-\infty}^{\infty} e^{ity}\, 1\!\!\mathrm{I}_\Delta(y)\, G(dy).$$

Hence, this equality is true for an arbitrary linear combination $\sum\limits_{k=1}^{m} b_k Z(\Delta_k)$ instead of $Z(\Delta)$. Now, by (9.5), this is true for the integral $Y(t)$:

$$\mathbf{E}\big\{X(t)\overline{Y(t)}\big\} = \int\limits_{-\infty}^{\infty} e^{ity} e^{-ity}\, G(dy) = R(0) = \mathbf{E}|X(t)|^2.$$

Hence, we get

$$\mathbf{E}|X(t)-Y(t)|^2 = \mathbf{E}\big\{(X(t)-Y(t))\overline{(X(t)-Y(t))}\big\} = \mathbf{E}\big\{X(t)\overline{X(t)}\big\} - \mathbf{E}\big\{X(t)\overline{Y(t)}\big\}$$

$$-\mathbf{E}\big\{Y(t)\overline{X(t)}\big\} + \mathbf{E}\big\{Y(t)\overline{Y(t)}\big\} = R(0) - R(0) - R(0) + R(0) = 0.$$

Thus $\mathbf{P}(X(t) = Y(t)) = 1$ for every $t \in \mathbf{R}$ and the theorem is proved. □

For a stationary sequence $X(k)$, $k \in \mathbb{Z}$, we have the following analogue of Theorem 9.2.

Theorem 9.3. *Let $X(k)$, $k \in \mathbb{Z}$, be a centered ($\mathbf{E}X(k) = 0$) stationary sequence. Then*

$$R(k) = \int\limits_{-\pi}^{\pi} e^{iky}\, dG(y), \qquad k \in \mathbb{Z}, \tag{9.9}$$

where $G(y)$ is a right continuous nondecreasing function, and

$$X(k) = \int\limits_{-\pi}^{\pi} e^{iky}\, Z(dy), \qquad k \in \mathbb{Z}, \qquad \text{a.s.}, \tag{9.10}$$

where $Z(dy)$ is an orthogonal stochastic measure such that $\mathbf{E}|Z(dy)|^2 = G(dy)$.

§ 10. Brownian motion process

A *Brownian motion* started at x is a stochastic process $W(t)$, $t \in [0,\infty)$, with $W(0) = x$ and with the finite-dimensional distributions

$$\mathbf{P}(W(t_1) \in \Delta_1, W(t_2) \in \Delta_2, \dots, W(t_n) \in \Delta_n) \tag{10.1}$$

$$= \int\limits_{\Delta_1} dx_1 \frac{e^{-(x_1-x)^2/2t_1}}{\sqrt{2\pi t_1}} \int\limits_{\Delta_2} dx_2 \frac{e^{-(x_2-x_1)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \int\limits_{\Delta_n} dx_n \frac{e^{-(x_n-x_{n-1})^2/2(t_n-t_{n-1})}}{\sqrt{2\pi(t_n-t_{n-1})}},$$

where $0 < t_1 < t_2 < \dots < t_n$, $\Delta_k \in \mathcal{B}(\mathbf{R})$, $k = 1,2,\dots,n$.

These finite-dimensional distributions determine a finite additive measure on the algebra of cylinder sets. The measure $\mathbf{P}_W$ that extends this measure from cylinder sets to the σ-algebra $\sigma(W(s), s \in [0,\infty))$, generated by the process W, is referred to as the *Wiener measure*.

The Brownian motion W can be considered on a finite time interval $[0,T]$. For the interval $[0,T]$, the Brownian motion is determined by the finite-dimensional distributions with the time from $[0,T]$ and the measure $\mathbf{P}_W^T$ associated with this process is defined on the σ-algebra $\sigma(W(s), s \in [0,T])$.

From (10.1) it follows that the random vector $(W(t_1), W(t_2), \dots, W(t_n))$ has the joint density

$$p_{\vec{t}}(\vec{x}) = \prod_{k=1}^{n} \varphi_{t_k - t_{k-1}}(x_k - x_{k-1}), \tag{10.2}$$

where $\varphi_t(x) = \dfrac{e^{-x^2/2t}}{\sqrt{2\pi t}}$ is the Gaussian density with mean 0 and variance t, $t_0 = 0$, $\vec{t} = (t_1,\dots,t_n)$, $x_0 = x$, $\vec{x} = (x_1,\dots,x_n) \in \mathbf{R}^n$. The vector of the increments

$$(W(t_1) - W(0), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1}))$$

has the joint density $q_{\vec{t}}(\vec{y}) = \prod\limits_{k=1}^{n} \varphi_{t_k - t_{k-1}}(y_k)$. Indeed, to compute this joint density one can use formula (1.11) with $\vec{g}(\vec{x}) = (x_1 - x, x_2 - x_1, \dots, x_n - x_{n-1})$, $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$. Here

$$\vec{g}^{\,(-1)}(\vec{y}) = \Big(x + y_1, x + y_1 + y_2, \dots, x + \sum_{k=1}^{n} y_k\Big),$$

and it is easy to see that $\det J_{\vec{g}^{\,(-1)}}(\vec{y}) = 1$.

Since this joint density is the product of the marginal densities, a Brownian motion is a process with independent increments.

From the expression for the marginal density it follows that for every $s < t$ the increment $W(t) - W(s)$ is normally distributed with

$$\mathbf{E}(W(t) - W(s)) = 0, \quad \mathbf{E}(W(t) - W(s))^2 = t - s. \tag{10.3}$$

The characteristic function of the increment has the form

$$\mathbf{E}e^{i\alpha(W(t)-W(s))} = e^{-\alpha^2(t-s)/2}, \qquad \alpha \in \mathbf{R}. \tag{10.4}$$

Notice that (10.4) is valid also for a complex α.

From (10.2) it follows that the Brownian motion W is a Gaussian process. Since $W(0) = x$, the process W has the mean $\mathbf{E}W(t) = x$ and the covariance function

$$\operatorname{Cov}(W(s), W(t)) := \mathbf{E}((W(s) - x)(W(t) - x)) = \min\{s, t\}. \tag{10.5}$$

By differentiating the characteristic function with respect to α, it is easy to compute all moments of the increments of Brownian motion. The odd moments are equal to zero, while the even moments are given by

$$\mathbf{E}(W(t) - W(s))^{2m} = (2m-1)!!(t-s)^m, \qquad m = 1, 2, \dots, \tag{10.6}$$

where $(2m-1)!!$ is the product of all odd numbers from 1 to $2m-1$.

Continuity. From (10.6) with $m = 2$ and Kolmogorov's continuity criterion it follows that the Brownian motion W has a continuous modification. It is natural to consider a Brownian motion with continuous paths. In order not to change the notation, we assume that the process W is itself continuous.

As a consequence, we can define a Brownian motion as follows:

A *Brownian motion*, $W(t)$, $t \in [0, \infty)$, started at x, is a continuous process with independent Gaussian increments, with mean zero and with the variance of an increment equal to the length of the interval for which the increment is considered.

The following construction of a Brownian motion W with the initial value $W(0) = 0$ was suggested by Paley and Wiener (1934).

Let ξ_k, $k = 1, 2, \dots$, be a sequence of independent Gaussian random variables with mean 0 and variance 1. Then the series

$$W(t) := \frac{t}{\sqrt{\pi}}\,\xi_0 + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} \frac{\sqrt{2}\sin(kt)}{k\sqrt{\pi}}\,\xi_k$$

converges a.s. uniformly in $t \in [0, \pi]$ and determines the Brownian motion for this interval. Such particular grouping of terms is needed to ensure the uniform convergence of the series.

The uniform convergence implies continuity of the process W, defined by this formula, and implies that it is a Gaussian process. Also, one can verify that

$$\mathbf{E}(W(s)W(t)) = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(ks)\sin(kt)}{k^2} = \min\{s, t\},$$

which is in accordance with (10.5).

For an estimation of the tail probabilities of increments of Brownian motion, i.e., the probabilities $\mathbf{P}(W(t) - W(s) > h)$, the following result is of key importance.

Lemma 10.1. *For any* $h > 0$

$$\frac{h}{h^2+1} e^{-h^2/2} < \int_h^\infty e^{-y^2/2}\,dy < \frac{1}{h} e^{-h^2/2}. \tag{10.7}$$

Proof. This result is a consequence of the relations

$$\int_h^\infty e^{-y^2/2}\,dy < \int_h^\infty \frac{y}{h} e^{-y^2/2}\,dy = \frac{1}{h} e^{-h^2/2}$$

$$= \int_h^\infty \Big(1 + \frac{1}{y^2}\Big) e^{-y^2/2}\,dy < \Big(1 + \frac{1}{h^2}\Big) \int_h^\infty e^{-y^2/2}\,dy.$$

□

Corollary 10.1. *As* $h/\sqrt{t-s} \to \infty$,

$$\mathbf{P}(W(t) - W(s) > h) = \int_h^\infty \frac{e^{-y^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\,dy \sim \frac{\sqrt{t-s}}{h\sqrt{2\pi}} e^{-h^2/2(t-s)}. \tag{10.8}$$

1. Characterizations of Brownian motion.

A *characterization by conditional characteristic function.*

Proposition 10.1. *A process* $X(t)$, $t \in [0,T]$, $X(0) = x$, *adapted to a filtration* $\{\mathcal{F}_t\}$, *is a Brownian motion if for any* $0 \le s < t \le T$ *and* $\alpha \in \mathbf{R}$,

$$\mathbf{E}\big\{e^{i\alpha\,(X(t)-X(s))}\big|\mathcal{F}_s\big\} = e^{-\alpha^2(t-s)/2} \quad \text{a.s.} \tag{10.9}$$

Proof. Taking the expectation of both sides of (10.9), we have

$$\mathbf{E}e^{i\alpha\,(X(t)-X(s))} = e^{-\alpha^2(t-s)/2},$$

and, consequently, the increments $X(t) - X(s)$ are normally distributed with mean zero and variance $t - s$. The even moments of these increments are given by (10.6). Therefore, by Kolmogorov's continuity criterion, it follows that X is a continuous process. Now it is sufficient to verify that increments of X are independent. For $0 = s_0 < s_1 < \cdots < s_m < s_{m+1}$ and $\alpha_k \in \mathbf{R}$, $k = 0, \dots, m$, we have

$$\mathbf{E}\exp\Big(i\sum_{k=0}^m \alpha_k(X(s_{k+1}) - X(s_k))\Big) = \mathbf{E}\Big\{\mathbf{E}\Big\{\exp\Big(i\sum_{k=0}^m \alpha_k(X(s_{k+1}) - X(s_k))\Big)\Big|\mathcal{F}_{s_m}\Big\}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(i\sum_{k=0}^{m-1} \alpha_k(X(s_{k+1}) - X(s_k))\Big)\mathbf{E}\Big\{\exp\Big(i\alpha_m(X(s_{m+1}) - X(s_m))\Big)\Big|\mathcal{F}_{s_m}\Big\}\Big\}$$

$$= \exp\Big(-\alpha_m^2(s_{m+1} - s_m)/2\Big)\mathbf{E}\exp\Big(i\sum_{k=0}^{m-1} \alpha_k(X(s_{k+1}) - X(s_k))\Big) = \dots$$

$$= \exp\Big(-\sum_{k=0}^m \alpha_k^2(s_{k+1} - s_k)/2\Big) = \prod_{k=0}^m \mathbf{E}\exp\Big(i\sum_{k=0}^m \alpha_k(X(s_{k+1}) - X(s_k))\Big).$$

This implies the independence of the increments of the process X and so, by definition, X is a Brownian motion. □

Lévy's characterization.

Theorem 10.1. *Let $X(t)$, $t \in [0,T]$, $X(0) = x$, be a continuous process adapted to a filtration $\{\mathcal{F}_t\}$. If for any $0 \le s < t \le T$,*

$$\mathbf{E}\{X(t) - X(s)|\mathcal{F}_s\} = 0 \qquad \text{a.s.,} \tag{10.10}$$

$$\mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\} = t - s \qquad \text{a.s.,} \tag{10.11}$$

then X is a Brownian motion.

Remark 10.1. Since equalities (10.10) and (10.11) can be written in the form

$$\mathbf{E}\{X(t)|\mathcal{F}_s\} = X(s), \qquad \mathbf{E}\{X^2(t) - t|\mathcal{F}_s\} = X^2(s) - s \quad \text{a.s.,} \tag{10.12}$$

Lévy's characterization can be formulated as follows: a continuous $\mathcal{F}_t$-adapted process $X(t)$, $X(0) = x$, is a Brownian motion if $X(t)$ and $X^2(t) - t$ are martingales with respect to the filtration $\{\mathcal{F}_t\}$. This is the *martingale characterization* of a Brownian motion.

Proof of Theorem 10.1. For arbitrary $h > 0$ and $t \in [0, T - h]$ we compute the conditional characteristic function of the variable $X(t+h) - X(t)$ with respect to the σ-algebra $\mathcal{F}_t$, i.e., $\mathbf{E}\big\{e^{i\alpha(X(t+h)-X(t))}\big|\mathcal{F}_t\big\}$, $\alpha \in \mathbf{R}$, and then apply Proposition 10.1.

Set $X_k^n := X\big(t + \frac{kh}{n}\big) - X\big(t + \frac{(k-1)h}{n}\big)$, $k = 1, 2, \dots, n$. For arbitrary $\alpha \in \mathbf{R}$,

$$\mathbf{E}\big\{\exp\big(i\alpha(X(t+h)-X(t))\big)\big|\mathcal{F}_t\big\} - \exp\big(-\alpha^2 h/2\big) \tag{10.13}$$

$$= \sum_{r=0}^{n-1} \mathbf{E}\bigg\{\exp\bigg(i\alpha\sum_{k=1}^{r+1} X_k^n\bigg) - \exp\bigg(i\alpha\sum_{k=1}^{r} X_k^n - \frac{h\alpha^2}{2n}\bigg)\bigg|\mathcal{F}_t\bigg\}\exp\Big(-\frac{n-r-1}{2n}h\alpha^2\Big).$$

In this sum, all terms except the first and the last ones cancel out. We represent the expectation of the difference of the exponents in the following form:

$$\mathbf{E}\bigg\{\exp\bigg(i\alpha\sum_{k=1}^{r} X_k^n\bigg)\Big(e^{i\alpha X_{r+1}^n} - e^{-h\alpha^2/2n}\Big)\bigg|\mathcal{F}_t\bigg\}$$

$$= \mathbf{E}\bigg\{\exp\bigg(i\alpha\sum_{k=1}^{r} X_k^n\bigg)\mathbf{E}\Big\{\Big(e^{i\alpha X_{r+1}^n} - e^{-h\alpha^2/2n}\Big)\Big|\mathcal{F}_{t+\frac{rh}{n}}\Big\}\bigg|\mathcal{F}_t\bigg\}$$

$$= \mathbf{E}\bigg\{\exp\bigg(i\alpha\sum_{k=1}^{r} X_k^n\bigg)\mathbf{E}\Big\{\Big(e^{\alpha X_{r+1}^n} - 1 - i\alpha X_{r+1}^n + \frac{\alpha^2}{2}(X_{r+1}^n)^2\Big)\Big|\mathcal{F}_{t+\frac{rh}{n}}\Big\}\bigg|\mathcal{F}_t\bigg\}$$

$$+\mathbf{E}\bigg\{\exp\bigg(i\alpha\sum_{k=1}^{r} X_k^n\bigg)\Big(1 - \frac{\alpha^2}{2n}h - e^{-h\alpha^2/2n}\Big)\bigg|\mathcal{F}_t\bigg\}. \tag{10.14}$$

To estimate the right-hand side of (10.14) we prove that

$$\mathbf{E}\Big\{(X(t+h)-X(t))^4\Big|\mathcal{F}_t\Big\}\le 3h^2. \tag{10.15}$$

Obviously,

$$\sum_{k=1}^{n}|X_k^n|^{2+\delta}\le \max_{1\le k\le n}|X_k^n|^{\delta}\sum_{k=1}^{n}(X_k^n)^2.$$

It is clear that $\max\limits_{1\le k\le n}|X_k^n|^{\delta}\to 0$ a.s., because of the continuity of X, while the variables $\sum\limits_{k=1}^{n}(X_k^n)^2$ are bounded in probability $\Big(\mathbf{P}\Big(\sum\limits_{k=1}^{n}(X_k^n)^2>C\Big)\underset{C\to\infty}{\longrightarrow}0\Big)$. Since $\mathbf{E}\sum\limits_{k=1}^{n}(X_k^n)^2=h$, we see that for any $\delta>0$ it holds that $\sum\limits_{k=1}^{n}|X_k^n|^{2+\delta}\to 0$ in probability. As a consequence we have

$$(X(t+h)-X(t))^4=\lim_{n\to\infty}\bigg\{\bigg(\sum_{k=1}^{n}X_k^n\bigg)^4+3\sum_{k=1}^{n}(X_k^n)^4-4\sum_{k=1}^{n}(X_k^n)^3\sum_{l=1}^{n}X_l^n\bigg\}. \tag{10.16}$$

Further we need an auxiliary result.

Proposition 10.2. *Let ξ and η be nonnegative random variables with $\mathbf{E}\xi<\infty$, $\mathbf{E}\eta<\infty$. Let ξ be $\mathcal{G}$-measurable and $\mathbf{E}\{\xi\mathbf{E}\{\eta|\mathcal{G}\}\}<\infty$.*

Then $\mathbf{E}\{\xi\eta\}<\infty$.

Proof. Set $\xi_N:=\xi\mathbb{1}_{\{\xi<N\}}+N\mathbb{1}_{\{\xi\ge N\}}$. It is clear that ξ_N is a $\mathcal{G}$-measurable increasing sequence of functions and $\lim\limits_{N\to\infty}\xi_N\uparrow\xi$. By the Lebesgue dominated convergence theorem,

$$\mathbf{E}\{\xi\eta\}=\lim_{N\to\infty}\mathbf{E}\{\xi_N\eta\}=\lim_{N\to\infty}\mathbf{E}\{\mathbf{E}\{\xi_N\eta|\mathcal{G}\}\}$$

$$=\lim_{N\to\infty}\mathbf{E}\{\xi_N\mathbf{E}\{\eta|\mathcal{G}\}\}=\mathbf{E}\{\xi\mathbf{E}\{\eta|\mathcal{G}\}\}<\infty.$$

□

On the right-hand side of (10.16) there are no terms $(X_k^n)^4$ and $(X_k^n)^3X_l^n$, because if we raise the first term to the fourth power, they disappear.

As to other terms we can apply Proposition 10.2, since by the Hölder inequality

$$\mathbf{E}\{|X_k^n||\mathcal{F}_{t+(k-1)h/n}\}\le(\mathbf{E}\{(X_k^n)^2|\mathcal{F}_{t+(k-1)h/n}\})^{1/2}\le(h/n)^{1/2}.$$

Using (10.11), we have

$$\mathbf{E}\{(X_k^n)^2(X_l^n)^2|\mathcal{F}_t\}=\mathbf{E}\{\mathbf{E}\{(X_k^n)^2(X_l^n)^2|\mathcal{F}_{t+kh/n}\}|\mathcal{F}_t\}$$

$$=\mathbf{E}\{(X_k^n)^2\mathbf{E}\{(X_l^n)^2|\mathcal{F}_{t+kh/n}\}|\mathcal{F}_t\}=(h/n)^2,\qquad k<l.$$

Using (10.10) and (10.11), we see that for $k < j < l < r$ or $k = j < l < r$ or $k < j = l < r$,

$$\mathbf{E}\{X_k^n X_j^n X_l^n X_r^n|\mathcal{F}_t\} = \mathbf{E}\{X_k^n X_j^n X_l^n \mathbf{E}\{X_r^n|\mathcal{F}_{t+(r-1)h/n}\}|\mathcal{F}_t\}\} = 0,$$

while for $k < j < l$

$$\mathbf{E}\{X_k^n X_j^n (X_l^n)^2|\mathcal{F}_t\} = \mathbf{E}\{X_k^n X_j^n \mathbf{E}\{(X_l^n)^2|\mathcal{F}_{t+(l-1)h/n}\}|\mathcal{F}_t\}\}$$

$$= \frac{h}{n}\mathbf{E}\{X_k^n \mathbf{E}\{X_j^n|\mathcal{F}_{t+(j-1)h/n}\}|\mathcal{F}_t\} = 0.$$

Now, taking into account (10.16), Fatou's Lemma yields

$$\mathbf{E}\Big\{(X(t+h) - X(t))^4\Big|\mathcal{F}_t\Big\} = \mathbf{E}\Big\{\liminf_{n\to\infty}\ 6\sum_{1\le k<j\le n}(X_k^n)^2(X_j^n)^2\Big|\mathcal{F}_t\Big\}$$

$$\le 6\liminf_{n\to\infty}\mathbf{E}\Big\{\sum_{1\le k<j\le n}(X_k^n)^2\mathbf{E}\big\{(X_j^n)^2\big|\mathcal{F}_{t+(j-1)h/n}\big\}\Big|\mathcal{F}_t\Big\}$$

$$= \liminf_{n\to\infty} 3n(n-1)\frac{h^2}{n^2} = 3h^2.$$

The estimate (10.15) is proved. As a consequence of (2.16) and Jensen's inequality, we have

$$\mathbf{E}\Big\{|X(t+h) - X(t)|^3\Big|\mathcal{F}_t\Big\} \le \Big(\mathbf{E}\Big\{(X(t+h) - X(t))^4\Big|\mathcal{F}_t\Big\}\Big)^{3/4} \le 3^{3/4}h^{3/2}.$$

Therefore,

$$\mathbf{E}\Big\{|X_r^n|^3\Big|\mathcal{F}_{t+(r-1)h/n}\Big\} \le \frac{3^{3/4}h^2}{n^{3/2}}.$$

Since

$$\Big|e^{i\alpha x} - 1 - i\alpha x + \frac{\alpha^2x^2}{2}\Big| \le \frac{|\alpha x|^3}{6},$$

we have

$$\mathbf{E}\Big\{\Big|e^{\alpha X_{r+1}^n} - 1 - i\alpha X_{r+1}^n + \frac{\alpha^2}{2}(X_{r+1}^n)^2\Big|\Big|\mathcal{F}_{t+rh/n}\Big\} \le \frac{|\alpha|^3h^{3/2}}{2n^{3/2}}.$$

From (10.14) and the inequality $|1 - x - e^{-x}| \le \frac{x^2}{2}$, $0 < x$, we deduce that

$$\mathbf{E}\Big\{\exp\Big(i\alpha\sum_{k=1}^r X_k^n\Big)\Big(e^{i\alpha X_{r+1}^n} - e^{-h\alpha^2/2n}\Big)\Big|\mathcal{F}_t\Big\} \le \frac{|\alpha|^3h^{3/2}}{2n^{3/2}} + \frac{\alpha^4h^2}{8n^2}.$$

Applying this estimate in (10.13), we obtain

$$\Big|\mathbf{E}\big\{e^{i\alpha(X(t+h)-X(t))}\big|\mathcal{F}_t\big\} - e^{-\alpha^2h/2}\Big| \le \frac{|\alpha|^3h^{3/2}}{2n^{1/2}} + \frac{\alpha^4h^2}{8n} \underset{n\to\infty}{\longrightarrow} 0.$$

Thus for arbitrary $0 \le t < v \le T$ and $\alpha \in \mathbf{R}$

$$\mathbf{E}\big\{e^{i\alpha(X(v)-X(t))}\big|\mathcal{F}_t\big\} = e^{-\alpha^2(v-t)/2} \qquad \text{a.s.},$$

and by the characterization property (10.9), the process $X(t)$, $t \in [0,T]$, is a Brownian motion. □

2. Basic properties of Brownian motion.

Hölder continuity. The mapping $t \to W(t)$ is locally Hölder continuous of order γ for every $0 < \gamma < 1/2$. In other words, for all $T > 0$, $0 < \gamma < 1/2$, and almost all ω there exists a coefficient $L_{T,\gamma}(\omega)$ such that for all $s, t \in [0, T]$,

$$|W(t) - W(s)| \le L_{T,\gamma}(\omega)|t - s|^{\gamma}. \tag{10.17}$$

Indeed, in view of (10.6), Kolmogorov's condition (3.4) holds with $\alpha = 2m$, $\beta = m - 1$ for an arbitrary nonnegative integer m. According to (3.5), the sample paths of the Brownian motion W are a.s. Hölder continuous of order γ for every $0 < \gamma < \dfrac{m-1}{2m}$. Since m is arbitrary, we have (10.17) for $0 < \gamma < 1/2$.

Nowhere differentiability. Brownian paths are a.s. nowhere locally Hölder continuous of order $\alpha \ge 1/2$. In particular, Brownian paths are nowhere differentiable for all time moments (see, for example, Bulinskii and Shiryaev (2003) Ch. III § 1).

The exact modulus of continuity of Brownian motion.

Theorem 10.2. *For any $T > 0$*

$$\limsup_{\Delta \downarrow 0} \frac{1}{\sqrt{2\Delta \ln(1/\Delta)}} \sup_{t \in [0,T]} |W(t + \Delta) - W(t)| = 1 \qquad \text{a.s.} \tag{10.18}$$

Proof. It can be assumed without loss of generality that $T = 1$ and $W(0) = 0$. Set $h(t) := \sqrt{2t \ln(1/t)}$, $0 < t < e^{-1}$. To establish (10.18) it suffices to prove that for any $\varepsilon > 0$

$$\limsup_{\Delta \downarrow 0} \frac{1}{h(\Delta)} \sup_{t \in [0,1]} |W(t + \Delta) - W(t)| \ge 1 - \varepsilon \qquad \text{a.s.} \tag{10.19}$$

$$\limsup_{\Delta \downarrow 0} \frac{1}{h(\Delta)} \sup_{t \in [0,1]} |W(t + \Delta) - W(t)| \le 1 + 3\varepsilon + 2\varepsilon^2 \qquad \text{a.s.} \tag{10.20}$$

We first prove (10.19). Using the independence of the increments of the Brownian motion W, (10.8), (10.7), and the estimate $1 - x < e^{-x}$, $0 < x < 1$, we have

$$\mathbf{P}\Bigl(\max_{1 \le k \le 2^n} \Bigl|W\Bigl(\frac{k}{2^n}\Bigr) - W\Bigl(\frac{k-1}{2^n}\Bigr)\Bigr| \le (1-\varepsilon) h\Bigl(\frac{1}{2^n}\Bigr)\Bigr) = \Bigl(1 - \int\limits_{(1-\varepsilon)\sqrt{2n \ln 2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dv\Bigr)^{2^n}$$

$$< \Bigl(1 - \frac{(1-\varepsilon)\sqrt{2n \ln 2}}{(1-\varepsilon)^2 2n \ln 2 + 1} \exp(-(1-\varepsilon)^2 n \ln 2)\Bigr)^{2^n} < \exp\Bigl(-\frac{(1-\varepsilon)\sqrt{2n \ln 2}}{(1-\varepsilon)^2 2n \ln 2 + 1} 2^{\varepsilon(2-\varepsilon)n}\Bigr).$$

The series of these terms converges. Then, by the first part of Borel–Cantelli lemma,

$$\lim_{n \to \infty} \frac{1}{h(1/2^n)} \max_{1 \le k \le 2^n} \Bigl|W\Bigl(\frac{k}{2^n}\Bigr) - W\Bigl(\frac{k-1}{2^n}\Bigr)\Bigr| > 1 - \varepsilon \qquad \text{a.s.,}$$

which obviously implies (10.19).

We next prove (10.20). Set $\delta := \dfrac{\varepsilon}{2+\varepsilon}$. By (10.7),

$$\mathbf{P}\Big(\max_{\substack{0\le k=j-i\le 2^{n\delta}\\ 0\le i<j\le 2^n}} |W(j2^{-n})-W(i2^{-n})| > (1+\varepsilon)h(k2^{-n})\Big)$$

$$\le \sum_{\substack{0\le k=j-i\le 2^{n\delta}\\ 0\le i<j\le 2^n}} \mathbf{P}\big(|W(j2^{-n})-W(i2^{-n})| > (1+\varepsilon)h(k2^{-n})\big)$$

$$\le 2^n\sum_{k=1}^{2^{n\delta}} \frac{\sqrt{k2^{-n}}}{\sqrt{2\pi}(1+\varepsilon)h(k2^{-n})}\exp\Big(-\frac{(1+\varepsilon)^2h^2(k2^{-n})}{2k2^{-n}}\Big) = \frac{2^n}{2\sqrt{\pi}}\sum_{k=1}^{2^{n\delta}} \frac{2^{-n(1+\varepsilon)^2}k^{(1+\varepsilon)^2}}{(1+\varepsilon)\ln(2^n/k)}$$

$$\le A_1 2^{n(1-(1+\varepsilon)^2+\delta((1+\varepsilon)^2+1))} = A_1 2^{-2n(1+\varepsilon)\varepsilon/(2+\varepsilon)}.$$

The probabilities are estimated by quantities forming a convergent series. Hence, by the first part of Borel–Cantelli lemma, there exists a.s. a number $m=m(\omega)$ such that for all $n\ge m$, $k=j-i\le 2^{n\delta}$, $0\le i<j\le 2^n$

$$|W(j2^{-n})-W(i2^{-n})| \le (1+\varepsilon)h(k2^{-n}). \tag{10.21}$$

Set $\Delta := t-s>0$. Since $2^{-l(1-\delta)}$ tends monotonically to zero as $l\to\infty$, no matter how small Δ is, it will always be between two sequential terms of this sequence. Let $2^{-(n+1)(1-\delta)} \le \Delta < 2^{-n(1-\delta)}$. Since we must consider only arbitrary small values of Δ, we can assume that $n\ge m$. We represent s and t in the binary rational form

$$s = i2^{-n} - \sum_{v=1}^{\infty} 2^{-p_v}, \qquad t = j2^{-n} + \sum_{v=1}^{\infty} 2^{-q_v},$$

where $n<p_1<p_2<\cdots$ and $n<q_1<q_2<\cdots$. Set $s_0 := i2^{-n}$, $s_l := i2^{-n} - \sum_{v=1}^{l} 2^{-p_v}$, $t_0 := j2^{-n}$, $t_l := j2^{-n} + \sum_{v=1}^{l} 2^{-q_v}$. The process $W(t)$, $t\in[0,1]$, is a.s. continuous in t, therefore,

$$W(s) = W(s_0) + \sum_{l=1}^{\infty}(W(s_l)-W(s_{l-1})),$$

$$W(t) = W(t_0) + \sum_{l=1}^{\infty}(W(t_l)-W(t_{l-1})).$$

By the triangle inequality,

$$|W(t)-W(s)| \le |W(t_0)-W(s_0)| + \sum_{l=1}^{\infty} |(W(s_l)-W(s_{l-1})|$$

$$+\sum_{l=1}^{\infty}|W(t_l)-W(t_{l-1})| \le (1+\varepsilon)\Big\{h(k2^{-n}) + \sum_{p>n} h(2^{-p}) + \sum_{q>n} h(2^{-q})\Big\}.$$

To estimate the differences $W(t_0) - W(s_0)$, $W(s_l) - W(s_{l-1})$, $W(t_l) - W(t_{l-1})$ we used inequality (10.21). This can be done, because the points s_l, t_l satisfy the conditions under which (10.21) holds. For some constant K and all sufficiently large n,

$$\sum_{p>n} h(2^{-p}) \le Kh(2^{-n}) \le \varepsilon h(2^{-(n+1)(1-\delta)}) \le \varepsilon h(\Delta),$$

because the function $h(\Delta)$, $0 < \Delta < e^{-1}$, is strictly increasing. Finally, since $\Delta \ge k2^{-n}$, we have

$$|W(t) - W(s)| \le (1 + 3\varepsilon + 2\varepsilon^2)h(\Delta)$$

and, consequently, (10.20) is valid. The theorem is proved. □

Stochastic exponent. For any α the process $M(t) := e^{\alpha W(t) - \alpha^2 t/2}$, $t \ge 0$, (a stochastic exponent) is a martingale with respect to the natural filtration $\mathcal{G}_0^t = \sigma\{W(v), 0 \le v \le t\}$, i.e., for every $s < t$,

$$\mathbf{E}\Big\{e^{\alpha(W(t)-W(s))}\Big|\mathcal{G}_0^s\Big\} = e^{\alpha^2(t-s)/2} \qquad \text{a.s.} \tag{10.22}$$

This follows from the facts that the Brownian motion W is a process with independent increments and that (10.4) is valid for $-i\alpha$ instead of α.

Strong Markov property. Let $W(t)$, $t \ge 0$, be a Brownian motion and $\mathcal{G}_0^t = \sigma\{W(s), 0 \le s \le t\}$ be the natural filtration. Let τ be a finite stopping time with respect to the filtration $\{\mathcal{G}_0^t\}$. Then the process $W(t+\tau) - W(\tau)$, $t \ge 0$, is a Brownian motion independent of $\mathcal{G}_0^\tau$.

This property follows from Proposition 7.4.

Quadratic variation. For any sequence of subdivisions $s = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = t$, satisfying the condition $\lim\limits_{n\to\infty} \max\limits_{0\le k\le n-1} |t_{n,k+1} - t_{n,k}| = 0$, the limit

$$\lim_{n\to\infty} \sum_{0\le k\le n-1} |W(t_{n,k+1}) - W(t_{n,k})|^2 = t - s \tag{10.23}$$

holds in mean square. We say that the quadratic variation of W on $[s,t]$ is $t - s$.

Proof. The random variable

$$V_n := \sum_{0\le k\le n-1} |W(t_{n,k+1}) - W(t_{n,k})|^2$$

has the mean

$$\mathbf{E}V_n = \sum_{0\le k\le n-1} (t_{n,k+1} - t_{n,k}) = t - s.$$

Using the fact that the variance of a sum of independent variables is equal to the sum of variances of the terms, we get

$$\mathbf{E}(V_n - (t-s))^2 = \operatorname{Var} V_n = \sum_{0\le k\le n-1} \operatorname{Var}(|W(t_{n,k+1}) - W(t_{n,k})|^2)$$

$$= \sum_{0\le k\le n-1} \mathbf{E}|W(t_{n,k+1}) - W(t_{n,k})|^4 - (t_{n,k+1} - t_{n,k})^2 = 2 \sum_{0\le k\le n-1} (t_{n,k+1} - t_{n,k})^2$$

$$\le 2 \max_{0\le k\le n-1} |t_{n,k+1} - t_{n,k}|\,(t-s). \qquad (10.24)$$

The right-hand side of this formula tends to zero as $n \to \infty$. This proves the statement. □

3. Basic properties of W with zero initial value ($W(0) = 0$).

Spatial homogeneity. For every $x \in \mathbf{R}$ the process $x + W$ is a Brownian motion starting at x.

This is a consequence of the fact that the finite-dimensional density, expressed by (10.2), depends only on the differences $x_k - x_{k-1}$, $k = 1, \dots, n$.

Indeed, using (10.1) with $x = 0$, we have

$$\mathbf{P}(x+W(t_1) \in A_1, \dots, x+W(t_n) \in A_n) = \mathbf{P}(W(t_1) \in A_1 - x, \dots, W(t_n) \in A_n - x)$$

$$= \int_{A_1-x} dx_1 \frac{e^{-x_1^2/2t_1}}{\sqrt{2\pi t_1}} \int_{A_2-x} dx_2 \frac{e^{-(x_2-x_1)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \int_{A_n-x} dx_n \frac{e^{-(x_n-x_{n-1})^2/2(t_n-t_{n-1})}}{\sqrt{2\pi(t_n-t_{n-1})}}$$

$$= \int_{A_1} dx_1 \frac{e^{-(x_1-x)^2/2t_1}}{\sqrt{2\pi t_1}} \int_{A_2} dx_2 \frac{e^{-(x_2-x_1)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \int_{A_n} dx_n \frac{e^{-(x_n-x_{n-1})^2/2(t_n-t_{n-1})}}{\sqrt{2\pi(t_n-t_{n-1})}}.$$

Comparing this with (10.1), we see that $x + W(t)$ is the Brownian motion starting at x.

Symmetry. The process $-W$ is a Brownian motion, since the Gaussian density (10.2) is even.

Scaling. For every $c > 0$ the process $\{\sqrt{c}\,W(t/c) :\ t \ge 0\}$ is a Brownian motion.

This statement follows from the fact that under such a transformation the equalities (10.3) remain valid.

Time reversibility. For a given $T > 0$ the processes $\{W(t), 0 \le t \le T\}$ and $\{W(T) - W(T-t), 0 \le t \le T\}$ are identical in law.

Indeed, these processes have independent increments and they satisfy (10.3).

Strong law of large numbers:

$$\lim_{t\to\infty} \frac{W(t)}{t} = 0 \qquad \text{a.s.}$$

This result, in particular, is a consequence of (10.25).

Time inversion. The process given by

$$Z(t) := \begin{cases} 0, & \text{if } t = 0, \\ t\,W(1/t), & \text{if } t > 0, \end{cases}$$

is a Brownian motion.

This is a Gaussian process with mean zero and $\mathrm{Cov}(Z(s), Z(t)) = st\min\{1/s, 1/\}$ $= \min\{s,t\}$. By the strong law of large numbers, it is continuous at 0. Thus, it is a Brownian motion.

The law of the iterated logarithm.

Theorem 10.3. *The following relation holds*

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\ln\ln t}} = 1 \qquad \text{a.s.} \tag{10.25}$$

As a consequence of the symmetry property of Brownian motion, we have the following result.

Corollary 10.2. *The following relation holds*:

$$\liminf_{t\to\infty} \frac{W(t)}{\sqrt{2t\ln\ln t}} = -1 \qquad \text{a.s.}$$

Using the time inversion property of a Brownian motion, we obtain one more corollary.

Corollary 10.3. *The following relation holds*:

$$\limsup_{t\downarrow 0} \frac{W(t)}{\sqrt{2t\ln\ln(1/t)}} = 1 \qquad \text{a.s.}$$

Proof of Theorem 10.3. We first prove that for any $0<\varepsilon<1$

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\ln\ln t}} \le 1+\varepsilon \qquad \text{a.s.} \tag{10.26}$$

By (10.22), $M(t) := e^{\alpha W(t)-\alpha^2 t/2}$, $t\ge 0$, is a martingale with respect to the natural filtration $\mathcal{G}_0^t = \sigma\{W(v), 0\le v\le t\}$. By the Doob inequality (5.11) with $p=1$, for any $\alpha>0$ and $\beta>0$ we have

$$\mathbf{P}\Big(\sup_{0\le s\le t}(W(s)-\alpha s/2) > \beta\Big) = \mathbf{P}\Big(\sup_{0\le s\le t} M(s) > e^{\alpha\beta}\Big) \le e^{-\alpha\beta}\mathbf{E}M(t) = e^{-\alpha\beta}. \tag{10.27}$$

Set $h(t) := \sqrt{2t\ln\ln t}$. For a fixed $0<\varepsilon<1$ set $\theta := \dfrac{1+2\varepsilon}{1+\varepsilon}$, $\alpha := (1+\varepsilon)\theta^{-n}h(\theta^n)$ and $\beta := h(\theta^n)/2$. By (10.27),

$$\mathbf{P}\Big(\sup_{0\le s\le \theta^{n+1}}(W(s)-\alpha s/2) > \beta\Big) \le e^{-\alpha\beta} = e^{-(1+\varepsilon)\ln(n\ln\theta)} = (\ln\theta)^{-(1+\varepsilon)}n^{-(1+\varepsilon)}.$$

This series of probabilities converges, and by the first part of the Borel–Cantelli lemma (see § 1), there exists a number $n_0 = n_0(\omega)$ such that for all $n\ge n_0$

$$\sup_{0\le s\le \theta^{n+1}}(W(s)-\alpha s/2) \le \beta \qquad \text{a.s.}$$

In particular, for $\theta^n \le t < \theta^{n+1}$, $n\ge n_0$,

$$W(t) \le \sup_{0\le s\le \theta^{n+1}} W(s) \le \frac{\alpha\theta^{n+1}}{2} + \beta = \Big(\frac{\theta(1+\varepsilon)}{2} + \frac{1}{2}\Big)h(\theta^n) < (1+\varepsilon)h(t). \tag{10.28}$$

This proves (10.26).

It now remains to prove that for any $0 < \varepsilon < 1$

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\ln\ln t}} \ge 1-\varepsilon \qquad \text{a.s.} \tag{10.29}$$

Choose an arbitrary $0 < \delta < \varepsilon$. Set $t_n := \theta^n$, with $\theta > 1$ such that $\dfrac{(1-\delta)^2\theta}{\theta-1} = 1$.

Consider the sequence of independent events

$$A_n := \big\{W(t_{n+1}) - W(t_n) \ge (1-\delta)h(t_{n+1})\big\}.$$

By (10.8),

$$\mathbf{P}(A_n) \sim \frac{\sqrt{\theta-1}}{2\sqrt{\pi}(1-\delta)\sqrt{\theta\ln((n+1)\ln\theta)}} \exp\Big(-\frac{(1-\delta)^2\theta\ln((n+1)\ln\theta)}{\theta-1}\Big)$$

$$\sim \frac{1}{2(n+1)\ln\theta\sqrt{\pi\ln(n+1)}}.$$

The series of these probabilities diverges, and by the second part of the Borel–Cantelli lemma (Lemma 2.2), we see that a.s. for infinitely many n

$$W(t_{n+1}) \ge (1-\delta)h(t_{n+1}) + W(t_n).$$

By the symmetry property of Brownian motion, (10.28) implies that for all $n \ge n_0$ we have $-W(t_n) < (1+\varepsilon)h(t_n)$. Therefore a.s. for infinitely many $n \ge n_0$,

$$W(t_{n+1}) \ge (1-\delta)h(t_{n+1}) - (1+\varepsilon)h(t_n) = h(t_{n+1})\Big(1-\delta-\frac{(1+\varepsilon)}{\sqrt{\theta}}\frac{\sqrt{\ln(n\ln\theta)}}{\sqrt{\ln((n+1)\ln\theta)}}\Big)$$

$$= h(t_{n+1})\Big(1-\delta-(1+\varepsilon)\sqrt{1-(1-\delta)^2}\frac{\sqrt{\ln(n\ln\theta)}}{\sqrt{\ln((n+1)\ln\theta)}}\Big) \ge (1-\varepsilon)h(t_{n+1})$$

for sufficiently large n_0 and small δ such that $\delta + (1+\varepsilon)\sqrt{2\delta-\delta^2} < \varepsilon$. □

Exercises.

Let $W(s)$, $s \ge 0$ be a Brownian motion with $W(0) = 0$.

10.1. Prove that

$$\mathbf{E}\exp\Big(\lambda\int_0^t f(s)W(s)\,ds\Big) = \exp\Big(\lambda^2\int_0^t ds f(s)\int_0^s vf(v)\,dv\Big), \qquad \lambda\in\mathbf{R},$$

for a continuous function $f(s)$, $s \ge 0$.

10.2. Compute

$$\mathbf{P}\Big(a < \int_0^t s^2W(s)\,ds < b\Big).$$

10.3. Let $\mathcal{G}_0^s = \sigma(W(v), 0 \le v \le s)$ be the σ-algebra of events generated by the Brownian motion W up to the time s. Compute

$$\mathbf{E}\Big\{\Big(\int_0^t v^3 W(v)\,dv\Big)^2\Big|\mathcal{F}_0^s\Big\}$$

for $0 < s < t$.

10.4. Prove that the process $V(t) := W^4(t) - 6tW^2(t) + 3t^2$, $t \ge 0$, is a martingale with respect to the natural filtration $\{\mathcal{G}_0^t\}$.

10.5. Prove that the process $V(t) := \Big(4+\frac{1}{3}W(t)\Big)^3 - \frac{1}{3}\int_0^t \Big(4+\frac{1}{3}W(s)\Big)\,ds$, $t \ge 0$, is a martingale with respect to the natural filtration $\{\mathcal{G}_0^t\}$.

10.6. Prove that the process $V(t) := \big(W(t) + 2t\big)\exp\big(-2W(t) - 2t\big)$, $t \ge 0$, is a martingale with respect to the natural filtration $\{\mathcal{G}_0^t\}$.

10.7. Prove that the process $V(t) := e^{t/2}\cos W(t)$, $t \ge 0$, is a martingale with respect to the natural filtration $\{\mathcal{G}_0^t\}$.

§ 11. Brownian bridge

A *bridge from* x *to* z *of a stochastic process* $X(s)$, $s \in [0,t]$, $X(0) = x$, is a process $X_{x,t,z}(s)$, $s \in [0,t]$, such that its finite-dimensional distributions coincide with those of $X(s)$, $s \in [0,t]$, given the condition $X(t) = z$, i.e., for any $0 < t_1 < t_2 < \cdots < t_n < t$ and $x_k \in \mathbf{R}$, $k = 1, 2, \dots, n$,

$$\mathbf{P}(X_{x,t,z}(t_1) < x_1, X_{x,t,z}(t_2) < x_2, \dots, X_{x,t,z}(t_n) < x_n)$$
$$= \mathbf{P}(X(t_1) < x_1, X(t_2) < x_2, \dots, X(t_n) < x_n | X(t) = z).$$

If the finite-dimensional distributions of a process X have a continuous joint density, then the right-hand side of this equality is

$$\frac{\frac{d}{dz}\mathbf{P}(X(t_1) < x_1, X(t_2) < x_2, \dots, X(t_n) < x_n, X(t) < z)}{\frac{d}{dz}\mathbf{P}(X(t) < z)},$$

and the definition of a bridge can be expressed as follows: $X_{x,t,z}(s)$, $s \in [0,t]$, is a process such that for any $0 < t_1 < t_2 < \cdots < t_n < t$ and $x_k \in \mathbf{R}$, $k = 1, 2, \dots, n$,

$$\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_n}\mathbf{P}(X_{x,t,z}(t_1) < x_1,\ X_{x,t,z}(t_2) < x_2,\ \dots, X_{x,t,z}(t_n) < x_n)$$
$$= \frac{\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_n}\frac{\partial}{\partial z}\mathbf{P}(X(t_1) < x_1, X(t_2) < x_2, \dots, X(t_n) < x_n, X(t) < z)}{\frac{d}{dz}\mathbf{P}(X(t) < z)}. \tag{11.1}$$

For a Brownian motion W the process $W_{x,t,z}(s)$, $s \in [0,t]$, is called a *Brownian bridge* from x to z on the interval $[0,t]$.

In view of (10.2), the joint density of the Brownian bridge $W_{x,t,z}(s)$, $s \in [0,t]$, with the starting point x and the end point z is given by

$$p_{t,z,\vec{t}}(\vec{x}) := \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \mathbf{P}(W_{x,t,z}(t_1) < x_1, W_{x,t,z}(t_2) < x_2, \dots, W_{x,t,z}(t_n) < x_n)$$

$$= \prod_{k=1}^{n} \varphi_{t_k - t_{k-1}}(x_k - x_{k-1}) \; \frac{\varphi_{t-t_n}(z - x_n)}{\varphi_t(z - x)}, \tag{11.2}$$

where $x_0 = x$, $t_0 = 0$.

For the time moments $0 < v < u < t$ the two-dimensional density of $W_{x,t,z}(s)$ has the form

$$p_{v,u}(x_1, x_2) = \frac{\sqrt{t}}{2\pi\sqrt{(t-u)(u-v)v}} \exp\Big(- \frac{(x_1 - x)^2}{2v} - \frac{(x_2 - x_1)^2}{2(u-v)} - \frac{(z - x_2)^2}{2(t-u)} + \frac{(z-x)^2}{2t} \Big).$$

Since in (11.2) the product of Gaussian densities is again Gaussian density, $p_{t,z,\vec{t}}(\vec{x})$ is a Gaussian density of n-dimensional random variable and hence the process $W_{x,t,z}(s)$, $s \in [0,t]$, is Gaussian.

The joint density (11.2) generates a measure on $C([0,t])$, the space of continuous functions on $[0,t]$ with the uniform norm. Then an equivalent definition of the Brownian bridge $W_{x,t,z}(s)$, $s \in [0,t]$, is the following: it is a process such that for any bounded continuous functional $\wp$ on $C([0,t])$,

$$\mathbf{E}\wp(W_{x,t,z}(s), 0 \le s \le t) = \mathbf{E}\{\wp(W(s), 0 \le s \le t) | W(t) = z\}.$$

A Brownian bridge is a *spatially homogeneous* process. For every $x \in \mathbf{R}$ the process $x + W_{0,t,z}(s)$, $s \in [0,t]$, is a Brownian bridge with the starting point x and the end point $z + x$.

This is a consequence of the fact that the finite-dimensional density, expressed by (11.2), depends only on the differences $x_k - x_{k-1}$, $k = 1, \dots, n$ and $z - x_n$, $z - x$.

Using the two-dimensional Gaussian density of $W_{x,t,z}$ one can compute its mean and covariance function.

Proposition 11.1. *The Gaussian process $W_{x,t,z}(s)$, $s \in [0,t]$, has the mean*

$$\mathbf{E}W_{x,t,z}(s) = x + \frac{s}{t}(z - x), \tag{11.3}$$

and the covariance function

$$\mathrm{Cov}(W_{x,t,z}(v), W_{x,t,z}(u)) = v - \frac{vu}{t} \quad \text{for} \quad 0 \le v < u \le t. \tag{11.4}$$

Proof. In order to compute the mean and the covariance of the Brownian bridge, we will represent its two-dimensional density (see the expression following (11.2)) in the standard form (8.3). Without loss of generality, we can assume that $x = 0$. It is easy to verify that

$$\frac{x_1^2}{2v} + \frac{(x_2 - x_1)^2}{2(u-v)} + \frac{(z - x_2)^2}{2(t-u)} - \frac{z^2}{2t} = \frac{u(x_1 - vz/t)^2}{2(u-v)v} - \frac{(x_1 - vz/t)(x_2 - uz/t)}{u-v} + \frac{(t-v)(x_2 - uz/t)^2}{2(t-u)(u-v)}.$$

Now (8.3) obviously implies (11.3) for $x = 0$. From (8.3) it is easy to deduce (11.4). □

Set

$$W^{\circ}_{x,t,z}(s) := W(s) - \frac{s}{t}(W(t) - z), \quad s \in [0,t].$$

It is obvious that $W^{\circ}_{x,t,z}$ is a Gaussian process with the mean

$$\mathbf{E}W^{\circ}_{x,t,z}(s) = x + \frac{s}{t}(z - x), \tag{11.5}$$

and the covariance function

$$\mathrm{Cov}(W^{\circ}_{x,t,z}(v), W^{\circ}_{x,t,z}(u)) = v - \frac{vu}{t} \quad \text{for} \quad 0 \le v < u \le t. \tag{11.6}$$

Thus the Gaussian processes $W^{\circ}_{x,t,z}$ and $W_{x,t,z}$ have the same finite-dimensional distributions, i.e., from the probabilistic point of view they are the same process.

Using the notation $W_{x,t,z}$ instead of $W^{\circ}_{x,t,z}$, the statement proved above can be formulated as follows: for the Brownian bridge $W_{x,t,z}$ the representation

$$W_{x,t,z}(s) = W(s) - \frac{s}{t}(W(t) - z), \qquad s \in [0,t], \tag{11.7}$$

holds true.

Time reversibility of Brownian bridge means that the finite-dimensional distributions of the processes $W_{x,t,z}(s)$ and $W_{z,t,x}(t-s)$, $s \in [0,t]$, coincide. To verify the validity of this property we can proceed as follows. Using the representation $W(s) = x + \widetilde{W}(s)$, where $\widetilde{W}$ is a Brownian motion with $\widetilde{W}(0) = 0$, formula (11.7), the symmetry and time reversibility properties of $\widetilde{W}$, we get

$$W_{x,t,z}(s) = x + \widetilde{W}(s) - \frac{s}{t}(\widetilde{W}(t) + x - z) \stackrel{\text{dist}}{=} x - (\widetilde{W}(t) - \widetilde{W}(t-s)) - \frac{s}{t}(x - z - \widetilde{W}(t))$$

$$= z + \widetilde{W}(t-s) - \frac{t-s}{t}(\widetilde{W}(t) + z - x) = W_{z,t,x}(t-s),$$

where $\stackrel{\text{dist}}{=}$ denotes the equality of finite-dimensional distributions of a processes.

We present another approach to the proof of (11.7) to illustrate some interesting properties of a Brownian bridge.

Proposition 11.2. *Let F be such that $|F(y)| \le e^{K|y|}$, $y \in \mathbf{R}$, for some $K > 0$. Then for any $0 < v < u < t$:*

$$\mathbf{E}F(W_{x,t,z}(u) - W_{x,t,z}(v))$$

$$= \frac{\sqrt{t}\, e^{(z-x)^2/2t}}{\sqrt{t-u+v}} \mathbf{E}\Big\{F(W(u) - W(v)) \exp\Big(- \frac{(W(u) - W(v) + x - z)^2}{2(t-u+v)}\Big)\Big\}. \qquad (11.8)$$

Proof. Using the expression for the two-dimensional density of the Brownian bridge $W_{x,t,z}$, we get

$$\mathbf{E}F(W_{x,t,z}(u) - W_{x,t,z}(v))$$

$$= \int_{-\infty}^{\infty} dx_1 \frac{e^{-(x_1-x)^2/2v}}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} dx_2 F(x_2 - x_1) \frac{e^{-(x_2-x_1)^2/2(u-v)}}{\sqrt{2\pi(u-v)}} \frac{\sqrt{2\pi t}\, e^{-(z-x_2)^2/2(t-u)}}{\sqrt{2\pi(t-u)}\, e^{-(z-x)^2/2t}}$$

$$= \sqrt{2\pi t}\, e^{(z-x)^2/2t} \int_{-\infty}^{\infty} d\alpha \frac{e^{-\alpha^2/2v}}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} d\gamma F(\gamma) \frac{e^{-\gamma^2/2(u-v)}}{\sqrt{2\pi(u-v)}} \frac{e^{-(\gamma+x-z+\alpha)^2/2(t-u)}}{\sqrt{2\pi(t-u)}}.$$

Since the convolution of two Gaussian densities is again a Gaussian density, with the variance equal to the sum of the variances, we have the equality

$$\int_{-\infty}^{\infty} d\alpha \frac{e^{-\alpha^2/2v}}{\sqrt{2\pi v}} \frac{e^{-(\gamma+x-z+\alpha)^2/2(t-u)}}{\sqrt{2\pi(t-u)}} = \frac{1}{\sqrt{2\pi(t-u+v)}} e^{-(\gamma+x-z)^2/2(t-u+v)}, \qquad (11.9)$$

from which (11.8) follows. □

We now compute the characteristic function of increments of a Brownian bridge. By (11.8), for $F(y) = e^{i\lambda y}$, $\lambda \in \mathbf{R}$,

$$\mathbf{E}e^{i\lambda(W_{x,t,z}(u) - W_{x,t,z}(v))} = \sqrt{t}\, e^{(z-x)^2/2t} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{e^{-(y+x-z)^2/2(t-u+v)}}{\sqrt{t-u+v}} \frac{e^{-y^2/2(u-v)}}{\sqrt{2\pi(u-v)}} dy$$

$$= \exp\Big(\frac{i\lambda(z-x)}{t}(u-v) - \frac{\lambda^2}{2t}(t-u+v)(u-v)\Big), \qquad 0 \le v < u \le t. \qquad (11.10)$$

This equality can be justified as follows. Under the integral sign there is a Gaussian density multiplied by an exponential. Since

$$\frac{(y+x-z)^2}{t-u+v} + \frac{y^2}{u-v} = \frac{t}{(t-u+v)(u-v)}\Big(y^2 - \frac{2y(z-x)(u-v)}{t}\Big) + \frac{(z-x)^2}{t-u+v},$$

the variance of this Gaussian distribution is equal to $(t-u+v)(u-v)/t$ and the mean is equal to $(z-x)(u-v)/t$. Then the characteristic function must be of the form (11.10). □

Proposition 11.3. *For any* $0 \le v \le u \le t$,

$$\mathbf{E}e^{i\lambda(W^{\circ}_{x,t,z}(u)-W^{\circ}_{x,t,z}(v))} = \exp\Big(\frac{i\lambda(z-x)}{t}(u-v) - \frac{\lambda^2}{2t}(t-u+v)(u-v)\Big).$$

This statement can be easily verified, since the Brownian motion W has independent Gaussian increments.

Thus the characteristic function of the increments $W^{\circ}_{x,t,z}(u) - W^{\circ}_{x,t,z}(v)$ coincide with (11.10). Then according to Proposition 8.3 the Gaussian processes $W^{\circ}_{x,t,z}$ and $W_{x,t,z}$ have the same finite-dimensional distributions.

We now prove (11.7) the third time, but in a very special way (see Billingsley (1968) p. 83).

Let $C([0,t])$ be the space of continuous functions on $[0,t]$ with the uniform norm. We prove that for any Borel set $\mathcal{E} \subset C([0,t])$

$$\mathbf{P}(W^{\circ}_{x,t,z} \in \mathcal{E}) = \mathbf{P}(W \in \mathcal{E} | W(t) = z). \tag{11.11}$$

Then, by definition, $W^{\circ}_{x,t,z}$ is a Brownian bridge.

Instead of the conditional measure in the function space $C([0,t])$ it is convenient to use the conditional expectation of bounded continuous functionals. To establish (11.11), one can prove that for any bounded continuous functional $\wp$ on $C([0,t])$

$$\mathbf{E}\wp(W^{\circ}_{x,t,z}(s), 0 \le s \le t) = \mathbf{E}\{\wp(W(s), 0 \le s \le t) | W(t) = z\}. \tag{11.12}$$

Using the definition of the conditional expectation (see 2.19), to establish (11.7) we must prove that

$$\mathbf{E}\wp(W^{\circ}_{x,t,z}(s), 0 \le s \le t) = \lim_{\delta \downarrow 0} \frac{\mathbf{E}\{\wp(W(s), 0 \le s \le t)\mathbb{1}_{[z,z+\delta)}(W(t))\}}{\mathbf{P}(W(t) \in [z, z+\delta))}. \tag{11.13}$$

An important property is that the process $W^{\circ}_{x,t,z}(s)$, $s \in [0,t]$, is independent of the variable $W(t)$. This statement is true, because for any $s \in [0,t]$

$$\mathbf{E}\big((W^{\circ}_{x,t,z}(s) - \mathbf{E}W^{\circ}_{x,t,z}(s))(W(t) - \mathbf{E}W(t))\big) = s - \frac{s}{t}t = 0,$$

i.e., the variable $W(t)$ is uncorrelated with each of the variables $W^{\circ}_{x,t,z}(t_1)$, $\dots$, $W^{\circ}_{x,t,z}(t_n)$ and all the variables are Gaussian.

Using this independence and continuity of the functional $\wp$, we get

$$\frac{\mathbf{E}\{\wp(W(s), 0 \le s \le t)\mathbb{1}_{[z,z+\delta)}(W(t))\}}{\mathbf{P}(W(t) \in [z, z+\delta))} - \mathbf{E}\wp\big(W^{\circ}_{x,t,z}(s), 0 \le s \le t\big)$$

$$= \frac{\mathbf{E}\{\big(\wp(W(s), 0 \le s \le t) - \wp(W^{\circ}_{x,t,z}(s), 0 \le s \le t)\big)\mathbb{1}_{[z,z+\delta)}(W(t))\}}{\mathbf{P}(W(t) \in [z, z+\delta))}$$

$$= \frac{\mathbf{E}\int_0^{\delta} \big(\wp(W^{\circ}_{x,t,z}(s) + \frac{s}{t}y, 0 \le s \le t) - \wp(W^{\circ}_{x,t,z}(s), 0 \le s \le t)\big)\frac{1}{\sqrt{2\pi t}}e^{-(y+z-x)^2/2t}\,dy}{\mathbf{P}(W(t) \in [z, z+\delta))}$$

$$= \mathbf{E}\Big(\wp\Big(W^{\circ}_{x,t,z}(s) + \frac{s}{t}\tilde{y}_{\delta}, 0 \le s \le t\Big) - \wp\big(W^{\circ}_{x,t,z}(s), 0 \le s \le t\big)\Big) \underset{\delta \to 0}{\longrightarrow} 0.$$

Here we applied the mean value theorem for integrals. The variable $\tilde{y}_\delta$ is some random point from the interval $(0,\delta)$. This completes the proof of (11.13) and, consequently, (11.7). □

Considering the space $C([v,u])$ of continuous functions on $[v,u]$ with the uniform norm, it is possible to generalize Proposition 11.2.

Proposition 11.4. *For any bounded measurable functional* $\wp$ *on* $C([v,u])$, $0 \le v < u < t$,

$$\mathbf{E}\wp(W_{x,t,z}(s) - W_{x,t,z}(v), v \le s \le u) \tag{11.14}$$

$$= \frac{\sqrt{t}\, e^{(z-x)^2/2t}}{\sqrt{t-u+v}} \mathbf{E}\Big\{\wp(W(s) - W(v), v \le s \le u) \exp\Big(-\frac{(W(u) - W(v) + x - z)^2}{2(t-u+v)}\Big)\Big\}.$$

Proof. Since the finite-dimensional distributions of a process can be expressed in terms of the finite-dimensional distributions of its increments, it is sufficient to prove that for an arbitrary bounded measurable function $F(\vec{x})$, $\vec{x} \in \mathbf{R}^m$, and $v = t_1 < t_2 < \cdots < t_m = u$

$$\mathbf{E}F(W_{x,t,z}(t_2) - W_{x,t,z}(t_1), W_{x,t,z}(t_3) - W_{x,t,z}(t_2), \dots, W_{x,t,z}(t_m) - W_{x,t,z}(t_{m-1}))$$

$$= \frac{\sqrt{t}\, e^{(z-x)^2/2t}}{\sqrt{t-u+v}} \mathbf{E}\Big\{F(W(t_2) - W(t_1), W(t_3) - W(t_2), \dots, W(t_m) - W(t_{m-1}))$$

$$\times \exp\Big(-\frac{(W(u) - W(v) + x - z)^2}{2(t-u+v)}\Big)\Big\}. \tag{11.15}$$

Using (11.2), $n = 2$, we get

$$\mathbf{E}F(W_{x,t,z}(t_2) - W_{x,t,z}(t_1), W_{x,t,z}(t_3) - W_{x,t,z}(t_2), \dots, W_{x,t,z}(t_m) - W_{x,t,z}(t_{m-1}))$$

$$= \int_{-\infty}^{\infty} dx_1 \frac{e^{-(x_1-x)^2/2v}}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} dx_2 \frac{e^{-(x_2-x_1)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \int_{-\infty}^{\infty} dx_m \frac{e^{-(x_m-x_{m-1})^2/2(t_m-t_{m-1})}}{\sqrt{2\pi(t_m-t_{m-1})}}$$

$$\times F(x_2 - x_1, x_3 - x_2, \dots, x_m - x_{m-1}) \frac{e^{-(z-x_m)^2/2(t-u)}}{\sqrt{2\pi(t-u)}} \frac{\sqrt{2\pi t}}{e^{-(z-x)^2/2t}}$$

$$= \sqrt{2\pi t}\, e^{(z-x)^2/2t} \int_{-\infty}^{\infty} d\alpha \frac{e^{-\alpha^2/2v}}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} d\gamma_2 \cdots \int_{-\infty}^{\infty} d\gamma_m F(\gamma_2, \dots, \gamma_m)$$

$$\times \frac{e^{-\gamma_2^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\gamma_m^2/2(t_m-t_{m-1})}}{\sqrt{2\pi(t_m-t_{m-1})}} \frac{\exp\big(-\big(\sum_{k=2}^m \gamma_k + x - z + \alpha\big)^2/2(t-u)\big)}{\sqrt{2\pi(t-u)}}$$

$$= \frac{\sqrt{2\pi t}}{e^{-(z-x)^2/2t}} \int_{-\infty}^{\infty} d\gamma_2 \cdots \int_{-\infty}^{\infty} d\gamma_m F(\gamma_2,\dots,\gamma_m) \frac{e^{-\gamma_2^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\gamma_m^2/2(t_m-t_{m-1})}}{\sqrt{2\pi(t_m-t_{m-1})}}$$

$$\times \frac{\exp\left(-\left(\sum_{k=2}^m \gamma_k + x - z\right)^2/2(t-u+v)\right)}{\sqrt{2\pi(t-u+v)}}. \tag{11.16}$$

Here in the last equality we used (11.9). Now, taking into account the independence of increments of a Brownian motion and the form of the density of increments, we see that the right-hand side of (11.16) is equal to the right-side side of (11.15). □

Remark 11.1. Let $\mathbf{P}_{\Delta W_{x,t,z}}$ and $\mathbf{P}_{\Delta W}$ be the measures associated with the processes $W_{x,t,z}(t) - W_{x,t,z}(v)$ and $W(t) - W(v)$, $t \in [v,u]$, respectively. Then from (11.14) for the choice

$$\wp(Z(s), v \le s \le u) = \mathbb{1}_A(Z(s), v \le s \le u), \qquad A \subseteq C([v,u]),$$

it follows that the measure $\mathbf{P}_{\Delta W_{x,t,z}}$ is absolutely continuous with respect to the measure $\mathbf{P}_{\Delta W}$ on the σ-algebra $\mathcal{Q}_v^u := \sigma(W(s) - W(v), v \le s \le u)$. The corresponding Radon–Nikodým derivative is

$$\frac{d\mathbf{P}_{\Delta W_{x,t,z}}}{d\mathbf{P}_{\Delta W}}\bigg|_{\mathcal{Q}_v^u} = \frac{\sqrt{t}\, e^{(z-x)^2/2t}}{\sqrt{t-u+v}} \exp\left(-\frac{(W(u)-W(v)+x-z)^2}{2(t-u+v)}\right) \qquad \text{a.s.}$$

A *Brownian motion with linear drift* μ is a process of the form $W^{(\mu)}(t) := \mu t + W(t)$, $t \in [0,\infty)$.

This is a process with independent Gaussian increments $W^{(\mu)}(t) - W^{(\mu)}(s)$, $s < t$, having mean $\mu(t-s)$ and variance $t-s$. The density of the increment has the form

$$\varphi_{t-s}^{(\mu)}(y) := \frac{d}{dy}\mathbf{P}(W^{(\mu)}(t) - W^{(\mu)}(s) < y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-\mu(t-s))^2/2(t-s)}$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} e^{\mu y - \mu^2(t-s)/2 - y^2/2(t-s)} = e^{\mu y - \mu^2(t-s)/2}\varphi_{t-s}(y), \tag{11.17}$$

where $\varphi_t(y)$ is the Gaussian density with mean 0 and variance t.

A Brownian motion with linear drift, as well as a Brownian motion, possesses the property of *spatial homogeneity.*

According to (11.1), the joint density of a *bridge of Brownian motion with linear drift* μ ($W_{x,t,z}^{(\mu)}(s)$, $s \in [0,t]$,) with starting point x and end point z has the following form: for any $0 = t_0 < t_1 < \cdots < t_n < t$, $x = x_0$, and $x_i \in \mathbf{R}$

$$\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \mathbf{P}(W_{x,t,z}^{(\mu)}(t_1) < x_1,\, W_{x,t,z}^{(\mu)}(t_2) < x_2,\, \dots,\, W_{x,t,z}^{(\mu)}(t_n) < x_n)$$

$$= \prod_{k=1}^n \varphi_{t_k - t_{k-1}}^{(\mu)}(x_k - x_{k-1}) \frac{\varphi_{t-t_n}^{(\mu)}(z-x_n)}{\varphi_t^{(\mu)}(z-x)}$$

$$= \prod_{k=1}^n \varphi_{t_k - t_{k-1}}(x_k - x_{k-1}) \frac{\varphi_{t-t_n}(z-x_n)}{\varphi_t(z-x)}. \tag{11.18}$$

The last equality follows from (11.17) and the equalities

$$\sum_{k=1}^{n}(t_k - t_{k-1}) + t - t_n = t, \qquad \sum_{k=1}^{n}(x_k - x_{k-1}) + z - x_n = z - x.$$

Comparing formulas (11.18) and (11.2), we come to the conclusion that the distributions of the bridge of Brownian motion with linear drift coincide with those of the Brownian bridge. Therefore the bridges $W_{x,t,z}(s)$ and $W^{(\mu)}_{x,t,z}(s)$, $s \in [0,t]$, are identical in law, i.e., from the probabilistic point of view they are the same process.

This can be also established in another way. Like for a Brownian motion, the analogue of the formula (11.7) holds true:

$$W^{(\mu)}_{x,t,z}(s) = W^{(\mu)}(s) - \frac{s}{t}(W^{(\mu)}(t) - z), \qquad s \in [0,t]. \tag{11.19}$$

Substituting in the right-hand side of (11.19) the expression $W^{(\mu)}(s) = \mu s + W(s)$, we obtain

$$W^{(\mu)}_{x,t,z}(s) = W(s) - \frac{s}{t}(W(t) - z), \tag{11.20}$$

and hence $W^{(\mu)}_{x,t,z}(s) = W_{x,t,z}(s)$, $s \in [0,t]$.

Proposition 11.5. *A bridge of a Gaussian process* $X(s)$, $s \in [0,t]$, *with* $X(0) = x$, *has the following representation*:

$$X_{x,t,z}(s) = X(s) - \frac{\mathrm{Cov}(X(s), X(t))}{\mathrm{Cov}(X(t), X(t))}(X(t) - z). \tag{11.20}$$

This statement can be proved analogously to the proof of (11.12), since the process $X_{x,t,z}(s)$, $s \in [0,t]$, is independent of the variable $X(t)$. The independence is due to the fact that the variable $X(t)$ is uncorrelated with each of the variables $X_{x,t,z}(s)$, $s \in [0,t]$, and all the variables are Gaussian.

Since, in general, a Gaussian process is not continuous, we have to prove directly the equality for the distributions involved in the definition of a bridge. In this case one should use the continuity of Gaussian finite-dimensional distributions with nondegenerate coordinates.

Exercises.

11.1. Let $W(s)$, $s \ge 0$ be a Brownian motion with $W(0) = x$. Compute the conditional distribution of $\int_0^t s^2 W(s)\,ds$ given that $W(t) = z$.

11.2. Let $W(s)$, $s \ge 0$ be a Brownian motion with $W(0) = x$. Compute

$$\mathbf{E}\Big\{\int_0^t W^2(s)\,ds \Big| W(t) = z\Big\}.$$

11.3. Let $W(s)$, $s \ge 0$ be a Brownian motion with $W(0) = 0$. Compute

$$\mathbf{E}\Big\{ \exp\Big(\int_0^t sW(s)\,ds \Big) \Big| W(t) = z \Big\}.$$

11.4. Let $W(s)$, $s \ge 0$ be a Brownian motion with $W(0) = x$. Compute $\mathbf{E}\big\{(W(s) - W(t/2))^2 \big| W(t) = z\big\}$ for $s \le t$.

11.5. Compute for $0 \le v < u \le t$ the second and the third moments of the increments of $W_{x,t,z}$, i.e., $\mathbf{E}(W_{x,t,z}(u) - W_{x,t,z}(v))^2$ and $\mathbf{E}(W_{x,t,z}(u) - W_{x,t,z}(v))^3$.

11.6. For what $Q(s)$, $s \in [0,t]$, is the process

$$Q_{x,t,z}(s) := Q(s) - \frac{\operatorname{sh} s}{\operatorname{sh} t}(Q(t) - z), \qquad s \in [0,t],$$

a bridge? Here $\operatorname{sh} t := \dfrac{e^t - e^{-t}}{2}$.

CHAPTER II

STOCHASTIC CALCULUS

§ 1. Stochastic integration with respect to Brownian motion

In this section we present the basic facts of the theory of stochastic integration in the case when the integrator is a Brownian motion W. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions (see § 4 Ch. I) and $W(t)$, $t \in [0,T]$, be an $\mathcal{F}_t$-measurable Brownian motion in this space. We also assume that for all $v > t$ the increments $W(v) - W(t)$ are independent of the σ-algebra $\mathcal{F}_t$. For $\{\mathcal{F}_t\}$ one can take the completed natural filtration, i.e., the family of the σ-algebras $\mathcal{G}_0^t = \sigma\{W(s), s \in [0,t]\}$, generated by the Brownian motion W up to the time t.

The goal is to give some meaning to the *stochastic integrals* of the type

$$\int_0^t f(s)\,dW(s). \tag{1.1}$$

Since the Brownian motion W has an infinite variation on any interval, it is not possible to define such integrals by means of classical approaches of the theory of integration. The approach proposed here is that the stochastic integral (1.1) can be defined via an isometry. The notion to which this approach leads us is called the *Itô integral* and the theory is called *stochastic calculus*. For a nonrandom function f, the integral (1.1) can be considered (see § 9 Ch. I) as the integral with respect to the orthogonal stochastic measure defined by $Z(\Delta) := W(b) - W(a)$, $\Delta = [a,b)$, and having the structure function $G(\Delta) = b - a$.

Consider the *class* $\mathcal{H}_2[0,T]$ of progressively measurable with respect to $\{\mathcal{F}_t\}$ stochastic processes $f(t)$, $t \in [0,T]$, satisfying the condition

$$\int_0^T \mathbf{E} f^2(s)\,ds < \infty. \tag{1.2}$$

In the present description we does not exclude the case $T = \infty$. In this case the interval $[0,T]$ is replaced by $[0,\infty)$.

Consider the class of *simple processes* of the form

$$\bar f(s) = \sum_{k=0}^{m-1} f_k \mathbb{I}_{[s_k, s_{k+1})}(s), \qquad s \in [0,T], \tag{1.3}$$

where $0 = s_0 < s_1 < \cdots < s_m = T$, the random variables f_k are $\mathcal{F}_{s_k}$-measurable, and $\mathbf{E} f_k^2 < \infty$, $k = 0, \dots m-1$. In the case $T = \infty$, we set $f_{m-1} \equiv 0$. Obviously, the function $\bar f$ belongs to $\mathcal{H}_2[0,T]$.

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_2

The stochastic integral of $\bar{f}$ with respect to W is defined to be

$$\int_0^T \bar{f}(s)\,dW(s) := \sum_{k=0}^{m-1} f_k(W(s_{k+1}) - W(s_k)). \tag{1.4}$$

For arbitrary constants α and β,

$$\int_0^T (\alpha\bar{f}_1(s) + \beta\bar{f}_2(s))\,dW(s) = \alpha\int_0^T \bar{f}_1(s)\,dW(s) + \beta\int_0^T \bar{f}_2(s)\,dW(s). \tag{1.5}$$

The mean of the stochastic integral defined by (1.4) equals zero, i.e.,

$$\mathbf{E}\int_0^T \bar{f}(s)\,dW(s) = 0. \tag{1.6}$$

Indeed, since f_k is $\mathcal{F}_{s_k}$-measurable, the variables f_k and $W(s_{k+1}) - W(s_k)$ are independent. Therefore, in view of (10.3) Ch. I, we have

$$\mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))\} = \mathbf{E}f_k\mathbf{E}(W(s_{k+1}) - W(s_k)) = 0.$$

Hence the expectation of the sum (1.4) is zero and (1.6) holds.

For the variance of the stochastic integral we have

$$\mathbf{E}\Bigl(\int_0^T \bar{f}(s)\,dW(s)\Bigr)^2 = \int_0^T \mathbf{E}\bar{f}^2(s)\,ds. \tag{1.7}$$

Indeed, since f_k and $W(s_{k+1}) - W(s_k)$ are independent, by (10.3) Ch. I, we have

$$\mathbf{E}\{f_k^2(W(s_{k+1}) - W(s_k))^2\} = \mathbf{E}f_k^2\mathbf{E}(W(s_{k+1}) - W(s_k))^2 = \mathbf{E}f_k^2(s_{k+1} - s_k).$$

For $k < l$ the random variables $f_k(W(s_{k+1}) - W(s_k))f_l$ are $\mathcal{F}_{s_l}$-measurable and the increments $W(s_{l+1}) - W(s_l)$ are independent of $\mathcal{F}_{s_l}$. Therefore,

$$\begin{aligned} I_{k,l} &:= \mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))f_l(W(s_{l+1}) - W(s_l))\} \\ &= \mathbf{E}\{f_k(W(s_{k+1}) - W(s_k))f_l\}\mathbf{E}(W(s_{l+1}) - W(s_l)) = 0. \end{aligned}$$

Here to prove that the expectation is finite we used the estimate

$$\begin{aligned} \mathbf{E}|f_k(W(s_{k+1}) - W(s_k))f_l| &\le \mathbf{E}^{1/2}\{f_k^2(W(s_{k+1}) - W(s_k))^2\}\mathbf{E}^{1/2}\{f_l^2\} \\ &= \mathbf{E}^{1/2}\{f_k^2\}(s_{k+1} - s_k)^{1/2}\mathbf{E}^{1/2}\{f_l^2\} < \infty. \end{aligned}$$

Now it is easy to check (1.7):

$$\mathbf{E}\Big(\int_0^T \bar{f}(s)\,dW(s)\Big)^2 = \mathbf{E}\Big(\sum_{k=0}^{m-1} f_k(W(s_{k+1}) - W(s_k))\Big)^2$$

$$= \sum_{k=0}^{m-1} \mathbf{E}\{f_k^2(W(s_{k+1}) - W(s_k))^2\} + 2 \sum_{0\le k<l\le m-1} I_{k,l}$$

$$= \sum_{k=0}^{m-1} \mathbf{E}f_k^2(s_{k+1} - s_k) = \int_0^T \mathbf{E}\bar{f}^2(s)\,ds.$$

Formula (1.7) is of key importance for the definition of the stochastic integral for the class of random processes $\mathcal{H}_2[0,T]$.

Let $L^2(\mathbf{P})$ be the space of square integrable random variables. Then $L^2(\mathbf{P})$ is a Hilbert space when equipped with the norm $(\mathbf{E}X^2)^{1/2}$, $X \in L^2(\mathbf{P})$.

For a function $f \in \mathcal{H}_2[0,T]$, the norm is $\Big(\int_0^T \mathbf{E}f^2(s)\,ds\Big)^{1/2}$.

In view of (1.7), for a class of simple processes $\bar{f} \in \mathcal{H}_2[0,T]$ the mapping

$$\bar{f} \to \int_0^T \bar{f}(s)\,dW(s) \tag{1.8}$$

is an *isometry* from a subset of $\mathcal{H}_2[0,T]$ into $L^2(\mathbf{P})$.

Proposition 1.1. *The set of simple processes is dense in the space* $\mathcal{H}_2[0,T]$*, i.e., for any process* $f \in \mathcal{H}_2[0,T]$ *there exists a sequence of simple processes* $\bar{f}_n \in \mathcal{H}_2[0,T]$ *such that*

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - \bar{f}_n(s))^2\,ds = 0. \tag{1.9}$$

Proof. Without loss of generality, we can assume that f is bounded. Otherwise we set $f_N(t) := f(t)\mathbb{1}_{[-N,N]}(f(t))$ and use the fact that

$$\lim_{N\to\infty} \int_0^T \mathbf{E}(f(s) - f_N(s))^2\,ds = 0.$$

For a continuous bounded f, set $\bar{f}_n(s) := f([ns]/n)$, where $[a]$ denotes the largest integer not exceeding a. Then (1.9) follows from the Lebesgue dominated convergence theorem for integrals of uniformly bounded functions.

Now to prove Proposition 1.1 it is enough to approximate a bounded progressively measurable process f by continuous processes. Such processes are

$$\widehat{f}_n(s) := n \int_{(s-1/n)^+}^{s} f(v)\,dv, \qquad n = 1, 2, \dots .$$

where $a^+ = \max\{0, a\}$. It is clear that $\widehat{f}_n$, $n = 1, 2, \dots$, are uniformly bounded progressively measurable processes, because they are continuous. Set $F(s) := \int_0^s f(v)\,dv$. Then F is a.s. a function of bounded variation. By the Lebesgue differentiation theorem, for almost all $s \in [0,T]$ there exists $F'(s)$ and the equality $f(s) = F'(s) = \lim_{n\to\infty} \widehat{f}_n(s)$ holds. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - \widehat{f}_n(s))^2\,ds = 0.$$

This completes the proof. □

In view of Proposition 1.1, the linear isometry (1.8) can be extended uniquely to a linear isometry from the whole $\mathcal{H}_2[0,T]$ into $L^2(\mathbf{P})$, thus defining the stochastic integral of $f \in \mathcal{H}_2[0,T]$ with respect to the Brownian motion.

This means the following. Consider the sequence $\{\bar{f}_n\}$ of functions, satisfying (1.9). Using the inequality

$$\int_0^T \mathbf{E}(\bar{f}_m(s) - \bar{f}_n(s))^2\,ds \le 2\int_0^T \mathbf{E}(f(s) - \bar{f}_m(s))^2\,ds + 2\int_0^T \mathbf{E}(f(s) - \bar{f}_n(s))^2\,ds$$

and formulas (1.5), (1.7), we have

$$\mathbf{E}\biggl(\int_0^T \bar{f}_m(s)\,dW(s) - \int_0^T \bar{f}_n(s)\,dW(s)\biggr)^2 = \int_0^T \mathbf{E}(\bar{f}_m(s) - \bar{f}_n(s))^2\,ds \underset{\substack{m\to\infty\\ n\to\infty}}{\longrightarrow} 0.$$

Thus the sequence $\int_0^T \bar{f}_n(s)\,dW(s)$ is Cauchy for the mean square convergence. Therefore, there exists a limit, which is assigned to be the stochastic integral of f with respect to the Brownian motion W.

Thus, for a function $f \in \mathcal{H}_2[0,T]$ such that (1.9) holds we set

$$\int_0^T f(s)\,dW(s) := \operatorname{l.i.m.} \int_0^T \bar{f}_n(s)\,dW(s), \tag{1.10}$$

where l. i. m. denotes the limit in mean square.

By (1.10), the properties (1.5)–(1.7) are valid for all processes from the space $\mathcal{H}_2[0,T]$:

1) for any constants α and β,

$$\int_0^T (\alpha f_1(s) + \beta f_2(s))\,dW(s) = \alpha\int_0^T f_1(s)\,dW(s) + \beta\int_0^T f_2(s)\,dW(s) \qquad \text{a.s.};$$

2) the mean of the stochastic integral equals zero, i.e.,

$$\mathbf{E}\int_0^T f(s)\,dW(s) = 0; \tag{1.11}$$

3) the variance of the stochastic integral satisfies the relation

$$\mathbf{E}\Big(\int_0^T f(s)\,dW(s)\Big)^2 = \int_0^T \mathbf{E}f^2(s)\,ds; \tag{1.12}$$

4) if

$$\lim_{n\to\infty}\int_0^T \mathbf{E}(f(s)-f_n(s))^2\,ds = 0,$$

then

$$\int_0^T f(s)\,dW(s) = \text{l. i. m.}\int_0^T f_n(s)\,dW(s). \tag{1.13}$$

In addition to the first property, from the construction of the stochastic integral one can deduce that for any bounded $\mathcal{F}_v$-measurable random variable ξ and any $t > v$

$$\int_0^T \xi \mathrm{1\!I}_{[v,t)}(s) f(s)\,dW(s) = \xi\int_0^T \mathrm{1\!I}_{[v,t)}(s) f(s)\,dW(s) \qquad \text{a.s.} \tag{1.14}$$

§ 2. Stochastic integrals with variable upper limit

Define a family of stochastic integrals with variable upper limit by setting

$$\int_0^t f(s)\,dW(s) := \int_0^T \mathrm{1\!I}_{[0,t)}(s) f(s)\,dW(s), \qquad \text{for every} \quad t\in[0,T]. \tag{2.1}$$

Then the following problem arises. Formula (1.10) defines the stochastic integral uniquely up to a set Λ_f of probability zero. This set depends on the integrand. Definition (2.1) involves a whole family of integrands depending on the time parameter t. Therefore, it is possible that the probability of the union of sets $\Lambda_{\mathrm{1\!I}_{[0,t)}f}$ is not zero. In this case the integrals are not determined as a function of t on a set of nonzero probability. We overcome this difficulty by proving that the stochastic integral, as a function of t, is a.s. continuous $\mathcal{F}_t$-measurable martingale.

For $v < t$ it is natural to set

$$\int_v^t f(s)\,dW(s) := \int_0^T \mathrm{1\!I}_{[v,t)}(s) f(s)\,dW(s). \tag{2.2}$$

Then

$$\int_v^t f(s)\,dW(s) = \int_0^t f(s)\,dW(s) - \int_0^v f(s)\,dW(s),$$

since $1\!\!\mathrm{I}_{[v,t)}(s) = 1\!\!\mathrm{I}_{[0,t)}(s) - 1\!\!\mathrm{I}_{[0,v)}(s)$ and the linearity property holds.

The following generalizations of the properties 2), 3) of § 1 hold: for every $v < t$

$$\mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\} = 0 \qquad \text{a.s.}, \tag{2.3}$$

$$\mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_v\Big\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_v\}\,ds \qquad \text{a.s.} \tag{2.4}$$

Indeed, for any $\mathcal{F}_v$-measurable bounded random variable ξ we have

$$\mathbf{E}\Big\{\xi\mathbf{E}\Big\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\}\Big\} = \mathbf{E}\Big\{\mathbf{E}\Big\{\xi\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_v\Big\}\Big\}$$

$$= \mathbf{E}\Big\{\xi\int_0^T 1\!\!\mathrm{I}_{[v,t)}(s)f(s)\,dW(s)\Big\} = \mathbf{E}\int_0^T \xi 1\!\!\mathrm{I}_{[v,t)}(s)f(s)\,dW(s) = 0,$$

where (1.14) and (1.11) were used. Since the random variable ξ is arbitrary, this implies (2.3).

Similarly, using (1.14) and (1.12), we have

$$\mathbf{E}\Big\{\xi^2\mathbf{E}\Big\{\Big(\int_v^t f(s)\,dW(s)\Big)^2\Big|\mathcal{F}_v\Big\}\Big\} = \mathbf{E}\Big(\int_0^T \xi 1\!\!\mathrm{I}_{[v,t)}(s)f(s)\,dW(s)\Big)^2$$

$$= \int_0^T \mathbf{E}\{\xi^2 1\!\!\mathrm{I}_{[v,t)}(s)f^2(s)\}\,ds = \mathbf{E}\Big\{\xi^2\int_v^t f^2(s)\,ds\Big\} = \mathbf{E}\Big\{\xi^2\int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_v\}\,ds\Big\}.$$

This implies (2.4).

Theorem 2.1. *Let* $f \in \mathcal{H}_2[0,T]$. *Then the process* $I(t) := \int_0^t f(s)\,dW(s)$, $t \in [0,T]$, *is an* a.s. *continuous martingale such that for any* $\varepsilon > 0$

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f(s)\,dW(s)\Big| \ge \varepsilon\Big) \le \frac{1}{\varepsilon^2}\int_0^T \mathbf{E}f^2(s)\,ds, \tag{2.5}$$

$$\mathbf{E}\sup_{0\le t\le T}\left|\int_0^t f(s)\,dW(s)\right|^2 \le 4\int_0^T \mathbf{E}f^2(s)\,ds. \tag{2.6}$$

Proof. The case $T=\infty$ can be considered as the limiting case for $T_n=n$. So we can assume that $T<\infty$. We first prove the theorem for the simple processes defined by (1.3). For such processes, for $t\in[s_l,s_{l+1})$, $l=0,\dots,m-1$, we have

$$I(t)=\int_0^T \mathbb{1}_{[0,t)}(s)\bar f(s)\,ds=\sum_{k=0}^{l-1} f_k(W(s_{k+1})-W(s_k))+f_l(W(t)-W(s_l)). \tag{2.7}$$

Since the Brownian motion is a.s. continuous, the process $I(t)$ is also continuous.

From (2.3) it follows that for $v<t$

$$\mathbf{E}\left\{\int_0^t \bar f(s)\,dW(s)\middle|\mathcal{F}_v\right\}=\int_0^v \bar f(s)\,dW(s),$$

i.e., for simple processes $I(t)$ is a martingale. By Doob's inequality for martingales (5.11), $p=2$, Ch. I,

$$\mathbf{P}\left(\sup_{0\le t\le T}\left|\int_0^t \bar f(s)\,dW(s)\right|\ge\varepsilon\right)\le\frac{1}{\varepsilon^2}\mathbf{E}\left(\int_0^T \bar f(s)\,dW(s)\right)^2=\frac{1}{\varepsilon^2}\int_0^T \mathbf{E}\bar f^2(s)\,ds.$$

The equality on the right-hand side is due to (1.7). This proves (2.5). Similarly, from the second Doob inequality for martingales (see (5.12), $p=2$, Ch. I) it follows that (2.6) is also valid. Thus for simple processes the theorem is proved.

For an arbitrary $f\in\mathcal{H}_2[0,T]$, using (1.9) we can choose a subsequence of the integer numbers n_k such that

$$\int_0^T \mathbf{E}(f(s)-\bar f_{n_k}(s))^2\,ds\le\frac{1}{2^k}.$$

Then

$$\int_0^T \mathbf{E}(\bar f_{n_{k+1}}(s)-\bar f_{n_k}(s))^2\,ds\le 2\int_0^T \mathbf{E}(f(s)-\bar f_{n_{k+1}}(s))^2\,ds$$

$$+2\int_0^T \mathbf{E}(f(s)-\bar f_{n_k}(s))^2\,ds\le\frac{3}{2^k}.$$

The process $\bar f_{n_{k+1}}(s)-\bar f_{n_k}(s)$ is simple, therefore, (2.5) applies. We have

$$\mathbf{P}\left(\sup_{0\le t\le T}\left|\int_0^t \bar f_{n_{k+1}}(s)\,dW(s)-\int_0^t \bar f_{n_k}(s)\,dW(s)\right|\ge\frac{1}{k^2}\right)$$

$$= \mathbf{P}\Big(\sup_{0\le t\le T} \Big| \int_0^t \big(\bar f_{n_{k+1}}(s) - \bar f_{n_k}(s)\big)\, dW(s) \Big| \ge \frac{1}{k^2} \Big)$$

$$\le k^4 \int_0^T \mathbf{E}(\bar f_{n_{k+1}}(s) - \bar f_{n_k}(s))^2\, ds \le \frac{3k^4}{2^k}.$$

Since the series of these probabilities converges, the first part of the Borel–Cantelli lemma, shows that there exists a.s. a number $k_0 = k_0(\omega)$ such that for all $k > k_0$

$$\sup_{0\le t\le T} \Big| \int_0^t \bar f_{n_{k+1}}(s)\, dW(s) - \int_0^t \bar f_{n_k}(s)\, dW(s) \Big| < \frac{1}{k^2}.$$

Then the sequence of integrals

$$\int_0^t \bar f_{n_m}(s)\, dW(s) = \int_0^t \bar f_{n_0}(s)\, dW(s) + \sum_{k=0}^{m-1} \Big(\int_0^t \bar f_{n_{k+1}}(s)\, dW(s) - \int_0^t \bar f_{n_k}(s)\, dW(s) \Big)$$

converges a.s. uniformly in $[0,T]$ to some limit, which, by definition, is a stochastic integral $I(t)$, i.e.,

$$\sup_{0\le t\le T} \Big| I(t) - \int_0^t \bar f_{n_m}(s)\, dW(s) \Big| \to 0, \qquad \text{as} \quad m \to \infty.$$

Since a uniform limit of continuous functions is continuous, the process $I(t)$, $t \in [0,T]$ is a.s. continuous. From (2.3) it follows that $I(t)$ is a martingale and the estimates (2.5), (2.6) hold. □

A very important property of stochastic integrals follows from (2.5) and (2.6). Let

$$\lim_{n\to\infty} \int_0^T \mathbf{E}(f(s) - f_n(s))^2\, ds = 0, \quad f_n, f \in \mathcal{H}_2[0,T].$$

Then

$$\sup_{t\in[0,T]} \Big| \int_0^t f(s)\, dW(s) - \int_0^t f_n(s)\, dW(s) \Big| \to 0 \quad \text{as } n \to \infty \tag{2.8}$$

in probability and in mean square.

This property enables us to justify the passage to a limit in different schemes involving stochastic integrals.

Here is a simple example of an interesting class of Gaussian processes expressed via the stochastic integral. For nonrandom functions $h(s)$ and $g(s)$, $s \in [0,T]$, set

$$\overline{W}(t) := x + h(t) + \int_0^t g(s)\, dW(s).$$

It is clear that $\overline{W}(t)$, $t \in [0,T]$, is a Gaussian process with the mean $x + h(t)$ and the covariance

$$\operatorname{Cov}(\overline{W}(s), \overline{W}(t)) = \int_0^s g^2(v)\,dv, \qquad \text{for } s \le t.$$

This is a process with independent increments, it is identical in law to the process $h(t) + W\Bigl(\int_0^t g^2(s)\,ds\Bigr)$, $W(0) = x$.

It is easy to understand (see (11.21) Ch. I) that for $h(0) = 0$ the process

$$\overline{W}^{\circ}_{x,t,z}(s) := \overline{W}(s) - \frac{\int_0^s g^2(v)\,dv}{\int_0^t g^2(v)\,dv}(\overline{W}(t) - z) \tag{2.9}$$

is the bridge from x to z of the process $\overline{W}$.

For every $\mu \in \mathbf{R}$, the process $\overline{W}$ with $h(t) = \mu \int_0^t g^2(s)\,ds$ has the same bridge as for $\mu = 0$.

Exercises.

2.1. Compute the conditional distribution of $\int_0^t s\,dW(s)$ given $W(t) = z$.

2.2. Check whether the following equalities hold true for some $\varepsilon > 0$:

1) $$\mathbf{E}\Bigl\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_{v+\varepsilon}\Bigr\} = 0 \qquad \text{a.s.}$$

2) $$\mathbf{E}\Bigl\{\Bigl(\int_v^t f(s)\,dW(s)\Bigr)^2\Big|\mathcal{F}_{v+\varepsilon}\Bigr\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_{v+\varepsilon}\}\,ds \qquad \text{a.s.}$$

3) $$\mathbf{E}\Bigl\{\int_v^t f(s)\,dW(s)\Big|\mathcal{F}_{v-\varepsilon}\Bigr\} = 0 \qquad \text{a.s.}$$

4) $$\mathbf{E}\Bigl\{\Bigl(\int_v^t f(s)\,dW(s)\Bigr)^2\Big|\mathcal{F}_{v-\varepsilon}\Bigr\} = \int_v^t \mathbf{E}\{f^2(s)|\mathcal{F}_{v-\varepsilon}\}\,ds \qquad \text{a.s.}$$

2.3. Prove directly from the definition of the Itô integral that

$$\int_0^t s\,dW(s) = tW(t) - \int_0^t W(s)\,ds$$

(the integration by parts formula).

2.4. Deduce directly from the definition of the Itô integral that

$$2\int_s^t W(v)\,dW(v) = W^2(t) - W^2(s) - (t-s).$$

Hint: Use the result about the quadratic variation of the Brownian motion.

§ 3. Extension of the class of integrands

The condition that processes from $\mathcal{H}_2[0,T]$ must have a finite second moment is rather restrictive. Using an approach based on the truncation of integrands, the definition of the stochastic integral can be generalized to a class of stochastic processes broader than $\mathcal{H}_2[0,T]$.

Let $\mathcal{L}_2[0,T]$ be a *class* of progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$ stochastic processes $f(t)$, $t \in [0,T]$, satisfying the condition

$$\mathbf{P}\Big(\int_0^T f^2(s)\,ds < \infty\Big) = 1. \tag{3.1}$$

Clearly, $\mathcal{H}_2[0,T] \subset \mathcal{L}_2[0,T]$.

For simple processes from $\mathcal{L}_2[0,T]$ of the form (1.3), where it is not supposed that the second moments of f_k, $k = 0,\dots m-1$, are finite, the stochastic integral with variable upper limit is defined by (2.7).

For further arguments we need the following estimate. For any simple process $\bar f \in \mathcal{L}_2[0,T]$ and any $C > 0$, $N > 0$,

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f(s)\,dW(s)\Big| \ge C\Big) \le \mathbf{P}\Big(\int_0^T \bar f^2(s)\,ds > N\Big) + \frac{N}{C^2}. \tag{3.2}$$

To prove this inequality define $f_N(t) := \bar f(t)\mathbb{1}_{[0,N]}\Big(\int_0^t \bar f^2(v)\,dv\Big)$. It is clear that the process $f_N(t)$ is progressively measurable with respect to the σ-algebras $\{\mathcal{F}_t\}$ and $\int_0^T \bar f_N^2(s)\,ds \le N$. Therefore, $f_N(t) \in \mathcal{H}_2[0,T]$ and the estimate (2.5) can be applied. Then

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f(s)\,dW(s)\Big| \ge C\Big) \le \mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f_N(s)\,dW(s)\Big| \ge C\Big)$$

$$+\mathbf{P}\big(\bar f(t) \ne f_N(t) \text{ for some } t \in [0,T]\big) \le \frac{N}{C^2} + \mathbf{P}\Big(\int_0^T \bar f^2(s)\,ds > N\Big).$$

Here the obvious inclusion

$$\Big\{\int_0^t \bar f^2(s)\,ds > N \text{ for some } t\in[0,T]\Big\} \subseteq \Big\{\int_0^T \bar f^2(s)\,ds > N\Big\}$$

was taken into account. The inequality (3.2) is proved.

Proposition 3.1. *The set of simple processes is dense in the space $\mathcal{L}_2[0,T]$, i.e., for any process $f\in\mathcal{L}_2[0,T]$ there exists a sequence of simple processes $\bar f_n\in\mathcal{L}_2[0,T]$ such that*

$$\lim_{n\to\infty}\int_0^T (f(s)-\bar f_n(s))^2\,ds = 0 \qquad \text{a.s.} \tag{3.3}$$

The proof of this statement is analogous to the proof of Proposition 1.1. It is only necessary to replace the mean square convergence by the a.s. convergence.

From (3.3) it follows that

$$\int_0^T (\bar f_m(s)-\bar f_n(s))^2\,ds \to 0, \qquad \text{as } \ m\to\infty, \quad n\to\infty,$$

in probability. For every m, n, $\varepsilon>0$ and $\delta>0$, letting in (3.2) $C=\varepsilon$, $N=\delta\varepsilon^2$, we have

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t \bar f_m(s)\,dW(s) - \int_0^t \bar f_n(s)\,dW(s)\Big| \ge \varepsilon\Big)$$
$$\le \mathbf{P}\Big(\int_0^T (\bar f_m(s)-\bar f_n(s))^2\,ds > \delta\varepsilon^2\Big) + \delta. \tag{3.4}$$

Letting first $m\to\infty$, $n\to\infty$, and then $\delta\to 0$, we obtain that the sequence of processes $\int_0^t \bar f_n(s)\,dW(s)$, $t\in[0,T]$, is Cauchy in the uniform norm $\sup_{t\in[0,T]}|\cdot|$ for the convergence in probability.

Therefore, there exists a stochastic process $I(t)$, $t\in[0,T]$, such that

$$\sup_{t\in[0,T]}\Big|I(t) - \int_0^t \bar f_n(s)\,dW(s)\Big| \to 0$$

in probability. We set $I(t) := \int_0^t f(s)\,dW(s)$.

Since according to Proposition 1.1 in Ch. I the convergence in probability is equivalent to a.s. convergence for some subsequences, we see that the process $I(t)$ is a.s. continuous.

Now we can prove by passage to the limit as $n\to\infty$ in (3.2), applied for the processes $\bar f_n$, that (3.2) is also valid for all processes $f\in\mathcal{L}_2[0,T]$.

As a result, we have the following theorem.

Theorem 3.1. *Let* $f \in \mathcal{L}_2[0,T]$*. Then the process* $I(t) = \int_0^t f(s) dW(s)$, $t \in [0,T]$, *is* a.s. *continuous, and for any* $C > 0$, $N > 0$,

$$\mathbf{P}\Big(\sup_{0\le t\le T}\Big|\int_0^t f(s)\,dW(s)\Big| \ge C\Big) \le \mathbf{P}\Big(\int_0^T f^2(s)\,ds > N\Big) + \frac{N}{C^2}. \tag{3.5}$$

We conclude by pointing out an important property following from (3.5). Let

$$\lim_{n\to\infty}\int_0^T (f(s)-f_n(s))^2\,ds = 0, \quad f_n, f \in \mathcal{L}_2[0,T],$$

in probability. Then

$$\sup_{t\in[0,T]}\Big|\int_0^t f(s)\,dW(s) - \int_0^t f_n(s)\,dW(s)\Big| \to 0 \qquad \text{as} \quad n\to\infty, \tag{3.6}$$

in probability.

In addition to the stochastic integral with variable upper limit, we define an integral with a random upper limit.

Let ρ be a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in [0,\infty)\}$. Let $f(s)$, $s \in [0,\infty)$, be a progressively measurable stochastic process satisfying the condition

$$\mathbf{P}\Big(\int_0^\infty f^2(s)\,ds < \infty\Big) = 1. \tag{3.7}$$

Then

$$\int_0^\rho f(s)\,dW(s) := \int_0^\infty \mathbb{1}_{[0,\rho)}(s) f(s)\,dW(s). \tag{3.8}$$

Note that, by the definition of a stopping time, $\{\rho \le s\} \in \mathcal{F}_s$ for every s. Then $\mathbb{1}_{[0,\rho)}(s) = 1 - \mathbb{1}_{[0,s]}(\rho)$ is an $\mathcal{F}_s$-measurable right continuous process. Therefore, it is progressively measurable and the stochastic integral on the right-hand side of (3.8) is well defined. The variable $\int_0^\rho f(s)\,dW(s)$ has mean zero, if $\int_0^\infty \mathbf{E}f^2(s)\,ds < \infty$, and it is $\mathcal{F}_\rho$-measurable, because the integral as the process of the upper limit is continuous.

For finite stopping times ($\mathbf{P}(\rho < \infty) = 1$) instead of (3.7) it is enough to assume that for any $T > 0$

$$\mathbf{P}\Big(\int_0^T f^2(s)\,ds < \infty\Big) = 1, \tag{3.9}$$

since in this case

$$\mathbf{P}\Big(\int_0^\infty 1\!\!\mathrm{I}_{[0,\rho)}(s) f^2(s)\,ds < \infty\Big) = 1.$$

§4. Itô's formula

It is often of interest to study the properties of the process $f(W(t))$, $t \geq 0$, where f is a given smooth function. For the investigation of such processes the technique of stochastic differentiation is very effective. Here we present some results due to K. Itô.

Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$ and let for all $v > t$ the increments $W(v) - W(t)$ be independent of the σ-algebra $\mathcal{F}_t$.

Let the stochastic processes $a(s)$, $b(s)$, $s \in [0,T]$, be progressively measurable with respect to the σ-algebras $\{\mathcal{F}_s\}$.

Assume that

$$\int_0^T |a(s)|\,ds < \infty, \qquad \int_0^T b^2(s)\,ds < \infty, \qquad \text{a.s.,}$$

i.e., $\sqrt{|a(\cdot)|} \in \mathcal{L}_2[0,T]$, $b(\cdot) \in \mathcal{L}_2[0,T]$.

Let $X(t)$, $t \in [0,T]$, be a stochastic process such that $X(0)$ is $\mathcal{F}_0$-measurable. If

$$X(t) = X(0) + \int_0^t a(v)\,dv + \int_0^t b(v)\,dW(v) \tag{4.1}$$

holds a.s. for all $t \in [0,T]$, then we say that $X(t)$ has a *stochastic differential* of the form

$$dX(t) = a(t)\,dt + b(t)\,dW(t). \tag{4.2}$$

Formula (4.2) is the brief symbolic notation for (4.1).

Theorem 4.1 (Itô's formula). *Let $f(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function. Then*

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2} f''(W(t))\,dt. \tag{4.3}$$

Proof. According to the definition of the stochastic differential, it is sufficient to prove that for all $0 \leq t \leq T$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(v))\,dW(v) + \frac{1}{2}\int_0^t f''(W(v))\,dv \qquad \text{a.s.} \tag{4.4}$$

The stochastic integral is well defined because $f'(W(\cdot)) \in \mathcal{L}_2[0,T]$. If equality (4.4) holds a.s. for a fixed t, then it holds a.s. for all $t \in [0,T]$, because all terms figuring in it are continuous processes.

We first assume that $f(x)$, $x \in \mathbf{R}$, is a three times continuously differentiable function with bounded derivatives f', f'', f'''.

Consider an arbitrary sequence of subdivisions $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = t$ of the interval $[0,t]$, satisfying the condition

$$\lim_{n\to\infty} \max_{0\le k\le n-1} |t_{n,k+1} - t_{n,k}| = 0. \tag{4.5}$$

We use the equality

$$f(W(t)) - f(W(0)) = \sum_{k=0}^{n-1} \big(f(W(t_{n,k+1})) - f(W(t_{n,k}))\big).$$

Applying Taylor's formula to the function $f(x)$, $x \in \mathbf{R}$, we have that for every $k = 0,\dots,n-1$

$$f(W(t_{n,k+1})) - f(W(t_{n,k})) = f'(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))$$

$$+\frac{1}{2} f''(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^2 + \frac{1}{6} f'''(W(\tilde t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^3,$$

where $\tilde t_{n,k}$ is some random point in the interval $[t_{n,k}, t_{n,k+1}]$.

By summing these expressions, we obtain

$$f(W(t)) - f(W(0)) = \sum_{k=0}^{n-1} f'(W(t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))$$

$$+\frac{1}{2}\sum_{k=0}^{n-1} f''(W(t_{n,k}))(t_{n,k+1} - t_{n,k}) + \frac{1}{6}\sum_{k=0}^{n-1} f'''(W(\tilde t_{n,k}))(W(t_{n,k+1}) - W(t_{n,k}))^3$$

$$+\frac{1}{2}\sum_{k=0}^{n-1} f''(W(t_{n,k}))\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big]$$

$$=: I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \tag{4.6}$$

Since

$$t_n(v) := \sum_{k=0}^{n-1} t_{n,k} 1\!\!\mathrm{I}_{[t_{n,k}, t_{n,k+1})}(v) \to v \qquad \text{as } n \to \infty$$

uniformly in $v \in [0,t]$, using the continuity of Brownian motion paths and of f', we get

$$\int_0^t \big(f'(W(v)) - f'(W(t_n(v)))\big)^2 dv \to 0 \qquad \text{as } n \to \infty, \qquad \text{a.s.}$$

By the definition of the stochastic integral,

$$I_{n,1}=\int_0^t f'(W(t_n(v)))\,dW(v)\to\int_0^t f'(W(v))\,dW(v) \qquad \text{as} \ \ n\to\infty, \tag{4.7}$$

in probability.

Since f'' is continuous,

$$I_{n,2}=\frac{1}{2}\int_0^t f''(W(t_n(v)))\,dv\to\frac{1}{2}\int_0^t f''(W(v))\,dv \qquad \text{as} \ \ n\to\infty, \qquad \text{a.s.} \tag{4.8}$$

Using the assumption that $|f'''(x)|\le C$ for all $x\in\mathbf{R}$, we obtain

$$|I_{n,3}|\le\frac{C}{6}\sum_{k=0}^{n-1}|W(t_{n,k+1})-W(t_{n,k})|^3$$

$$\le\frac{C}{6}\max_{0\le k\le n-1}|W(t_{n,k+1})-W(t_{n,k})|\sum_{k=0}^{n-1}|W(t_{n,k+1})-W(t_{n,k})|^2.$$

By the continuity of Brownian motion paths and condition (4.5) on the sequence of subdivisions of $\{t_{n,k}\}$, we have

$$\max_{0\le k\le n-1}|W(t_{n,k+1})-W(t_{n,k})|\to 0 \qquad \text{a.s.}$$

Since the Brownian motion W has the finite quadratic variation (see (10.23) Ch. I),

$$\sum_{k=0}^{n-1}|W(t_{n,k+1})-W(t_{n,k})|^2\to t \qquad \text{as} \ \ n\to\infty,$$

in mean square. Therefore, $I_{n,3}\to 0$ in probability.

To prove the convergence $I_{n,4}\to 0$ in probability we estimate $\mathbf{E}I_{n,4}^2$:

$$\mathbf{E}I_{n,4}^2\le\frac{1}{4}\sum_{k=0}^{n-1}\mathbf{E}\big\{(f''(W(t_{n,k})))^2\big[(W(t_{n,k+1})-W(t_{n,k}))^2-(t_{n,k+1}-t_{n,k})\big]^2\big\}$$

$$+\frac{1}{2}\sum_{0\le k<l\le n-1}\mathbf{E}\big\{f''(W(t_{n,k}))\big[(W(t_{n,k+1})-W(t_{n,k}))^2-(t_{n,k+1}-t_{n,k})\big]f''(W(t_{n,l}))$$

$$\times\big[(W(t_{n,l+1})-W(t_{n,l}))^2-(t_{n,l+1}-t_{n,l})\big]\big\}. \tag{4.9}$$

For $k<l$ the random variables

$$f''(W(t_{n,k}))\big[(W(t_{n,k+1})-W(t_{n,k})^2-(t_{n,k+1}-t_{n,k})\big]f''(W(t_{n,l})) \tag{4.10}$$

are $\mathcal{F}_{t_{n,l}}$-measurable and the increments $W(t_{n,l+1}) - W(t_{n,l})$ are independent of $\mathcal{F}_{t_{n,l}}$. Therefore, the expectation after the sign of the double sum is equal to the product of the expectations of the random variables (4.10), and the expectation

$$\mathbf{E}\{(W(t_{n,l+1}) - W(t_{n,l}))^2 - (t_{n,l+1} - t_{n,l})\} = 0.$$

Thus the second sum on the right-hand side of (4.9) equals zero. Since $|f''(x)| \le C$, $x \in \mathbf{R}$, we obtain

$$\mathbf{E}I_{n,4}^2 \le \frac{C^2}{4} \sum_{k=0}^{n-1} \mathbf{E}\big[(W(t_{n,k+1}) - W(t_{n,k}))^2 - (t_{n,k+1} - t_{n,k})\big]^2$$

$$= \frac{C^2}{4} \sum_{k=0}^{n-1} \mathrm{Var}\{(W(t_{n,k+1}) - W(t_{n,k}))^2\} \le \frac{C^2}{2} \max_{0 \le k \le n-1} |t_{n,k+1} - t_{n,k}|\, t.$$

Here we used the estimate (10.24) Ch. I. Using condition (4.5), we finally have

$$I_{n,4} \to 0 \tag{4.11}$$

in mean square and, consequently, in probability.

From (4.6), using the limits (4.7), (4.8) and the convergence of the random variables $I_{n,3}$, $I_{n,4}$ to zero in probability, we get (4.4).

The convergence (4.11) plays a very important role in the whole theory of stochastic differentiation, because it enables us to replace the second-order increments $(W(t_{n,k+1}) - W(t_{n,k}))^2$ by the first-order ones $t_{n,k+1} - t_{n,k}$, when applying Taylor's formula. In the limiting case this can be expressed as follows: the square of the differential of the Brownian motion $\big((dW(t))^2\big)$ coincides with dt, i.e., $(dW(t))^2 = dt$.

To prove (4.4) without the assumption that the derivatives f', f'', and f''' are bounded, we can use the approximation of f by a sequence of functions with bounded derivatives up to the third order.

We first prove (4.4) for a twice continuously differentiable function f with bounded support. Set

$$\hat{f}_n(x) = n \int_{x-1/n}^{x} f(v)\, dv, \qquad n = 1, 2, \dots.$$

These functions are three times continuously differentiable. They have bounded support and bounded third derivative. The first and the second derivatives are uniformly bounded and

$$\hat{f}_n(x) \to f(x), \quad \hat{f}'_n(x) \to f'(x), \quad \hat{f}''_n(x) \to f''(x), \qquad \text{as } n \to \infty$$

uniformly in $x \in \mathbf{R}$.

Indeed, by the mean value theorem for integrals, we have $\hat{f}_n(x) = f(x_n)$,

$$\hat{f}'_n(x) = \frac{f(x) - f(x - 1/n)}{1/n} = f'(\tilde{x}_n), \quad \hat{f}''_n(x) = \frac{f'(x) - f'(x - 1/n)}{1/n} = f''(\hat{x}_n),$$

where x_n, $\tilde{x}_n$, $\hat{x}_n$, are some points from the interval $[x, x-1/n]$. Using the fact that f and its derivatives f', f'' are uniformly continuous because they have bounded support, we obtain the desired approximation.

For the functions $\hat{f}_n(x)$ equality (4.4) holds. Now, taking into account (3.6) and the continuity of the Brownian motion, we can pass to the limit in (4.4) for the functions $\hat{f}_n(x)$. This proves (4.4) for twice continuously differentiable functions f with bounded support.

As the second step we approximate a twice continuously differentiable function f by the functions

$$f_n(x) = f(x)\mathbb{1}_{[-n,n]}(x) + g_n(x)\mathbb{1}_{(n,n+1]}(x) + g_n(x)\mathbb{1}_{[-n-1,-n)}(x)$$

with bounded support. Here the functions $g_n(x)$ are such that $f_n(x)$, $x \in \mathbf{R}$, is twice continuously differentiable function for every n.

From (2.5) for $f \equiv 1$ it follows that

$$\mathbf{P}\Bigl(\sup_{0\le t\le T} |W(t)| \ge n\Bigr) \le \frac{T}{n^2}. \tag{4.12}$$

Then for any $\varepsilon > 0$

$$\mathbf{P}\Bigl(\int_0^T (f'(W(v)) - f_n'(W(v)))^2 \, dv > \varepsilon\Bigr) \le \mathbf{P}\Bigl(\sup_{0\le t\le T} |W(t)| \ge n\Bigr) \le \frac{T}{n^2} \to 0 \tag{4.13}$$

as $n \to \infty$. Similarly,

$$\mathbf{P}\Bigl(\int_0^T |f''(W(v)) - f_n''(W(v))| \, dv > \varepsilon\Bigr) \to 0 \qquad \text{as } n \to \infty. \tag{4.14}$$

Taking into account these estimates and (3.6), we can pass to the limit in (4.4) for functions f_n. Thus (4.4) holds for twice continuously differentiable functions and this completes the proof. □

Remark 4.1. The main feature of Itô's formula is that the second derivative appears in the expression for the first differential. This is impossible in the standard analysis. In stochastic analysis it is the consequence of the properties of Brownian motion.

The analog of (4.4) holds even if the function f has no second derivative.

Theorem 4.2. *Let $f(x)$, $x \in \mathbf{R}$, be a differentiable function, whose first derivative has the form*

$$f'(x) = f'(0) + \int_0^x g(y)dy, \tag{4.15}$$

where $g(x)$, $x \in \mathbf{R}$, *is a measurable function bounded on any finite interval. Then* a.s. *for all* $0 \le t \le T$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(v))\,dW(v) + \frac{1}{2}\int_0^t g(W(v))\,dv. \qquad (4.16)$$

Proof. It is sufficient to prove (4.16) for a function g with bounded support. Otherwise $g(x)$, $x \in \mathbf{R}$, can be approximated by the functions $g_n(x) = g(x)\mathbb{1}_{[-n,n]}(x)$ and we can apply the arguments used before in (4.12)–(4.14) for the proof of Theorem 4.1.

Assume that $\{x : g(x) \neq 0\} \subseteq [a,b]$ for some $a < b$. Set

$$\hat{f}_n(x) := n \int_{x-1/n}^{x} f(y)\,dy, \qquad n = 1, 2, \dots.$$

These are the twice continuously differentiable functions and

$$\hat{f}_n(x) \to f(x), \quad \hat{f}_n'(x) \to f'(x)$$

uniformly in $x \in \mathbf{R}$. Moreover,

$$\hat{f}_n''(x) = n \int_{x-1/n}^{x} g(y)\,dy \to g(x)$$

for almost all x.

Then

$$\int_0^T \mathbf{E}|g(W(s)) - \hat{f}_n''(W(s))|\,ds \le \int_0^T \int_a^b |g(x) - \hat{f}_n''(x)| \frac{e^{-(x-x_0)^2/2s}}{\sqrt{2\pi s}}\,dx ds \to 0$$

as $n \to \infty$, where $W(0) = x_0$.

For the functions $\hat{f}_n(x)$ equality (4.4) holds and we can pass to the limit. This proves the theorem. □

Further, we derive Itô's formula for the case when f depends also on the time parameter t.

Theorem 4.3. *Let* $f(t,x)$, $(t,x) \in [0,T]\times\mathbf{R}$, *be a continuous function with continuous partial derivatives* $\frac{\partial}{\partial t} f(t,x)$, $\frac{\partial}{\partial x} f(t,x)$ *and with continuous partial derivatives* $\frac{\partial^2}{\partial x^2} f(t,x)$ *for* $x \neq x_k$, *where* $\min_{k\in\mathbb{Z}}(x_{k+1} - x_k) \ge \delta > 0$ *for some* $\delta > 0$. *Assume that at the points* x_k *the second order partial derivatives have left and right limits uniformly bounded in* $[0,T]$. *Then*

$$df(t,W(t)) = \frac{\partial}{\partial t} f(t,W(t))\,dt + \frac{\partial}{\partial x} f(t,W(t))\,dW(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t,W(t))\,dt, \quad (4.17)$$

where at the points x_k the second partial derivatives are treated as the left limits of the corresponding derivatives.

Proof. According to the definition of a stochastic differential it is sufficient to prove that for all $0 \le t \le T$,

$$f(t, W(t)) - f(0, W(0)) = \int_0^t \frac{\partial}{\partial v} f(v, W(v))\, dv$$

$$+ \int_0^t \frac{\partial}{\partial x} f(v, W(v))\, dW(v) + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(v, W(v))\, dv. \tag{4.18}$$

We first prove (4.18) for the case when $f(t, x) = \sigma(t)g(x)$ and there exists the continuous derivatives σ' and g'. Moreover, we assume that exists the continuous derivative $g''(x)$ for $x \in \mathbf{R} \setminus \{x_k\}_{k\in\mathbb{Z}}$ with bounded left and right limits at the points x_k. Using subdivisions of the interval $[0, t]$, as in the proof of Theorem 4.2, we can write

$$\sigma(t)g(W(t)) - \sigma(0)g(W(0)) = \sum_{k=0}^{n-1} \big(\sigma(t_{n,k+1})g(W(t_{n,k+1})) - \sigma(t_{n,k})g(W(t_{n,k}))\big)$$

$$= \sum_{k=0}^{n-1} g(W(t_{n,k+1}))(\sigma(t_{n,k+1}) - \sigma(t_{n,k})) + \sum_{k=0}^{n-1} \sigma(t_{n,k})(g(W(t_{n,k+1})) - g(W(t_{n,k}))). \tag{4.19}$$

By Theorem 4.2,

$$g(W(t_{n,k+1})) - g(W(t_{n,k})) = \int_{t_{n,k}}^{t_{n,k+1}} g'(W(v))\, dW(v) + \frac{1}{2} \int_{t_{n,k}}^{t_{n,k+1}} g''(W(v))\, dv.$$

Set

$$t_n^+(v) := \sum_{k=0}^{n-1} t_{n,k+1} \mathbb{1}_{[t_{n,k}, t_{n,k+1})}(v).$$

Then using the representation

$$\sigma(t_{n,k+1}) - \sigma(t_{n,k}) = \int_{t_{n,k}}^{t_{n,k+1}} \sigma'(v)\, dv$$

and the notation $t_n(v)$ introduced in the proof of Theorem 4.1, one can write (4.19) in the form

$$\sigma(t)g(W(t)) - \sigma(0)g(W(0)) = \int_0^t \sigma'(v) g(W(t_n^+(v))\, dv$$

$$+\int_0^t \sigma(t_n(v))g'(W(v))\,dW(v)+\frac{1}{2}\int_0^t \sigma(t_n(v))g''(W(v))\,dv. \qquad (4.20)$$

Since $t_n(v)\to v$ and $t_n^+(v)\to v$ uniformly in $v\in[0,t]$, the passage to the limit in (4.20) proves (4.18) for the special case $f(t,x)=\sigma(t)g(x)$. Here to justify the passage to the limit for the stochastic integral we can apply (3.6).

It is clear that (4.18) is valid for the functions

$$f_n(t,x):=\sum_{k=0}^{n}\sigma_{n,k}(t)g_{n,k}(x), \qquad (4.21)$$

where the functions $g_{n,k}$ have the same properties as the function g above.

For an arbitrary smooth function $f(t,x)$ there exists a sequence of functions $f_n(t,x)$, of the form (4.21), such that for any $N>0$

$$\lim_{n\to 0}\sup_{0\le t\le T}\sup_{|x|\le N}\Bigl(|f(t,x)-f_n(t,x)|+\Bigl|\frac{\partial}{\partial t}f(t,x)-\frac{\partial}{\partial t}f_n(t,x)\Bigr|\Bigr)=0,$$

$$\lim_{n\to 0}\sup_{0\le t\le T}\sup_{|x|\le N}\Bigl|\frac{\partial}{\partial x}f(t,x)-\frac{\partial}{\partial x}f_n(t,x)\Bigr|=0,$$

$$\lim_{n\to 0}\sup_{0\le t\le T}\sup_{|x|\le N,x\notin D}\Bigl|\frac{\partial^2}{\partial x^2}f(t,x)-\frac{\partial^2}{\partial x^2}f_n(t,x)\Bigr|=0,$$

where $D:=\{x_k\}_{k\in\mathbb{Z}}$. Using arguments similar to those stated in (4.12)–(4.14), it is not difficult to complete the proof of the theorem for the general case. □

We now consider the general form of the Itô formula for twice continuously differentiable functions of several arguments.

Theorem 4.4. *Let $f(t,\vec{x})$, $(t,\vec{x})\in[0,T]\times\mathbf{R}^d$, be a continuous function with continuous partial derivatives $\frac{\partial}{\partial t}f(t,\vec{x})$, $\frac{\partial}{\partial x_i}f(t,\vec{x})$, $\frac{\partial^2}{\partial x_i\partial x_j}f(t,\vec{x})$, $i,j=1,\dots,d$.*

Suppose that the coordinates of the vector process $\vec{X}(t)$, $x\in[0,T]$, have the stochastic differentials

$$dX_i(t)=a_i(t)\,dt+b_i(t)\,dW(t), \qquad i=1,\dots,d,$$

where the functions $a_i(t)$ and $b_i(t)$, $t\in[0,T]$, are right continuous and have left limits.

Then the process $f(t,\vec{X}(t))$, $x\in[0,T]$, has the stochastic differential given by

$$df(t,\vec{X}(t))=\frac{\partial}{\partial t}f(t,\vec{X}(t))\,dt+\sum_{i=1}^{d}a_i(t)\frac{\partial}{\partial x_i}f(t,\vec{X}(t))\,dt$$

$$+\sum_{i=1}^{d}b_i(t)\frac{\partial}{\partial x_i}f(t,\vec{X}(t))\,dW(t)+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}b_i(t)b_j(t)\frac{\partial^2}{\partial x_i\partial x_j}f(t,\vec{X}(t))\,dt. \qquad (4.22)$$

Remark 4.2. One can prove (4.22) under the assumption that the second-order partial derivatives $\frac{\partial^2}{\partial x_i \partial x_j} f(t, \vec{x})$, $i, j = 1, \dots, d$, do not exist at vector points $\vec{x}_k$, $k \in \mathbb{Z}$, with coordinates satisfying for some $\delta > 0$ the inequality

$$\min_{1 \le i \le d, k \in \mathbb{Z}} (x_{i,k+1} - x_{i,k}) \ge \delta > 0.$$

Proof of Theorem 4.4. According to the definition of the stochastic differential, it is sufficient to prove that a.s. for all $0 \le t \le T$

$$f(t, \vec{X}(t)) - f(0, \vec{X}(0)) = \int_0^t \frac{\partial}{\partial v} f(v, \vec{X}(v))\, dv + \sum_{i=1}^d \int_0^t a_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dv$$

$$+ \sum_{i=1}^d \int_0^t b_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dW(v) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t b_i(v) b_j(v) \frac{\partial^2}{\partial x_i \partial x_j} f(v, \vec{X}(v))\, dv. \tag{4.23}$$

We prove first (4.23) in the case when the processes a_i and b_i, $i = 1, \dots, d$, are simple. Without loss of generality we can assume that the intervals of constancy are the same for all processes a_i, b_i, i.e.,

$$a_i(s) = \sum_{k=0}^{m-1} a_{i,k} \mathbb{1}_{[s_k, s_{k+1})}(s), \qquad b_i(s) = \sum_{k=0}^{m-1} b_{i,k} \mathbb{1}_{[s_k, s_{k+1})}(s), \qquad i = 1, \dots, d,$$

where $0 = s_0 < s_1 < \cdots < s_l < \cdots < s_m = T$, and the random variables $a_{i,k}$, $b_{i,k}$ are $\mathcal{F}_{s_k}$-measurable, $k = 0, \dots m - 1$, $i = 1, \dots, d$.

In this case the coordinate process X_i for $v \in [s_k, s_{k+1})$ has the form

$$X_i(v) = X_i(s_k) + a_{i,k}(v - s_k) + b_{i,k}(W(v) - W(s_k)), \qquad i = 1, \dots, d.$$

Set for $v \in [s_k, s_{k+1})$

$$g(v, x) := f(v, \vec{X}(s_k) + \vec{a}_k(v - s_k) + \vec{b}_k(x - W(s_k))),$$

where $\vec{a}_k = (a_{1,k}, \dots, a_{d,k})$, $\vec{b}_k = (b_{1,k}, \dots, b_{d,k})$.

We can apply Theorem 4.3, although in the definition of the function g we have the random variables $X_i(s_k)$, $W(s_k)$, $a_{i,k}$, and $b_{i,k}$ (however, it is important that these random variables are $\mathcal{F}_{s_k}$-measurable). Since

$$\frac{\partial}{\partial v} g = \frac{\partial}{\partial v} f + \sum_{i=1}^d a_{i,k} \frac{\partial}{\partial x_i} f, \quad \frac{\partial}{\partial x} g = \sum_{i=1}^d b_{i,k} \frac{\partial}{\partial x_i} f, \quad \frac{\partial^2}{\partial x^2} g = \sum_{i=1}^d \sum_{j=1}^d b_{i,k} b_{j,k} \frac{\partial^2}{\partial x_i \partial x_j} f$$

for $v \in [s_k, s_{k+1})$, using (4.18) we have

$$f(s_{k+1}, \vec{X}(s_{k+1})) - f(s_k, \vec{X}(s_k)) = g(s_{k+1}, W(s_{k+1})) - g(s_k, W(s_k))$$
$$= \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial v} g(v, W(v))\, dv + \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial x} g(v, W(v))\, dW(v) + \frac{1}{2} \int_{s_k}^{s_{k+1}} \frac{\partial^2}{\partial x^2} g(v, W(v))\, dv$$
$$= \int_{s_k}^{s_{k+1}} \frac{\partial}{\partial v} f(v, \vec{X}(v)) dv + \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} a_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v))\, dv$$
$$+ \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} b_i(v) \frac{\partial}{\partial x_i} f(v, \vec{X}(v)) dW(v) + \frac{1}{2} \int_{s_k}^{s_{k+1}} \sum_{i=1}^{d} \sum_{i=j}^{d} b_i(v) b_j(v) \frac{\partial^2}{\partial x_i \partial x_j} f(v, \vec{X}(v)) dv.$$

If $t \in [s_l, s_{l+1})$ for some l, then summing these equalities for $k = 0, \dots, l-1$, and adding the analogous equality for the interval $[s_l, t)$, we obtain (4.23) in the case when a_i and b_i, $i = 1, \dots, d$, are simple processes.

In the general case we can approximate X_i, $i = 1, \dots, d$, by the processes

$$X_{i,n}(t) = X_i(0) + \int_0^t a_{i,n}(v)\, dv + \int_0^t b_{i,n}(v)\, dW(v),$$

where the simple processes $a_{i,n}$ and $b_{i,n}$ are such that

$$\int_0^t |a_i(v) - a_{i,n}(v)|\, dv \to 0, \qquad \int_0^t (b_i(v) - b_{i,n}(v))^2\, dv \to 0, \qquad \text{as } n \to \infty \qquad \text{a.s.}$$

Passage to the limit as $n \to \infty$ in (4.23), done for $\vec{X}_n(t) = (X_{1,n}(t), \dots, X_{d,n}(t))$, completes the proof. □

Notice that for $b_i(t) \equiv 0$, $t \in [0, T]$, $i = 1, \dots, d$, formula (4.22) turns into the classical formula of differentiation of composition of functions. However, in the case when the stochastic differential is included, the second derivatives of functions with respect to the spatial variables play an important role. This is due to the fact that when computing the principal values of the increments of functions of stochastic processes one can use Taylor's formula. Thus, when considering the squares of stochastic differentials, the term $(dW(t))^2$ has, in fact, the first order equal to dt.

We now give an informal description of the generalized Itô's formula, using the following rule:
the differential of function of several stochastic processes is computed by applying Taylor's formula, where one sets $(dt)^2 = 0$, $dt\, dW(t) = 0$, $(dW(t))^2 = dt$, and the differentials of higher orders must be equal to zero.

To illustrate this rule, consider a function with two spatial variables. Let $f(t, x, y)$, $t \in [0, \infty)$, $x, y \in \mathbf{R}$, be a continuous function with continuous partial derivatives f'_t, f'_x, f'_y, $f''_{x,x}$, $f''_{x,y}$, and $f''_{y,y}$.

Suppose that the processes X and Y have the stochastic differentials

$$dX(t) = a(t)\,dt + b(t)\,dW(t), \qquad dY(t) = c(t)\,dt + q(t)\,dW(t).$$

Then according to the rule stated above,

$$(dX(t))^2 = (a(t))^2(dt)^2 + 2a(t)b(t)\,dt\,dW(t) + (b(t))^2(dW(t))^2 = b^2(t)\,dt.$$

Similarly, $(dY(t))^2 = q^2(t)\,dt$, $dX(t)dY(t) = b(t)q(t)\,dt$. It is clear that the differentials of higher orders of the processes X, Y are equal to zero.

Applying Taylor's formula, we obtain

$$\begin{aligned} df(t,X(t),Y(t)) &= f'_t(t,X(t),Y(t))\,dt + f'_x(t,X(t),Y(t))\,dX(t) \\ &+ f'_y(t,X(t),Y(t))\,dY(t) + \frac{1}{2}f''_{x,x}(t,X(t),Y(t))(dX(t))^2 \\ &+ f''_{x,y}(t,X(t),Y(t))dX(t)dY(t) + \frac{1}{2}f''_{y,y}(t,X(t),Y(t))(dY(t))^2. \end{aligned}$$

Therefore,

$$\begin{aligned} df(t,X(t),Y(t)) &= f'_t(t,X(t),Y(t))\,dt + f'_x(t,X(t),Y(t))\{a(t)\,dt + b(t)\,dW(t)\} \\ &+ f'_y(t,X(t),Y(t))\{c(t)\,dt + q(t)\,dW(t)\} + \frac{1}{2}f''_{x,x}(t,X(t),Y(t))b^2(t)\,dt \\ &+ f''_{x,y}(t,X(t),Y(t))b(t)q(t)dt + \frac{1}{2}f''_{y,y}(t,X(t),Y(t))q^2(t)\,dt. \end{aligned} \tag{4.24}$$

Remark 4.3. One can consider independent Brownian motions $W_1(t)$ and $W_2(t)$, $t \ge 0$. Suppose that the processes X the Y have the stochastic differentials

$$dX(t) = a(t)\,dt + b(t)\,dW_1(t), \qquad dY(t) = c(t)\,dt + q(t)\,dW_2(t).$$

In this case $dX(t)dY(t) = 0$, since one must set $dW_1(t)dW_2(t) = 0$. This is a consequence of the fact that for any $s < t$

$$\mathbf{E}\{(W_1(t) - W_1(s))(W_2(t) - W_2(s))\} = \mathbf{E}(W_1(t) - W_1(s))\mathbf{E}(W_2(t) - W_2(s)) = 0.$$

This feature must be taken into account when applying Taylor's formula for computing the differential $df(t,X(t),Y(t))$.

As an application of Theorem 4.4, we derive the *Burkholder–Davis–Gundy inequality* for stochastic integrals.

Lemma 4.1. *Let $h(v)$, $v \in [s,t]$, be a progressively measurable process. Then for $k = 1,2,\dots$ the inequality*

$$\mathbf{E}\sup_{s\le u\le t}\Bigl(\int_s^u h(v)\,dW(v)\Bigr)^{2k} \le 2^k k^{2k}\Bigl(\frac{2k}{2k-1}\Bigr)^{(2k-1)k}\mathbf{E}\Bigl(\int_s^t h^2(v)\,dv\Bigr)^k \tag{4.25}$$

holds.

Proof. Set

$$Z(u) := \int_s^u h(v)\,dW(v), \qquad s \le u \le t,$$

and $\tau_N := \inf\{u \ge s : |Z(u)| = N\}$, assuming $\tau_N = t$ for $\sup_{s\le u\le t} |Z(u)| < N$. Then $\{\tau_N \ge v\} = \Big\{ \sup_{s\le u\le v} |Z(u)| \le N \Big\} \in \mathcal{F}_v$ for every $v \in [0,t]$.

For a fixed s the process

$$Z(u \wedge \tau_N) = \int_s^u 1\!\!\mathrm{I}_{\{v\le\tau_N\}} h(v)\,dW(v), \qquad s \le u \le t,$$

is a martingale with respect to the family of σ-algebras $\{\mathcal{F}_u\}$. By Doob's inequality for martingales (see (5.12) Ch. I),

$$\mathbf{E} \sup_{s\le u\le t\wedge\tau_N} Z^{2k}(u) = \mathbf{E} \sup_{s\le u\le t} Z^{2k}(u\wedge\tau_N) \le \Big(\frac{2k}{2k-1}\Big)^{2k} \mathbf{E} Z^{2k}(t\wedge\tau_N). \tag{4.26}$$

Applying to the process $Z^{2k}(t)$ Itô's formula and substituting $t\wedge\tau_N$ instead of t, we have

$$Z^{2k}(t\wedge\tau_N) = 2k \int_s^t 1\!\!\mathrm{I}_{\{v\le\tau_N\}} Z^{2k-1}(v)h(v)\,dW(v) + k(2k-1) \int_s^{t\wedge\tau_N} Z^{2k-2}(v)h^2(v)\,dv.$$

Since the expectation of the stochastic integral is zero,

$$\mathbf{E} Z^{2k}(t\wedge\tau_N) = k(2k-1)\mathbf{E}\bigg(\int_s^{t\wedge\tau_N} Z^{2k-2}(v)h^2(v)\,dv \bigg).$$

Next applying Hölder's inequality, we obtain

$$\mathbf{E} Z^{2k}(t\wedge\tau_N) \le k(2k-1)\mathbf{E}\bigg(\sup_{s\le u\le t\wedge\tau_N} Z^{2k-2}(u) \int_s^t h^2(v))\,dv \bigg)$$

$$\le k(2k-1)\mathbf{E}^{(k-1)/k}\Big(\sup_{s\le u\le t\wedge\tau_N} Z^{2k-2}(u)\Big)^{k/(k-1)} \mathbf{E}^{1/k}\bigg(\int_s^t h^2(v))\,dv \bigg)^k.$$

In view of (4.26), this yields

$$\mathbf{E} \sup_{s\le u\le t\wedge\tau_N} Z^{2k}(u) \le 2k^2 \Big(\frac{2k}{2k-1}\Big)^{2k-1} \mathbf{E}^{(k-1)/k} \sup_{s\le u\le t\wedge\tau_N} Z^{2k}(u)\, \mathbf{E}^{1/k}\bigg(\int_s^t h^2(v))\,dv \bigg)^k,$$

or

$$\mathbf{E}^{1/k} \sup_{s\le u\le t\wedge\tau_N} Z^{2k}(u) \le 2k^2\Big(\frac{2k}{2k-1}\Big)^{2k-1}\mathbf{E}^{1/k}\Big(\int_s^t h^2(v))\,dv\Big)^k.$$

By raising both sides of this inequality to the power k, letting $N\to\infty$ and applying Fatou's lemma (see (5.18) Ch. I), we get (4.25). □

As it was noticed by R. L. Stratonovich (1966), for special integrands it is possible to define a stochastic integral different from Itô's integral.

Example 4.1. Let $f(x)$, $x\in\mathbf{R}$, be a continuously differentiable function. Let $0=t_{n,0}<t_{n,1}<\cdots<t_{n,n}=T$ be an arbitrary sequence of subdivisions of the interval $[0,T]$, satisfying (4.5). Then the limits in probability

$$\int_0^T f(W(t))\diamond dW(t) := \lim_{n\to\infty}\sum_{k=0}^{n-1} f(W(t_{n,k+1}))(W(t_{n,k+1})-W(t_{n,k})), \tag{4.27}$$

$$\int_0^T f(W(t))\circ dW(t) := \lim_{n\to\infty}\sum_{k=0}^{n-1} f\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big)\Big)(W(t_{n,k+1})-W(t_{n,k})) \tag{4.28}$$

exist.

The existence is due to (4.11). Indeed, assuming that f is a twice continuously differentiable function with bounded second derivative f'' and applying Taylor's formula, we have

$$\sum_{k=0}^{n-1} f(W(t_{n,k+1}))(W(t_{n,k+1})-W(t_{n,k})) = \sum_{k=0}^{n-1} f(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\sum_{k=0}^{n-1} f'(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))^2+\frac12\sum_{k=0}^{n-1} f''(W(\tilde t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))^3.$$

The last sum tends to zero in probability analogously to $I_{n,3}$. In view of (4.11), the second sum on the right-hand side of this equality tends to $\int_0^T f'(W(t))\,dt$. We conclude that the limit (4.27) exists and

$$\int_0^T f(W(t))\diamond dW(t) = \int_0^T f(W(t))\,dW(t) + \int_0^T f'(W(t))\,dt. \tag{4.29}$$

Analogously,

$$\sum_{k=0}^{n-1} f\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big)\Big)(W(t_{n,k+1})-W(t_{n,k})) = \sum_{k=0}^{n-1} f(W(t_{n,k}))(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\sum_{k=0}^{n-1} f'(W(t_{n,k}))\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big)-W(t_{n,k})\Big)(W(t_{n,k+1})-W(t_{n,k}))$$
$$+\frac12\sum_{k=0}^{n-1} f''(W(\tilde t_{n,k}))\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big)-W(t_{n,k})\Big)^2(W(t_{n,k+1})-W(t_{n,k})).$$

The main quantities on the right-hand side of this equality are the first sum, which tends to Itô's integral, and the term

$$\sum_{k=0}^{n-1} f'(W(t_{n,k}))\Big(W\Big(\frac{t_{n,k}+t_{n,k+1}}{2}\Big)-W(t_{n,k})\Big)^2,$$

which tends to $\frac{1}{2}\int_0^T f'(W(t))\,dt$. Therefore,

$$\int_0^T f(W(t))\circ dW(t)=\int_0^T f(W(t))\,dW(t)+\frac{1}{2}\int_0^T f'(W(t))\,dt. \qquad (4.30)$$

Exercises.

4.1. Use Itô's formula to prove that for a Brownian motion W with $W(0)=0$,

$$\int_0^t W^4(s)\,dW(s)=\frac{1}{5}W^5(t)-2\int_0^t W^4(s)\,ds.$$

4.2. Use Itô's formula to compute the differentials:

1) $d\Big(W^3(t)-\frac{t^2}{2}+\int_0^t W^2(s)\,dW(s)\Big)$;

2) $d\big(W(t)\operatorname{sh}W(t)\big)$, where $\operatorname{sh}x:=\dfrac{e^x-e^{-x}}{2}$;

3) $d\exp\big(W^2(t)+W^3(t)\big)$.

4.3. Prove that the process $e^{t/2}\cos W(t)$, $t\ge 0$, is a martingale.

4.4. Use Itô's formula to compute the differentials:

1) $d\exp\Big(W^5(t)+\int_0^t W^4(s)\,dW(s)\Big)$;

2) $d\big(W^3(t)\exp\big(W^2(t)\big)\big)$.

4.5. Suppose that the process V has the differential

$$dV(t)=\mu V(t)\,dt+\sigma V(t)\,dW(t), \qquad V(0)=x>0.$$

Write out $\ln V(t)$.

4.6. Suppose that the process Z has the differential

$$dZ(t)=\big(n\sigma^2-2\gamma Z(t)\big)dt+2\sigma\sqrt{Z(t)}\,dW(t).$$

Compute $d\sqrt{Z(t)}$.

4.7. Prove that the following stochastic processes are martingales:

1) $\Bigl(c+\frac{1}{3}W(t)\Bigr)^3-\frac{1}{3}\int\limits_0^t\bigl(c+\frac{1}{3}W(s)\bigr)\,ds$ for any $c\in\mathbf{R}$;

2) $(W(t)+t)\exp\bigl(-W(t)-\frac{1}{2}t\bigr)$.

§ 5. Brownian local time. Tanaka's formula

Let $X(t)$, $t\in[0,T]$, be a progressively measurable with respect to a filtration $\{\mathcal{F}_t\}$ stochastic process. The *occupation measure* of the process X up to the time t is the measure μ_t defined by

$$\mu_t(\Delta):=\int\limits_0^t \mathbb{1}_\Delta(X(s))ds,\qquad \Delta\in\mathcal{B}(\mathbf{R}),\qquad 0\le t\le T, \tag{5.1}$$

where $\mathbb{1}_\Delta(\cdot)$ is the indicator function.

In other words, $\mu_t(\Delta)$ is equal to the Lebesgue measure (mes) of the time spent by a sample path of the process X in the set Δ up to the time t ($\mu_t(\Delta)=\mathrm{mes}\{s: X(s)\in\Delta,\ s\in[0,t]\}$). This is a random measure that depends on the path of the process.

If a.s. for every t the measure μ_t has a density, i.e., there exists a nonnegative random function $\ell(t,x)$ such that

$$\mu_t(\Delta)=\int\limits_\Delta \ell(t,x)\,dx \tag{5.2}$$

for any Borel set Δ, then the density $\ell(t,x)$ is called the *local time* of the process X at the level x up to the time t.

In the special case when in (5.2) the process $\ell(t,x)$ is continuous in x, one has the following equivalent definition: if a.s. for all $(t,x)\in[0,T]\times\mathbf{R}$ there exists the limit

$$\ell(t,x)=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\delta+\varepsilon}\int\limits_0^t \mathbb{1}_{[x-\delta,x+\varepsilon)}(X(s))ds \qquad \text{a.s.}, \tag{5.3}$$

then $\ell(t,x)$ is called the *local time* of the process X.

From (5.3) it follows that for any fixed x the local time $\ell(t,x)$ is a nondecreasing random function with respect to t, which increases only on the set $\{t: X(t)=x\}$. As a rule, the Lebesgue measure of this set is zero and the most natural measure for such a set turned out to be the local time at the level x.

From the definition of μ_t it obviously follows that the support of μ_t is included in the set

$$\Bigl\{x: \inf_{0\le s\le t}X(s)\le x\le \sup_{0\le s\le t}X(s)\Bigr\}.$$

If the process X has continuous paths, the support of μ_t is a.s. finite. Then

$$\int_0^t f(X(s))ds = \int_{-\infty}^{\infty} f(x)\mu_t(dx) \qquad \text{a.s.} \tag{5.4}$$

for any locally integrable function f. Indeed, by (5.1),

$$\int_0^t \mathbb{I}_\Delta(X(s))ds = \int_{-\infty}^{\infty} \mathbb{I}_\Delta(x)\mu_t(dx)$$

and f can be approximated by the functions $\sum_{k=1}^{n} c_{n,k}\mathbb{I}_{\Delta_{n,k}}(x)$, $\Delta_{n,k} \in \mathcal{B}(\mathbf{R})$.

In particular, if the local time $\ell(t,x)$ exists, then

$$\int_0^t f(X(s))ds = \int_{-\infty}^{\infty} f(x)\ell(t,x)dx \qquad \text{a.s.} \tag{5.5}$$

Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to a filtration $\{\mathcal{F}_t\}$ and let for all $v > t$ the increment $W(v) - W(t)$ be independent of the σ-algebra $\mathcal{F}_t$. Assume that $W(0) = x_0$.

The concept of a local time was introduced by P. Lévy (1939). G. Trotter (1958) proved that for a Brownian motion there exists a continuous local time (the *Brownian local time*). The following result is due to H. Tanaka.

Theorem 5.1 (Tanaka's formula). *The Brownian local time $\ell(t,x)$ exists. The local time $\ell(t,x)$ is an* a.s. *jointly continuous process in $(t,x) \in [0,T] \times \mathbf{R}$, and*

$$(W(t) - x)^+ - (W(0) - x)^+ = \int_0^t \mathbb{I}_{[x,\infty)}(W(s))\,dW(s) + \frac{1}{2}\ell(t,x), \tag{5.6}$$

where $a^+ = \max\{a, 0\}$.

Proof. We prove first that for the process

$$J_x(t) := \int_0^t \mathbb{I}_{[x,\infty)}(W(s))\,dW(s)$$

there exist a modification that is continuous in $(t,x) \in [0,T] \times \mathbf{R}$.

Note first that for a fixed x the process $J_x(t)$ is continuous in t by the property of the stochastic integral as a function of the upper limit. Let us consider $J_x(\cdot)$ as a random variable taking values in the space of continuous functions on $[0,T]$. This space is a Banach space when equipped with the norm $\|f\| := \sup_{t\in[0,T]} |f(t)|$.

Analogously to the proof of Theorem 3.2 Ch. I for real-valued processes one can derive Kolmogorov's continuity criterion for processes with values in a Banach space.

This criterion implies that for any $N > 0$ the process J_x, $x \in [-N, N]$ is a.s. continuous with respect to the norm $\|\cdot\|$ if there exist positive constants α, β, and M_N such that

$$\mathbf{E}\|J_x - J_y\|^\alpha \le M_N |x-y|^{1+\beta}, \qquad |x|, |y| \le N. \tag{5.7}$$

For any $0 < \gamma < \beta/\alpha$, the sample paths of the process J_x, $x \in [-N, N]$ a.s. satisfy the Hölder condition

$$\|J_x - J_y\| \le L_{N,\gamma}(\omega)|x-y|^\gamma. \tag{5.8}$$

Indeed, from the proof of the analog of Theorem 3.2 Ch. I it follows that (5.8) is true for the set D of dyadic rational points. By Cauchy's criterion, the process J_y, $y \in D\bigcap[-N, N]$, can be extended by continuity to the whole interval $[-N, N]$. Since

$$\lim_{y \to x} \int_0^T \mathbf{E}\big(\mathbb{1}_{[y,\infty)}(W(s)) - \mathbb{1}_{[x,\infty)}(W(s))\big)^2 \, ds = 0,$$

we have by (2.8) that for all $x \in [-N, N]$ the process J_x has the desired form as a stochastic integral. Moreover, in view of the a.s. continuity of stochastic integrals for a countable number of particular integrands and the uniform convergence in $t \in [0, T]$, the process $J_x(t)$, $t \in [0, T]$, is a.s. continuous with respect to t for all x simultaneously.

We now prove (5.7). We have

$$\mathbf{E}\|J_x - J_y\|^4 = \mathbf{E} \sup_{t \in [0,T]} \left| \int_0^t \mathbb{1}_{[x,y)}(W(s)) \, dW(s) \right|^4 \qquad \text{for} \quad x < y.$$

By (4.25), $k = 2$,

$$\mathbf{E}\|J_x - J_y\|^4 \le 360 \, \mathbf{E}\left| \int_0^T \mathbb{1}_{[x,y)}(W(s)) \, ds \right|^2$$

$$= 720 \int_0^T ds \int_s^T du \mathbf{E}\big[\mathbb{1}_{[x,y)}(W(s))\mathbb{1}_{[x,y)}(W(u))\big]$$

$$= 720 \int_0^T ds \int_s^T du \int_x^y \int_x^y \frac{e^{-(x_1 - x_0)^2/2s}}{\sqrt{2\pi s}} \frac{e^{-(x_2-x_1)^2/2(u-s)}}{\sqrt{2\pi(u-s)}} dx_1 dx_2$$

$$\le \frac{360}{\pi}|x-y|^2 \int_0^T ds \int_s^T du \frac{1}{\sqrt{s(u-s)}} = M_T|x-y|^2.$$

Thus for the process J_x the Hölder condition (5.8) holds for $0 < \gamma < 1/4$.

Applying (4.25) for an arbitrary even power, we can prove the estimate

$$\mathbf{E}\|J_x - J_y\|^{2k} \le M_{k,T}|x-y|^k, \qquad k = 1, 2, \dots .$$

Therefore (5.8) holds for any $0 < \gamma < 1/2$.

The continuity of $J_x(t)$ in (t,x) follows from (5.8), because

$$|J_x(t) - J_y(s)| \le |J_x(t) - J_x(s)| + \|J_x(\cdot) - J_y(\cdot)\|.$$

We now prove that for arbitrary $r \in \mathbf{R}$ there the limit

$$\ell(t,r) := \lim_{\alpha\uparrow r}\lim_{\beta\downarrow r} \frac{1}{\beta-\alpha}\int_0^t \mathbb{1}_{[\alpha,\beta)}(W(s))ds \qquad \text{a.s.} \tag{5.9}$$

exists uniformly in $t \in [0,T]$ and (5.6) holds for $x = r$.

Set

$$f_{\alpha,\beta}(x) := \int_{-\infty}^{x}\int_{-\infty}^{z} \frac{\mathbb{1}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy\,dz.$$

By the formula of stochastic differentiation (4.16), a.s. for all $t \in [0,T]$,

$$\frac{1}{2}\int_0^t \frac{\mathbb{1}_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\,ds = f_{\alpha,\beta}(W(t)) - f_{\alpha,\beta}(W(0)) - \int_0^t f'_{\alpha,\beta}(W(s))\,dW(s). \tag{5.10}$$

It is clear that

$$f'_{\alpha,\beta}(x) = \int_{-\infty}^{x} \frac{\mathbb{1}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy = \begin{cases} 1, & \beta \le x, \\ \dfrac{x-\alpha}{\beta-\alpha}, & \alpha < x < \beta, \\ 0, & x \le \alpha, \end{cases} \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} \mathbb{1}_{[r,\infty)}(x), \quad \text{for } x \ne r,$$

$$f_{\alpha,\beta}(x) = \int_{-\infty}^{x}\int_{-\infty}^{z} \frac{\mathbb{1}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy\,dz = \begin{cases} x - \dfrac{\beta+\alpha}{2}, & \beta \le x, \\ \dfrac{(x-\alpha)^2}{2(\beta-\alpha)}, & \alpha < x < \beta, \\ 0, & x \le \alpha, \end{cases} \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} (x-r)^+.$$

Since

$$|\mathbb{1}_{[r,\infty)}(x) - f'_{\alpha,\beta}(x)| \le \mathbb{1}_{[\alpha,\beta)}(x), \qquad \alpha < r < \beta,$$

and, consequently,

$$|(x-r)^+ - f_{\alpha,\beta}(x)| \le |\beta - \alpha|,$$

we have

$$\sup_{t\in[0,T]} \bigl|(W(t)-r)^+ - f_{\alpha,\beta}(W(t))\bigr| \le |\beta-\alpha| \underset{\alpha\uparrow r,\ \beta\downarrow r}{\longrightarrow} 0 \qquad \text{a.s.} \tag{5.11}$$

Let us prove that

$$\int_0^t f'_{\alpha,\beta}(W(s))\,dW(s) = \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy \qquad \text{a.s.} \tag{5.12}$$

It is clear that

$$f'_{\alpha,\beta}(x) = \int_{-\infty}^x \frac{\mathbb{1}_{[\alpha,\beta)}(y)}{\beta-\alpha}\,dy = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbb{1}_{[y,\infty)}(x)\,dy.$$

Then (5.12) can be written in the form

$$\frac{1}{\beta-\alpha}\int_0^t\int_\alpha^\beta \mathbb{1}_{[y,\infty)}(W(s))\,dy\,dW(s) = \frac{1}{\beta-\alpha}\int_\alpha^\beta\int_0^t \mathbb{1}_{[y,\infty)}(W(s))\,dW(s)\,dy$$

and this is the switching the order of integration formula (analog of Fubini's theorem) and for the stochastic integral such formula must be proved.

Set

$$q_n(x) = \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta} \mathbb{1}_{[k/n,\infty)}(x)\frac{1}{n}.$$

Since

$$\left| q_n(x) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathbb{1}_{[[yn]/n,\infty)}(x)\,dy\right| \le \frac{2}{n(\beta-\alpha)},$$

we have

$$|f'_{\alpha,\beta}(x) - q_n(x)| \le \frac{3}{n(\beta-\alpha)}. \tag{5.13}$$

Using the continuity of J_x in x, we obtain

$$\int_0^t q_n(W(s))\,dW(s) = \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta}\int_0^t \mathbb{1}_{[k/n,\infty)}(W(s))\,dW(s)\frac{1}{n}$$

$$= \frac{1}{\beta-\alpha}\sum_{\alpha\le k/n\le\beta} J_{k/n}(t)\frac{1}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy.$$

This together with (5.13) imply (5.12).

Substituting (5.12) into (5.10), we have

$$\frac{1}{2}\int_0^t \frac{\mathbb{1}_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\,ds = f_{\alpha,\beta}(W(t)) - f_{\alpha,\beta}(W(0)) - \frac{1}{\beta-\alpha}\int_\alpha^\beta J_y(t)\,dy.$$

Applying (5.11) and taking into account the continuity of J_x (see (5.8)), we see that the limit (5.9) exists uniformly in $t \in [0,T]$, and (5.6) holds for $x = r$. The statement that equality (5.6) holds for all t and x simultaneously follows from the continuity of $J_x(t)$ and $(W(t)-x)^+$ in (t,x). This also implies the continuity of $\ell(t,x)$ in $(t,x) \in [0,T] \times \mathbf{R}$.

Moreover, since

$$\sup_{z\in\mathbf{R}} |(z-x)^+ - (z-y)^+| \le |x-y|,$$

from (5.6) and (5.8), it follows that for any $0 < \gamma < 1/2$ and $N > 0$

$$\|\ell(\cdot,x) - \ell(\cdot,y)\| \le B_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x,y \in [-N,N]. \tag{5.14}$$

We can prove that Brownian local time paths with respect to x are a.s. nowhere locally Hölder continuous of order $\gamma \ge 1/2$ (see Ch. V § 11). In particular, they are nowhere differentiable in x. The theorem is proved. □

Since the local time has the finite support

$$\Big\{x : \inf_{0\le s\le t} W(s) \le x \le \sup_{0\le s\le t} W(s)\Big\},$$

from (5.5) it follows that for any locally integrable function f and any $t > 0$,

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)\ell(t,x)\,dx \qquad \text{a.s.}, \tag{5.15}$$

and the integral on the right-hand side is finite.

From (5.9) we have

$$\mathbf{E}\ell(t,x) = \lim_{\alpha\uparrow x}\lim_{\beta\downarrow x} \int_0^t \mathbf{E}\Big(\frac{1\!\!1_{[\alpha,\beta)}(W(s))}{\beta-\alpha}\Big)ds = \int_0^t \frac{1}{\sqrt{2\pi s}} e^{-(x-x_0)^2/2s}\,ds. \tag{5.16}$$

Here $\frac{1}{\sqrt{2\pi s}} e^{-(x-x_0)^2/2s}$ is the density of the variable $W(s)$, $W(0) = x_0$.

Using Tanaka's formula (5.6) one can generalize Itô's formula (4.16) as follows.

Theorem 5.2. *Let b be a function of bounded variation on any finite interval. Set*

$$f(x) := f_0 + \int_0^x b(y)dy, \tag{5.17}$$

where f_0 is a constant.

Then a.s. *for all $t \in [0,T]$,*

$$f(W(t)) - f(W(0)) = \int_0^t b(W(s))\,dW(s) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(t,x)\,b(dx), \tag{5.18}$$

where $b(dx)$ is the signed measure (charge) associated to b via its representation as a difference of two nondecreasing functions.

Remark 5.1. The differential form of (5.18) is the following formula:

$$df(W(t)) = b(W(t))\,dW(t) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(dt,x)\,b(dx).$$

Remark 5.2. If $b(dx)$ has a bounded density, then $b(dx) = g(x)\,dx$, $b(x) = f'(x)$ and, in view of (5.15), formula (5.18) transforms into (4.16).

Remark 5.3. Let the function f be twice continuously differentiable except at the finite number of points $x_1 < x_2 < \cdots < x_m$, in which f is assumed to have the right and left derivatives. Then from (5.18) it follows that

$$\begin{aligned} f(W(t)) - f(W(0)) &= \int_0^t \sum_{k=0}^{m} f'(W(s))\mathbb{1}_{(x_k,x_{k+1})}(W(s))\,dW(s) \\ &+ \frac{1}{2}\int_0^t \sum_{k=0}^{m} f''(W(s))\mathbb{1}_{(x_k,x_{k+1})}(W(s))\,ds \\ &+ \frac{1}{2}\sum_{k=1}^{m}(f'(x_k+0) - f'(x_k-0))\ell(t,x_k) \qquad \text{a.s.,} \end{aligned} \tag{5.19}$$

where we set $x_0 = -\infty$, $x_{m+1} = \infty$.

Proof of Theorem 5.2. It suffices to prove (5.18) only for a nondecreasing function b, since any function of bounded variation is the difference of two nondecreasing functions.

For the functions

$$b_n(x) := \sum_{k=1}^{n} c_{n,k}\,\mathbb{1}_{[r_{n,k},\infty)}(x) \tag{5.20}$$

equality (5.18) follows from Tanaka's formula (5.6).

Now set

$$f_n(x) := f_0 + \int_0^x b_n(y)dy = f_0 + \sum_{k=1}^{n} c_{n,k}(x - r_{n,k})^+.$$

Then, by (5.6),

$$f_n(W(t)) - f_n(W(0)) = \int_0^t b_n(W(s))\,dW(s) + \frac{1}{2}\int_{-\infty}^{\infty} \ell(t,x)\,b_n(dx) \qquad \text{a.s.} \tag{5.21}$$

It is clear that any nondecreasing function can be uniformly approximated on any compact set by functions of the form (5.20), i.e., for any $N > 0$

$$\sup_{|x|\le N} |b(x) - b_n(x)| \to 0 \qquad \text{as } n \to \infty. \tag{5.22}$$

One can ensure that $b_n(N) = b(N)$ and $b_n(-N) = b(-N)$. Of course, the sequence of functions b_n depends on N. It is clear that

$$\sup_{|x|\le N} |f(x) - f_n(x)| \le 2N \sup_{|x|\le N} |b(x) - b_n(x)|. \tag{5.23}$$

Since, by (4.12),

$$\mathbf{P}\Big(\sup_{0\le t\le T} |W(t)| \ge N\Big) \le \frac{T}{N^2}, \tag{5.24}$$

and by the choice of N, this probability can be made sufficiently small, we can restrict ourselves to the consideration of the set $\Omega_N = \Big\{\sup_{0\le t\le T} |W(t)| < N\Big\}$. From (5.23), (5.22), and (3.6) it follows that

$$\sup_{t\in[0,T]} |f(W(t)) - f_n(W(t))| \underset{n\to\infty}{\longrightarrow} 0, \qquad |f(W(0)) - f_n(W(0))| \underset{n\to\infty}{\longrightarrow} 0, \tag{5.25}$$

$$\sup_{t\in[0,T]} \Big| \int_0^t b(W(s))\, dW(s) - \int_0^t b_n(W(s))\, dW(s) \Big| \underset{n\to\infty}{\longrightarrow} 0 \tag{5.26}$$

in probability given the set Ω_N.

Let us prove that

$$\sup_{t\in[0,T]} \Big| \int_{-N}^N \ell(t,x)\, b(dx) - \int_{-N}^N \ell(t,x)\, b_n(dx) \Big| \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.} \tag{5.27}$$

By (5.14),

$$\sup_{t\in[0,T]} \Big| \int_{-N}^N \ell(t,x)\, b(dx) - \int_{-N}^N \ell(t, \tfrac{[xm]}{m})\, b(dx) \Big|$$

$$\le \sup_{|x|\le N} \|\ell(\cdot,x) - \ell(\cdot, \tfrac{[xm]}{m})\| (b(N) - b(-N)) \le \frac{B_{N,\gamma}(\omega)}{m^\gamma} (b(N) - b(-N)).$$

Analogously, in view of (5.22),

$$\sup_{t\in[0,T]} \Big| \int_{-N}^N \ell(t,x)\, b_n(dx) - \int_{-N}^N \ell(t, \tfrac{[xm]}{m})\, b_n(dx) \Big|$$

$$\le \sup_{|x|\le N} \|\ell(\cdot,x)-\ell(\cdot,\tfrac{[xm]}{m})\|(b_n(N)-b_n(-N)) \le \frac{B_{N,\gamma}(\omega)}{m^\gamma}(b(N)-b(-N)).$$

In addition, we have the estimate

$$\sup_{t\in[0,T]}\biggl|\int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\,b(dx) - \int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\,b_n(dx)\biggr|$$

$$= \sup_{t\in[0,T]}\biggl|\int_{-N}^{N} \ell(t,\tfrac{[xm]}{m})\,(b(dx)-b_n(dx))\biggr|$$

$$\le \sum_{k=-[Nm]}^{[Nm]} \ell(T,\tfrac{k}{m})\bigl|b(\tfrac{k+1}{m})-b(\tfrac{k}{m})-b_n(\tfrac{k+1}{m})+b_n(\tfrac{k}{m})\bigr|.$$

Now letting first $n\to\infty$ and then $m\to\infty$, we obtain (5.27).

Taking into account (5.24)–(5.27) we see that the passage to the limit in (5.21) leads to (5.18). □

Similarly, we can prove the following generalization of the special case of Theorem 4.3 where $f(t,x)=\sigma(t)f(x)$.

Theorem 5.3. *Let f be the function defined by* (5.17) *and $\sigma(t)$, $t\ge 0$, be a function with locally integrable derivative.*

Then a.s. *for all $t\in[0,T]$,*

$$\sigma(t)f(W(t))-\sigma(0)f(W(0)) = \int_0^t \sigma'(s)f(W(s))\,ds$$

$$+\int_0^t \sigma(s)b(W(s))\,dW(s) + \frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \sigma(s)\ell(ds,x)\,b(dx). \tag{5.28}$$

Proof. We can apply the method used to establish formulas (4.19) and (4.20). Considering subdivisions of $[0,t]$, as in the proof of Theorem 4.1, we can write, according to (5.18), that

$$f(W(t_{n,k+1}))-f(W(t_{n,k})) = \int_{t_{n,k}}^{t_{n,k+1}} f'(W(v))\,dW(v)+\frac{1}{2}\int_{-\infty}^{\infty}(\ell(t_{n,k+1},x)-\ell(t_{n,k}))\,b(dx).$$

The analog of (4.20) is the relation

$$\sigma(t)f(W(t))-\sigma(0)f(W(0)) = \int_0^t \sigma'(v)f(W(t_n^+(v))\,dv$$

$$+\int_0^t \sigma(t_n(v))f'(W(v))\,dW(v)+\frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \sigma(t_n(v))\ell(dv,x)\,b(dx). \tag{5.29}$$

The function $\sigma(v)$, $v\in[0,T]$, is uniformly continuous with a modulus of continuity $\Delta(\delta)\to 0$ as $\delta\to 0$. Using this, we have

$$\Bigl|\int_0^t(\sigma(t_n(v))-\sigma(v))\ell(dv,x)\Bigr|\le \Delta\Bigl(\max_{1\le k\le n}|t_{n,k}-t_{n,k-1}|\Bigr)\ell(t,x).$$

The subdivisions of the interval $[0,t]$ satisfy (4.5), therefore in (5.29) we can pass to the limit and get (5.28). □

Example 5.1. Compute for $b>0$ the stochastic differential $d|\,|W(t)-a|-b\,|$. It is obvious that

$$|\,|x-a|-b\,|=(a-x-b)\mathbb{1}_{(-\infty,a-b)}(x)+(x-a+b)\mathbb{1}_{[a-b,a)}(x)$$
$$+(a-x+b)\mathbb{1}_{[a,a+b)}(x)+(x-a-b)\mathbb{1}_{[a+b,\infty)}(x).$$

Applying (5.19), we have

$$d|\,|W(t)-a|-b\,|=\bigl(\mathbb{1}_{(a+b,\infty)}(W(t))-\mathbb{1}_{(a,a+b)}(W(t))+\mathbb{1}_{(a-b,a)}(W(t))$$
$$-\mathbb{1}_{(-\infty,a-b)}(W(t))\bigr)dW(t)+\ell(dt,a+b)-\ell(dt,a)+\ell(dt,a-b).$$

Since the expectation of a stochastic integral equals zero, from (5.16) it follows that

$$\frac{d}{dt}\mathbf{E}_{x_0}||W(t)-a|-b|=\frac{1}{\sqrt{2\pi t}}e^{-(a-b-x_0)^2/2t}+\frac{1}{\sqrt{2\pi t}}e^{-(a+b-x_0)^2/2t}-\frac{1}{\sqrt{2\pi t}}e^{-(a-x_0)^2/2t},$$

where the subscript in the expectation means that it is computed with respect to the process W with $W(0)=x_0$.

Exercises.

5.1. Compute the differentials

1) $d\exp\Bigl(|W(t)|^3+\int_0^t W^2(s)\,dW(s)\Bigr)$;

2) $d\bigl(|W(t)|e^{|W(t)-r|}\bigr)$;

3) $d|\,|W(t)-a|^3-b^3|, \qquad 0<b<a.$

§ 6. Stochastic exponent

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space and $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$. Let for all $v > t$ the increments $W(v) - W(t)$ be independent of the σ-algebra $\mathcal{F}_t$.

For an arbitrary $b \in \mathcal{L}_2[0,T]$, consider the *stochastic exponent*

$$\rho(t) := \exp\biggl(\int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b^2(s)\,ds\biggr), \qquad t \in [0,T]. \tag{6.1}$$

Let us compute the stochastic differential of the process ρ. Applying Itô's formula (4.22), $d = 1$, for $f(t,x) = e^x$ and the process

$$X(t) = \int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b^2(s)\,ds,$$

we have

$$d\rho(t) = \rho(t)\Bigl[b(t)\,dW(t) - \frac{1}{2}b^2(t)\,dt + \frac{1}{2}b^2(t)\,dt\Bigr] = \rho(t)b(t)\,dW(t).$$

Therefore,

$$d\rho(t) = \rho(t)b(t)\,dW(t), \qquad \rho(0) = 1. \tag{6.2}$$

The process ρ is called the stochastic exponent by analogy with the classical exponent $\tilde{\rho}(t) = \exp\bigl(\int_0^t b(s)\,ds\bigr)$, which is the solution of the equation

$$d\tilde{\rho}(t) = \tilde{\rho}(t)b(t)\,dt, \qquad \tilde{\rho}(0) = 1.$$

Equation (6.2) is the simplest form of so-called *stochastic differential equation* (see § 7). According to the definition of stochastic differentials, (6.2) is equivalent to the equation

$$\rho(t) = 1 + \int_0^t \rho(s)b(s)\,dW(s), \qquad t \in [0,T]. \tag{6.3}$$

We will prove that, under some conditions, $\rho(t)$ is a nonnegative martingale with respect to the filtration $\{\mathcal{F}_t\}$, with mean value $\mathbf{E}\rho(t) = 1$ for every $t \in [0,T]$.

Proposition 6.1. *Let b be a continuous stochastic process from $\mathcal{L}_2[0,T]$. Suppose that for some $\delta > 0$*

$$\mathbf{E}\exp\biggl((1+\delta)\int_0^T b^2(s)\,ds\biggr) < \infty, \tag{6.4}$$

or

$$\sup_{0\le s\le T} \mathbf{E}e^{\delta b^2(s)} < \infty. \tag{6.5}$$

Then for any $0 \le t_1 < t_2 \le T$,

$$\mathbf{E}\exp\Big(\int_{t_1}^{t_2} b(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} b^2(s)\,ds\Big) = 1, \tag{6.6}$$

and, in addition,

$$\mathbf{E}\Big\{\exp\Big(\int_{t_1}^{t_2} b(s)\,dW(s) - \frac{1}{2}\int_{t_1}^{t_2} b^2(s)\,ds\Big)\Big|\mathcal{F}_{t_1}\Big\} = 1 \qquad \text{a.s.} \tag{6.7}$$

Remark 6.1. The relations (6.6) and (6.7) are valid (see Novikov (1972), Liptser and Shiryaev (1974)) for an arbitrary process from $\mathcal{L}_2[0,T]$ under weaker assumptions than (6.4), (6.5), which are taken from Gihman and Skorohod (1972). In (6.4) the factor $1+\delta$ can be replaced by the factor $1/2$, but to improve it to the factor $1/2-\delta$ is not possible.

Proof of Proposition 6.1. We assume first that $b(s) = \bar b(s)$, $s\in[0,T]$, is a simple process defined by (1.3) and $\sup_{0\le s\le T}|\bar b(s)| \le M$, where M is nonrandom. Then

$$\mathbf{E}\exp\Big(\int_{t_1}^{t_2} \bar b(s)\,dW(s)\Big) \le e^{M^2(t_2-t_1)/2}.$$

This means that for every $m>0$

$$\mathbf{E}\exp\Big(m\int_{t_1}^{t_2} \bar b(s)\,dW(s)\Big) \le e^{m^2M^2(t_2-t_1)/2}. \tag{6.8}$$

Indeed, since on the interval $[s_k, s_{k+1})$, $s_0 = t_1$, $s_m = t_2$, $k = 1,\dots,m-1$, the process $\bar b$ is equal to the $\mathcal{F}_{s_k}$-measurable random variable b_k, using the properties of conditional expectations, and (10.9) Ch. I we have

$$\mathbf{E}\exp\Big(\int_{t_1}^{t_2} \bar b(s)\,dW(s)\Big) = \mathbf{E}\Big\{\mathbf{E}\Big\{\exp\Big(\sum_{k=0}^{m-1} b_k(W(s_{k+1}) - W(s_k))\Big)\Big|\mathcal{F}_{s_{m-1}}\Big\}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(\sum_{k=0}^{m-2} b_k(W(s_{k+1}) - W(s_k))\Big)\exp\big(b_{m-1}^2(s_m - s_{m-1})/2\big)\Big\}$$

$$\le e^{M^2(s_m-s_{m-1})/2}\mathbf{E}\exp\Big(\sum_{k=0}^{m-2} b_k(W(s_{k+1}) - W(s_k))\Big) \le e^{M^2(t_2-t_1)/2}.$$

By (6.3),

$$\exp\Big(\int_{t_1}^{t_2}\bar b(s)\,dW(s)-\frac12\int_{t_1}^{t_2}\bar b^2(s)\,ds\Big)-1$$
$$=\int_{t_1}^{t_2}\exp\Big(\int_{t_1}^{t}\bar b(s)\,dW(s)-\frac12\int_{t_1}^{t}\bar b^2(s)\,ds\Big)\bar b(t)\,dW(t).$$

Since

$$\int_{t_1}^{t_2}\mathbf{E}\Big\{\Big(\exp\Big(\int_{t_1}^{t}\bar b(s)\,dW(s)-\frac12\int_{t_1}^{t_2}\bar b^2(s)\,ds\Big)\bar b(t)\Big)^2\Big\}dt<\infty,$$

the expectation of the stochastic integral is zero, and

$$\mathbf{E}\exp\Big(\int_{t_1}^{t_2}\bar b(s)\,dW(s)-\frac12\int_{t_1}^{t_2}\bar b^2(s)\,ds\Big)=1. \tag{6.9}$$

Thus (6.6) is proved for bounded simple processes. Now, by (2.3), equation (6.3) and inequality (6.8), we get $\mathbf{E}\{\rho(t_2)|\mathcal{F}_{t_1}\}=\rho(t_1)$ a.s. Hence, (6.7) holds for the simple processes.

We turn to the proof of the statement for a continuous process b. Let $b(s)=0$ for $s<0$. We construct for the process b a sequence of bounded simple processes $\bar b_n(s)$, $s\in[0,T]$, such that

$$\bar b_n^2(s)\le b^2(s-1/n) \tag{6.10}$$

and

$$\lim_{n\to\infty}\int_0^T(b(s)-\bar b_n(s))^2\,dt=0 \qquad \text{a.s.} \tag{6.11}$$

For $s\in[0,1/n)$, we set $\bar b_n(s)=0$. For $s\in[k/n,(k+1)/n)$, $k=1,2,\dots,[nT]$, we set $\bar b_n(s):=\min\Big\{\inf_{(k-1)/n\le s\le k/n}b(s),n\Big\}$ if $\inf_{(k-1)/n\le s\le k/n}b(s)>0$, we set $\bar b_n(s):=\max\Big\{\sup_{(k-1)/n\le s\le k/n}b(s),-n\Big\}$, if $\sup_{(k-1)/n\le s\le k/n}b(s)<0$, and we set $\bar b_n(s)=0$ if in at least one point of the interval $[(k-1)/n,k/n]$ the process b becomes equal zero. The simple bounded processes $\bar b_n$ are adapted to the filtration $\{\mathcal{F}_t\}$ and (6.10) is satisfied. Then (6.11) holds, because the process b is uniformly continuous on $[0,T]$.

In view of (6.11) and (3.6), the sequence of random variables

$$\exp\Big(\int_{t_1}^{t_2}\bar b_n(s)\,dW(s)-\frac12\int_{t_1}^{t_2}\bar b_n^2(s)\,ds\Big)$$

converges in probability to the variable

$$\exp\Big(\int_{t_1}^{t_2}b(s)\,dW(s)-\frac12\int_{t_1}^{t_2}b^2(s)\,ds\Big).$$

If we prove that this sequence of random variables is uniformly integrable (see § 1 Ch. I), then by Proposition 1.3 of Ch. I, we can pass to the limit in (6.9), applied to the simple processes $\overline{b}_n$ and get (6.6). Equality (6.7) is proved analogously with the help of property 7′) of the conditional expectations (see § 2 Ch. I).

Choose $\gamma > 0$ such that $(1+\gamma)^2(1+\gamma/2) = 1+\delta$. Using Hölder's inequality and (6.9), we get

$$\mathbf{E}\Bigl(\exp\Bigl(\int_{t_1}^{t_2}\overline{b}_n(s)\,dW(s)-\frac{1}{2}\int_{t_1}^{t_2}\overline{b}_n^2(s)\,ds\Bigr)\Bigr)^{1+\gamma}$$

$$=\mathbf{E}\Bigl\{\exp\Bigl((1+\gamma)\int_{t_1}^{t_2}\overline{b}_n(s)\,dW(s)-\frac{(1+\gamma)^3}{2}\int_{t_1}^{t_2}\overline{b}_n^2(s)\,ds+\frac{\gamma(1+\gamma)(2+\gamma)}{2}\int_{t_1}^{t_2}\overline{b}_n^2(s)\,ds\Bigr)\Bigr\}$$

$$\le\Bigl[\mathbf{E}\Bigl(\exp\Bigl((1+\gamma)^2\int_{t_1}^{t_2}\overline{b}_n(s)\,dW(s)-\frac{(1+\gamma)^4}{2}\int_{t_1}^{t_2}\overline{b}_n^2(s)\,ds\Bigr)\Bigr]^{1/(1+\gamma)}$$

$$\times\Bigl[\mathbf{E}\exp\Bigl(\frac{(1+\gamma)^2(2+\gamma)}{2}\int_{t_1}^{t_2}\overline{b}_n^2(s)\,ds\Bigr)\Bigr]^{\gamma/(1+\gamma)}$$

$$\le\Bigl[\mathbf{E}\exp\Bigl((1+\delta)\int_{t_1-1/n}^{t_2-1/n}b^2(s)\,ds\Bigr)\Bigr]^{\gamma/(1+\gamma)}<\infty.$$

By Proposition 1.2 Ch. I with $G(x) = x^{1+\gamma}$, this implies that the corresponding sequence of random variables is uniformly integrable. Proposition 6.1 is proved under the condition (6.4).

We turn to the proof of this assertion under the condition (6.5).

Since the function $g(x) := e^x$ is convex, by Jensen's inequality for the integral of the normalized measure (see (1.4) Ch. I), we have that for $v < u$ and $\delta > 0$

$$\exp\Bigl((1+\delta)\int_v^u b^2(s)\,ds\Bigr)=\exp\Bigl(\int_v^u(1+\delta)(u-v)b^2(s)\frac{ds}{u-v}\Bigr)\le\int_v^u e^{(1+\delta)(u-v)b^2(s)}\,\frac{ds}{u-v}.$$

By (6.5), for any $0 < u-v \le \dfrac{\delta}{1+\delta}$ there holds the estimate

$$\mathbf{E}\exp\Bigl((1+\delta)\int_v^u b^2(s)\,ds\Bigr)\le\frac{1}{u-v}\int_v^u\mathbf{E}e^{\delta b^2(s)}\,ds<\infty.$$

This is exactly the condition (6.4), therefore by the assertion proved above, we have for any $0 < u-v \le \dfrac{\delta}{1+\delta}$ the equality

$$\mathbf{E}\Bigl\{\exp\Bigl(\int_v^u b(s)\,dW(s)-\frac{1}{2}\int_v^u b^2(s)\,ds\Bigr)\Big|\mathcal{F}_v\Bigr\}=1. \qquad (6.12)$$

Divide the interval $[t_1, t_2]$ by points $t_1 = v_0 < v_1 < \dots < v_m = t_2$ such that $\max_{1\le r\le m}(t_k - t_{k-1}) \le \frac{\delta}{1+\delta}$. Under the assumption that (6.12) is proved for $v = v_0$, $u = v_{m-1}$, we prove (6.12) for $v = v_0$, $u = v_m$. Since (6.12) holds for $v = v_{m-1}$, $u = v_m$, we have

$$\mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_m} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_m} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_0}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_{m-1}} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_{m-1}} b^2(s)\,ds\Big)$$

$$\times\,\mathbf{E}\Big\{\exp\Big(\int_{v_{m-1}}^{v_m} b(s)\,dW(s) - \frac{1}{2}\int_{v_{m-1}}^{v_m} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_{m-1}}\Big\}\Big|\mathcal{F}_{v_0}\Big\}$$

$$= \mathbf{E}\Big\{\exp\Big(\int_{v_0}^{v_{m-1}} b(s)\,dW(s) - \frac{1}{2}\int_{v_0}^{v_{m-1}} b^2(s)\,ds\Big)\Big|\mathcal{F}_{v_0}\Big\} = 1 \quad .$$

The induction base for $v = v_0$, $u = v_1$ is also valid. Therefore (6.7) holds. Proposition 6.1 is proved. □

Remark 6.2. Suppose that the process $b(s)$, $s \in [0, T]$, is adapted to the filtration $\{\mathcal{F}_s\}$, and $\sup_{0\le s\le T} |b(s)| \le M$ for some nonrandom constant M. Then for any $m > 0$

$$\mathbf{E}\exp\Big(m\int_{t_1}^{t_2} b(s)\,dW(s)\Big) \le e^{m^2M^2(t_2-t_1)/2}. \tag{6.13}$$

Indeed, according to Proposition 1.1, the process b can be approximated by a sequence of bounded simple processes $\bar b_n$ such that (2.8) holds. For a simple processes $\bar b_n$ we have (6.8). By Proposition 1.2 Ch. I with $G(x) = x^{1+\gamma}$, $\gamma > 0$, the corresponding sequence of random variables is uniformly integrable. Therefore, we can pass to the limit under the expectation sign in (6.8) applied for $\bar b_n$. This implies (6.13).

Equation (6.3) gives us the iterative procedure

$$\rho(t) = 1 + \int_0^t \Big(1 + \int_0^{t_1} \rho(s)b(s)\,dW(s)\Big)b(t_1)\,dW(t_1) = 1 + \int_0^t b(t_1)\,dW(t_1)$$

$$+ \int_0^t\int_0^{t_1}\Big(1 + \int_0^{t_2} \rho(s)b(s)\,dW(s)\Big)b(t_2)\,dW(t_2)b(t_1)\,dW(t_1)$$

$$= 1 + \int_0^t dW(t_1)\, b(t_1) + \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2)$$

$$+ \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2) \int_0^{t_2} dW(t_3)\, b(t_3) + \cdots .$$

Formally, we have the series

$$\rho(t) = \sum_{n=0}^{\infty} \rho_n(t), \tag{6.14}$$

where $\rho_0(t) \equiv 1$ and

$$\rho_n(t) := \int_0^t dW(t_1)\, b(t_1) \int_0^{t_1} dW(t_2)\, b(t_2) \cdots \int_0^{t_{n-1}} dW(t_n)\, b(t_n).$$

This is equivalent to the equality

$$\rho_n(t) = \int_0^t \rho_{n-1}(t_1) b(t_1)\, dW(t_1). \tag{6.15}$$

Of course, we need to prove that the series (6.14) converges a.s. We assume this first. Therefore, the stochastic exponent is represented as the sum of multiple Itô integrals of the process $b(t)$, $t \in [0,T]$.

The usual multiple integral has a simple expression, i.e.,

$$\int_0^t dt_1\, b(t_1) \int_0^{t_1} dt_2\, b(t_2) \cdots \int_0^{t_{n-1}} dt_n\, b(t_n) = \frac{1}{n!}\Bigl(\int_0^t b(s)\, ds \Bigr)^n .$$

For a multiple stochastic integral $\rho_n(t)$ the formula is not so simple. To derive it, we proceed as follows.

For further purposes we consider the *Hermite polynomials*

$$\mathrm{He}_n(t,x) := (-t)^n e^{x^2/2t} \frac{d^n}{dx^n} e^{-x^2/2t} = n! \sum_{0 \le k \le n/2} \frac{(-1)^k x^{n-2k} t^k}{2^k k! (n-2k)!}, \qquad n = 0, 1, 2, \ldots .$$

As to the right-hand side of this equality, see the corresponding example of formula 5 in Appendix 6. We set $\mathrm{He}_0(t,x) := 1$. It is easy to compute that $\mathrm{He}_1(t,x) = x$, $\mathrm{He}_2(t,x) = x^2 - t$, $\mathrm{He}_3(t,x) = x^3 - 3xt$, $\mathrm{He}_4(t,x) = x^4 - 6x^2 t + 3t^2$.

The generating function of the Hermite polynomials is determined by the formula

$$\sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \mathrm{He}_n(t,x) = e^{\gamma x - \gamma^2 t/2}, \qquad \gamma \in \mathbf{R}. \tag{6.16}$$

To prove (6.16), we note that the Taylor expansion of the function $e^{-(x+\Delta)^2/2t}$ is

$$e^{-(x+\Delta)^2/2t} = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} \frac{d^n}{dx^n} e^{-x^2/2t}, \qquad x \in \mathbf{R}.$$

Multiplying this equality by $e^{x^2/2t}$ and setting $\Delta = -\gamma t$, we have (6.16):

$$e^{\gamma x - \gamma^2 t/2} = \sum_{n=0}^{\infty} \frac{(-\gamma t)^n}{n!} e^{x^2/2t} \frac{d^n}{dx^n} e^{-x^2/2t}.$$

Using the generating function (6.16), it is easy to derive the formulas

$$\frac{\partial}{\partial t} \operatorname{He}_n(t,x) = -\frac{n(n-1)}{2} \operatorname{He}_{n-2}(t,x),$$

$$\frac{\partial^k}{\partial x^k} \operatorname{He}_n(t,x) = \frac{n!}{(n-k)!} \operatorname{He}_{n-k}(t,x), \qquad k = 1,2,\dots. \tag{6.17}$$

Substituting $x = \int_0^t b(s)\,dW(s)$, $t = \int_0^t b^2(s)\,ds$ in (6.16), we have

$$\exp\Bigl(\gamma \int_0^t b(s)\,dW(s) - \frac{\gamma^2}{2} \int_0^t b^2(s)\,ds\Bigr) = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \operatorname{He}_n\Bigl(\int_0^t b^2(s)\,ds, \int_0^t b(s)\,dW(s)\Bigr). \tag{6.18}$$

The series on the right-hand side converges a.s., since (6.16) converge for all $x \in \mathbf{R}$ and $t > 0$. The left-hand side of (6.18) is the stochastic exponent $\rho^{(\gamma)}(t)$ defined in (6.1) with the function $\gamma b(t)$ instead of $b(t)$. For this stochastic exponent the equality (6.14) has the form

$$\rho^{(\gamma)}(t) = \sum_{n=0}^{\infty} \gamma^n \rho_n(t), \tag{6.19}$$

where $\rho_n(t)$ is defined by (6.15). Comparing (6.19) with (6.18) we come to the conclusion that the multiple Itô integral $\rho_n(t)$ must be equal to

$$\rho_n(t) = \frac{1}{n!} \operatorname{He}_n\Bigl(\int_0^t b^2(s)\,ds, \int_0^t b(s)\,dW(s)\Bigr), \qquad n = 1,2,\dots. \tag{6.20}$$

Below we prove (6.20) directly, using Itô's differentiation formula. Then this implies that the series (6.19) converges a.s for arbitrary γ, since the series (6.18) converges a.s., and our assumption on the convergence of the series (6.14) will be proved.

We prove (6.20) by induction. It is clear that (6.20) holds for $n = 0$ and $n = 1$. Suppose that it holds for index $n-1$ and let us prove it for index n. It is also evident that (6.20) holds for $t = 0$, since $\operatorname{He}_n(0,0) = 0$, $n = 1,2,\dots$. The last equality follows from (6.16). Now it is sufficient to prove that the stochastic differentials of both sides of (6.20) coincide.

According to (6.15) and the induction hypothesis, the stochastic differential on the left-hand side of (6.20) equals

$$d\rho_n(t) = \rho_{n-1}(t)b(t)\,dW(t) = \frac{1}{(n-1)!}\,\mathrm{He}_{n-1}\;\; b(t)\,dW(t). \tag{6.21}$$

Here and in what follows we omit the arguments $\int_0^t b^2(s)\,ds$, $\int_0^t b(s)\,dW(s)$ in the notation of the Hermite polynomials in (6.20).

Applying Itô's formula (4.22) for $\vec{X} = \Big(\int_0^t b^2(s)\,ds, \int_0^t b(s)\,dW(s)\Big)$ and taking into account formulae (6.17), we obtain the following expression for the differential on the right-hand side (6.20):

$$d\,\mathrm{He}_n = \frac{\partial}{\partial t}\,\mathrm{He}_n\;\; b^2(t)\,dt + \frac{\partial}{\partial x}\,\mathrm{He}_n\;\; b(t)\,dW(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\,\mathrm{He}_n\;\; b^2(t)\,dt$$

$$= -\frac{n(n-1)}{2}\,\mathrm{He}_{n-2}\;\; b^2(t)\,dt + n\,\mathrm{He}_{n-1}\;\; b(t)\,dW(t) + \frac{n(n-1)}{2}\,\mathrm{He}_{n-2}\;\; b^2(t)\,dt$$

$$= n\;\mathrm{He}_{n-1}\;\; b(t)\,dW(t).$$

After normalization by $n!$ this stochastic differential coincides with (6.21) and, consequently, (6.20) is proved, because $\rho_n(0) = 0$ and $\mathrm{He}_n(0,0) = 0$ for $n \ge 1$. □

Proposition 6.2. *Let $b(s)$, $s \in [0,t]$, be a stochastic process from $\mathcal{L}_2[0,T]$. Then*

$$0.3\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2 \le \mathbf{E}\Big(\int_0^t b(s)\,dW(s)\Big)^4 \le 30\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2. \tag{6.22}$$

Proof. Since $\mathrm{He}_4(t,x) = x^4 - 6x^2t + 3t^2$, we have

$$24\rho_4(t) = \Big(\int_0^t b(s)\,dW(s)\Big)^4 - 6\Big(\int_0^t b(s)\,dW(s)\Big)^2 \int_0^t b^2(s)\,ds + 3\Big(\int_0^t b^2(s)\,ds\Big)^2.$$

We can assume that the function b is bounded, otherwise we can apply the truncation procedure. Let $\sup_{0\le s\le t} |b(s)| \le M$. According to (6.15) and the definition of the stochastic integral, in order to take the expectation of $\rho_4(t)$ we need to be sure that $\int_0^t \mathbf{E}(\rho_3(t_1)b(t_1))^2\,dt_1 < \infty$. In view of (6.15) and (1.12), the required estimate follows from the inequalities

$$\int_0^t \mathbf{E}(\rho_3(t_1)b(t_1))^2\,dt_1 \le M^2 \int_0^t \mathbf{E}(\rho_3(t_1))^2\,dt_1 = M^2 \int_0^t dt_1 \int_0^{t_1} \mathbf{E}(\rho_2(t_2)b(t_2))^2\,dt_2$$

$$\le M^6 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \mathbf{E}(\rho_1(t_3))^2\,dt_3 \le M^8 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 = \frac{M^8 t^4}{4!}.$$

Since the expectation of a stochastic integral is zero, we have $\mathbf{E}\rho_4(t) = 0$. Now, from the expression for $24\rho_4(t)$ it follows that

$$\mathbf{E}\Big(\int_0^t b(s)\,dW(s)\Big)^4 = 6\mathbf{E}\Big\{\Big(\int_0^t b(s)\,dW(s)\Big)^2 \int_0^t b^2(s)\,ds\Big\} - 3\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2.$$

Applying Hölder's inequality, we get

$$\mathbf{E}\Big(\int_0^t b(s)\,dW(s)\Big)^4 \le 6\mathbf{E}^{1/2}\Big(\int_0^t b(s)\,dW(s)\Big)^4 \mathbf{E}^{1/2}\Big(\int_0^t b^2(s)\,ds\Big)^2 - 3\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^2.$$

Set

$$z := \mathbf{E}^{1/2}\Big(\int_0^t b(s)\,dW(s)\Big)^4 \Big/ \mathbf{E}^{1/2}\Big(\int_0^t b^2(s)\,ds\Big)^2.$$

Then the previous inequality can be written in the form $z^2 - 6z + 3 \le 0$. This is equivalent to $3 - \sqrt{6} \le z \le 3 + \sqrt{6}$. For nonnegative z this is equivalent to $15 - 6\sqrt{6} \le z^2 \le 15 + 6\sqrt{6}$. Finally this implies $0.3 \le z^2 \le 30$, and hence (6.22) is proved. □

Exercises.

6.1. Let $b(s)$, $s \in [0,t]$, be a stochastic process from $\mathcal{L}_2[0,T]$. Prove the estimate

$$c_1\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^3 \le \mathbf{E}\Big(\int_0^t b(s)\,dW(s)\Big)^6 \le c_2\,\mathbf{E}\Big(\int_0^t b^2(s)\,ds\Big)^3$$

for some positive constants c_1 and c_2.

§ 7. Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $W(t)$, $t \in [0,T]$, be a Brownian motion with a starting point $x \in \mathbf{R}$, and $\xi \in \mathbf{R}$ be a random variable independent of W. Let $\mathcal{F}_t := \sigma\{\xi, W(s), 0 \le s \le t\}$ be the σ-algebra of events generated by the random variable ξ and by the Brownian motion in the interval $[0,t]$.

Let $a(t,x)$ and $b(t,x)$, $t \in [0,T]$, $x \in \mathbf{R}$, be measurable functions.

A process $X(t)$, $t \in [0,T]$, $X(0) = \xi$, is said to be a *strong solution* of the *stochastic differential equation*

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t), \qquad X(0) = \xi, \tag{7.1}$$

if X is a continuous $\mathcal{F}_t$-adapted process such that a.s. for all $t \in [0,T]$

$$\int_0^t (|a(s,X(s))| + |b(s,X(s))|^2)\,ds < \infty \tag{7.2}$$

and

$$X(t) = \xi + \int_0^t a(s,X(s))\,ds + \int_0^t b(s,X(s))\,dW(s). \tag{7.3}$$

Note that due to (7.2) the integrals in (7.3) are well defined. In this section we follow the presentation in the book Gihman and Skorohod (1972).

1. Existence and uniqueness of solution.

Theorem 7.1. *Suppose that functions a and b satisfy the Lipschitz condition: there exists a constant C_T such that for all $t \in [0,T]$ and $x, y \in \mathbf{R}$,*

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le C_T|x - y|, \tag{7.4}$$

and the linear growth condition: for all $t \in [0,T]$ and $x \in \mathbf{R}$

$$|a(t,x)| + |b(t,x)| \le C_T(1 + |x|). \tag{7.5}$$

Let $\mathbf{E}\xi^2 < \infty$.

Then there exists a unique strong solution of (7.1) satisfying the condition

$$\sup_{0\le t\le T} \mathbf{E}X^2(t) < \infty. \tag{7.6}$$

Remark 7.1. Condition (7.5) follows from (7.4) if $|a(t,0)| + |b(t,0)| \le C_T$ for all $t \in [0,T]$.

Remark 7.2. Conditions (7.4) and (7.5) are rather essential even for deterministic equations.

Indeed, the equation

$$\frac{dX(t)}{dt} = X^2(t), \qquad X(0) = 1$$

has the unique solution $X(t) = \frac{1}{1-t}$, $t \in [0,1]$. Thus it is impossible to find a solution, for example, in the interval $[0,2]$.

Generally speaking, condition (7.5) that the functions increase no faster than linearly guarantees that the solution X of (7.3) does not *explode*, i.e., $|X(t)|$ does not tend to ∞ in a finite time.

Another important example concerns the fact that for $t \in [0,T]$ the equation

$$\frac{dX(t)}{dt} = 3X^{2/3}(t), \qquad X(0) = 0,$$

has more than one solution. Indeed, for any $t_0 \in [0,T]$ the function

$$X(t) = \begin{cases} 0, & \text{for } 0 \le t \le t_0, \\ (t-t_0)^3, & \text{for } t_0 \le t \le T, \end{cases}$$

is a solution. In this case the Lipschitz condition (7.4) is not satisfied.

Proof of Theorem 7.1. We first prove the uniqueness. Suppose that there are two continuous solutions satisfying (7.3) and (7.6), i.e.,

$$X_l(t) = \xi + \int_0^t a(s, X_l(s))\,ds + \int_0^t b(s, X_l(s))\,dW(s), \qquad l = 1, 2.$$

Then using the inequality $(g+h)^2 \le 2g^2 + 2h^2$, we get

$$\mathbf{E}(X_1(t) - X_2(t))^2 \le 2\mathbf{E}\Big(\int_0^t (a(s, X_1(s)) - a(s, X_2(s)))\,ds\Big)^2 + 2\mathbf{E}\Big(\int_0^t (b(s, X_1(s)) - b(s, X_2(s)))\,dW(s)\Big)^2.$$

Applying the Hölder inequality for the first term and the isometry property (1.12) for the second one, we have

$$\mathbf{E}(X_1(t) - X_2(t))^2 \le 2t\int_0^t \mathbf{E}(a(s, X_1(s)) - a(s, X_2(s))^2\,ds + 2\int_0^t \mathbf{E}(b(s, X_1(s)) - b(s, X_2(s)))^2\,ds.$$

Now using the Lipschitz condition (7.4) we obtain

$$\mathbf{E}(X_1(t) - X_2(t))^2 \le L\int_0^t \mathbf{E}(X_1(s) - X_2(s))^2\,ds \qquad \text{for all } t \in [0,T], \tag{7.7}$$

where $L = 2(T+1)C_T^2$.

We will often use Gronwall's lemma.

Lemma 7.1 (Gronwall). *Let $g(t)$ and $h(t)$, $0 \le t \le T$, be bounded measurable functions and let for some $K > 0$ and all $t \in [0,T]$*

$$g(t) \le h(t) + K\int_0^t g(s)\,ds.$$

Then

$$g(t) \le h(t) + K \int_0^t e^{K(t-s)} h(s)\, ds, \qquad t \in [0,T]. \tag{7.8}$$

If h is nondecreasing, then

$$g(t) \le h(t)\, e^{Kt}, \qquad t \in [0,T]. \tag{7.9}$$

Proof. Set

$$\psi(t) := h(t) + K \int_0^t e^{K(t-s)} h(s)\, ds, \qquad \Delta(t) := \psi(t) - g(t),$$

and note that the function $\Delta(t)$, $t \in [0,T]$, is bounded. Since

$$\Big(\int_0^t e^{K(t-s)} h(s)\, ds \Big)' = h(t) + K \int_0^t e^{K(t-s)} h(s)\, ds = \psi(t),$$

the function ψ satisfies the equation

$$\psi(t) = h(t) + K \int_0^t \psi(s)\, ds,$$

and

$$\Delta(t) \ge K \int_0^t \Delta(s)\, ds, \qquad t \in [0,T].$$

Since $K > 0$, by iteration, we get

$$\Delta(t) \ge K^2 \int_0^t \int_0^s \Delta(u)\, du\, ds = K^2 \int_0^t (t-u)\Delta(u)\, du \ge K^3 \int_0^t (t-u) \int_0^u \Delta(s)\, ds\, du$$

$$= K^3 \int_0^t \frac{(t-u)^2}{2} \Delta(s)\, ds \ge \cdots \ge \frac{K^{n+1}}{n!} \int_0^t (t-s)^n \Delta(s)\, ds.$$

The last term tends to zero as $n \to \infty$, consequently, $\Delta(t) \ge 0$, $t \in [0,T]$, and (7.8) holds. For a nondecreasing function h inequality (7.9) is a simple consequence of (7.8), because

$$g(t) \le h(t) + K h(t) \int_0^t e^{K(t-s)}\, ds = h(t)\, e^{Kt}.$$

□

Since $\sup\limits_{0\le s\le T} \mathbf{E}\{X_1^2(t)+X_2^2(t)\}<\infty$, using Gronwall's lemma for $h\equiv 0$, we deduce from (7.7) that $\mathbf{E}(X_1(t)-X_2(t))^2=0$ and, consequently, $\mathbf{P}(X_1(t)=X_2(t))=1$ for every $t\in[0,T]$. Therefore, the solutions X_1, X_2 coincide a.s. for all rational moments of time, and by the continuity of paths $\mathbf{P}\Bigl(\sup\limits_{0\le t\le T}|X_1(t)-X_2(t)|=0\Bigr)=1$. The uniqueness is proved.

To prove that there exists a solution of the stochastic equation (7.3) we apply the method of successive approximations.

Set $X_0(t):=\xi$,

$$X_n(t):=\xi+\int_0^t a(s,X_{n-1}(s))\,ds+\int_0^t b(s,X_{n-1}(s))\,dW(s). \tag{7.10}$$

Note that $X_n(t)$ is a continuous $\mathcal{F}_t$-adapted process for every n.

By the linear growth condition (7.5), analogously to (7.7), we have

$$\mathbf{E}(X_1(t)-X_0(t))^2\le 2t\int_0^t \mathbf{E}a^2(s,\xi)\,ds+2\int_0^t \mathbf{E}b^2(s,\xi)\,ds\le Lt(1+\mathbf{E}\xi^2)=KLt.$$

We now make the inductive assumption that for $k=n-1$

$$\mathbf{E}(X_{k+1}(t)-X_k(t))^2\le \frac{K(Lt)^{k+1}}{(k+1)!}\qquad \text{for all } t\in[0,T]. \tag{7.11}$$

Then analogously to (7.7) we have

$$\mathbf{E}(X_{n+1}(t)-X_n(t))^2\le L\int_0^t \mathbf{E}(X_n(s)-X_{n-1}(s))^2\,ds$$

$$\le \frac{KL^{n+1}}{n!}\int_0^t s^n\,ds=\frac{K(Lt)^{n+1}}{(n+1)!}.$$

Thus (7.11) holds for $k=n$ and the proof of (7.11) for all $k=0,1,2\dots$ is completed by induction.

The estimate (7.11) will enable us to prove that the processes $X_n(t)$ converge a.s. uniformly in $t\in[0,T]$ to a limit. We apply the estimate

$$\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|\le \int_0^T |a(s,X_n(s))-a(s,X_{n-1}(s))|\,ds$$
$$+\sup_{0\le t\le T}\biggl|\int_0^t (b(s,X_n(s))-b(s,X_{n-1}(s)))\,dW(s)\biggr|.$$

Then using Doob's inequality (2.6) and (1.12), we obtain

$$\mathbf{E}\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|^2\le 2\mathbf{E}\Big(\int_0^T|a(s,X_n(s))-a(s,X_{n-1}(s))|ds\Big)^2$$

$$+8\int_0^T\mathbf{E}(b(s,X_n(s))-b(s,X_{n-1}(s)))^2\,ds\le 4L\int_0^t\mathbf{E}(X_n(s)-X_{n-1}(s))^2\,ds$$

$$\le\frac{KL^{n+1}T^{n+1}}{(n+1)!}.$$

By the Chebyshev inequality,

$$\mathbf{P}\Big(\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|>\frac{1}{n^2}\Big)\le 4n^4K\frac{L^{n+1}T^{n+1}}{(n+1)!}.$$

Since the series of these probabilities converge, by the first part of the Borel–Cantelli lemma, there exists a.s. a number $n_0=n_0(\omega)$ such that for all $n>n_0$

$$\sup_{0\le t\le T}|X_{n+1}(t)-X_n(t)|\le\frac{1}{n^2}.$$

This implies that the random variables

$$X_n(t)=\xi+\sum_{k=0}^{n-1}\big(X_{k+1}(t)-X_k(t)\big)\tag{7.12}$$

converge uniformly in $t\in[0,T]$ to

$$X(t)=\xi+\sum_{k=0}^{\infty}\big(X_{k+1}(t)-X_k(t)\big)$$

i.e.,

$$\mathbf{P}\Big(\lim_{n\to\infty}\sup_{0\le t\le T}|X_n(t)-X(t)|=0\Big)=1.$$

Therefore $X(t)$, $t\in[0,T]$, is a continuous $\mathcal{F}_t$-adapted process and, in view of (7.5), we see that (7.2) holds.

Using (7.4) and the uniform convergence of X_n to X, one can pass to the limit in (7.10). We have $a(t,X_n(t))\to a(t,X(t))$ and $b(t,X_n(t))\to b(t,X(t))$ a.s. uniformly in $t\in[0,T]$, and

$$\int_0^T\big(b(t,X_n(t))-b(t,X(t))\big)^2\,dt\to 0\qquad\text{a.s.}$$

We now can apply (3.6) and, by passage to the limit in (7.10), prove that the process X is the solution of equation (7.3).

To complete the proof of Theorem 7.1 it remains to prove that this solution satisfies (7.6).

From the inequality $\Big(\sum_{k=1}^{n} c_k\Big)^2 \le n \sum_{k=1}^{n} c_k^2$, and (7.11), (7.12) we get that $\mathbf{E}X_n^2(t) \le C(n+1)$ for some C. From (7.10) we have

$$\mathbf{E}X_{n+1}^2(t) \le 3\mathbf{E}\xi^2 + 3\mathbf{E}\Big(\int_0^t a(s, X_n(s))\,ds\Big)^2 + 3\mathbf{E}\Big(\int_0^t b(s, X_n(s))\,dW(s)\Big)^2.$$

Applying Hölder's inequality to the second term and the isometry property (1.12) to the third term, and using (7.5), we obtain

$$\mathbf{E}X_{n+1}^2(t) \le 3\mathbf{E}\xi^2 + 3T\int_0^t \mathbf{E}a^2(s, X_n(s))\,ds + 3\int_0^t \mathbf{E}b^2(s, X_n(s))\,ds$$

$$\le 3\mathbf{E}\xi^2 + 3L\int_0^t (1 + \mathbf{E}X_n^2(s))\,ds \le M + M\int_0^t \mathbf{E}X_n^2(s)\,ds.$$

for some constant M. By iteration, we get that for all $t \in [0, T]$

$$\mathbf{E}X_{n+1}^2(t) \le M + M^2 t + M^3\frac{t^2}{2!} + \cdots + M^{n+2}\frac{t^{n+1}}{(n+1)!}.$$

Therefore, $\mathbf{E}X_{n+1}^2(t) \le Me^{Mt}$. By Fatou's lemma,

$$\mathbf{E}X^2(t) \le Me^{Mt}. \qquad (7.13)$$

This proves (7.6). □

2. Local dependence of solutions on coefficients.

The meaning of the assertions presented below is the following. If for two stochastic differential equations with the same initial value the coefficients coincide for all time moments and for the spatial variable from some interval, then the solutions of these equations coincide up to the first exit time from this interval.

Theorem 7.2. *Suppose that the coefficients a_1, b_1 and a_2, b_2 of the stochastic differential equations*

$$dX_l(t) = a_l(t, X_l(t))\,dt + b_l(t, X_l(t))\,dW(t), \qquad X_l(0) = \xi, \qquad (7.14)$$

$l = 1, 2$, *satisfy conditions* (7.4), (7.5) *and* $a_1(t, x) = a_2(t, x)$, $b_1(t, x) = b_2(t, x)$, *for* $(t, x) \in [0, T] \times [-N, N]$ *with some* $N > 0$. *Let* $\mathbf{E}\xi^2 < \infty$.

Let X_l, $l = 1, 2$, *be the strong solutions of* (7.14) *and* $H_l := \max\{t \in [0, T] : \sup_{0\le s\le t} |X_l(s)| \le N\}$. *Then* $\mathbf{P}(H_1 = H_2) = 1$ *and*

$$\mathbf{P}\Big(\sup_{0\le s\le H_1} |X_1(s) - X_2(s)| = 0\Big) = 1. \qquad (7.15)$$

Proof. Set

$$\varphi_1(t) = \begin{cases} 1, & \text{if} \quad \sup\limits_{0\le s\le t} |X_1(s)| \le N, \\ 0, & \text{if} \quad \sup\limits_{0\le s\le t} |X_1(s)| > N. \end{cases}$$

It is clear that $\varphi_1(t) = 1$ iff $t \in [0, H_1]$. Since given the event $\{\varphi_1(t) = 1\}$ we have $a_1(s, X_1(s)) = a_2(s, X_1(s))$ and $b_1(s, X_1(s)) = b_2(s, X_1(s))$ for all $s \in [0, t]$, one can write

$$\varphi_1(t)(X_1(t) - X_2(t)) = \varphi_1(t) \int_0^t (a_2(s, X_1(s)) - a_2(s, X_2(s)))\, ds$$

$$+ \varphi_1(t) \int_0^t (b_2(s, X_1(s)) - b_2(s, X_2(s)))\, dW(s).$$

Since the equality $\varphi_1(t) = 1$ implies $\varphi_1(s) = 1$ for all $s \le t$, using the Lipschitz condition (7.4), we obtain analogously to (7.7) that for all $t \in [0, T]$

$$\mathbf{E}\big\{\varphi_1(t)(X_1(t) - X_2(t))^2\big\} \le L \int_0^t \mathbf{E}\big\{\varphi_1(s)(X_1(s) - X_2(s))^2\big\}\, ds. \tag{7.16}$$

By Gronwall's lemma, $\mathbf{E}\big\{\varphi_1(t)(X_1(t) - X_2(t))^2\big\} = 0$. Since the processes $X_1(t)$ and $X_2(t)$ are continuous, we get

$$\mathbf{P}\Big(\sup_{0\le t\le T} (\varphi_1(t)(X_1(t) - X_2(t))^2) = 0\Big) = 1.$$

This implies that in the interval $[0, H_1]$ the processes $X_1(t)$ and $X_2(t)$ coincide a.s. Therefore, $H_2 \ge H_1$ a.s. Switching the indices 1 and 2, we have that $H_1 \ge H_2$ a.s. Consequently, $H_1 = H_2$ a.s. and (7.15) holds. □

3. Local Lipschitz condition.

In Theorem 7.1 condition (7.4) can be weakened to the local Lipschitz condition.

Theorem 7.3. *Suppose that the functions $a(t, x)$ and $b(t, x)$ satisfy the local Lipschitz condition: for every $N > 0$ there exists a constant $C_{N,T}$ such that for all $t \in [0, T]$ and $x, y \in [-N, N]$*

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \le C_{N,T}|x - y|, \tag{7.17}$$

and the linear growth condition: for all $t \in [0, T]$ and $x \in \mathbf{R}$

$$|a(t, x)| + |b(t, x)| \le C_T(1 + |x|). \tag{7.18}$$

Then there exists a unique strong solution of (7.1).

Remark 7.3. The condition (7.17) holds if there exists $\frac{\partial}{\partial x} a(t, x)$ and $\frac{\partial}{\partial x} b(t, x)$ continuous in $(t, x) \in [0, T] \times \mathbf{R}$.

Proof of Theorem 7.3. The proof involves a truncation procedure. We prove first the existence of a solution of (7.1).

Set

$$\xi_N := \xi \mathbb{1}_{\{|\xi|\le N\}} + N \operatorname{sign}\xi \mathbb{1}_{\{|\xi|>N\}},$$
$$a_N(t,x) := a(t,x)\mathbb{1}_{[0,N]}(|x|) + a(t, N\operatorname{sign} x)\mathbb{1}_{(N,\infty)}(|x|)$$

and

$$b_N(t,x) := b(t,x)\mathbb{1}_{[0,N]}(|x|) + b(t, N\operatorname{sign} x)\mathbb{1}_{(N,\infty)}(|x|).$$

Let $X_N(t)$ be a solution of the stochastic differential equation

$$dX_N(t) = a_N(t, X_N(t))\,dt + b_N(t, X_N(t))\,dW(t), \qquad X_N(0) = \xi_N. \tag{7.19}$$

For equation (7.19) all conditions of Theorem 7.1 holds. Therefore there exists a unique continuous solution of this equation satisfying the estimate

$$\sup_{0\le t\le T} \mathbf{E}X_N^2(t) < \infty.$$

Set $H_N := \max\{t \in [0,T] : \sup\limits_{0\le s\le t} |X_N(s)| \le N\}$. Since for $N' > N$ we have $a_N(t,x) = a_{N'}(t,x)$ and $b_N(t,x) = b_{N'}(t,x)$ for $x \in [-N,N]$, by Theorem 7.2, $X_N(t) = X_{N'}(t)$ for all $t \in [0, H_N]$ a.s. Therefore,

$$\{H_N = T\} \subseteq \Big\{ \sup_{N'>N} \sup_{0\le t\le T} |X_N(t) - X_{N'}(t)| = 0\Big\}$$

and, consequently,

$$\mathbf{P}\Big(\sup_{N'>N} \sup_{0\le t\le T} |X_N(t) - X_{N'}(t)| > 0\Big) \le \mathbf{P}(H_N < T) = \mathbf{P}\Big(\sup_{0\le t\le T} |X_N(t)| > N\Big).$$

Next we will prove that

$$\lim_{N\to\infty} \mathbf{P}\Big(\sup_{0\le t\le T} |X_N(t)| > N\Big) = 0. \tag{7.20}$$

Once this is done, then from the previous estimate and the first part of the Borel–Cantelli lemma, it follows that for a sufficiently scarce subsequence N_n there exists a.s. a number $n_0 = n_0(\omega)$ such that for all $N_n \ge N_{n_0}$

$$\sup_{N'>N_n} \sup_{0\le t\le T} |X_{N_n}(t) - X_{N'}(t)| = 0.$$

Therefore, by Cauchy's criterion, the sequence of processes $X_{N_n}(t)$, $t \in [0,T]$, is Cauchy in the uniform norm for the a.s. convergence. Thus, $X_{N_n}(t)$ converges to a limit process $X(t)$ uniformly in $t \in [0,T]$. In the stochastic equation

$$X_N(t) = \xi_N + \int_0^t a_N(s, X_N(s))\,ds + \int_0^t b_N(s, X_N(s))\,dW(s). \tag{7.21}$$

we can pass to the limit as $N_n \to \infty$. The usual integral converges in view of the estimates (7.17), (7.18). To justify the convergence of the stochastic integral we can use the same estimates and (3.6). As a result, we see that the process $X(t)$, $t \in [0,T]$, is the strong solution of equation (7.1).

Thus it is enough to prove (7.20). Using (7.21), it is easy to prove that for any $t \in [0,T]$

$$\sup_{0\le s\le t} |X_N(s)|^2 \le 3\xi_N^2 + 3T\int_0^t a_N^2(s, X_N(s))\,ds + 3\sup_{0\le s\le t}\biggl(\int_0^s b_N(u, X_N(u))\,dW(u)\biggr)^2.$$

We multiply this inequality by $\psi(\xi)$, where $\psi(x) = \dfrac{1}{1+x^2}$. Then using (2.6), (7.18) and the estimate $\xi_N^2\psi(\xi) \le 1$, we get

$$\mathbf{E}\bigl\{\psi(\xi)X_N^2(t)\bigr\} \le \mathbf{E}\Bigl\{\psi(\xi)\sup_{0\le s\le t} X_N^2(s)\Bigr\} \le 3 + 3TC_T^2\int_0^t \mathbf{E}\bigl\{\psi(\xi)(1+X_N^2(s))\bigr\}\,ds$$

$$+\,12C_T^2\int_0^t \mathbf{E}\bigl\{\psi(\xi)(1+X_N^2(s))\bigr\}\,ds.$$

By Gronwall's lemma, we have $\mathbf{E}\bigl\{\psi(\xi)X_N^2(t)\bigr\} \le C$ for $t \in [0,T]$ and for some constant C. Consequently,

$$\mathbf{E}\Bigl\{\psi(\xi)\sup_{0\le s\le T} X_N^2(s)\Bigr\} \le C_1$$

for some constant C_1 independent of N.

One has the estimates

$$\mathbf{P}\Bigl(\sup_{0\le t\le T}|X_N(t)| > N\Bigr) = \mathbf{P}\Bigl(\psi(\xi)\sup_{0\le t\le T} X_N^2(t) > N^2\psi(\xi)\Bigr)$$

$$\le \mathbf{P}\Bigl(\psi(\xi)\sup_{0\le t\le T} X_N^2(t) > \delta N^2\Bigr) + \mathbf{P}(\psi(\xi)\le\delta) \le \frac{C_1}{\delta N^2} + \mathbf{P}(\psi(\xi)\le\delta),$$

for any $\delta > 0$. This implies

$$\limsup_{N\to\infty}\mathbf{P}\Bigl(\sup_{0\le t\le T}|X_N(t)| > N\Bigr) \le \mathbf{P}(\psi(\xi)\le\delta) = \mathbf{P}(\xi^2 \ge (1-\delta)/\delta).$$

But $\lim\limits_{\delta\downarrow 0}\mathbf{P}(\xi^2 \ge (1-\delta)/\delta) = 0$. This proves (7.20) and, consequently, the existence of the solution of (7.1).

We now prove uniqueness of a solution of (7.1). Let $X_1(t)$ and $X_2(t)$ be two a.s. continuous solutions of (7.1), satisfying the initial condition $X_1(0) = X_2(0) = \xi$. Set

$$\varphi_N(t) := \mathbb{1}_{[0,N]}\Bigl(\sup_{0\le v\le t}|X_1(v)|\Bigr)\mathbb{1}_{[0,N]}\Bigl(\sup_{0\le v\le t}|X_2(v)|\Bigr).$$

Using the local Lipshitz condition, the isometry property (1.12) for stochastic integrals, and the fact that the equality $\varphi_N(t) = 1$ implies the equalities $\varphi_N(s) = 1$ for all $s \le t$, we can obtain (see the analogous estimate (7.7))

$$\mathbf{E}\{\varphi_N(t)(X_1(t) - X_2(t))^2\} \le 2t \int_0^t \mathbf{E}\{\varphi_N(s)(a(s, X_1(s)) - a(s, X_2(s)))^2\}\, ds$$

$$+ 2\int_0^t \mathbf{E}\{\varphi_N(s)(b(s, X_1(s)) - b(s, X_2(s)))^2\}\, ds$$

$$\le 2(T+1)C_N^2 \int_0^t \mathbf{E}\{\varphi_N(s)(X_1(s) - X_1(s))^2\}\, ds.$$

By Gronwall's lemma, $\mathbf{E}\{\varphi_N(t)(X_1(t) - X_2(t))^2\} = 0$. Therefore, for arbitrary N

$$\mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t) - X_2(t)| > 0\Big) \le \mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t)| > N\Big) + \mathbf{P}\Big(\sup_{0\le t\le T} |X_2(t)| > N\Big).$$

From the continuity of the solutions X_1 and X_2 it follows that their suprema are finite. This implies that the probabilities on the right-hand side of this estimate tend to zero as $N \to \infty$. Thus $\mathbf{P}\Big(\sup_{0\le t\le T} |X_1(t) - X_2(t)| = 0\Big) = 1$, and this means uniqueness of the solution of (7.1). □

It is convenient to have estimates for the moments of even order of the solution of the stochastic differential equation (7.1).

Theorem 7.4. *Suppose that the functions $a(t, x)$ and $b(t, x)$ satisfy the conditions of Theorem 7.3. Let $\mathbf{E}\xi^{2m} < \infty$, where m is a positive integer. Then*

$$\mathbf{E}X^{2m}(t) \le \big(\mathbf{E}\xi^{2m} + Kt\big)e^{2Kt}, \tag{7.22}$$

and for $s < t$

$$\mathbf{E}(X(t) - X(s))^{2m} \le \widetilde{K}T\big(1 + Kt + \mathbf{E}\xi^{2m}\big)(1 + (t-s)^m)(t-s)^m e^{2Kt}, \tag{7.23}$$

for some constants K and $\widetilde{K}$, depending only on m and C_T.

Proof. We use the notations from the proof of Theorem 7.3. Since, by (7.18), the variables ξ_N and the functions a_N and b_N are bounded by the constant $C_T(1+N)$, then from (7.21) and (4.25) it follows that $\mathbf{E}X_N^{2m}(t) \le C_T N^{2m}(1 + t^{2m})$. Applying Itô's formula to $X_N^{2m}(t)$, we get

$$X_N^{2m}(t) = \xi_N^{2m}$$

$$+ \int_0^t \Big(2mX_N^{2m-1}(s)a_N(s, X_N(s)) + m(2m-1)X_N^{2m-2}(s)b_N^2(s, X_N(s))\Big)ds$$

$$+\int_0^t 2mX_N^{2m-1}(s)b_N(s,X_N(s))\,dW(s).$$

Then

$$\mathbf{E}X_N^{2m}(t)=\mathbf{E}\xi_N^{2m}+\int_0^t\Big(2m\mathbf{E}\big\{X_N^{2m-1}(s)a_N(s,X_N(s))\big\}$$

$$+m(2m-1)\mathbf{E}\big\{X_N^{2m-2}(s)b_N^2(s,X_N(s))\big\}\Big)ds$$

$$\le\mathbf{E}\xi^{2m}+(2m+3)mC_T^2\int_0^t\mathbf{E}\big\{\big(1+X_N^2(s)\big)X_N^{2m-2}(s)\big\}\,ds.$$

Applying the obvious inequality $x^{2m-2}\le 1+x^{2m}$, we have

$$\mathbf{E}X_N^{2m}(t)\le\mathbf{E}\xi^{2m}+(2m+3)mC_T^2t+2(2m+3)mC_T^2\int_0^t\mathbf{E}X_N^{2m}(s)\,ds.$$

By Gronwall's lemma (see (7.9)),

$$\mathbf{E}X_N^{2m}(t)\le\big(\mathbf{E}\xi^{2m}+(2m+3)mC_T^2t\big)\exp\big(2(2m+3)mC_T^2t\big).$$

Therefore, by Fatou's lemma this implies (7.22), since $X_N(t)\to X(t)$.

We now prove (7.23). Obviously

$$\mathbf{E}(X(t)-X(s))^{2m}\le\mathbf{E}\bigg(\int_s^t a(v,X(v))\,dv+\int_s^t b(v,X(v))\,dW(v)\bigg)^{2m}$$

$$\le 2^{2m-1}\bigg(\mathbf{E}\bigg(\int_s^t a(v,X(v))\,dv\bigg)^{2m}+\mathbf{E}\bigg(\int_s^t b(v,X(v))\,dW(v)\bigg)^{2m}\bigg).$$

Using (4.25) with $L_k=2^kk^{2k}\Big(\frac{2k}{2k-1}\Big)^{(2k-1)k}$, Hölder's inequality, (7.18), and (7.22), we get

$$\mathbf{E}(X(t)-X(s))^{2m}\le(2(t-s))^{2m-1}\int_s^t\mathbf{E}a^{2m}(v,X(v))\,dv+L_m\mathbf{E}\bigg(\int_s^t b^2(v,X(v))\,dv\bigg)^m$$

$$\le(2(t-s))^{2m-1}\int_s^t\mathbf{E}a^{2m}(v,X(v))\,dv+L_m(t-s)^{m-1}\int_s^t\mathbf{E}b^{2m}(v,X(v))\,dv$$

$$\le 2^{2m-1}C_T^{2m}(L_m+(t-s)^m)(t-s)^{m-1}\int_s^t (1+\mathbf{E}X^{2m}(v))\,dv$$

$$\le 2^{4m-2}(t-s)^{m-1}C_T^{2m}(L_m+(t-s)^m)\int_s^t \big(1+(\mathbf{E}\xi^{2m}+Kv)e^{2Kv}\big)\,dv$$

$$\le C_T^{2m}L_mK2^{4m-1}(1+(t-s)^m)(t-s)^m(1+Kt+\mathbf{E}\xi^{2m})e^{2Kt}.$$

□

4. Multi-dimensional stochastic differential equations.

Consider the vector-valued stochastic differential equations.

Let $\vec{W}(t)=(W_1(t),\dots,W_m(t))$, $t\in[0,T]$, be m-dimensional Brownian motion with independent coordinates, which are one-dimensional Brownian motions with the initial values $W_k(0)=x_k$, $k=1,2,\dots,m$. Let the random vector $\vec{\xi}\in\mathbf{R}^n$ be independent of the process $\vec{W}$ and let $\mathcal{F}_t:=\sigma\{\vec{\xi},\vec{W}(s),0\le s\le t\}$ be the σ-algebra of events generated by $\vec{\xi}$ and the Brownian motions W_k, $k=1,2,\dots,m$, in $[0,t]$.

Let $\vec{a}(t,\vec{x})$, $t\in[0,T]$, $\vec{x}\in\mathbf{R}^n$, be a measurable function with the state space $\mathbf{R}^n$ and $\mathbb{B}(t,\vec{x})$ be an $n\times m$ matrix with measurable real-valued functions as elements. Denote by $|\vec{a}|$ the Euclidean norm of the vector $\vec{a}$. Set $|\mathbb{B}|:=\Big(\sum_{k=1}^n\sum_{l=1}^m b_{k,l}^2\Big)^{1/2}$ for matrixes $\mathbb{B}$ with elements $\{b_{k,l}\}_{k=1,l=1}^{n,m}$.

Consider the n-dimensional stochastic differential equation

$$d\vec{X}(t)=\vec{a}(t,\vec{X}(t))\,dt+\mathbb{B}(t,\vec{X}(t))\,d\vec{W}(t),\qquad \vec{X}(0)=\vec{\xi}. \tag{7.24}$$

In coordinates this equation becomes the system of stochastic differential equations

$$dX_k(t)=a_k(t,X_1(t),\dots,X_n(t))dt$$

$$+\sum_{l=1}^m b_{k,l}(t,X_1(t),\dots,X_n(t))dW_l(t),\quad X_k(0)=\xi_k,\qquad k=1,2,\dots,n.$$

The process $\vec{X}$ is a *strong solution* of (7.24) if it is a continuous $\mathcal{F}_t$-adapted process such that a.s. for all $t\in[0,T]$,

$$\int_0^t \big(|\vec{a}(s,\vec{X}(s))|+|\mathbb{B}(s,\vec{X}(s))|^2\big)\,ds<\infty \tag{7.25}$$

and

$$\vec{X}(t)=\vec{\xi}+\int_0^t \vec{a}(s,\vec{X}(s))\,ds+\int_0^t \mathbb{B}(s,\vec{X}(s))\,d\vec{W}(s). \tag{7.26}$$

Theorem 7.5. *Suppose that $\vec{a}(t,\vec{x})$ and $\mathbb{B}(t,\vec{x})$ satisfy the Lipschitz condition: there exists a constant C_T such that for all $t \in [0,T]$ and $\vec{x},\vec{y} \in \mathbf{R}^n$*

$$|\vec{a}(t,\vec{x}) - \vec{a}(t,\vec{y})| + |\mathbb{B}(t,\vec{x}) - \mathbb{B}(t,\vec{y})| \le C_T|\vec{x}-\vec{y}|, \tag{7.27}$$

and the condition: for all $t \in [0,T]$ and $\vec{x} \in \mathbf{R}^n$

$$|\vec{a}(t,\vec{x})| + |\mathbb{B}(t,\vec{x})| \le C_T(1+|\vec{x}|). \tag{7.28}$$

Let $\mathbf{E}|\vec{\xi}|^2 < \infty$. Then there exists a unique strong solution of (7.24) satisfying the condition

$$\sup_{0\le t\le T} \mathbf{E}|\vec{X}(t)|^2 < \infty. \tag{7.29}$$

By standard techniques of linear algebra, the proof of this theorem follows essentially the proof of Theorem 7.1 for the one-dimensional case.

Remark 7.4. The m-dimensional Brownian motion $\vec{W}_\circ(t)$, $t \in [0,T]$, with dependent coordinates can be obtained from $\vec{W}(t)$ with independent coordinates by a linear transformation. This means that there exists an $m \times m$ matrix $\mathbb{C}$ such that $\vec{W}_\circ(t) = \mathbb{C}\vec{W}(t)$.

This is due to the fact that the matrix of variances of the Brownian motion $\vec{W}_\circ(t)$ is positive definite, and then

$$\mathrm{Var}(\vec{W}_\circ(1)) = \mathbb{C}^T\,\mathbb{C}$$

for some matrix $\mathbb{C}$. Here the symbol T stands for the transposition of matrices. It is unnecessary to consider the analog of equation (7.24) for the Brownian motion $\vec{W}_\circ$, since

$$\mathbb{B}(t,\vec{X}(t))\,d\vec{W}_\circ(t) = \mathbb{B}(t,\vec{X}(t))\mathbb{C}\,d\vec{W}(t).$$

Exercises.

7.1. Let $\vec{W}(t)$, $t \in [0,T]$, be an m-dimensional Brownian motion with independent coordinates and $\mathcal{F}_t := \sigma\{\vec{W}(s), 0 \le s \le t\}$ be the σ-algebra of events generated by the Brownian motions W_k, $k = 1,2,\dots,m$, in $[0,t]$.

Let $\mathbb{B}(t)$, be an $n\times m$ matrix with progressively measurable processes as elements. Let

$$d\vec{X}(t) = \mathbb{B}(t)\,d\vec{W}(t), \qquad \vec{X}(0) = \vec{x}_0.$$

Prove that the process

$$M(t) := |\vec{X}(t)|^2 - \int_0^t |\mathbb{B}(s)|^2 ds$$

is a martingale with respect to the filtration $\mathcal{F}_t$.

7.2. Under the assumptions of Exercise 7.1, prove for $r \in \mathbb{N}$ the formula

$$\begin{aligned} d|\vec{X}(t)|^{2r} &= 2r|\vec{X}(t)|^{2r-2}(\vec{X}(t))^T\mathbb{B}(t)\,d\vec{W}(t) \\ &\quad + \Big(2r(r-1)|\vec{X}(t)|^{2r-4}|(\vec{X}(t))^T\mathbb{B}(t)|^2 + r|\vec{X}(t)|^{2r-2}|\mathbb{B}(t)|^2\Big)dt. \end{aligned}$$

§ 8. Methods of solving of stochastic differential equations

1. Stochastic exponent. We already have an example of a stochastic differential equation and its solution. This is the stochastic exponent (see § 6)

$$\rho(t) = \exp\biggl(\int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b^2(s)\,ds\biggr),$$

which is the solution of the equation

$$d\rho(t) = b(t)\rho(t)\,dW(t), \qquad \rho(0) = 1. \tag{8.1}$$

The state space of the solution of this stochastic equation is the positive real line.

2. Linear stochastic differential equation. The general form of the linear stochastic differential equation is

$$dX(t) = (a(t)X(t) + r(t))\,dt + (b(t)X(t) + q(t))\,dW(t), \qquad X(0) = x_0. \tag{8.2}$$

This equation also has an explicit solution.

For the product of the stochastic exponent $\rho(t)$ and the ordinary exponent $\rho_0(t) = \exp\Bigl(\int_0^t a(s)\,ds\Bigr)$ we have

$$d\bigl(\rho_0(t)\rho(t)\bigr) = \rho(t)\,d\rho_0(t) + \rho_0(t)\,d\rho(t) = \rho_0(t)\rho(t)\bigl(a(t)\,dt + b(t)\,dW(t)\bigr).$$

Therefore, the solution of the homogeneous linear stochastic differential equation

$$dY(t) = a(t)Y(t)\,dt + b(t)Y(t)\,dW(t), \qquad Y(0) = 1. \tag{8.3}$$

is the product of the ordinary exponent and the stochastic one:

$$Y(t) = \exp\biggl(\int_0^t b(s)\,dW(s) + \int_0^t \bigl(a(s) - \tfrac{1}{2}b^2(s)\bigr)\,ds\biggr).$$

It is well known how the solutions of the ordinary nonhomogeneous linear equations are expressed via the solutions of homogeneous ones. We can expect that the solution of equation (8.2) has the same structure. It can be checked by direct computation that the solution of (8.2) is

$$X(t) = Y(t)\biggl\{x_0 + \int_0^t q(s)Y^{-1}(s)\,dW(s) + \int_0^t (r(s) - b(s)q(s))Y^{-1}(s)\,ds\biggr\}. \tag{8.4}$$

Indeed,

$$dX(t) = X(t)\{a(t)\,dt + b(t)\,dW(t)\} + q(t)\,dW(t) + (r(t) - b(t)q(t))\,dt$$

$$+b(t)q(t)\,dt = (a(t)X(t) + r(t))\,dt + (b(t)X(t) + q(t))\,dW(t).$$

Note that for the case $q(t) \equiv 0$ and $r(t) \geq 0$, $t \geq 0$, the state space of the linear stochastic differential equation with initial value $x_0 > 0$ is the positive real line.

If the ratio $q(t)/b(t)$ is well defined for all $t \geq 0$, then equation (8.2) can be transformed to the equation with $q(t) \equiv 0$ by means of a shift of the variable X. Indeed, (8.2) can be rewritten in the form

$$dX(t)=\Big\{a(t)\big(X(t)+\tfrac{q(t)}{b(t)}\big)+r(t)-a(t)\tfrac{q(t)}{b(t)}\Big\}dt+b(t)\Big(X(t)+\tfrac{q(t)}{b(t)}\Big)dW(t), \quad X(0)=x_0.$$

Setting $Z(t) := X(t) + \dfrac{q(t)}{b(t)}$ and assuming that $\dfrac{q(t)}{b(t)}$ is a differentiable function, we have

$$dZ(t) = \Big\{a(t)Z(t)+r(t)-a(t)\tfrac{q(t)}{b(t)}+\Big(\tfrac{q(t)}{b(t)}\Big)'\Big\}\,dt+b(t)Z(t)\,dW(t), \quad Z(0) = x_0+\frac{q(0)}{b(0)}. \tag{8.5}$$

Therefore, by (8.4) with $q(s) \equiv 0$, the solution of (8.2) can be written in the form

$$X(t) = Y(t)\Big\{x_0 + \frac{q(0)}{b(0)} + \int_0^t \Big(r(s) - a(s)\tfrac{q(s)}{b(s)} + \Big(\tfrac{q(s)}{b(s)}\Big)'\Big)Y^{-1}(s)\,ds\Big\} - \frac{q(t)}{b(t)}. \tag{8.6}$$

For the case when $r(t) - a(t)\frac{q(t)}{b(t)} + \Big(\frac{q(t)}{b(t)}\Big)' \equiv 0$, $t \geq 0$, or, equivalently,

$$\frac{q(t)}{b(t)} = \frac{q(0)}{b(0)} \exp\Big(\int_0^t a(s)\,ds\Big) - \int_0^t r(s)\exp\Big(\int_s^t a(v)\,dv\Big)\,ds,$$

we have the *shifted stochastic exponent*

$$X(t) = \Big(x_0 + \tfrac{q(0)}{b(0)}\Big)\exp\Big(\int_0^t b(s)\,dW(s) + \int_0^t \Big(a(s) - \tfrac{1}{2}b^2(s)\Big)\,ds\Big) - \frac{q(t)}{b(t)}$$

as the solution of equation (8.2).

3. Nonrandom time change. For a nonrandom function $h(t)$ that is different from zero for all $t \geq 0$ and satisfies $\int_0^t h^2(s)\,ds < \infty$, the processes $\int_0^t h(s)\,dW(s)$ and $W\Big(\int_0^t h^2(s)\,ds\Big)$, $W(0) = 0$, are identical in law. Indeed, they are Gaussian processes with mean 0 and the variance $\int_0^t h^2(s)\,ds$. Also, these processes have independent increments.

Moreover, we can write

$$\int_0^t h(s)\,d\widetilde{W}(s) = W\Big(\int_0^t h^2(s)\,ds\Big)$$

for some new Brownian motion $\widetilde{W}(t)$, $t \ge 0$. Indeed,

$$\widetilde{W}(t) = \int_0^t h^{-1}(s)\,dW\Big(\int_0^s h^2(v)\,dv\Big).$$

The process $W\Big(\int_0^t h^2(s)\,ds\Big)$ has independent increments, and there are well-developed techniques of integration with respect to such processes (see §9 Ch. I).

Making the time substitution $t \to \int_0^t h^2(s)\,ds$ in the stochastic differential equation

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t), \qquad X(0) = x_0, \tag{8.7}$$

we have that the process $V(t) := X\Big(\int_0^t h^2(s)\,ds\Big)$ satisfies the following stochastic differential equation:

$$dV(t) = a\Big(\int_0^t h^2(s)\,ds, V(t)\Big)h^2(t)\,dt + b\Big(\int_0^t h^2(s)\,ds, V(t)\Big)h(t)\,d\widetilde{W}(t), \quad V(0) = x_0. \tag{8.8}$$

Such a time substitution enables us to change the coefficients $a(t,x)$ and $b(t,x)$ as functions of the time variable.

4. Random time change. Let $W(t)$, $t \in [0,\infty)$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Suppose that the increments $W(v) - W(t)$ are independent of the σ-algebra $\mathcal{F}_t$ for all $v > t$.

Let $b(t)$, $t \in [0,\infty)$, be a progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ process such that $\theta(t) := \int_0^t b^2(s)\,ds < \infty$ a.s. for every $t > 0$. Consider the stochastic integral $Y(t) := \int_0^t b(s)\,dW(s)$, $t \in [0,\infty)$. Suppose that $\int_0^\infty b^2(s)\,ds = \infty$. Let $\theta^{(-1)}(t) := \min\{s : \theta(s) = t\}$ be the left continuous inverse function to θ, defined for all $t \ge 0$. Since for every s

$$\{\theta^{(-1)}(t) > s\} = \Big\{\int_0^s b^2(v)\,dv < t\Big\} \in \mathcal{F}_s,$$

the variable $\theta^{(-1)}(t)$ is a stopping time with respect to the filtration $\{\mathcal{F}_s\}_{s\ge 0}$.

Theorem 8.1 (Lévy). *The process* $\widetilde{W}(t) := Y(\theta^{(-1)}(t))$, $t \geq 0$ *is a Brownian motion.*

Proof. For $\vec{\gamma} := (\gamma_1, \dots, \gamma_n) \in \mathbf{R}^n$ and $0 < t_1 < \dots < t_n$ denote $B := \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l (t_k \wedge t_l)$, where $s \wedge t := \min\{s, t\}$. To prove that $\widetilde{W}(t)$, $t \in [0, \infty)$, is a Brownian motion, it is enough to verify the following formula for the characteristic function:

$$\varphi(\vec{\gamma}) := \mathbf{E} \exp\Big(i \sum_{k=1}^{n} \gamma_k \widetilde{W}(t_k)\Big) = e^{-B/2},$$

since then $\widetilde{W}$ is a Gaussian process with mean zero and the covariance function $\operatorname{Cov}(\widetilde{W}(s), \widetilde{W}(t)) = s \wedge t$, but it is a Brownian motion.

We have

$$\varphi(\vec{\gamma}) e^{B/2} = \mathbf{E} \exp\Big(i \sum_{k=1}^{n} \gamma_k \int_0^{\theta^{(-1)}(t_k)} b(s)\, dW(s) + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l (t_k \wedge t_l)\Big)$$

$$= \mathbf{E} \exp\Big(i \sum_{k=1}^{n} \gamma_k \int_0^{\theta^{(-1)}(t_k)} b(s)\, dW(s) + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_k \gamma_l \int_0^{\theta^{(-1)}(t_k) \wedge \theta^{(-1)}(t_l)} b^2(s)\, ds\Big)$$

$$= \mathbf{E} \exp\Big(i \int_0^{\infty} g(s)\, dW(s) + \frac{1}{2} \int_0^{\infty} g^2(s)\, ds\Big),$$

where $g(s) := b(s) \sum_{k=1}^{n} \gamma_k \mathbb{1}_{[0, \theta^{(-1)}(t_k)]}(s)$. Note that from the above transformations it follows that $\int_0^{\infty} g^2(s)\, ds = B$. The process

$$\rho(t) := \mathbf{E} \exp\Big(i \int_0^{t} g(s)\, dW(s) + \frac{1}{2} \int_0^{t} g^2(s)\, ds\Big)$$

is a uniformly integrable stochastic exponent (in formula (6.1) instead of $b(s)$ we have $ig(s)$). By (6.3) and (6.4), it is a complex-valued martingale (the real and the imaginary parts are martingales), and $\mathbf{E}\rho(t) = 1$ for every t. Letting here $t \to \infty$, we get $\mathbf{E}\rho(\infty) = 1$. As a result, we have that $\varphi(\vec{\gamma}) e^{B/2} = 1$. This proves the theorem. □

Consider a nonhomogeneous stochastic differential equation

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.9}$$

with $b(t, x) > 0$ for all $(t, x) \in [0, \infty) \times \mathbf{R}$.

Let $\theta(t) := \int_0^t b^2(s, X(s))\,ds < \infty$ a.s. for every $t > 0$ and $\theta(\infty) = \infty$. Let $\theta^{(-1)}(t) := \min\{s : \theta(s) = t\}$, $t \ge 0$, be the left continuous function inverse of θ.

Set $Y(t) := X(\theta^{(-1)}(t))$, $\widetilde{W}(t) := \int_0^{\theta^{(-1)}(t)} b(s, X(s))\,dW(s)$. By Theorem 8.1, the process $\widetilde{W}$ is a Brownian motion. In equation (8.9) we can make the random time substitution. This yields

$$Y(t) - Y(0) = \int_0^{\theta^{(-1)}(t)} a(s, X(s))\,ds + \int_0^{\theta^{(-1)}(t)} b(s, X(s))\,dW(s)$$

$$= \int_0^t a(\theta^{(-1)}(s), X(\theta^{(-1)}(s)))\,d\theta^{(-1)}(s) + \widetilde{W}(t) = \int_0^t \frac{a(\theta^{(-1)}(s), Y(s))}{b^2(\theta^{(-1)}(s), Y(s))}\,ds + \widetilde{W}(t),$$

because

$$d\theta^{(-1)}(s) = \frac{ds}{\theta'(\theta^{(-1)}(s))} = \frac{ds}{b^2(\theta^{(-1)}(s), Y(s))}.$$

Thus in equation (8.9), by the random time substitution, the coefficient before the stochastic differential is transformed to 1. We get the equation

$$dY(t) = \frac{a(\theta^{(-1)}(t), Y(t))}{b^2(\theta^{(-1)}(t), Y(t))}\,dt + d\widetilde{W}(t), \qquad Y(0) = x_0. \tag{8.10}$$

A feature of this equation is that the coefficient $\frac{a(\theta^{(-1)}(t), Y(t))}{b^2(\theta^{(-1)}(t), Y(t))}$ depends on the stopping time $\theta^{(-1)}(t)$. Since these stopping times are increasing in t, there exists an increasing family of σ-algebras $\mathcal{A}_t := \mathcal{F}_{\theta^{(-1)}(t)}$ connected with them (see § 4 Ch. I). The process $\theta^{(-1)}(t)$, $t \ge 0$, is adapted (see property 8 § 4 Ch. I) to the filtration $\{\mathcal{A}_t\}_{t\ge0}$. Since the process $\int_0^t b(s, X(s))\,dW(s)$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\ge0}$, the Brownian motion $\widetilde{W}(t)$ is also adapted to the filtration $\{\mathcal{A}_t\}_{t\ge0}$ (see § 4 Ch. I). In addition, for all $v > t$ the increments $\widetilde{W}(v) - \widetilde{W}(t)$ are independent of the σ-algebra $\mathcal{A}_t$. This can be proved in the following way. Analogously to the proofs of properties (2.3) and (2.4), we can verify that a.s.

$$\mathbf{E}\{\widetilde{W}(v) - \widetilde{W}(t)\big|\mathcal{A}_t\} = \mathbf{E}\left\{\int_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b(s, X(s))\,dW(s)\right|\left.\mathcal{A}_t\right\} = 0,$$

$$\mathbf{E}\{(\widetilde{W}(v) - \widetilde{W}(t))^2\big|\mathcal{A}_t\} = \mathbf{E}\left\{\left(\int_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b(s, X(s))\,dW(s)\right)^2\right|\left.\mathcal{A}_t\right\}$$

$$= \mathbf{E}\Bigg\{ \int\limits_{\theta^{(-1)}(t)}^{\theta^{(-1)}(v)} b^2(s, X(s))\, ds \Bigg| \mathcal{A}_t \Bigg\} ds = v - t.$$

Since the process of $\widetilde{W}$ is continuous, by Lévy's characterization (see Theorem 10.1 Ch. I) these two properties guarantee that $\widetilde{W}$ is a Brownian motion. This is a different proof than the one given above in Theorem 8.1. At the end of the proof of Theorem 10.1 Ch. I it was established that for any $0 \le t < v$ and $\alpha \in \mathbf{R}$,

$$\mathbf{E}\big\{ e^{i\alpha(\widetilde{W}(v) - \widetilde{W}(t))} \big| \mathcal{A}_t \big\} = e^{-\alpha^2 (v-t)/2} \qquad \text{a.s.}$$

This equality is equivalent to the fact that the random variables $\widetilde{W}(v) - \widetilde{W}(t)$ are independent of the σ-algebra $\mathcal{A}_t$.

The adaptivity of the random coefficient of equation (8.10) with the filtration $\{\mathcal{A}_t\}_{t \ge 0}$ and the independence of $\widetilde{W}(v) - \widetilde{W}(t)$ for every $v > t$ of the σ-algebra $\mathcal{A}_t$ enables us to prove that there exists a unique solution of such an equation.

5. Reduction to linear stochastic differential equations. Consider the homogeneous stochastic differential equation

$$dX(t) = a(X(t))\, dt + b(X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.11}$$

where the coefficients a and b are independent of the time variable.

For this equation we describe a method of reduction to a linear stochastic differential equation. Let $f(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function which has the inverse $f^{(-1)}(x)$.

Set $\widetilde{X}(t) := f(X(t))$, $t \ge 0$. Write out the stochastic differential equation for the process $\widetilde{X}(t)$. Applying Itô's formula, we have

$$df(X(t)) = f'(X(t))a(X(t))\, dt + f'(X(t))b(X(t))\, dW(t) + \frac{1}{2} f''(X(t))b^2(X(t))\, dt.$$

Setting

$$\tilde{a}(x) := f'(f^{(-1)}(x))a(f^{(-1)}(x)) + \frac{1}{2} f''(f^{(-1)}(x))b^2(f^{(-1)}(x)), \tag{8.12}$$

$$\tilde{b}(x) := f'(f^{(-1)}(x))b(f^{(-1)}(x)), \tag{8.13}$$

and using the equality $X(t) = f^{(-1)}(\widetilde{X}(t))$, we obtain

$$d\widetilde{X}(t) = \tilde{a}(\widetilde{X}(t))\, dt + \tilde{b}(\widetilde{X}(t))\, dW(t), \qquad \widetilde{X}(0) = f(x_0). \tag{8.14}$$

Therefore, the substitution $\widetilde{X}(t) = f(X(t))$ reduces equation (8.11) to equation (8.14).

We derive conditions on the coefficients $a(x)$ and $b(x)$, $x \in \mathbf{R}$, under which equation (8.11) can be further reduced to a linear one, i.e., $\tilde{a}(x) = \alpha x + r$ and

$\tilde{b}(x) = \beta x + q$ for some constants α, β, r, and q. For these coefficients the linear stochastic differential equation has (see (8.6)) the solution of the form

$$\widetilde{X}(t) = \Big(f(x_0) + \frac{q}{\beta}\Big) e^{\beta(W(t)-W(0))+(\alpha-\beta^2/2)t}$$

$$+ \frac{\beta r - \alpha q}{\beta} \int_0^t e^{\beta(W(t)-W(s))+(\alpha-\beta^2/2)(t-s)}\, ds - \frac{q}{\beta}.$$

Using expressions (8.12) and (8.13), for the coefficients $\tilde{a}$ and $\tilde{b}$, we get

$$f'(x)a(x) + \frac{1}{2} f''(x)b^2(x) = \alpha f(x) + r, \tag{8.15}$$

$$f'(x)b(x) = \beta f(x) + q. \tag{8.16}$$

Suppose that $b(x)$ is a continuously differentiable function such that $b(x) \neq 0$ for all x from the state space of the process X.

In the case $\beta \neq 0$ the first-order differential equation (8.16) has the solution

$$f(x) = \frac{c}{\beta} \exp\Big(\beta \int_{x_0}^{x} \frac{dy}{b(y)}\Big) - \frac{q}{\beta}, \tag{8.17}$$

where c is some constant and x_0 is the starting point of X.

By (8.16),

$$f''(x)b(x) + b'(x)f'(x) = \beta f'(x),$$

or

$$f''(x)b^2(x) = (\beta - b'(x))(\beta f(x) + q). \tag{8.18}$$

Substituting this expression in equation (8.15), we have

$$\Big(\frac{a(x)}{b(x)} + \frac{1}{2}(\beta - b'(x))\Big)(\beta f(x) + q) = \alpha f(x) + r, \tag{8.19}$$

or

$$\Big(\frac{a(x)}{b(x)} - \frac{b'(x)}{2} + \frac{\beta}{2} - \frac{\alpha}{\beta}\Big)\Big(f(x) + \frac{q}{\beta}\Big) = \frac{\beta r - \alpha q}{\beta^2}.$$

In the case $\beta = 0$ equation (8.16) has the solution

$$f(x) = q \int_{x_0}^{x} \frac{dy}{b(y)} + h, \tag{8.20}$$

where h is some constant, and (8.19) for $\beta = 0$ is transformed to the equality

$$\frac{a(x)}{b(x)} - \frac{b'(x)}{2} = \alpha \int_{x_0}^{x} \frac{dy}{b(y)} + \frac{\alpha h + r}{q}. \tag{8.21}$$

Thus, we can formulate the following statement.

Proposition 8.1. *For a continuously differentiable coefficient $b(x)$ that is different from zero for all x from the state space of X, the homogeneous stochastic differential equation* (8.11) *is reducible to the homogeneous linear stochastic differential equation in the following cases.*

If for some constants α, r, q and $\beta \neq 0$, $c \neq 0$

$$a(x) = b(x)\biggl(\frac{b'(x)}{2} + \frac{\alpha}{\beta} - \frac{\beta}{2} + \frac{\beta r - \alpha q}{\beta c}\exp\biggl(-\beta\int_{x_0}^{x}\frac{dy}{b(y)}\biggr)\biggr), \tag{8.22}$$

then the process $\widetilde{X}(t) := \frac{c}{\beta}\exp\biggl(\beta\int_{x_0}^{X(t)}\frac{dy}{b(y)}\biggr) - \frac{q}{\beta}$ *satisfies the equation*

$$d\widetilde{X}(t) = (\alpha\widetilde{X}(t) + r)\,dt + (\beta\widetilde{X}(t) + q)\,dW(t), \qquad \widetilde{X}(0) = \frac{c-q}{\beta}. \tag{8.23}$$

If for some constants α, r, h, and $q \neq 0$,

$$a(x) = b(x)\biggl(\frac{b'(x)}{2} + \alpha\int_{x_0}^{x}\frac{dy}{b(y)} + \frac{\alpha h + r}{q}\biggr), \tag{8.24}$$

then the process $\widetilde{X}(t) := q\int_{x_0}^{X(t)}\frac{dy}{b(y)} + h$ *satisfies the equation*

$$d\widetilde{X}(t) = (\alpha\widetilde{X}(t) + r)\,dt + q\,dW(t), \qquad \widetilde{X}(0) = h. \tag{8.25}$$

6. Reduction to ordinary differential equations. Consider a nonhomogeneous stochastic differential equation

$$dX(t) = a(t, X(t))\,dt + b(t)X(t)\,dW(t), \qquad X(0) = x_0. \tag{8.26}$$

Here it is important that the coefficient in front of the stochastic differential is linear. Let $\rho(t)$ be the stochastic exponent

$$\rho(t) = \exp\biggl(\int_0^t b(s)\,dW(s) - \frac{1}{2}\int_0^t b^2(s)\,ds\biggr).$$

Compute the stochastic differential of the process $X(t)\rho^{-1}(t)$. By Itô's formula,

$$d(X(t)\rho^{-1}(t)) = \rho^{-1}(t)\{a(t, X(t))\,dt + b(t)X(t)\,dW(t)\}$$
$$-X(t)\rho^{-2}(t)b(t)\rho(t)\,dW(t) + X(t)\rho^{-3}(t)b^2(t)\rho^2(t)\,dt$$
$$-\rho^{-2}(t)b(t)X(t)b(t)\rho(t)\,dt = \rho^{-1}(t)a(t, X(t))\,dt.$$

Setting $Z(t) := X(t)\rho^{-1}(t)$, we see that this process satisfies the ordinary differential equation

$$Z'(t) = \rho^{-1}(t)\, a(t, \rho(t)Z(t)), \qquad Z(0) = x_0. \tag{8.27}$$

Therefore, multiplying the solution of equation (8.26) by the factor $\rho^{-1}(t)$, we transform this stochastic differential equation into the "deterministic" differential equation, which is valid for each sample path of the process $\rho(t)$, $t \geq 0$.

Equation (8.27) is rather complicated for the investigations, since it has nowhere differentiable coefficient $\rho(t)$. An example of the class of functions $a(t, x)$, $t \geq 0$, $x \geq 0$, for which it has an explicit solution is

$$a(t, x) = a(t)x^{\gamma}.$$

Nevertheless, equation (8.27) can be useful for numerical computations.

Combining the approaches described above and in Subsection 5, we can reduce an arbitrary homogeneous stochastic differential equation

$$dX(t) = a(X(t))\, dt + b(X(t))\, dW(t), \qquad X(0) = x_0, \tag{8.28}$$

with continuously differentiable coefficient $b(x)$ that is different from zero for all x from the state space of X, to a "deterministic" differential equation, which is valid for each sample path of the process.

Consider the function f, defined by (8.17) for $q = 0$, $\beta = 1$ and $c = 1$, i.e.,

$$f(x) = \exp\left(\int_{x_0}^{x} \frac{dy}{b(y)}\right). \tag{8.29}$$

The function $B(x) := \int_{x_0}^{x} \frac{dy}{b(y)}$ has the inverse $B^{(-1)}(x)$. This implies that the function $f(x)$ has the inverse $f^{(-1)}(x) = B^{(-1)}(\ln x)$.

According to (8.12), (8.16), and (8.18), the process $\widetilde{X}(t) = \exp(B(X(t)))$ satisfies the stochastic differential equation

$$d\widetilde{X}(t) = \tilde{a}(\widetilde{X}(t))\, dt + \widetilde{X}(t)\, dW(t), \qquad \widetilde{X}(0) = 1, \tag{8.30}$$

where

$$\tilde{a}(x) = x\left(\frac{a(B^{(-1)}(\ln x))}{b(B^{(-1)}(\ln x))} + \frac{1}{2} - \frac{b'(B^{(-1)}(\ln x))}{2}\right). \tag{8.31}$$

Then equation (8.30) takes the form (8.26) with $b(t) \equiv 1$. Therefore, the process $Z(t) = \widetilde{X}(t)\, e^{W(0)-W(t)+t/2}$ satisfies the ordinary differential equation

$$Z'(t) = e^{W(0)-W(t)+t/2}\, \tilde{a}(Z(t)\, e^{W(t)-W(0)-t/2}), \qquad Z(0) = 1. \tag{8.32}$$

Finally,

$$X(t) = B^{(-1)}(\ln \widetilde{X}(t)) = B^{(-1)}(\ln Z(t) + W(t) - W(0) - t/2).$$

Exercises.

8.1. Solve the linear stochastic differential equation

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dW(t), \qquad V(0) = x_0 > 0.$$

8.2. Solve the linear stochastic differential equation

$$dU(t) = -\gamma U(t)\,dt + \sigma\,dW(t), \qquad U(0) = x_0 > 0.$$

8.3. Solve the linear stochastic differential equation

$$dZ(t) = (\beta Z(t) + \sigma)\,dW(t), \qquad Z(0) = x_0 > 0.$$

8.4. Solve the stochastic differential equation

$$dX(t) = \frac{1}{X(t)}\,dt + \beta X(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

8.5. Solve the stochastic differential equation

$$dX(t) = X^{-\gamma}(t)\,dt + \beta X(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

8.6. Solve the stochastic differential equation

$$dX(t) = aX(t)(1 - gX(t))\,dt + bX(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

8.7. Let $\sigma(x)$, $x \in \mathbf{R}$, be a function with bounded derivative such that the integral

$$S(x) := \int_{x_0}^{x} \frac{1}{\sigma(y)}\,dy$$

is finite for all $x \in \mathbf{R}$.

Let the process $X(t)$, $t \in [0,T]$, be a solution of the stochastic differential equation

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))\,dt + \sigma(X(t))\,dW(t), \qquad X(0) = x_0.$$

Prove that $Z(t) := S(X(t))$, $t \in [0,T]$, is a Brownian motion.

8.8. Solve for $\gamma \neq 1$ the stochastic differential equation

$$dX(t) = \big(aX(t) - gX^{\gamma}(t)\big)\,dt + bX(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

8.9. Solve for integer $m \neq 1$ the stochastic differential equation

$$dX(t) = \Big(\frac{m}{2}X^{2m-1}(t) + \mu X^{m}(t)\Big)\,dt + X^{m}(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

8.10. Under what conditions on the parameters a, b, n, for $m \neq 1$ is the stochastic differential equation

$$dX(t) = aX^n(t)\,dt + bX^m(t)\,dW(t), \qquad X(0) = x_0 > 0.$$

reducible to the linear one? What is the solution?

8.11. Let $X(t)$ be a solution of the stochastic differential equation

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \qquad X(0) = x_0,$$

with a continuously differentiable coefficient $b(x)$ different from zero for all x from the state space of X.

Compute the function $f(x)$ such that the process $\widetilde{X}(t) = f(X(t))$, $t \geq 0$, satisfies the equation containing only the pure stochastic differential (the factor before dt is zero).

§ 9. Dependence of solutions of stochastic differential equations on initial values

Consider a stochastic differential equation with nonrandom initial value:

$$dX_x(t) = a(t, X_x(t))\,dt + b(t, X_x(t))\,dW(t), \qquad X_x(0) = x. \tag{9.1}$$

The integral analog of this equation is the following:

$$X_x(t) = x + \int_0^t a(s, X_x(s))\,ds + \int_0^t b(s, X_x(s))\,dW(s). \tag{9.2}$$

1. Continuous dependence of solutions on initial values. We want to consider solutions of (9.2) for all $x \in \mathbf{R}$ simultaneously. Moreover, it is better to consider them as a process of (t, x).

Then the following problem arises. A solution of the stochastic differential equation (9.2), as it was proved in § 7, exists a.s. It can depend on the initial value. Therefore, there is a set Λ_x of probability zero, where the solution does not exists. The probability of the union of the sets Λ_x can be nonzero. In this case the solutions of (9.2) are not determined as a function of x on a set of nonzero probability. We had the analogous situation when considering the stochastic integral as a function of the variable upper limit.

The main approach to overcome this difficulty is to prove that the process $X_x(t)$, $(t, x) \in [0, T] \times \mathbf{R}$, can be chosen to be continuous.

Theorem 9.1. *Suppose that the functions $a(t, x)$ and $b(t, x)$ satisfy the conditions* (7.4) *and* (7.5). *Then there exist a modification $X_x(t)$ of a solution of* (9.1) a.s. *jointly continuous in $(t, x) \in [0, T] \times \mathbf{R}$.*

If $X(t)$, $t \in [0,T]$, is the solution of the equation

$$X(t) = \xi + \int_0^t a(s,X(s))\,ds + \int_0^t b(s,X(s))\,dW(s), \tag{9.3}$$

with a square integrable random initial value ξ independent of the process $W(t)$, $t \in [0,T]$, then

$$\mathbf{P}\Big(\sup_{0\le t\le T} |X(t) - X_\xi(t)| = 0\Big) = 1. \tag{9.4}$$

Proof. From (9.2) it is easy to derive that

$$\sup_{0\le v\le t} |X_x(v) - X_y(v)| \le |x-y| + \int_0^t |a(s,X_x(s)) - a(s,X_y(s))|\,ds$$

$$+ \sup_{0\le v\le t} \Big| \int_0^v (b(s,X_x(s)) - b(s,X_y(s)))\,dW(s)\Big|.$$

Applying Doob's inequality (2.6) and estimating as in (7.7), we get

$$\mathbf{E} \sup_{0\le v\le t} (X_x(v) - X_y(v))^2 \le 3|x-y|^2 + 3\mathbf{E}\Big(\int_0^t |a(s,X_x(s)) - a(s,X_y(s))|\,ds\Big)^2$$

$$+12\int_0^t \mathbf{E}(b(s,X_x(s)) - b(s,X_y(s)))^2 ds \le 3|x-y|^2 + 3(T+4)C_T^2\int_0^t \mathbf{E}(X_x(s) - X_y(s))^2 ds$$

$$\le 3|x-y|^2 + 3(T+4)C_T^2 \int_0^t \mathbf{E} \sup_{0\le v\le s} (X_x(v) - X_y(v))^2 ds.$$

By (7.6), from the second inequality it follows that $\mathbf{E} \sup\limits_{0\le v\le t} (X_x(v) - X_y(v))^2$, $t \in [0,T]$, is bounded. Then by Gronwall's lemma (see (7.9)),

$$\mathbf{E} \sup_{0\le v\le t} (X_x(v) - X_y(v))^2 \le 3|x-y|^2 e^{3(T+4)C_T^2 t}, \qquad t \in [0,T]. \tag{9.5}$$

Now we can apply arguments similar to those used to prove the continuity of $J_x(t)$, $(t,x) \in [0,T]\times\mathbf{R}$, in § 5. For every fixed x the process $X_x(t)$ is continuous in t. Let us consider $X_x(\cdot)$ as a random variable taking values in the space $C([0,T])$ of continuous functions on $[0,T]$. This space, when equipped with the uniform norm $\|f\| := \sup\limits_{t\in[0,T]} |f(t)|$ is a Banach space. Then (9.5) can be written as

$$\mathbf{E}\|X_x - X_y\|^2 \le M_T|x-y|^2. \tag{9.6}$$

Applying Kolmogorov's continuity criterion in the form (5.7), (5.8), we obtain that for any $0 < \gamma < 1/2$ and $N > 0$ a.s.

$$\|X_x - X_y\| \le L_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x, y \in D\bigcap[-N, N], \tag{9.7}$$

where D is the set of dyadic (binary rational) points $k/2^n$ of $\mathbf{R}$. Since D is countable, Theorem 7.1 shows that a.s. for every $y \in D$ and for all $t \in [0, T]$ there exists a unique solution $X_y(t)$ of the equation

$$X_y(t) = y + \int_0^t a(s, X_y(s))\, ds + \int_0^t b(s, X_y(s))\, dW(s). \tag{9.8}$$

Using (9.7) we can a.s. extend the processes X_y from the dyadic set D to the whole real line by setting $X_x(t) = \lim_{y\to x, y\in D} X_y(t)$. This limit is uniform in $t \in [0, T]$, therefore, $X_x(t)$ is a.s. continuous in $t \in [0, T]$ simultaneously for all x. Due to this, (7.4), and (3.6), the passage to the limit in (9.8) as $y \to x$, $y \in D$, proves that $X_x(t)$ is a.s. the solution of equation (9.2) for all $(t, x) \in [0, T] \times \mathbf{R}$.

From (9.7) we get that for any $0 < \gamma < 1/2$ the processes X_x a.s. satisfy the Hölder condition

$$\sup_{0\le t\le T} |X_x(t) - X_y(t)| \le L_{N,\gamma}(\omega)|x-y|^\gamma, \qquad x, y \in [-N, N], \tag{9.9}$$

for all integer N.

The joint continuity of the solution $X_x(t)$ in (t, x) follows from (9.9) and from the continuity of $X_x(t)$ in t for all x. Indeed, for arbitrary $x, y \in [-N, N]$ and $s, t \in [0, T]$

$$|X_x(t) - X_y(s)| \le |X_x(t) - X_x(s)| + \|X_x(\cdot) - X_y(\cdot)\|. \tag{9.10}$$

Substituting in (9.2) instead of x the random variable ξ, satisfying the condition of Theorem 9.1, we have a.s for all $t \in [0, T]$

$$X_\xi(t) = \xi + \int_0^t a(s, X_\xi(s))\, ds + \int_0^t b(s, X_\xi(s))\, dW(s). \tag{9.11}$$

To explain this equality we do the following. Set $\xi_n(\omega) := \sum_{-\infty}^{\infty} \frac{k}{n} \mathbb{1}_{\Omega_{k,n}}(\omega)$, where $\Omega_{k,n} = \bigl\{\omega : \xi(\omega) \in \bigl[\frac{k}{n}, \frac{k+1}{n}\bigr)\bigr\}$. Then from (9.2), applied for $x = \frac{k}{n}$ and a.s. all $\omega \in \Omega_{k,n}$, we have for all $t \in [0, T]$ the equation

$$X_{\xi_n}(t) = \xi_n + \int_0^t a(s, X_{\xi_n}(s))\, ds + \int_0^t b(s, X_{\xi_n}(s))\, dW(s).$$

Since $|\xi - \xi_n| \le 1/n$, applying (9.9), we can pass to the limit in this equation. As a result, we have (9.11). This stochastic equation coincides with (9.3), therefore by the uniqueness of the solution, the processes $X_\xi(t)$ and $X(t)$, $t \in [0,T]$, are indistinguishable in the sense of (9.4). □

Now we are going to prove a generalization of Theorem 9.1. Consider the family $X_{s,x}(t)$, $0 \le s \le t \le T$, of solutions of the stochastic differential equation

$$X_{s,x}(t) = x + \int_s^t a(q, X_{s,x}(q))\, dq + \int_s^t b(q, X_{s,x}(q))\, dW(q), \qquad t \in [s,T]. \quad (9.12)$$

Theorem 9.2. *Suppose that the functions* $a(t,x)$ *and* $b(t,x)$ *satisfy conditions* (7.4), (7.5). *Then there exists a modification* $X_{s,x}(t)$ *of a solution of* (9.12) a.s. *jointly continuous in* (s,t,x), $0 \le s \le t \le T$, $x \in \mathbf{R}$. *Moreover, for the solution* $X(t)$, $t \in [0,T]$, *of equation* (9.3) *the following equality holds* a.s. *for all* $s \in [0,T]$:

$$X(t) = X_{s,X(s)}(t), \qquad t \in [s,T], \quad (9.13)$$

Proof. The main approach to the proof of this result is the same as for Theorem 9.1, but there are some technical differences. Set $a_s(q,x) := a(q,x)\mathbb{1}_{[s,\infty)}(q)$, $b_s(q,x) := b(q,x)\mathbb{1}_{[s,\infty)}(q)$. Consider the stochastic differential equation

$$\widetilde{X}_{s,x}(t) = x + \int_0^t a_s(q, \widetilde{X}_{s,x}(q))\, dq + \int_0^t b_s(q, \widetilde{X}_{s,x}(q))\, dW(q), \qquad t \in [0,T]. \quad (9.14)$$

The conditions of Theorem 7.1 and 7.4 are satisfied, so there exists a unique strong solution of (9.14), obeying the estimate

$$\sup_{0\le t\le T} \mathbf{E}\widetilde{X}^{2m}_{s,x}(t) < K_{m,x,T}, \quad (9.15)$$

where m is a positive integer. It is clear that $\widetilde{X}_{s,x}(t) = x\mathbb{1}_{[0,s]}(t) + X_{s,x}(t)\mathbb{1}_{[s,T]}(t)$.

From (9.14) we deduce the estimate

$$\sup_{0\le v\le t} |\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v)| \le |x-y| + \int_0^t |a_s(q, \widetilde{X}_{s,x}(q)) - a_u(q, \widetilde{X}_{u,y}(q))|\, dq$$

$$+ \sup_{0\le v\le t} \left| \int_0^v (b_s(q, \widetilde{X}_{s,x}(q)) - b_u(q, \widetilde{X}_{u,y}(q)))\, dW(q) \right|.$$

Note that for $s < u$

$$a_s(q,x) - a_u(q,y) = a(q,x)\mathbb{1}_{[s,u]}(q) + (a(q,x) - a(q,y))\mathbb{1}_{[u,T]}(q).$$

Using conditions (7.4), (7.5), we have

$$|a_s(q,x) - a_u(q,y)| \le C_T((1+|x|)\mathbb{1}_{[s,u]}(q) + |x-y|).$$

Then, applying the analogous inequality for $b_s(q,x)$, (4.25), (9.15) and the Hölder inequality, we get for $x, y \in [-N, N]$ and for $t \in [0,T]$ the estimate

$$\mathbf{E}\sup_{0\le v\le t}(\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v))^{2m}$$

$$\le K_{m,T,N}\biggl(|x-y|^{2m} + |u-s|^m + \int_0^t \mathbf{E}\bigl(\widetilde{X}_{s,x}(q) - \widetilde{X}_{u,y}(q)\bigr)^{2m} dq\biggr).$$

$$\le K_{m,T,N}\biggl(|x-y|^{2m} + |u-s|^m + \int_0^t \mathbf{E}\sup_{0\le v\le q}\bigl(\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v)\bigr)^{2m} dq\biggr).$$

Finally, by Gronwall's lemma (see (7.9)),

$$\mathbf{E}\sup_{0\le v\le t}(\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v))^{2m} \le \widetilde{K}_{m,T,N}\bigl(|x-y|^{2m} + |u-s|^m\bigr).$$

Applying Kolmogorov's continuity criterion (the analog of Theorem 3.3 for processes with values in Banach spaces), we deduce that for every $0 < \gamma < 1/2$ and $N > 0$, a.s. for all $s, u \in D\bigcap[0,T]$ and $x, y \in D\bigcap[-N,N]$

$$\sup_{0\le v\le t}\bigl|\widetilde{X}_{s,x}(v) - \widetilde{X}_{u,y}(v)\bigr| \le L_{N,T,\gamma}(\omega)\bigl(|u-s|^\gamma + |x-y|^\gamma\bigr), \tag{9.16}$$

where D is the set of dyadic (binary rational) points.

Using (9.16) we can a.s. extend the processes $\widetilde{X}_{u,y}$ from the dyadic set $D \times D$ to the whole real plane by setting

$$\widetilde{X}_{s,x}(t) = \lim_{y\to x, y\in D}\ \lim_{u\to s, u\in D}\widetilde{X}_{u,y}(t), \qquad s\in[0,T],\ x\in[-N,N].$$

This limit is uniform in $t \in [0,T]$, therefore $\widetilde{X}_{s,x}(t)$ is a.s. continuous in $t \in [0,T]$ for all s, x simultaneously. Due to this, (7.4), and (3.6), the passage to the limit in (9.14) as $s_n \to s$, $x_n \to x$, $(s_n, x_n) \in D \times D$, proves that $\widetilde{X}_{s,x}(t)$ is a.s. the solution of equation (9.14) for all $(s,t,x) \in [0,T]^2 \times \mathbf{R}$. Analogously to the proof of Theorem 9.1, one establishes that the process $\widetilde{X}_{s,x}(t)$, $(s,t,x) \in [0,T]^2 \times \mathbf{R}$, is continuous and, consequently, the same is true for the process $X_{s,x}(t)$, $0 \le s \le t \le T$, $x \in \mathbf{R}$. The equality (9.13) also holds (see the end of the proof of Theorem 9.1). □

Remark 9.1. Let τ be a stopping time with respect to the filtration $\mathcal{F}_t := \sigma\{\xi, W(s), 0 \le s \le t\}$, where ξ is a random variable independent of the Brownian motion W. Then for the set $\{\tau \le t \le T\}$ the following stochastic differential equation

$$X_{\tau,x}(t) = x + \int_\tau^t a(q, X_{\tau,x}(q))\, dq + \int_\tau^t b(q, X_{\tau,x}(q))\, dW(q), \qquad t \in [\tau, T], \quad (9.17)$$

makes sense.

This is due to the fact that for a fixed x the processes $a_\tau(q,x) := a(q,x)\mathbb{1}_{[\tau,\infty)}(q)$ and $b_\tau(q,x) := b(q,x)\mathbb{1}_{[\tau,\infty)}(q)$ are adapted to the filtration $\mathcal{F}_q$, $q \ge 0$.

2. Differentiability of solutions with respect to the initial value. Consider the question of differentiability of the solution X_x with respect to the initial value x. Since X_x is a random function, we should treat the derivative with respect to x in the mean square sense. If for a random function $Z(x)$, $x \in \mathbf{R}$, there exists the random function $V(x)$ such that

$$\lim_{\Delta \to 0} \mathbf{E}\Big(\frac{Z(x+\Delta) - Z(x)}{\Delta} - V(x)\Big)^2 = 0,$$

we call $V(x)$ the *mean square derivative* of Z and set $\dfrac{d}{dx}Z(x) := V(x)$.

The mean square differentiability is important, for example, for the proof that the function $u(x) := \mathbf{E}f(X_x(t))$ is continuously differentiable, where $f(x)$, $x \in \mathbf{R}$, has a continuous bounded first derivative.

Theorem 9.3. *Suppose that the functions $a(t,z)$ and $b(t,z)$, $(t,z) \in [0,T] \times \mathbf{R}$, are continuous, and have continuous bounded partial derivatives $a'_z(t,z)$ and $b'_z(t,z)$ with respect to z.*

Then the continuous solution $X_x(t)$ of (9.1) *has a stochastically continuous in $(t,x) \in [0,T] \times \mathbf{R}$ mean square derivative $X_x^{(1)}(t) := \dfrac{d}{dx}X_x(t)$, which satisfies the equation*

$$X_x^{(1)}(t) = 1 + \int_0^t a'_z(s, X_x(s)) X_x^{(1)}(s)\, ds + \int_0^t b'_z(s, X_x(s)) X_x^{(1)}(s)\, dW(s). \quad (9.18)$$

Remark 9.2. Under the assumptions of Theorem 9.3, the functions a and b satisfy the Lipschitz condition (7.4) and the linear growth condition (7.5).

Remark 9.3. The function $X_x^{(1)}(t)$ satisfying equation (9.18) has the form (see (8.3))

$$X_x^{(1)}(t) = \exp\bigg(\int_0^t b'_z(s, X_x(s))\, dW(s) + \int_0^t \Big\{a'_z(s, X_x(s)) - \tfrac{1}{2}\big(b'_z(s, X_x(s))\big)^2\Big\} ds\bigg). \quad (9.19)$$

This derivative is positive and, therefore for every fixed t, the process $X_x(t)$ is a.s. an increasing function with respect to x.

Proof of Theorem 9.3. We start with an auxiliary result.

Lemma 9.1. *Let $a_\Delta(t)$ and $b_\Delta(t)$, $t \in [0,T]$, $\Delta \in [-1,1]$, be a family of uniformly bounded progressively measurable processes, i.e., $|a_\Delta(t)| \le K$, $|b_\Delta(t)| \le K$ for all $t \in [0,T]$ and some nonrandom constant K. Suppose that for every Δ a progressively measurable process $Y_\Delta(t)$, $t \in [0,T]$, satisfies the equation*

$$Y_\Delta(t) = 1 + \int_0^t a_\Delta(s) Y_\Delta(s)\, ds + \int_0^t b_\Delta(s) Y_\Delta(s)\, dW(s). \tag{9.20}$$

Then for any $p \in \mathbf{R}$,

$$\mathbf{E} Y_\Delta^p(t) \le e^{|p|(|p|K+1)Kt}. \tag{9.21}$$

If $a_\Delta(t) \to a_0(t)$ and $b_\Delta(t) \to b_0(t)$ as $\Delta \to 0$ in probability for every $t \in [0,T]$, then $Y_\Delta(t) \to Y_0(t)$ in probability and in mean square, where Y_0 is the solution of (9.20) for $\Delta = 0$.

Proof. We note first that $Y_\Delta(t)$ can be represented (see (8.3)) in the form

$$Y_\Delta(t) = \exp\Bigl(\int_0^t b_\Delta(s)\, dW(s) + \int_0^t \bigl(a_\Delta(s) - \tfrac{1}{2} b_\Delta^2(s)\bigr)\, ds \Bigr) \tag{9.22}$$

and, consequently, $Y_\Delta(t)$ is a nonnegative process.

Using the Hölder inequality and (6.13), we have

$$\mathbf{E} Y_\Delta^p(t) \le \mathbf{E}^{1/2} \exp\Bigl(2p \int_0^t b_\Delta(s)\, dW(s) \Bigr) \mathbf{E}^{1/2} \exp\Bigl(2p \int_0^t a_\Delta(s)\, ds \Bigr) \le e^{p^2K^2t} e^{|p|Kt}.$$

Note that the estimate (9.21) is valid for both positive and negative p.

Since the coefficients a_Δ and b_Δ, $\Delta \in [-1,1]$, are uniformly bounded and they converge in probability, they converge also in mean square. Therefore,

$$\lim_{\Delta \to 0} \int_0^t \mathbf{E}(b_\Delta(s) - b_0(s))^2\, ds = 0, \qquad \lim_{\Delta \to 0} \int_0^t \mathbf{E}|a_\Delta(s) - a_0(s)|\, ds = 0.$$

Then, in view of (2.8), we can pass to the limit in (9.22) and get that $Y_\Delta(t) \to Y_0(t)$ in probability.

For arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{E}(Y_\Delta(t) - Y_0(t))^2 &= \mathbf{E}\{ 1\!\!1_{[0,\varepsilon]}(|Y_\Delta(t) - Y_0(t)|)(Y_\Delta(t) - Y_0(t))^2 \} \\ &\quad + \mathbf{E}\{ 1\!\!1_{(\varepsilon,\infty)}(|Y_\Delta(t) - Y_0(t)|)(Y_\Delta(t) - Y_0(t))^2 \} =: I_{1,\Delta} + I_{2,\Delta}. \end{aligned}$$

Using the convergence $Y_\Delta(t) \to Y_0(t)$ in probability, we see that the first term on the right-hand side of this equality tends to zero by the Lebesgue dominated convergence theorem. The second term is estimated by Hölder's inequality as follows:

$$I_{2,\Delta} \le \mathbf{P}^{1/2}(|Y_\Delta(t) - Y_0(t)| > \varepsilon) \mathbf{E}^{1/2}(Y_\Delta(t) - Y_0(t))^4.$$

This term also tends to zero in view of (9.21), $p = 4$, and the convergence $Y_\Delta(t) \to Y_0(t)$ in probability. Consequently, $Y_\Delta(t) \to Y_0(t)$ in mean square. Lemma 9.1 is proved. □

We continue the proof of the theorem. Since $a(t,x)$, $b(t,x)$ have bounded derivatives with respect to x, the functions

$$\tilde{a}(t,x,y) := \frac{a(t,y)-a(t,x)}{y-x}, \qquad \tilde{b}(t,x,y) := \frac{b(t,y)-b(t,x)}{y-x}, \qquad x \neq y,$$

can be extended continuously to the diagonal $x = y$ by setting $\tilde{a}(t,z,z) := a'_z(t,z)$, $\tilde{b}(t,z,z) := b'_z(t,z)$.

For a fixed x and $\Delta \neq 0$, set $Y_\Delta(t) := \frac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$, $t \in [0,T]$. Since

$$X_{x+\Delta}(t) - X_x(t) = \Delta + \int_0^t (a(s, X_{x+\Delta}(s)) - a(s, X_x(s)))\, ds$$
$$+ \int_0^t (b(s, X_{x+\Delta}(s)) - b(s, X_x(s)))\, dW(s),$$

the process $Y_\Delta(t)$ satisfies equation (9.20) with the coefficients

$$a_\Delta(t) := \tilde{a}(t, X_{x+\Delta}(t), X_x(t)), \qquad b_\Delta(t) := \tilde{b}(t, X_{x+\Delta}(t), X_x(t)).$$

These coefficients are uniformly bounded, because the functions $a(t,z)$ and $b(t,z)$ have continuous bounded derivatives with respect to z.

By (9.9), $X_{x+\Delta}(t) \longrightarrow X_x(t)$ as $\Delta \to 0$ a.s. Therefore,

$$a_\Delta(t) \to a'_z(t, X_x(t)), \qquad b_\Delta(t) \to b'_z(t, X_x(t)) \qquad \text{as } \Delta \to 0 \qquad \text{a.s.}$$

Finally, applying Lemma 9.1, we have that as $\Delta \to 0$ the variables $Y_\Delta(t)$ converge in probability and in mean square to the limit $Y_0(t)$ which is said to be the derivative $X^{(1)}_x(t) = \frac{d}{dx} X_x(t)$. The limit process $X^{(1)}_x(t)$ satisfies (9.19) and (9.18).

In view of (9.9), the conditions on the coefficients a, b, and (3.6),

$$\sup_{t\in[0,T]} \left| \int_0^t b'_z(t, X_y(t))\, dW(s) - \int_0^t b'_z(t, X_x(t))\, dW(s) \right| \longrightarrow 0 \qquad \text{as } y \to x$$

in probability. The ordinary integral in (9.19) is also continuous with respect to x, uniformly in $t \in [0,T]$. This and (9.19) imply that the process $X^{(1)}_x(t)$, $(t,x) \in [0,T] \times \mathbf{R}$, is stochastically continuous, and so continuous in mean square. □

Remark 9.4. Under the assumptions of Theorem 9.3,

$$\mathbf{E}\big(X^{(1)}_x(t)\big)^p \le e^{|p|(|p|K+1)Kt} \tag{9.23}$$

for any $p \in \mathbf{R}$.

Indeed, (9.23) is a consequence of (9.21) and Fatou's lemma.

3. Second derivative of a solution with respect to the initial value. Similarly to the proof of Theorem 9.3, we can prove the following result concerning the second-order derivative of $X_x(t)$ with respect to the initial value x.

Theorem 9.4. *Suppose that the functions $a(t,z)$ and $b(t,z)$, $(t,z)\in[0,T]\times\mathbf{R}$, are continuous, and have continuous partial derivatives $a'_z(t,z)$, $b'_z(t,z)$, $a''_{zz}(t,z)$, $b''_{zz}(t,z)$ with respect to z.*

Then the continuous solution $X_x(t)$ of equation (9.1) *has a stochastically continuous in $(t,x)\in[0,T]\times\mathbf{R}$ mean square second-order derivative $X^{(2)}_x(t):=\frac{d^2}{dx^2}X_x(t)=\frac{d}{dx}X^{(1)}_x(t)$, which satisfies the equation*

$$X^{(2)}_x(t)=\int_0^t a''_{zz}(s,X_x(s))(X^{(1)}_x(s))^2\,ds+\int_0^t a'_z(s,X_x(s))X^{(2)}_x(s)\,ds$$
$$+\int_0^t b''_{zz}(s,X_x(s))(X^{(1)}_x(s))^2\,dW(s)+\int_0^t b'_z(s,X_x(s))X^{(2)}_x(s)\,dW(s). \tag{9.24}$$

Remark 9.5. The function $X^{(2)}_x(t)$ satisfying this equation has (see (8.4)) the form

$$X^{(2)}_x(t)=X^{(1)}_x(t)\biggl\{\int_0^t b''_{zz}(s,X_x(s))X^{(1)}_x(s)\,dW(s)$$
$$+\int_0^t\bigl(a''_{zz}(s,X_x(s))-b'_z(s,X_x(s))b''_{zz}(s,X_x(s))\bigr)X^{(1)}_x(s)\,ds\biggr\}. \tag{9.25}$$

Proof of Theorem 9.4. From (9.18) for $\Delta\in[-1,1]$, it follows that

$$X^{(1)}_{x+\Delta}(t)-X^{(1)}_x(t)=\int_0^t\bigl(a'_z(s,X_{x+\Delta}(s))\,X^{(1)}_{x+\Delta}(s)-a'_z(s,X_x(s))\,X^{(1)}_x(s)\bigr)ds$$
$$+\int_0^t\bigl(b'_z(s,X_{x+\Delta}(s))\,X^{(1)}_{x+\Delta}(s)-b'_z(s,X_x(s))\,X^{(1)}_x(s)\bigr)dW(s). \tag{9.26}$$

Since $a(t,x)$ and $b(t,x)$ have bounded second derivatives with respect to x, the functions

$$\tilde a'(t,x,y):=\frac{a'_y(t,y)-a'_x(t,x)}{y-x},\qquad \tilde b'(t,x,y):=\frac{b'_y(t,y)-b'_x(t,x)}{y-x},\qquad x\ne y,$$

can be extended continuously to the diagonal $x=y$ by the equalities $\tilde a'(t,z,z):=a''_{zz}(t,z)$, $\tilde b'(t,z,z):=b''_{zz}(t,z)$. We also denote

$$a'_\Delta(t):=\tilde a'(t,X_{x+\Delta}(t),X_x(t)),\qquad b'_\Delta(t):=\tilde b'(t,X_{x+\Delta}(t),X_x(t)).$$

These coefficients are uniformly bounded, because the functions $a(t,x)$ and $b(t,x)$ have continuous bounded second derivatives with respect to x.

For a fixed x and $\Delta \neq 0$ we set $Z_\Delta(t) := \dfrac{X^{(1)}_{x+\Delta}(t) - X^{(1)}_x(t)}{\Delta}$ and, as in the proof of Theorem 9.3, we set $Y_\Delta(t) := \dfrac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$, $t \in [0,T]$.

Then (9.26) can be rewritten in the form

$$Z_\Delta(t) = \int_0^t a'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s)\,ds + \int_0^t b'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s)\,dW(s)$$

$$+ \int_0^t a'_z(s, X_x(s)) Z_\Delta(s)\,ds + \int_0^t b'_z(s, X_x(s)) Z_\Delta(s)\,dW(s). \tag{9.27}$$

This stochastic differential equation, as equation for the process Z_Δ, has the form (8.2). The coefficients of its linear homogeneous part are of the same form as in (9.18). Therefore, according to (8.4) and the fact that in this case (8.3) is exactly (9.18), the solution (9.27) has the form

$$Z_\Delta(t) = X^{(1)}_x(t) \biggl\{ \int_0^t b'_\Delta(s) Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \bigl(X^{(1)}_x(s)\bigr)^{-1} dW(s)$$

$$+ \int_0^t \bigl(a'_\Delta(s) - b'_z(s, X_x(s))\, b'_\Delta(s)\bigr) Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \bigl(X^{(1)}_x(s)\bigr)^{-1} ds \biggr\}. \tag{9.28}$$

Using this representation, it is not hard to get the estimate

$$\sup_{0\le\Delta\le1}\ \sup_{0\le t\le T} \mathbf{E} Z^{2n}_\Delta(t) < \infty \tag{9.29}$$

for any positive integer n.

Indeed, taking into account the boundedness of the functions a'_Δ, b'_Δ, b'_z, the estimate (4.25), and the nonnegativity of the processes Y_Δ, $X^{(1)}_x$, we get

$$\mathbf{E} Z^{2n}_\Delta(t) \le 2^{2n-1} \widetilde{C} \biggl\{ \mathbf{E} \biggl(\int_0^t Y^2_\Delta(s) \Bigl(X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} \Bigr)^2 ds \biggr)^n$$

$$+ \mathbf{E} \biggl(\int_0^t Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} ds \biggr)^{2n} \biggr\} \le C_n \int_0^t \mathbf{E} \Bigl(Y_\Delta(s) X^{(1)}_{x+\Delta}(s) \frac{X^{(1)}_x(t)}{X^{(1)}_x(s)} \Bigr)^{2n} ds.$$

Applying Hölder's inequality, we obtain

$$\mathbf{E} Z^{2n}_\Delta(t) \le C \int_0^t \mathbf{E}^{1/4} Y^{8n}_\Delta(s) \mathbf{E}^{1/4} \bigl(X^{(1)}_{x+\Delta}(s)\bigr)^{8n} \mathbf{E}^{1/4} \bigl(X^{(1)}_x(t)\bigr)^{8n} \mathbf{E}^{1/4} \bigl(X^{(1)}_x(s)\bigr)^{-8n} ds.$$

Now we can use estimates (9.21), (9.23), which leads to (9.29). The estimate (9.29) guarantees (see Proposition 1.1 Ch. I) the uniform integrability of the family of random variables $\{Z^2_\Delta(t)\}_{\Delta>0}$ for every $t \in [0,T]$.

By (9.9), $X_{x+\Delta}(t) \to X_x(t)$ as $\Delta \to 0$ a.s. Therefore,

$$a'_\Delta(t) \to a''_{zz}(t, X_x(t)), \qquad b'_\Delta(t) \to b''_{zz}(t, X_x(t)) \qquad \text{as } \Delta \to 0 \qquad \text{a.s.}$$

In turn, $Y_\Delta(t)$ converges as $\Delta \to 0$ in probability and in mean square to the derivative $X^{(1)}_x(t)$, and $X^{(1)}_{x+\Delta}(t) \to X^{(1)}_x(t)$. Consequently, in (9.28) we can pass to the limit as $\Delta \to 0$. Then we see that the processes $Z_\Delta(t)$ converge as $\Delta \to 0$ in probability and in mean square to the limit $Z_0(t)$, which is called the second-order derivative $X^{(2)}_x(t) = \dfrac{\partial^2}{\partial x^2} X_x(t)$. The limit process $X^{(2)}_x(t)$ satisfies (9.25), and hence it satisfies (9.24). By (9.25) and (9.29), $X^{(2)}_x(t)$ is stochastically continuous in $(t,x) \in [0,T] \times \mathbf{R}$, and it is continuous in the mean square, since the processes $X_x(t)$ and $X^{(1)}_x(t)$ are stochastically continuous with respect to x uniformly in $t \in [0,T]$. □

Remark 9.6. Under the assumptions of Theorem 9.4,

$$\sup_{0 \le t \le T} \mathbf{E}\big(X^{(2)}_x(t)\big)^{2n} < \infty \tag{9.30}$$

for any positive integer n.

Indeed, (9.30) is a consequence of (9.29) and Fatou's lemma.

§ 10. Girsanov's transformation

To clarify the subject of this section we start with a simple example.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\zeta = \zeta(\omega)$ be a Gaussian random variable with mean zero and variance 1. The characteristic function of this variable is given by the formula

$$\mathbf{E}e^{iz\zeta} = \int_\Omega e^{iz\zeta(\omega)} \mathbf{P}(d\omega) = \int_{-\infty}^{\infty} e^{izx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-z^2/2}, \qquad z \in \mathbf{R}. \tag{10.1}$$

This equality holds also for a complex z.

Define a new probability measure by setting

$$\widetilde{\mathbf{P}}(A) := \int_A \exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big) \mathbf{P}(d\omega)$$

for sets $A \in \mathcal{F}$. This relation has a brief expression in terms of the Radon–Nikodým derivative

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} := \frac{\widetilde{\mathbf{P}}(d\omega)}{\mathbf{P}(d\omega)} = \exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big).$$

Note that $\widetilde{\mathbf{P}}(\Omega) = 1$, since by (10.1), for $z = i\mu$ we have $\widetilde{\mathbf{P}}(\Omega) = e^{-\mu^2/2}\mathbf{E}e^{-\mu\zeta} = 1$.

Proposition 10.1. *The random variable* $\tilde{\zeta} = \zeta + \mu$ *with respect to the measure* $\widetilde{\mathbf{P}}$ *is the Gaussian random variable with mean zero and variance* 1.

Proof. Denote by $\widetilde{\mathbf{E}}$ the expectation with respect to the measure $\widetilde{\mathbf{P}}$. Then for an arbitrary bounded random variable η,

$$\widetilde{\mathbf{E}}\eta := \int_\Omega \eta(\omega)\,\widetilde{\mathbf{P}}(d\omega) = \int_\Omega \eta(\omega)\exp\Big(-\mu\zeta(\omega) - \frac{\mu^2}{2}\Big)\mathbf{P}(d\omega) = \mathbf{E}\Big\{\eta\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}.$$

Using this, we have

$$\begin{aligned}\widetilde{\mathbf{E}}e^{iz\tilde{\zeta}} &= \widetilde{\mathbf{E}}e^{iz(\zeta+\mu)} = \mathbf{E}\Big\{e^{iz(\zeta+\mu)}\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}\\ &= e^{-\mu^2/2}\int_{-\infty}^{\infty} e^{iz(x+\mu)}e^{-\mu x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx\\ &= \int_{-\infty}^{\infty} e^{iz(x+\mu)}\frac{1}{\sqrt{2\pi}}e^{-(x+\mu)^2/2}\,dx = \int_{-\infty}^{\infty} e^{izy}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy = e^{-z^2/2}.\end{aligned}$$

This proves the statement. □

The main point of Proposition 10.1 can be formulated as follows: a special choice of the probability measure can compensate the shift of a Gaussian random variable.

The distribution of a random variable is uniquely determined by the characteristic function or by the family of expectations of a bounded measurable functions of this variable. The statement that the random variable $\tilde{\zeta} = \zeta + \mu$ with respect to the measure $\widetilde{\mathbf{P}}$ is again distributed as ζ can be expressed as follows: for an arbitrary bounded measurable function f we have $\widetilde{\mathbf{E}}f(\tilde{\zeta}) = \mathbf{E}f(\zeta)$, or, in view of the definitions of $\widetilde{\mathbf{E}}$ and $\tilde{\zeta}$,

$$\mathbf{E}\Big\{f(\zeta+\mu)\exp\Big(-\mu\zeta - \frac{\mu^2}{2}\Big)\Big\} = \mathbf{E}f(\zeta). \tag{10.2}$$

As we saw, if instead of the abstract expectation with respect to the probability measure $\mathbf{P}$ we write (10.2) in terms of integrals with respect to the Gaussian distribution function $dG(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$, then formula (10.2) turns to the integration by substitution formula. We can rewrite (10.2) in another way. We apply (10.2) to the function $f(x)e^{\mu x - \mu^2/2}$ instead of $f(x)$ and get

$$\mathbf{E}f(\zeta+\mu) = \mathbf{E}\Big\{f(\zeta)\exp\Big(\mu\zeta - \frac{\mu^2}{2}\Big)\Big\}.$$

For $f(x) = \mathbb{I}_A(x)$, $A \in \mathcal{F}$, this formula has the brief equivalent

$$\frac{d\mathbf{P}_{\zeta+\mu}}{d\mathbf{P}_\zeta} = \exp\Big(\mu\zeta - \frac{\mu^2}{2}\Big),$$

where $\mathbf{P}_{\zeta+\mu}$ is the measure corresponding to the variable $\zeta + \mu$ and $\mathbf{P}_\zeta$ is the measure corresponding to the variable ζ.

Results analogous to Proposition 10.1 hold also for some random variables taking values in functional spaces, i.e., for stochastic processes. The next result concerns the Brownian motion.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space. Let $W(t)$, $t \in [0,T]$, be a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$. Suppose that for all $v > t$ the increments $W(v) - W(t)$ are independent of the σ-algebra $\mathcal{F}_t$.

For an arbitrary $b \in \mathcal{L}_2[0,T]$, consider the stochastic exponent

$$\rho(t) := \exp\Big(- \int_0^t b(s)\, dW(s) - \frac{1}{2} \int_0^t b^2(s)\, ds \Big), \qquad t \in [0,T].$$

Here compared with the exponent from § 6 we take the process $-b(s)$ instead of $b(s)$. The stochastic differential of ρ is

$$d\rho(t) = -\rho(t) b(t)\, dW(t). \tag{10.3}$$

Therefore,

$$\rho(t) = 1 - \int_0^t \rho(v) b(v)\, dW(v).$$

Suppose that for some $\delta > 0$

$$\mathbf{E} \exp\Big((1+\delta) \int_0^T b^2(s)\, ds \Big) < \infty,$$

or

$$\sup_{0 \le s \le T} \mathbf{E} e^{\delta b^2(s)} < \infty.$$

Then the stochastic exponent $\rho(t)$, $t \in [0,T]$, is (see Proposition 6.1) a nonnegative martingale with the mean $\mathbf{E}\rho(t) = 1$ for every $t \in [0,T]$.

Define the probability measure $\widetilde{\mathbf{P}}$ by setting

$$\widetilde{\mathbf{P}}(A) := \int_A \rho(T, \omega) \mathbf{P}(d\omega)$$

for $A \in \mathcal{F}$. Note that $\widetilde{\mathbf{P}}(\Omega) = \mathbf{E}\rho(T) = 1$.

Denote by $\widetilde{\mathbf{E}}$ the expectation with respect to the measure $\widetilde{\mathbf{P}}$. Then

$$\widetilde{\mathbf{E}}\eta := \int_\Omega \eta(\omega)\, \widetilde{\mathbf{P}}(d\omega) = \int_\Omega \eta(\omega) \rho(T,\omega)\, \mathbf{P}(d\omega) = \mathbf{E}\{\eta \rho(T)\}.$$

Proposition 10.2. *For any bounded $\mathcal{F}_t$-measurable random variable η the following equalities hold*:

$$\widetilde{\mathbf{E}}\eta = \mathbf{E}\{\eta\rho(t)\}, \tag{10.4}$$

and for $s < t$,

$$\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\} = \frac{1}{\rho(s)}\mathbf{E}\{\eta\rho(t)|\mathcal{F}_s\}. \tag{10.5}$$

Proof. Indeed, using the martingale property of $\rho(t)$, $t \in [0, T]$, we have

$$\widetilde{\mathbf{E}}\eta = \mathbf{E}\{\mathbf{E}\{\eta\rho(T)|\mathcal{F}_t\}\} = \mathbf{E}\{\eta\mathbf{E}\{\rho(T)|\mathcal{F}_t\}\} = \mathbf{E}\{\eta\rho(t)\}.$$

To prove (10.5) we consider an arbitrary bounded $\mathcal{F}_s$-measurable random variable ξ. We compute the expectation $\widetilde{\mathbf{E}}\{\xi\eta\}$ in two different ways. Using the properties of the conditional expectation and (10.4), we have

$$\widetilde{\mathbf{E}}\{\xi\eta\} = \widetilde{\mathbf{E}}\{\widetilde{\mathbf{E}}\{\xi\eta|\mathcal{F}_s\}\} = \widetilde{\mathbf{E}}\{\xi\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\}\} = \mathbf{E}\{\xi\rho(s)\widetilde{\mathbf{E}}\{\eta|\mathcal{F}_s\}\}. \tag{10.6}$$

On the other hand, first applying (10.4) and then using the properties of the conditional expectation, we obtain

$$\widetilde{\mathbf{E}}\{\xi\eta\} = \mathbf{E}\{\xi\eta\rho(t)\} = \mathbf{E}\{\mathbf{E}\{\xi\eta\rho(t)|\mathcal{F}_s\}\} = \mathbf{E}\{\xi\mathbf{E}\{\eta\rho(t)|\mathcal{F}_s\}\}. \tag{10.7}$$

Since ξ is an arbitrary bounded $\mathcal{F}_s$-measurable random variable, the coincidence of the right-hand sides of (10.6) and (10.7) implies (10.5). □

The following result is due to I. V. Girsanov (1960) (for nonrandom b see Cameron and Martin (1945)).

Theorem 10.1. *The process $\widetilde{W}(t) = W(t) + \int_0^t b(s)\,ds$ is a Brownian motion with respect to the measure $\widetilde{\mathbf{P}}$.*

Proof. By the characterization property (10.9) Ch. I, to prove that the process $\widetilde{W}$ is a Brownian motion it is sufficient to verify that for any $s < t$ and $z \in \mathbf{R}$,

$$\widetilde{\mathbf{E}}\{\exp(iz(\widetilde{W}(t) - \widetilde{W}(s)))|\mathcal{F}_s\} = e^{-z^2(t-s)/2} \qquad \text{a.s.} \tag{10.8}$$

We prove (10.8). We first assume that $\sup_{0\le s\le T} |b(s)| \le M$ for some nonrandom constant M. For any fixed s and $t \ge s$, we set

$$\eta(t) := \exp\bigl(iz(\widetilde{W}(t) - \widetilde{W}(s))\bigr) = \exp\Bigl(iz(W(t) - W(s)) + iz\int_s^t b(u)\,du\Bigr).$$

Note that $\eta(s) = 1$. According to (10.5),

$$g(t) := \widetilde{\mathbf{E}}\{\eta(t)|\mathcal{F}_s\} = \frac{1}{\rho(s)}\mathbf{E}\{\eta(t)\rho(t)|\mathcal{F}_s\}.$$

We fix s and for $t > s$ apply Itô's formula (4.24) for

$$f(t,x,y) = e^x y, \qquad X(t) = iz(W(t) - W(s)) + iz\int_s^t b(u)\,du, \qquad Y(t) = \rho(t).$$

Then taking into account (10.3), we obtain

$$\begin{aligned} d(\eta(t)\rho(t)) &= \eta(t)\rho(t)\{iz\,dW(t) + izb(t)\,dt\} \\ &\quad - \eta(t)\rho(t)b(t)\,dW(t) - \frac{1}{2}z^2\eta(t)\rho(t)\,dt - iz\eta(t)\rho(t)b(t)\,dt. \end{aligned}$$

In the integral form this is written as follows: for every $t \ge s$,

$$\eta(t)\rho(t) = \rho(s) + iz\int_s^t \eta(u)\rho(u)\,dW(u) - \int_s^t \eta(u)\rho(u)b(u)\,dW(u) - \frac{z^2}{2}\int_s^t \eta(u)\rho(u)\,du.$$

Since $|\eta(t)| \le 1$ and, by (6.13), the estimate $\mathbf{E}(\rho(u)b(u))^2 \le M^2 e^{2M^2 u}$ holds, we can use (2.3) and get

$$\mathbf{E}\{\eta(t)\rho(t)|\mathcal{F}_s\} = \rho(s) - \frac{z^2}{2}\int_s^t \mathbf{E}\{\eta(u)\rho(u)|\mathcal{F}_s\}\,du \qquad \text{a.s.}$$

Using the definition of the function g, this can be written in the form

$$g(t) = 1 - \frac{z^2}{2}\int_s^t g(u)\,du, \qquad t \ge s.$$

The solution of this differential equation is $g(t) = e^{-z^2(t-s)/2}$, which is the required result (10.8).

We will prove (10.8) for an arbitrary process $b \in \mathcal{L}_2[0,T]$, satisfying the assumptions stated above. There exists a sequence of bounded processes $b_n \in \mathcal{L}_2[0,T]$, such that

$$\lim_{n\to\infty}\int_0^T (b(s) - b_n(s))^2\,ds = 0 \qquad \text{a.s.}$$

Then the processes $\widetilde{W}_n(t) = W(t) + \int_0^t b_n(s)\,ds$ converge to the process $\widetilde{W}$. We denote $\rho_n(t)$ the stochastic exponent corresponding to the process b_n. Then, in view of (3.6), $\rho_n(t) \to \rho(t)$ in probability for every t. Since $\mathbf{E}\rho_n(t) = \mathbf{E}\rho(t) = 1$, we have

$$\mathbf{E}|\rho_n(t) - \rho(t)| = \mathbf{E}\big(|\rho(t) - \rho_n(t)| + \rho(t) - \rho_n(t)\big) = 2\mathbf{E}(\rho(t) - \rho_n(t))^+.$$

Since $(\rho(t)-\rho_n(t))^+ \le \rho(t)$, by the Lebesgue dominated convergence theorem, $\rho_n(t)\to\rho(t)$ in mean. For the processes b_n the equality (10.8) has been already proved, i.e., in view of (10.5),

$$\mathbf{E}\{\exp(iz(\widetilde{W}_n(t)-\widetilde{W}_n(s)))\rho_n(t)|\mathcal{F}_s\} = \rho_n(s)e^{-z^2/2(t-s)}.$$

By property 7′) of conditional expectations (see § 2 Ch. I), we can pass to the limit in this equality and obtain (10.8) for the process b. □

Remark 10.1. For a nonrandom function b Theorem 10.1 was first proved by Cameron and Martin (1945).

Girsanov's theorem can be presented in another form. Let $C([0,T])$ be the space of continuous functions on $[0,T]$. When equipped with the uniform norm $C([0,T])$ becomes a Banach space. Instead of an abstract probability measure $\mathbf{P}$ we can consider the Wiener measure $\mathbf{P}_W$, which for cylinder sets is determined by (10.1) Ch. I. Although the Wiener measure is concentrated on the sets of nowhere differentiable paths, it can be extended to the σ-algebra $\mathcal{B}(C([0,T]))$ of Borel sets of the space $C([0,T])$. This measure can be characterized by the expectations of a bounded measurable functionals of Brownian motion.

Girsanov's theorem can be recast as.

Theorem 10.2. *Let $\wp(X(s), 0\le s\le t)$ be a bounded measurable functional on $C([0,t])$. Then*

$$\mathbf{E}\Big\{\wp\Big(W(s)+\int_0^s b(u)\,du, 0\le s\le t\Big)\rho(t)\Big\} = \mathbf{E}\wp(W(s), 0\le s\le t). \tag{10.9}$$

Proof. Indeed, the statement of Theorem 10.1 is equivalent to the following: for any bounded measurable functional $\wp(X(s), 0\le s\le t)$,

$$\widetilde{\mathbf{E}}\wp(\widetilde{W}(s), 0\le s\le t) = \mathbf{E}\wp(W(s), 0\le s\le t).$$

In view of (10.4), the left-hand side of this equality coincides with the left-hand side of (10.9). □

Let us consider a very important application of Girsanov's transformation.

Let $X(t)$ and $Y(t)$, $t\in[0,T]$, be solutions of the stochastic differential equations

$$dX(t) = \sigma(t, X(t))\,dW(t) + \mu_1(t, X(t))\,dt, \tag{10.10}$$

$$dY(t) = \sigma(t, Y(t))\,dW(t) + \mu_2(t, Y(t))\,dt, \tag{10.11}$$

with the same nonrandom initial values. Suppose that the coefficients σ, μ_1, μ_2 satisfy the conditions of Theorem 7.1. Assume also that $\sigma(t,x)\ne 0$ for all $(t,x)\in[0,T]\times\mathbf{R}$.

Let $\mathcal{G}_0^t = \sigma(W(s), 0\le s\le t)$ be the σ-algebra of events generated by the Brownian motion up to the time t. It was proved in § 7 that the processes X and Y are adapted to the natural filtration $\mathcal{G}_0^t$, i.e., for every t the variables $X(t)$ and $Y(t)$ are measurable with respect to $\mathcal{G}_0^t$.

Theorem 10.3. *Let* $\alpha(t,x) := \dfrac{\mu_1(t,x)-\mu_2(t,x)}{\sigma(t,x)}$, *be a continuous function of the variables* $(t,x) \in [0,T] \times \mathbf{R}$. *Denote*

$$\rho(t) := \exp\Biggl(-\int_0^t \alpha(s,X(s))\,dW(s) - \frac{1}{2}\int_0^t \alpha^2(s,X(s))\,ds\Biggr)$$

and suppose that for some $\delta > 0$

$$\mathbf{E}\exp\Biggl((1+\delta)\int_0^T \alpha^2(t,X(t))\,dt\Biggr) < \infty \qquad \text{or} \qquad \sup_{0\le t\le T} \mathbf{E}e^{\delta\alpha^2(t,X(t))} < \infty.$$

Then for any bounded measurable functional $\wp(Z(s), 0 \le s \le t)$ *on* $C([0,t])$,

$$\mathbf{E}\wp(Y(s), 0 \le s \le t) = \mathbf{E}\bigl\{\wp\bigl(X(s), 0 \le s \le t\bigr)\rho(t)\bigr\} \tag{10.12}$$

for every $t \in [0,T]$.

Remark 10.2. Let $\mathbf{P}_X$ and $\mathbf{P}_Y$ be the measures associated with the processes $X(t)$ and $Y(t)$, $t \in [0,T]$, respectively. Then from (10.12) for the functional

$$\wp(Z(s), 0 \le s \le t) = \mathbb{1}_A(Z(s), 0 \le s \le t), \qquad A \in \mathcal{B}(C[0,t]),$$

it follows that the measure $\mathbf{P}_Y$ is absolutely continuous with respect to $\mathbf{P}_X$ when restricted to $\mathcal{G}_0^t$ and there exists the Radon–Nikodým derivative

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_X}\bigg|_{\mathcal{G}_0^t} = \rho(t) \qquad \text{a.s.} \tag{10.13}$$

Proof of Theorem 10.3. Since α is a continuous function, the process $\alpha(t,X(t))$, $t \in [0,T]$, is progressively measurable with respect to the filtration $\{\mathcal{G}_0^t\}$.

By Theorem 10.1, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \alpha(s,X(s))\,ds$$

is a Brownian motion with respect to the measure $\widetilde{\mathbf{P}}$. Since

$$dX(t) = \sigma(t,X(t))\,d\widetilde{W}(t) + \mu_2(t,X(t))\,dt$$

and this stochastic differential equation coincides with (10.11), the finite-dimensional distributions of the process X with respect to the measure $\widetilde{\mathbf{P}}$ coincide with those of the process Y with respect to the measure $\mathbf{P}$. This implies that

$$\mathbf{E}\wp(Y(s), 0 \le s \le t) = \widetilde{\mathbf{E}}\wp(X(s), 0 \le s \le t).$$

In view of (10.4), the right-hand side of this equality coincides with the right-hand side of (10.12). □

The Radon–Nikodým derivative (10.13) can be rewritten as a functional of X. Indeed, from (10.10) it follows that

$$dW(t) = \frac{1}{\sigma(t,X(t))}(dX(t) - \mu_1(t,X(t))\,dt).$$

Then

$$\int_0^t \alpha(s,X(s))\,dW(s) = \int_0^t \frac{\alpha(s,X(s))}{\sigma(t,X(t))}\,dX(s) - \int_0^t \frac{\alpha(s,X(s))\mu_1(s,X(s))}{\sigma(t,X(t))}\,ds.$$

As a result, we have

$$\rho(t) = \exp\biggl(\int_0^t \frac{\mu_2(s,X(s)) - \mu_1(s,X(s))}{\sigma^2(s,X(s))}\,dX(s) - \frac{1}{2}\int_0^t \frac{\mu_2^2(s,X(s)) - \mu_1^2(s,X(s))}{\sigma^2(s,X(s))}\,ds\biggr).$$

Consider the particular case when $\sigma(t,x) \equiv 1$, $\mu_1(t,x) \equiv 0$, $\mu_2(t,x) \equiv \mu(x)$. Suppose that for some $\delta > 0$

$$\mathbf{E}\exp\biggl((1+\delta)\int_0^T \mu^2(W(t))\,dt\biggr) < \infty \qquad \text{or} \qquad \sup_{0\le t\le T} \mathbf{E}e^{\delta\mu^2(W(t))} < \infty.$$

Then for the process $Y(t) = W(t) + \int_0^t \mu(Y(s))\,ds$, $t \in [0,T]$, we have

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_W}\bigg|_{\mathcal{G}_0^t} = \exp\biggl(\int_0^t \mu(W(s))\,dW(s) - \frac{1}{2}\int_0^t \mu^2(W(s))\,ds\biggr)$$

$$= \exp\biggl(\int_{W(0)}^{W(t)} \mu(y)dy - \frac{1}{2}\int_0^t \mu^2(W(s))\,ds - \frac{1}{2}\int_0^t \mu'(W(s))ds\biggr) \qquad \text{a.s.} \qquad (10.14)$$

Here the second equality follows from the Itô formula under the assumption that the function μ is differentiable.

In particular, for the Brownian motion with linear drift $\mu(x) \equiv \mu$, i.e., for the process $W^{(\mu)}(t) = \mu t + W(t)$ with $W(0) = x$, formula (10.14) has the form

$$\mathbf{E}\wp\bigl(W^{(\mu)}(s), 0 \le s \le t\bigr) = e^{-\mu x - \mu^2 t/2}\mathbf{E}\bigl\{e^{\mu W(t)}\wp(W(s), 0 \le s \le t)\bigr\}. \qquad (10.15)$$

Exercises.

10.1. Let $X(t)$ be a solution of the stochastic differential equation

$$dX(t) = a(X(t))\,dt + dW(t), \qquad X(0) = x_0.$$

Use the Girsanov theorem to prove that for all K, $x_0 \in \mathbf{R}$ and $t > 0$

$$\mathbf{P}(X(t) \geq K) > 0.$$

10.2. Let $Y(t) = W(t) + \mu t + \eta t^2$ be the Brownian motion with the quadratic drift, $W(0) = x$. Check that

$$\frac{d\mathbf{P}_Y}{d\mathbf{P}_W}\bigg|_{\mathcal{G}_0^t} = \exp\biggl(-\mu x - \frac{\mu^2 t}{2} - \mu\eta t^2 - \frac{2\eta^2 t^3}{3} + (\mu + 2\eta t)W(t) - 2\eta\int_0^t W(s)\,ds\biggr).$$

10.3. Let $Y(t)$ be a solution of the stochastic differential equation

$$dY(t) = -\theta Y(t)\,dt + dW(t), \qquad Y(0) = x, \quad \theta \in \mathbf{R}.$$

Compute $\dfrac{d\mathbf{P}_Y}{d\mathbf{P}_W}\bigg|_{\mathcal{G}_0^t}$.

§ 11. Probabilistic solution of the Cauchy problem

Let $X(t)$, $t \geq 0$, be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))\,dW(t) + \mu(X(t))\,dt, \qquad X(0) = x. \tag{11.1}$$

Suppose that for every $N > 0$ there exists a constant K_N such that

$$|\sigma(x) - \sigma(y)| + |\mu(x) - \mu(y)| \leq K_N|x - y| \tag{11.2}$$

for all $x, y \in [-N, N]$. Introduce, in addition, the following restriction on the growth of the coefficients σ and μ: there exists a constant K such that

$$|\sigma(x)| + |\mu(x)| \leq K(1 + |x|) \tag{11.3}$$

for all $x \in \mathbf{R}$. Then, by Theorem 7.3, equation (11.1) has a unique continuous solution defined for all $t \geq 0$.

Denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and the expectation with respect to the process X with the starting point $X(0) = x$.

Let $H_{a,b} := \min\{s : X(s) \notin (a, b)\}$ be the *first exit time* of the process X from the interval (a, b).

Theorem 11.1. *Let $\Phi(x)$, $f(x)$, $x \in [a,b]$, be continuous functions, and let f be nonnegative. Suppose that the coefficients σ, μ satisfy condition* (11.2) *for $N = \max\{|a|,|b|\}$, and $\sigma(x) > 0$ for $x \in [a,b]$.*

Let $u(t,x)$, $(t,x) \in [0,\infty) \times [a,b]$, be a solution of the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x) + \mu(x)\frac{\partial}{\partial x}u(t,x) - f(x)u(t,x), \tag{11.4}$$

$$u(0,x) = \Phi(x), \tag{11.5}$$

$$u(t,a) = \Phi(a), \qquad u(t,b) = \Phi(b). \tag{11.6}$$

Then

$$u(t,x) = \mathbf{E}_x\Big\{\Phi(X(t\wedge H_{a,b}))\exp\Big(-\int\limits_0^{t\wedge H_{a,b}} f(X(s))\,ds\Big)\Big\}. \tag{11.7}$$

Proof. We extend the functions σ and μ outside the interval $[a,b]$ such that they satisfy (11.2), (11.3), and the condition $\sigma(x) > 0$ for $x \in \mathbf{R}$. In this case, by Theorem 7.2, the process X is not changed in the interval $[0, H_{a,b}]$. We also extend f to be a nonnegative continuous function outside $[a,b]$. One can extend the solution u of the problem (11.4)–(11.6) outside $[a,b]$ such that it will be continuously differentiable in $(t,x) \in (0,\infty) \times \mathbf{R}$. Moreover, there exist the continuous second derivative in x except in the points $a - k(b-a)$ and $b + k(b-a)$, $k \in \mathbb{N}$, and this derivative has the left and right limits at its points of discontinuity. It is not stated that $u(t,x)$ satisfies the equation (11.4) for $x \notin [a,b]$. For example, we can set $u(t,x) := -u(t,2a-x)$ for $x \in [2a-b,a]$, $u(t,x) := -u(t,2b-x)$ for $x \in [b,2b-a]$, $u(t,x) := u(t,x+2b-2a)$ for $x \in [3a-2b, 2a-b]$ and so on.

For a fixed t set

$$\eta(s) := u(t-s, X(s))\exp\Big(-\int\limits_0^s f(X(v))\,dv\Big), \qquad s \in [0,t].$$

Applying Itô's formula (4.22) for $d = 1$ in the integral form together with the Remark 4.2, we have for every $0 \le q \le t$ that

$$\eta(q) - \eta(0) = \int\limits_0^q \exp\Big(-\int\limits_0^s f(X(v))\,dv\Big)\Big[\Big(\frac{\partial}{\partial s}u(t-s,X(s))$$
$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s)) + \mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))$$
$$- f(X(s))u(t-s,X(s))\Big)ds + \sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\Big].$$

Replacing q by the stopping time $t\wedge H_{a,b}$, we get

$$\eta(t\wedge H_{a,b}) - \eta(0) = \int\limits_0^{t\wedge H_{a,b}} \exp\Big(-\int\limits_0^s f(X(v))\,dv\Big)\Big[\Big(-\frac{\partial}{\partial t}u(t-s,X(s))$$

$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s))+\mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))$$
$$-f(X(s))u(t-s,X(s))\Big)ds+\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\Big].$$

Using the fact that $u(t,x)$ satisfies equation (11.4) for $x\in[a,b]$, we have

$$\eta(t\wedge H_{a,b})-\eta(0)=\int_0^{t\wedge H_{a,b}}\exp\Big(-\int_0^s f(X(v))\,dv\Big)\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)$$

$$=\int_0^t \mathbb{1}_{[0,H_{a,b})}(s)\exp\Big(-\int_0^s f(X(v))\,dv\Big)\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s). \quad (11.8)$$

It is important that $H_{a,b}$ is a stopping time with respect to the filtration $\mathcal{G}_0^t=\sigma(W(s),0\le s\le t)$. This ensures that the stochastic integral is well defined (see (3.8)). Note also that all integrands are bounded, because the process X does not leave the interval (a,b) up to the time $H_{a,b}$.

The expectation of the stochastic integral equals zero, and therefore

$$\mathbf{E}_x\eta(t\wedge H_{a,b})=\mathbf{E}_x\eta(0).$$

It is clear that

$$\mathbf{E}_x\eta(0)=\mathbf{E}\{u(t,X(0))|X(0)=x\}=u(t,x).$$

By the boundary conditions (11.5) and (11.6), we have

$$u(t-(t\wedge H_{a,b}),X(t\wedge H_{a,b}))=\Phi(X(t\wedge H_{a,b})),$$

and so

$$\mathbf{E}_x\eta(t\wedge H_{a,b})=\mathbf{E}_x\Big\{\Phi(X(t\wedge H_{a,b}))\exp\Big(-\int_0^{t\wedge H_{a,b}}f(X(v))\,dv\Big)\Big\}.$$

Thus (7.11) holds. □

Remark 11.1. It is very important that in Theorem 11.1 and in the following results of this section we assume that the solution of the corresponding differential problem exists.

The following generalization of Theorem 11.1 gives the probabilistic solution for the nonhomogeneous Cauchy problem.

Theorem 11.2. *Let $\Phi(x)$, $f(x)$ and $g(x)$, $x\in[a,b]$, be continuous functions, and f be nonnegative. Suppose that the coefficients σ, μ satisfy condition* (11.2) *with $N=\max\{|a|,|b|\}$, and $\sigma(x)>0$ for $x\in[a,b]$.*

Let $u(t,x)$, $(t,x)\in[0,\infty)\times[a,b]$, be a solution of the nonhomogeneous Cauchy problem

$$\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x)+\mu(x)\frac{\partial}{\partial x}u(t,x)-f(x)u(t,x)+g(x), \tag{11.9}$$

$$u(0,x)=\varPhi(x), \tag{11.10}$$

$$u(t,a)=\varPhi(a), \qquad u(t,b)=\varPhi(b). \tag{11.11}$$

Then

$$u(t,x)=\mathbf{E}_x\bigg\{\varPhi(X(t\wedge H_{a,b}))\exp\bigg(-\int_0^{t\wedge H_{a,b}} f(X(s))\,ds\bigg)$$
$$+\int_0^{t\wedge H_{a,b}} g(X(s))\exp\bigg(-\int_0^s f(X(v))\,dv\bigg)ds\bigg\}. \tag{11.12}$$

Proof. As in the proof of Theorem 11.1, we consider the extension of the solution of the problem (11.9)–(11.11) outside the interval $[a,b]$, assuming that the functions f and g are continuously extended outside $[a,b]$ so that g is bounded and f is nonnegative. For a fixed t, set

$$\eta(s):=u(t-s,X(s))\exp\bigg(-\int_0^s f(X(v))\,dv\bigg)+\int_0^s g(X(v))\exp\bigg(-\int_0^v f(X(q))\,dq\bigg)dv.$$

Applying Itô's formula and then substituting in it the stopping time $t\wedge H_{a,b}$, we get

$$\eta(t\wedge H_{a,b})-\eta(0)=\int_0^{t\wedge H_{a,b}}\exp\bigg(-\int_0^s f(X(v))\,dv\bigg)\bigg[\bigg(-\frac{\partial}{\partial t}u(t-s,X(s))$$

$$+\frac{1}{2}\sigma^2(X(s))\frac{\partial^2}{\partial x^2}u(t-s,X(s))+\mu(X(s))\frac{\partial}{\partial x}u(t-s,X(s))-f(X(s))u(t-s,X(s))$$
$$+g(X(s))\bigg)ds+\sigma(X(s))\frac{\partial}{\partial x}u(t-s,X(s))\,dW(s)\bigg].$$

We use the fact that for $x\in[a,b]$ the function $u(t,x)$ satisfies equation (11.9). As a result, we have (11.8). Now the proof is completed analogous to the proof of Theorem 11.1. □

Taking the Laplace transform with respect to t, we can reduce the problem (11.9)–(11.11) to a problem for an ordinary differential equation.

For any $\lambda>0$, set

$$U(x):=\lambda\int_0^\infty e^{-\lambda t}u(t,x)\,dt. \tag{11.13}$$

Then for every x the function $u(t,x)$, $t \geq 0$, is uniquely determined by the function U, $\lambda > 0$, as the inverse Laplace transform.

Applying the integration by parts formula, taking into account the boundary condition (11.10) and the fact that the function $u(t,x)$, by virtue of the representation (11.12), obeys for some K the estimate $|u(t,x)| \leq K(1+t)$, we get

$$\lambda \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} u(t,x)\,dt = -\lambda \Phi(x) + \lambda^2 \int_0^\infty e^{-\lambda t} u(t,x)\,dt = -\lambda \Phi(x) + \lambda U(x).$$

In addition, we have

$$U'(x) = \lambda \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial x} u(t,x)\,dt, \qquad U''(x) = \lambda \int_0^\infty e^{-\lambda t} \frac{\partial^2}{\partial x^2} u(t,x)\,dt.$$

Now integrating both sides of (11.9) with the weight function $\lambda\, e^{-\lambda t}$, we get

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x) - g(x), \quad x \in (a,b). \tag{11.14}$$

The boundary conditions (11.11) are transformed to the conditions

$$U(a) = \Phi(a), \qquad U(b) = \Phi(b). \tag{11.15}$$

One can give a natural probabilistic interpretation to the Laplace transform (formula (11.13)). Namely, let τ be an exponentially distributed random time independent of the Brownian motion W and, consequently, of the process X. Let the density of τ have the form $\lambda e^{-\lambda t} \mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbf{R}$, $\lambda > 0$. Then applying Fubini's theorem, we get

$$U(x) = \mathbf{E}u(\tau,x) = \mathbf{E}_x\biggl\{\Phi(X(\tau \wedge H_{a,b})) \exp\biggl(-\int_0^{\tau\wedge H_{a,b}} f(X(s))\,ds\biggr)$$
$$+ \int_0^{\tau\wedge H_{a,b}} g(X(s)) \exp\biggl(-\int_0^s f(X(v))\,dv\biggr) ds\biggr\}. \tag{11.16}$$

Therefore the function U is equal to the expectation of the same random process as in formula (11.12), with the random time τ instead of a fixed time t.

As a result, we can formulate the following analog of Theorem 11.2.

Theorem 11.3. *Let $\Phi(x)$, $f(x)$ and $g(x)$, $x \in [a,b]$, be continuous functions and let f be nonnegative. Suppose that the coefficients σ and μ satisfy condition* (11.2) *with $N = \max\{|a|,|b|\}$ and $\sigma(x) > 0$ for $x \in [a,b]$.*

Then the function $U(x)$, $x \in [a,b]$, defined by (11.16) *is the unique continuous solution of the problem* (11.14), (11.15).

§ 12. Ordinary differential equations, probabilistic approach

The proof of Theorem 11.2 is based on the result that the nonhomogeneous Cauchy problem (11.9)–(11.11) has a solution. The proof of this result is very complicated and requires additional conditions on the functions σ, μ, f and g. As it was proved, by the Laplace transform with respect to t the problem (11.9)–(11.11) is reduced to the ordinary differential equation (11.14) with the boundary conditions (11.15). The ordinary differential problem has a unique solution, which is not difficult to prove. The solution of (11.14), (11.15) has the probabilistic expression (11.16). Our aim, in particular, is to give a direct probabilistic proof for this expression, which is not based on the solution of the Cauchy problem.

We consider first the preliminary results concerning the solutions of ordinary second-order differential equations.

Proposition 12.1. *Let $g(x)$, $x \in (l,r)$, be a nonnegative continuous function that does not vanish identically, $l \geq -\infty$, $r \leq \infty$. Then the homogeneous equation*

$$\phi''(x) - g(x)\phi(x) = 0, \qquad x \in (l,r), \tag{12.1}$$

has two nonnegative convex linearly independent solutions ψ and φ such that $\psi(x)$, $x \in (l,r)$, is increasing, and $\varphi(x)$, $x \in (l,r)$, is decreasing.

Proof. Without loss of generality, we assume that $0 \in (l,r)$ and $g(0) > 0$. Consider for $x \in [0,r)$ the solution ψ_+ of equation (12.1) with the initial values $\psi_+(0) = 1$, $\psi_+'(0) = 1$. This solution is a convex function, therefore $\psi_+(x) \geq 1+x$. Another linearly independent solution, as is easily seen, has the form

$$\varphi(x) = \psi_+(x) \int_x^r \frac{dv}{\psi_+^2(v)} \leq \psi_+(x) \int_x^r \frac{dv}{(1+v)^2}, \qquad x \in [0,r).$$

Since $\psi_+'(x)$ is nondecreasing, the following estimates hold for $x \in [0,r)$:

$$\varphi'(x) = \psi_+'(x) \int_x^r \frac{dv}{\psi_+^2(v)} - \frac{1}{\psi_+(x)} < \int_x^r \frac{\psi_+'(v)}{\psi_+^2(v)}\,dv - \frac{1}{\psi_+(x)} = -\frac{1}{\psi_+(r)} \leq 0.$$

It follows that $\varphi(x)$, $x \in [0,r)$, is a nonnegative, convex, nonincreasing function, and $\varphi'(0) < 0$. We continue the solution φ to the interval $(l,0]$ so that it satisfies (12.1). Since the solution is convex, $\varphi(x) \geq \varphi(0) + \varphi'(0)x$ for $x \in (l,0]$. Another linearly independent solution in this interval is given by

$$\psi(x) = \varphi(x) \int_l^x \frac{dv}{\varphi^2(v)} \leq \varphi(x) \int_l^x \frac{dv}{(\varphi(0)+\varphi'(0)v)^2}.$$

Arguing similarly, we find that this solution obeys the estimates

$$\psi'(x) = \frac{1}{\varphi(x)} + \varphi'(x) \int_l^x \frac{dv}{\varphi^2(v)} > \frac{1}{\varphi(x)} + \int_l^x \frac{\varphi'(v)}{\varphi^2(v)}\,dv = \frac{1}{\varphi(l)} \geq 0$$

and $\psi'(0) > 0$. We continue the solution ψ to the interval $[0, r)$ so that it satisfies equation (12.1). Since the solution is convex, it is strictly increasing on the interval. Proposition 12.1 is proved. □

Consider the homogeneous equation

$$\phi''(x) + q(x)\phi'(x) - h(x)\phi(x) = 0, \qquad x \in \mathbf{R}. \tag{12.2}$$

We set

$$p(x) := \exp\Bigl(\int_0^x q(v)\,dv\Bigr), \qquad y(x) := \int_0^x \frac{dv}{p(v)}, \qquad x \in \mathbf{R}.$$

The function $y(x)$, $x \in \mathbf{R}$, is strictly increasing and $y(0) = 0$; hence, it has the strictly increasing inverse function $y^{(-1)}(y)$, $y \in (l, r)$, where

$$l := -\int_{-\infty}^0 \frac{dv}{p(v)} \ge -\infty, \qquad r := \int_0^\infty \frac{dv}{p(v)} \le \infty.$$

By a change of variables, equation (12.2) can be reduced to the form (12.1) with

$$g(y) = p^2(y^{(-1)}(y))h(y^{(-1)}(y)), \qquad y \in (l, r). \tag{12.3}$$

Indeed, we change x to $y(x)$. For this choice $\bar\phi(y) := \phi(y^{(-1)}(y))$, $y \in (l, r)$, i.e., we consider the function $\bar\phi$ such that $\phi(x) = \bar\phi(y(x))$, $x \in \mathbf{R}$. Since (12.2) can be written as

$$(p(x)\phi'(x))' - p(x)h(x)\phi(x) = 0, \qquad x \in \mathbf{R},$$

and $p(x)\phi'(x) = \bar\phi'(y(x))$, equation (12.2) is transformed to the following one

$$\frac{\bar\phi''(y(x))}{p(x)} - p(x)h(x)\phi(x) = 0, \qquad x \in \mathbf{R},$$

or, equivalently, to the equation

$$\bar\phi''(y(x)) - p^2(y^{(-1)}(y(x)))h(y^{(-1)}(y(x)))\bar\phi(y(x)) = 0, \qquad x \in \mathbf{R}.$$

This equation for the new variable y is in the form (12.1).

An important question is when does equation (12.2) have nonzero bounded solutions on the whole real line? Since equation (12.1) considered on the whole real line does not have nonzero bounded solutions, the same is true for (12.2) if $l = -\infty$ and $r = \infty$.

Thus, we have proved the following statement.

Proposition 12.2. *Let $q(x)$ and $h(x)$, $x \in \mathbf{R}$, be continuous functions, and let h be a nonnegative function that does not vanish identically. Then equation* (12.2) *has two nonnegative linearly independent solutions ψ and φ such that $\psi(x)$, $x \in \mathbf{R}$, is an increasing and $\varphi(x)$, $x \in \mathbf{R}$, is a decreasing solution.*

If $l = -\infty$ and $r = \infty$, then equation (12.2) *does not have nonzero bounded solutions.*

The functions $\psi(x)$ and $\varphi(x)$, $x \in \mathbf{R}$, are called *fundamental solutions* of (12.2). Their *Wronskian* $w(x) := \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$ has the form

$$w(x) = w(0) \exp\Big(- \int_0^x q(y)\, dy \Big)$$

and it is a positive function.

Indeed, from (12.2) it follows that the Wronskian satisfies the equation

$$w'(x) = -q(x)\, w(x), \qquad w(0) > 0, \qquad x \in \mathbf{R},$$

which yields the desired formula.

If either $l > -\infty$ or $r < \infty$, then the answer to our question depends on the functions $q(x)$ and $h(x)$, $x \in \mathbf{R}$. Thus, if $l = -\infty$, $r < \infty$ and $\lim_{y \uparrow r} g(y) < \infty$, where g is defined by (12.3), equation (12.2) has a bounded solution on the whole real line, because in this case g can be continued beyond the boundary r. Then the solution $\bar{\psi}(x)$ of equation (12.1) is bounded for $x \in (l, r)$ and $\psi(x) = \bar{\psi}(y(x))$, $x \in \mathbf{R}$, is a bounded solution of (12.2). If in this case $\lim_{y \uparrow r} g(y) = \infty$, the equation may or may not have a nonzero bounded solution. The left boundary l is treated analogously.

Let us give some examples. Consider for $\alpha \in \mathbf{R}$ the equation

$$\phi''(x) - \phi'(x) - e^{\alpha x} \phi(x) = 0, \qquad x \in \mathbf{R}.$$

By the change of variable $x = \ln(y+1)$, $y \in (-1, \infty)$, this equation is transformed to

$$\bar{\phi}''(y) - (y+1)^{\alpha-2} \bar{\phi}(y) = 0, \qquad y \in (-1, \infty).$$

If $\alpha = 2$, there exists the limit $\lim_{y \downarrow -1} g(y) = 1$. The fundamental solutions of the transformed equation have the form $\bar{\varphi}(y) = e^{-y}$ and $\bar{\psi}(y) = e^{y}$. The solutions of the original equation then are $\varphi(x) = \exp\big(1 - e^x\big)$ and $\psi(x) = \exp\big(e^x - 1\big)$, $x \in \mathbf{R}$. The solution φ is bounded.

If $\alpha = 0$, the fundamental solutions are $\bar{\varphi}(y) = (y+1)^{\sqrt{5/4}+1/2}$ and $\bar{\psi}(y) = (y+1)^{-\sqrt{5/4}+1/2}$, and the solutions of the original equation are $\varphi(x) = e^{x(\sqrt{5/4}+1/2)}$ and $\psi(x) = e^{-x(\sqrt{5/4}-1/2)}$, $x \in \mathbf{R}$. Hence, there are no nontrivial bounded solutions.

If $\alpha = 1$, the fundamental solutions of the transformed equation have the form $\bar{\varphi}(y) = \sqrt{y+1}\, K_1(2\sqrt{y+1})$ and $\bar{\psi}(y) = \sqrt{y+1}\, I_1(2\sqrt{y+1})$ (see Appendix 4, equation 6a for $p = 1/2$). The solutions of the original equation are

$\varphi(x) = e^{x/2}K_1(2e^{x/2})$ and $\psi(x) = e^{x/2}I_1(2e^{x/2})$, $x \in \mathbf{R}$. The solution φ is bounded, according to the asymptotic behavior of the modified Bessel function K_1 (see Appendix 2).

The conditions $l = -\infty$ and $r = \infty$, which guarantee unboundedness of nonzero solutions, are not always easy to check. In addition, they do not cover all cases. We prove the following useful result.

Proposition 12.3. *Let $q(x)$ and $h(x)$, $x \in \mathbf{R}$, be continuous functions and let the function h be nonnegative. Suppose that for some $C > 0$*

$$|q(x)| \le C(1 + |x|) \qquad \text{for all} \quad x \in \mathbf{R}, \tag{12.4}$$

$$\liminf_{y\to\infty} \frac{1}{y}\int_0^y h(x)\,dx > 0, \qquad \liminf_{y\to\infty} \frac{1}{y}\int_{-y}^0 h(x)\,dx > 0. \tag{12.5}$$

Then the homogeneous equation (12.2) *has no nonzero bounded solutions.*

Proof. By Proposition 12.2, the homogeneous equation (12.2) has two linearly independent nonnegative solutions ψ and φ such that $\psi(x)$, $x \in \mathbf{R}$, is increasing and $\varphi(x)$, $x \in \mathbf{R}$, is decreasing. Assume that $\psi_+ := \lim_{x\to\infty} \psi(x) < \infty$. The left condition in (12.5) implies the existence of $y_0 > 1$ and $h_0 > 0$ such that

$$\frac{1}{y}\int_0^y h(x)\,dx \ge h_0$$

for all $y > y_0$. Set $\varepsilon := \dfrac{h_0\psi_+}{2h_0 + 4C}$, where the constant C is taken from condition (12.4). Let y_0 be so large that

$$\psi(x) \in (\psi_+ - \varepsilon, \psi_+)$$

for $x \ge y_0$. Set $y_1 := \dfrac{h_0 + 4C}{h_0(h_0 + 2C)}\displaystyle\int_0^{y_0} h(x)\,dx$. If $y > \max\{y_0, y_1\}$, then

$$\int_{y_0}^{y} \psi''(x)\,dx = \int_{y_0}^{y} (h(x)\psi(x) - q(x)\psi'(x))\,dx \ge \int_{y_0}^{y} (h(x)\psi(x) - 2Cx\psi'(x))\,dx$$

$$\ge \int_{y_0}^{y} h(x)\psi(x)\,dx - 2Cy(\psi(y) - \psi(y_0)) \ge (\psi_+ - \varepsilon)\Big(yh_0 - \int_0^{y_0} h(x)\,dx\Big) - 2Cy\varepsilon$$

$$= y\psi_+\frac{h_0}{2} - \psi_+\Big(\frac{h_0 + 4C}{2h_0 + 4C}\Big)\int_0^{y_0} h(x)\,dx = \psi_+\frac{h_0}{2}(y - y_1) > 0.$$

Consequently,

$$\psi'(y) - \psi'(y_0) = \int\limits_{y_0}^{y} \psi''(x)\,dx \ge \psi_+ \frac{h_0}{2}(y - y_1)$$

for $y > \max\{y_0, y_1\}$, which contradicts the relation

$$\int\limits_{y_0}^{\infty} \psi'(y)\,dy = \psi_+ - \psi(y_0) < \infty.$$

Therefore, the limit of the function ψ_+ cannot be finite and the solution ψ cannot be bounded. A similar reasoning shows that the solution φ cannot be bounded if the right condition in (12.5) holds. Hence, only the trivial solution can be bounded. □

We return to the problem (11.14), (11.15). Let X be the solution of the stochastic differential equation (11.1) with coefficients satisfying (11.2) and (11.3). Suppose that $\sigma^2(x) > 0$ for all $x \in \mathbf{R}$. Let $H_{a,b} := \min\{s : X(s) \notin (a,b)\}$ be the first exit time of the process X from the interval (a,b).

We start with a simpler problem than (11.14), (11.15).

Theorem 12.1. *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be continuous functions. Suppose that Φ is bounded and f is nonnegative. Let $U(x)$ be a bounded solution of the equation*

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{12.6}$$

Then

$$U(x) = \mathbf{E}_x\biggl\{\Phi(X(\tau))\exp\biggl(-\int\limits_0^{\tau} f(X(s))\,ds\biggr)\biggr\}. \tag{12.7}$$

Remark 12.1. Under the conditions of Theorem 12.1 equation (12.6) has a unique bounded solution in $\mathbf{R}$, because it must have the probabilistic expression of the form (12.7). Consequently, the corresponding homogeneous equation has only the trivial bounded solution.

Proof of Theorem 12.1. Set

$$\eta(t) := U(X(t))\exp\biggl(-\lambda t - \int\limits_0^t f(X(v))\,dv\biggr).$$

Applying Itô's formula, we see that

$$\eta(r) - \eta(0) = \int\limits_0^r \exp\biggl(-\lambda t - \int\limits_0^t f(X(v))\,dv\biggr)\Bigl[U'(X(t))\sigma(X(t))\,dW(t)$$

$$+\Big(U'(X(t))\mu(X(t)+\tfrac{1}{2}U''(X(t))\sigma^2(X(t))-(\lambda+f(X(t)))U(X(t))\Big)dt\Big]$$

for any $0<r$. Taking into account (12.6), we can write

$$\eta(r\wedge H_{a,b})-U(x)=\int_0^r \mathbb{1}_{[0,H_{a,b})}(t)e^{-\lambda t}\exp\Big(-\int_0^t f(X(v))\,dv\Big)$$
$$\times\big[U'(X(t))\sigma(X(t))\,dW(t)-\lambda\Phi(X(t))\,dt\big]. \qquad (12.8)$$

By reasons similar to those given for equation (11.8), one can take the expectation to the stochastic integral. Now, computing the expectation of both sides of (12.8) and taking into account that the expectation of the stochastic integral is equal to zero, we obtain

$$U(x)=\mathbf{E}_x\eta(r\wedge H_{a,b})+\mathbf{E}_x\int_0^{r\wedge H_{a,b}}\lambda e^{-\lambda t}\Phi(X(t))\exp\Big(-\int_0^t f(X(v))\,dv\Big)\,dt. \qquad (12.9)$$

Since the diffusion process X is continuous and defined for all time moments, $H_{a,b}\to\infty$ as $a\to-\infty$ and $b\to\infty$. By the Lebesgue dominated convergence theorem, one can pass to the limit in (12.9) as $a\to-\infty$ and $b\to\infty$. Next, we let $r\to\infty$. By the definition of the process η, the term $\mathbf{E}_x\eta(r)$ tends to zero. Hence, it follows from (12.9) that

$$U(x)=\mathbf{E}_x\int_0^\infty\lambda e^{-\lambda t}\Phi(X(t))\exp\Big(-\int_0^t f(X(v))\,dv\Big)\,dt.$$

Then, using the assumption that τ does not depend on the diffusion X and has the density $\lambda e^{-\lambda t}\mathbb{1}_{[0,\infty)}(t)$, we conclude by Fubini's theorem that the above equality is identical to (12.7). □

We have the following version of Theorem 12.1.

Theorem 12.2. *Let $\Phi(x)$, $f(x)$ and $F(x)$, $x\in\mathbf{R}$, be continuous functions. Suppose that Φ, F are bounded and f is nonnegative. Let $U(x)$, $x\in\mathbf{R}$, be a bounded solution of the equation*

$$\frac{1}{2}\sigma^2(x)U''(x)+\mu(x)U'(x)-(\lambda+f(x))U(x)=-\lambda\Phi(x)-F(x), \qquad x\in\mathbf{R}. \qquad (12.10)$$

Then

$$U(x)=\mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau f(X(s))\,ds\Big)+\int_0^\tau F(X(s))\exp\Big(-\int_0^s f(X(v))\,dv\Big)ds\Big\}.$$

Proof. This result is a corollary of Theorem 12.1. Indeed, since τ is independent of X, Fubini's theorem shows that

$$\mathbf{E}_x\Big\{\int_0^\tau F(X(s))\exp\Big(-\int_0^s f(X(v))\,dv\Big)ds\Big\}$$

$$= \lambda \int_0^\infty e^{-\lambda t} \int_0^t \mathbf{E}_x \Big\{ F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv \Big) \Big\} ds\,dt$$

$$= \int_0^\infty e^{-\lambda s} \mathbf{E}_x \Big\{ F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv \Big) \Big\} ds$$

$$= \frac{1}{\lambda} \mathbf{E}_x \Big\{ F(X(\tau)) \exp\Big(-\int_0^\tau f(X(v))\,dv \Big) \Big\}.$$

Now we can apply Theorem 12.1 with the function $\Phi(x) + \frac{1}{\lambda} F(x)$ instead of $\Phi(x)$. □

In the following result we can assume initially that the functions μ and σ satisfy condition (11.2) only on the interval (a, b), because one can continue μ and σ outside (a, b) in such a way that conditions (11.2) and (11.3) hold. In this case, by Theorem 7.2, the process $X(t)$ is not changed for $t \in [0, H_{a,b}]$.

Theorem 12.3. *Let $\Phi(x)$, $f(x)$ and $F(x)$, $x \in [a, b]$, be continuous functions, and let f be nonnegative. Let $U(x)$, $x \in [a, b]$, be a solution of the problem*

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x) - F(x), \quad x \in (a, b), \tag{12.11}$$

$$U(a) = \Phi(a), \qquad U(b) = \Phi(b). \tag{12.12}$$

Then

$$U(x) = \mathbf{E}_x \Big\{ \Phi(X(\tau \wedge H_{a,b})) \exp\Big(-\int_0^{\tau \wedge H_{a,b}} f(X(s))\,ds \Big)$$

$$+ \int_0^{\tau \wedge H_{a,b}} F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv \Big) ds \Big\}. \tag{12.13}$$

Proof. We continue the solution of the problem (12.11), (12.12) to the whole real line. One can extend the functions Φ, f and F to the real line so that the extensions are bounded and f is nonnegative.

Let $\psi(x)$, $x \in \mathbf{R}$, and $\varphi(x)$, $x \in \mathbf{R}$, be linearly independent solutions of the homogeneous equation corresponding to (12.11), with ψ increasing and φ decreasing and nonnegative. Then $\psi(b)\varphi(a) - \psi(a)\varphi(b) > 0$.

The extension of the solution of (12.11), (12.12) to the real line can be written as

$$U(x) = U_p(x) + A_{a,b}\psi(x) + B_{a,b}\varphi(x), \tag{12.14}$$

where U_p is a particular solution of equation (12.11) for $x \in \mathbf{R}$, and the constants $A_{a,b}$, $B_{a,b}$ satisfy the system of algebraic equations

$$\Phi(a) = U_p(a) + A_{a,b}\psi(a) + B_{a,b}\varphi(a),$$
$$\Phi(b) = U_p(b) + A_{a,b}\psi(b) + B_{a,b}\varphi(b).$$

This system has the following unique solution:

$$A_{a,b} = \frac{(\Phi(b) - U_p(b))\varphi(a) - (\Phi(a) - U_p(a))\varphi(b)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}, \tag{12.15}$$

$$B_{a,b} = \frac{\psi(b)(\Phi(a) - U_p(a)) - \psi(a)(\Phi(b) - U_p(b))}{\psi(b)\varphi(a) - \psi(a)\varphi(b)}. \tag{12.16}$$

We set

$$\eta(t) := U(X(t)) \exp\Big(- \lambda t - \int_0^t f(X(v))\, dv\Big)$$

$$+ e^{-\lambda t} \int_0^t F(X(s)) \exp\Big(- \int_0^s f(X(v))\, dv\Big)\, ds.$$

Applying Itô's formula, we see that

$$\eta(r) - \eta(0) = \int_0^r \exp\Big(- \lambda t - \int_0^t f(X(v))\, dv\Big) \Big[U'(X(t))\, \sigma(X(t))\, dW(t)$$

$$+ \Big(U'(X(t))\mu(X(t)) + \frac{1}{2} U''(X(t))\, \sigma^2(X(t)) - (\lambda + f(X(t)))U(X(t)) + F(X(t))\Big) dt\Big]$$

$$- \lambda \int_0^r e^{-\lambda t} \int_0^t F(X(s)) \exp\Big(- \int_0^s f(X(v))\, dv\Big)\, ds\, dt$$

for every $r > 0$. Taking into account (12.11), we get the equality

$$\eta(H_{a,b}) - U(x) = \int_0^{H_{a,b}} \exp\Big(- \lambda t - \int_0^t f(X(v))\, dv\Big) \big[U'(X(t))\, \sigma(X(t))\, dW(t)$$

$$- \lambda \Phi(X(t))\big)\, dt\big] - \lambda \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Big(- \int_0^s f(X(v))\, dv\Big)\, ds\, dt.$$

It is important that the process $\mathbb{1}_{[0,H_{a,b}]}(t)$, $t \geq 0$, is adapted to the filtration $\mathcal{G}_0^t = \sigma(W(s), 0 \leq s \leq t)$ and for $t \leq H_{a,b}$ the functions $U'(X(t))$ and $\sigma(X(t))$ are bounded by a constant depending on a, b. Therefore, we can take the expectation of the stochastic integral and this expectation is equal to zero.

Since $\sup_{x\in[a,b]} |\Phi(x)| < \infty$ and $\sup_{x\in[a,b]} |F(x)| < \infty$, the expectations of the other terms of the difference $\eta(H_{a,b}) - U(x)$ are finite. Applying the expectation, we derive the equality

$$U(x) = \mathbf{E}_x \eta(H_{a,b}) + \lambda \mathbf{E}_x \int_0^{H_{a,b}} \exp\Big(- \lambda t - \int_0^t f(X(v))\, dv\Big) \Phi(X(t))\, dt$$

$$+\lambda \mathbf{E}_x \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv\Big)\,ds\,dt.$$

Let us consider each of the terms on the right-hand side of this equality. We use the equality $U(X(H_{a,b})) = \Phi(X(H_{a,b}))$, and the fact that τ is independent of X and has exponential distribution. By Fubini's theorem, these terms can be represented as follows:

$$\mathbf{E}_x \eta(H_{a,b}) = \mathbf{E}_x\Big\{\Phi(X(H_{a,b})) \exp\Big(-\int_0^{H_{a,b}} f(X(s))\,ds\Big) 1\!\!\mathrm{I}_{\{\tau > H_{a,b}\}}\Big\}$$

$$+\mathbf{E}_x\Big\{\int_0^{H_{a,b}} F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv\Big)\,ds\, 1\!\!\mathrm{I}_{\{\tau > H_{a,b}\}}\Big\},$$

$$\lambda \mathbf{E}_x \int_0^{H_{a,b}} \exp\Big(-\lambda t - \int_0^t f(X(v))\,dv\Big)\Phi(X(t))\,dt$$

$$= \mathbf{E}_x\Big\{\Phi(X(\tau)) \exp\Big(-\int_0^{\tau} f(X(s))\,ds\Big) 1\!\!\mathrm{I}_{\{\tau \le H_{a,b}\}}\Big\},$$

and

$$\lambda \mathbf{E}_x \int_0^{H_{a,b}} e^{-\lambda t} \int_0^t F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv\Big)\,ds\,dt$$

$$= \mathbf{E}_x\Big\{\int_0^{\tau} F(X(s)) \exp\Big(-\int_0^s f(X(v))\,dv\Big)\,ds\, 1\!\!\mathrm{I}_{\{\tau \le H_{a,b}\}}\Big\}.$$

Summing these equalities, we see that U takes the form (12.13). □

The analogs of Theorem 12.3 are of special interest in the cases when either $\tau \to \infty$, or $H_{a,b} \to \infty$, and when both limits take place. We begin with the analysis of the results for the second case.

Theorem 12.4. *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be continuous functions, with Φ bounded and f nonnegative.*

Suppose that there exists the bounded on any finite interval derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$, $x \in \mathbf{R}$. Then

$$U(x) = \mathbf{E}_x\Big\{\Phi(X(\tau)) \exp\Big(-\int_0^{\tau} f(X(s))\,ds\Big)\Big\}, \qquad x \in \mathbf{R}, \tag{12.17}$$

is the unique bounded solution of the equation

$$\frac{1}{2}\,\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{12.18}$$

Remark 12.2. In contrast to Theorem 12.1, in this result it is not assumed that equation (12.18) has a bounded solution; instead, we state that the function (12.17) is such a solution.

Proof of Theorem 12.4. We set

$$U_{a,b}(x) := \mathbf{E}_x\Big\{\Phi(X(\tau\wedge H_{a,b}))\exp\Big(-\int_0^{\tau\wedge H_{a,b}} f(X(s))\,ds\Big)\Big\} \tag{12.19}$$

for $a < x < b$. By Theorem 12.3, the function $U_{a,b}(x)$, $x \in (a,b)$, is the solution of (12.11), (12.12) with $F \equiv 0$. We extend $U_{a,b}(x)$ to the whole real line by formula (12.14).

As it was mentioned above, $H_{a,b} \to \infty$ as $a \to -\infty$ and $b \to \infty$.

By the Lebesgue dominated convergence theorem,

$$\lim_{a\to-\infty,b\to\infty} U_{a,b}(x) = U(x), \qquad x \in \mathbf{R}. \tag{12.20}$$

We can assume that $a < 0 < b$. Integrating (12.11), we get the equation

$$\frac{1}{2}\,(U'_{a,b}(x) - U'_{a,b}(0)) + \frac{\mu(x)}{\sigma^2(x)}\,U_{a,b}(x) - \frac{\mu(0)}{\sigma^2(0)}\,U_{a,b}(0) - \int_0^x \Big(\frac{\mu(y)}{\sigma^2(y)}\Big)' U_{a,b}(y)\,dy$$

$$- \int_0^x \Big(\frac{\lambda + f(y)}{\sigma^2(y)}\Big) U_{a,b}(y)\,dy = -\lambda\int_0^x \frac{\Phi(y)}{\sigma^2(y)}\,dy. \tag{12.21}$$

Integrating this equation, we find that for $x \in \mathbf{R}$

$$\frac{1}{2}(U_{a,b}(x) - U_{a,b}(0)) - \frac{1}{2}\,U'_{a,b}(0)\,x + \int_0^x \frac{\mu(z)}{\sigma^2(z)}\,U_{a,b}(z)\,dz - \frac{\mu(0)}{\sigma^2(0)}\,U_{a,b}(0)x$$

$$- \int_0^x\int_0^z \Big(\Big(\frac{\mu(y)}{\sigma^2(y)}\Big)' + \frac{\lambda + f(y)}{\sigma^2(y)}\Big) U_{a,b}(y)\,dy\,dz = -\lambda\int_0^x\int_0^z \frac{\Phi(y)}{\sigma^2(y)}\,dy\,dz. \tag{12.22}$$

From (12.19) it follows that the functions $U_{a,b}(x)$, $x \in (a,b)$, are bounded by the same constant as the function Φ. Using (12.20) and applying the Lebesgue dominated convergence theorem, we deduce from (12.22) that there exists the limit $\widetilde{U}_0 := \lim_{a\to-\infty,b\to\infty} U'_{a,b}(0)$, and

$$\frac{1}{2}(U(x) - U(0)) - \frac{1}{2}\widetilde{U}_0\,x + \int_0^x \frac{\mu(z)}{\sigma^2(z)}\,U(z)\,dz - \frac{\mu(0)}{\sigma^2(0)}U(0)\,x$$

$$-\int_0^x\int_0^z\Big(\Big(\frac{\mu(y)}{\sigma^2(y)}\Big)'+\frac{\lambda+f(y)}{\sigma^2(y)}\Big)U(y)\,dy\,dz=-\lambda\int_0^x\int_0^z\frac{\Phi(y)}{\sigma^2(y)}dy\,dz. \tag{12.23}$$

From this equality it follows that $U(x)$, $x\in\mathbf{R}$, is a continuous function. In addition, U is differentiable for all x including zero, and $\widetilde U_0=U'(0)$. Differentiating (12.23) with respect to x and applying the integration by parts formula, we see that U satisfies the equation

$$\frac12(U'(x)-U'(0))+\int_0^x\frac{\mu(y)}{\sigma^2(y)}\,U'(y)\,dy-\int_0^x\frac{\lambda+f(y)}{\sigma^2(y)}U(y)\,dy=-\lambda\int_0^x\frac{\Phi(y)}{\sigma^2(y)}dy.$$

Differentiating this equation with respect to x, we get that U is the solution of (12.18).

Now the fact that such bounded solution is unique follows from Remark 12.1. □

Consider the transformation of Theorem 12.3 as $\tau\to\infty$ and $H_{a,b}\to\infty$ simultaneously.

Theorem 12.5. *Let $f(x)$, $x\in\mathbf{R}$, be a nonnegative continuous function. Suppose that there exists the bounded on any finite interval derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$, $x\in\mathbf{R}$. Then the function*

$$L(x):=\mathbf{E}_x\exp\Big(-\int_0^\infty f(X(s))\,ds\Big),\qquad x\in\mathbf{R}, \tag{12.24}$$

is the solution of the homogeneous equation

$$\frac12\sigma^2(x)\phi''(x)+\mu(x)\phi'(x)-f(x)\phi(x)=0,\qquad x\in\mathbf{R}. \tag{12.25}$$

To prove this result one can repeat the proof of Theorem 12.4 for $\Phi\equiv1$, adding to it the passage to the limit as $\lambda\to0$. In this case, $\lim_{\lambda\to0}\tau=\infty$ in probability, since $\mathbf{P}(\tau>t)=e^{-\lambda t}$ for $t\ge0$.

This result has an important consequence.

Corollary 12.1. *Let $f(x)$, $x\in\mathbf{R}$, be a nonnegative continuous function. Suppose that there exists the bounded derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$, $x\in\mathbf{R}$, and*

$$\liminf_{y\to\infty}\frac1y\int_0^y\frac{f(x)}{\sigma^2(x)}\,dx>0,\qquad\liminf_{y\to\infty}\frac1y\int_{-y}^0\frac{f(x)}{\sigma^2(x)}\,dx>0. \tag{12.26}$$

Then

$$\int_0^\infty f(X(s))\,ds=\infty\qquad\text{a.s.} \tag{12.27}$$

Indeed, according to Proposition 12.3, under these assumptions equation (12.25) has no nonzero bounded solutions. Therefore, $L\equiv0$ and we have (12.27).

Propositions 12.2 and 12.3 imply the following result.

Corollary 12.2. *Let $f(x)$, $x \in \mathbf{R}$, be a nonnegative continuous function. Suppose that conditions* (12.26) *hold and*

$$\left|\frac{\mu(x)}{\sigma^2(x)}\right| \le C(1+|x|) \qquad \text{for all} \quad x \in \mathbf{R} \tag{12.28}$$

for some $C > 0$.

Then the homogeneous equation (12.25) *has two nonnegative linearly independent solutions $\psi(x)$ and $\varphi(x)$ such that $\psi(x)$ is increasing and $\lim_{x\to\infty} \psi(x) = \infty$, while $\varphi(x)$ is decreasing and $\lim_{x\to-\infty} \varphi(x) = \infty$.*

Another extreme version of Theorem 12.3 is the case when only $\tau \to \infty$. As we have seen, this happens if $\lambda \to 0$. We precede the consideration of this case by the following important result.

Lemma 12.1. *For every $x \in [a, b]$,*

$$\mathbf{E}_x H_{a,b} < \infty. \tag{12.29}$$

Remark 12.3. From (12.29) it follows that $\mathbf{P}_x(H_{a,b} < \infty) = 1$ for $x \in [a, b]$.

Proof of Lemma 12.1. We consider the family $\{U_\lambda(x), x \in [a, b]\}_{\lambda \ge 0}$ of solutions of the problem

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - \lambda U(x) = -1, \qquad x \in (a, b), \tag{12.30}$$

$$U(a) = 0, \qquad U(b) = 0. \tag{12.31}$$

From Theorem 12.3 with $f \equiv 0$, $\varPhi \equiv 0$ and $F \equiv 1$ it follows that $U_\lambda(x) = \mathbf{E}_x\{\tau \wedge H_{a,b}\}$ for $\lambda > 0$.

We will prove that for all $x \in [a, b]$

$$\sup_{\lambda>0} U_\lambda(x) \le U_0(x), \tag{12.32}$$

where $U_0(x)$ is the solution of (12.30), (12.31) for $\lambda = 0$. This estimate is useful for us due to the following reason. Since $\lim_{\lambda\to 0} \tau = \infty$ in probability, $\lim_{\lambda\to 0}\{\tau \wedge H_{a,b}\} = H_{a,b}$. Now from (12.32), by Fatou's lemma, it follows that

$$\mathbf{E}_x H_{a,b} \le \sup_{\lambda>0} \mathbf{E}_x\{\tau \wedge H_{a,b}\} \le U_0(x),$$

and this is what we want to prove.

To prove (12.32), we use the following result.

Proposition 12.4. *The solution of the problem*

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) = -F(x), \qquad x \in (a, b), \tag{12.33}$$

$$Q(a) = \varPhi(a), \qquad Q(b) = \varPhi(b), \tag{12.34}$$

has the form

$$Q(x) = \frac{S(b)-S(x)}{S(b)-S(a)}\Bigl(\Phi(a) + \int_a^x (S(y)-S(a))F(y)\,dM(y)\Bigr)$$

$$+\frac{S(x)-S(a)}{S(b)-S(a)}\Bigl(\Phi(b) + \int_x^b (S(b)-S(y))F(y)\,dM(y)\Bigr), \qquad (12.35)$$

where

$$S(x) := \int^x \exp\Bigl(-\int^y \frac{2\mu(z)}{\sigma^2(z)}\,dz\Bigr)\,dy, \qquad dM(x) = \frac{2}{\sigma^2(x)}\exp\Bigl(\int^x \frac{2\mu(z)}{\sigma^2(z)}\,dz\Bigr)\,dx.$$

In the definition of the functions $S(x)$ and $M(x)$, the lower limit of integration can be arbitrary, but the same.

Formula (12.35) can be verified by direct differentiation since the function S satisfies the equation

$$\frac{1}{2}\,\sigma^2(x)S''(x) + \mu(x)S'(x) = 0.$$

For the derivation of (12.35) see also the proof of formula (15.13) of Ch. IV.

The difference $U_0(x) - U_\lambda(x)$ is the solution of (12.33), (12.34) with $F(x) = \lambda U_\lambda(x)$, $\Phi(a) = 0$ and $\Phi(b) = 0$. Therefore, this difference is nonnegative. This proves (12.32) and thus completes the proof of Lemma 12.1. □

It is possible to pass to the limit as $\lambda \to 0$ in the problem (12.11), (12.12) and in (12.13) and get the following result.

Theorem 12.6. *Let $f(x)$ and $F(x)$, $x \in [a,b]$, be continuous functions and let f be nonnegative. Let the function Φ be defined only at two points a and b.*

Then the function

$$Q(x) := \mathbf{E}_x\Bigl\{\Phi(X(H_{a,b}))\exp\Bigl(-\int_0^{H_{a,b}} f(X(s))\,ds\Bigr)$$

$$+\int_0^{H_{a,b}} F(X(s))\exp\Bigl(-\int_0^s f(X(v))\,dv\Bigr)\,ds\Bigr\}$$

is the solution of the problem

$$\frac{1}{2}\,\sigma^2(x)Q''(x) + \mu(x)Q'(x) - f(x)Q(x) = -F(x), \quad x \in (a,b), \qquad (12.36)$$

$$Q(a) = \Phi(a), \qquad Q(b) = \Phi(b), \qquad x \in [a,b]. \qquad (12.37)$$

The proof of this theorem repeats the proof of Theorem 12.3 for $\lambda = 0$ ($\tau = \infty$) with the function $U(x)$ replaced by $Q(x)$. Here an important point is the finiteness of the integral $\int_0^\infty \mathbf{E}_x \mathbb{1}_{[0,H_{a,b}]}(t)\,dt = \mathbf{E}_x H_{a,b}$. This enables us to take the expectation of the difference $\eta(H_{a,b}) - Q(x)$, which is expressed as a stochastic integral. As a result, this expectation is equal to zero, and we get the required equality $Q(x) = \mathbf{E}_x\eta(H_{a,b})$.

Theorem 12.6 and Proposition 12.4 imply the following assertions.

Proposition 12.5. *The probabilities of the first exit from the interval* $[a,b]$ *have the form*

$$\mathbf{P}_x(X(H_{a,b})=a)=\frac{S(b)-S(x)}{S(b)-S(a)},\qquad \mathbf{P}_x(X(H_{a,b})=b)=\frac{S(x)-S(a)}{S(b)-S(a)}. \quad (12.38)$$

This corollary is obtained from Theorem 12.6 with $F\equiv 0$, $f\equiv 0$. For $\Phi(a)=1$ and $\Phi(b)=0$ we have the left equality in (12.38), while for $\Phi(a)=0$ and $\Phi(b)=1$ we have the right one.

Proposition 12.6. *The expectation* $\mathbf{E}_x H_{a,b}$ *is expressed by the formula*

$$\mathbf{E}_x H_{a,b}=\frac{S(b)-S(x)}{S(b)-S(a)}\int_a^x (S(y)-S(a))\,dM(y)+\frac{S(x)-S(a)}{S(b)-S(a)}\int_x^b (S(b)-S(y))\,dM(y). \quad (12.39)$$

To derive this result, we should use Theorem 12.6 with $F\equiv 1$, $f\equiv 0$, $\Phi(a)=0$ and $\Phi(b)=0$. Then $Q(x)=\mathbf{E}_x H_{a,b}$ is the solution of the problem (12.33), (12.34).

Now we consider another stopping time: the first hitting time of a level z by the process X, i.e., $H_z=\min\{s: X(s)=z\}$. This stopping time can be either finite or infinite.

Theorem 12.7. *Let* $f(x)$, $x\in\mathbf{R}$, *be a nonnegative continuous function. Then*

$$L_z(x):=\mathbf{E}_x\biggl\{\exp\biggl(-\int_0^{H_z} f(X(s))\,ds\biggr)\mathbb{1}_{\{H_z<\infty\}}\biggr\}=\begin{cases}\psi(x), & \text{for } x\le z,\\ \varphi(x), & \text{for } z\le x,\end{cases} \quad (12.40)$$

where φ *is a positive decreasing solution and* ψ *is a positive increasing solution of the homogeneous equation* (12.25) *that satisfy the equalities* $\varphi(z)=\psi(z)=1$.

Proof. It is clear that a.s.

$$H_z=\begin{cases}\lim\limits_{a\to-\infty} H_{a,z}, & \text{for } x\le z,\\ \lim\limits_{b\to\infty} H_{z,b}, & \text{for } z\le x.\end{cases} \quad (12.41)$$

Denote

$$Q^{(y)}_{a,b}(x):=\mathbf{E}_x\biggl\{\mathbb{1}_y(W(H_{a,b}))\exp\biggl(-\int_0^{H_{a,b}} f(X(s))\,ds\biggr)\biggr\}.$$

Here the presence of the indicator function of a one-point set reduces the expectation to the set of sample paths leaving the interval through the upper boundary ($y=b$) or the lower boundary ($y=a$).

Since a.s.

$$\mathbb{1}_{\{H_z<\infty\}}=\begin{cases}\lim\limits_{a\to-\infty}\mathbb{1}_{\{z\}}(W(H_{a,z})), & \text{for } x\le z,\\ \lim\limits_{b\to\infty}\mathbb{1}_{\{z\}}(W(H_{z,b})), & \text{for } z\le x,\end{cases}$$

the Lebesgue dominated convergence theorem shows that

$$L_z(x) = \begin{cases} \lim\limits_{a\to-\infty} Q^{(z)}_{a,z}(x), & \text{for } x \le z, \\ \lim\limits_{b\to\infty} Q^{(z)}_{z,b}(x), & \text{for } z \le x. \end{cases} \tag{12.42}$$

We apply Theorem 12.6 with $F \equiv 0$. The solution of the problem (12.36), (12.37) has the form

$$Q^{(a)}_{a,b}(x) = \frac{\psi_0(b)\varphi_0(x) - \psi(x)\varphi_0(b)}{\psi_0(b)\varphi_0(a) - \psi_0(a)\varphi_0(b)} \tag{12.43}$$

for $\Phi(a) = 1$ and $\Phi(b) = 0$, and

$$Q^{(b)}_{a,b}(x) = \frac{\psi_0(x)\varphi_0(a) - \psi_0(a)\varphi_0(x)}{\psi_0(b)\varphi_0(a) - \psi_0(a)\varphi_0(b)} \tag{12.44}$$

for $\Phi(a) = 0$ and $\Phi(b) = 1$, where $\varphi_0(x)$ and $\psi_0(x)$, $x \in \mathbf{R}$, are fundamental solutions of the homogeneous equation (12.25) such that $\varphi_0(z) = \psi_0(z) = 1$.

From (12.44) for $b = z$ it follows that $\lim\limits_{a\to-\infty} Q^{(z)}_{a,z}(x) = \dfrac{\psi_0(x) - \rho_-\varphi_0(x)}{1-\rho_-}$, where $\rho_- = \lim\limits_{a\to-\infty} \dfrac{\psi_0(a)}{\varphi_0(a)}$. This limit exists and it is less than 1, because the ratio $\dfrac{\psi_0(a)}{\varphi_0(a)}$ is an increasing function. We set $\psi(x) := \dfrac{\psi_0(x) - \rho_-\varphi_0(x)}{1-\rho_-}$, $x \in \mathbf{R}$. It is clear that $\psi(x)$ is an increasing function. By (12.42) for $x \le z$, we have that $L_z(x) = \psi(x)$ and this function is positive.

We use similar arguments for the domain $x \ge z$. In this case from (12.43) for $a = z$ it follows that $\lim\limits_{b\to\infty} Q^{(z)}_{z,b}(x) = \dfrac{\varphi_0(x) - \rho_+\psi_0(x)}{1-\rho_+}$, where $\rho_+ = \lim\limits_{b\to\infty} \dfrac{\varphi_0(b)}{\psi_0(b)}$. This limit exists and it is less than 1. We set $\varphi(x) := \dfrac{\varphi_0(x) - \rho_+\psi_0(x)}{1-\rho_+}$, $x \in \mathbf{R}$. Then $L_z(x) = \varphi(x)$ for $x \ge z$ and the function $\varphi(x)$, $x \in \mathbf{R}$ is decreasing and positive.□

Corollary 12.3. *The following equality*

$$\mathbf{P}_x(H_z < \infty) = \begin{cases} \dfrac{S(x) - S(-\infty)}{S(z) - S(-\infty)}, & \text{for } x \le z, \\ \dfrac{S(\infty) - S(x)}{S(\infty) - S(z)}, & \text{for } z \le x, \end{cases}$$

holds, where for $S(-\infty) = -\infty$ or $S(\infty) = \infty$ the corresponding ratio equals to 1.

This follows from (12.40), (12.42) with $f \equiv 0$, and (12.35) with $F \equiv 0$.

§ 13. The Cauchy problem, existence of a solution

As mentioned above, the proof of existence of a solution of the Cauchy problem is very complicated. In this section we give a probabilistic proof of this existence.

Let the process $X_x(t)$, $t \in [0, T]$, be the solution of the homogeneous stochastic differential equation

$$X_x(t) = x + \int_0^t a(X_x(s))\,ds + \int_0^t b(X_x(s))\,dW(s). \tag{13.1}$$

We assume that the coefficients $a(x)$ and $b(x)$, $x \in \mathbf{R}$, are continuous, bounded, and have continuous bounded derivatives $a'(x)$, $b'(x)$, $a''(x)$, $b''(x)$.

Theorem 13.1. *Let $\varPhi(x)$, $x \in \mathbf{R}$, be a continuous bounded function with continuous bounded derivatives $\varPhi'(x)$, $\varPhi''(x)$. Then the function*

$$u(t,x) := \mathbf{E}\,\varPhi(X_x(t)) \tag{13.2}$$

is differentiable with respect to t, twice continuously differentiable with respect to x, and it is the solution of the problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\,b^2(x)\frac{\partial^2}{\partial x^2}u(t,x) + a(x)\frac{\partial}{\partial x}u(t,x), \tag{13.3}$$

$$u(0,x) = \varPhi(x), \tag{13.4}$$

$(t,x) \in (0,T) \times \mathbf{R}$.

Remark 1.13. Equation (13.3) is called the *backward Kolmogorov equation.* For nonhomogeneous stochastic differential equations the analogue of Theorem 13.1 will be considered in § 2 Ch. IV. We also refer to Gihman and Skorohod (1969).

Proof of Theorem 13.1. Clearly, the function u is bounded by the same constant as the function $\varPhi$.

The initial condition (13.4) is easily verified by passage to the limit under the expectation sign in (13.2) as $t \downarrow 0$.

Let us verify that for every fixed t the function $u(t,x)$ is twice continuously differentiable with respect to x. By Theorems 9.2 and 9.3, the process $X_x(t)$, $t \in [0,T]$, has stochastically continuous mean square derivatives of the first and the second order with respect to x, which we denote by $X_x^{(1)}(t)$ and $X_x^{(2)}(t)$.

Denote $u_x^{(1)}(t,x) := \mathbf{E}\big\{\varPhi'(X_x(t))\,X_x^{(1)}(t)\big\}$ and prove that $\dfrac{\partial}{\partial x}u(t,x) = u_x^{(1)}(t,x)$. Set $Y_\Delta(t) := \dfrac{X_{x+\Delta}(t) - X_x(t)}{\Delta}$. We have

$$\begin{aligned}\Big|\frac{u(t,x+\Delta)-u(t,x)}{\Delta} - u_x^{(1)}(t,x)\Big| &\le \mathbf{E}\Big|\frac{\varPhi(X_{x+\Delta}(t)) - \varPhi(X_x(t))}{\Delta} - \varPhi'(X_x(t))\,X_x^{(1)}(t)\Big| \\ &\le \mathbf{E}\Big|\frac{\varPhi(X_{x+\Delta}(t)) - \varPhi(X_x(t))}{X_{x+\Delta}(t) - X_x(t)}\big(Y_\Delta(t) - X_x^{(1)}(t)\big)\Big| \\ &\quad + \mathbf{E}\Big\{\Big|\frac{\varPhi(X_{x+\Delta}(t)) - \varPhi(X_x(t))}{X_{x+\Delta}(t) - X_x(t)} - \varPhi'(X_x(t))\Big|\,\big|X_x^{(1)}(t)\big|\Big\} \to 0 \quad \text{as } \Delta \to 0.\end{aligned}$$

This relation is due to the fact that the ratio $\dfrac{\varPhi(y) - \varPhi(x)}{y - x}$ is bounded and converges to $\varPhi'(x)$ as $y \to x$, while the function $X_x(t)$ is continuous in x (Theorem 9.1), and the fact that $\mathbf{E}\big(Y_\Delta(t) - X_x^{(1)}(t)\big)^2 \to 0$ (Theorem 9.3).

Thus we proved that the function $u(t,x)$ has a derivative

$$\frac{\partial}{\partial x}u(t,x) = \mathbf{E}\big\{\varPhi'(X_x(t))\,X_x^{(1)}(t)\big\}. \tag{13.5}$$

This derivative is continuous in (t,x) thanks to the continuity of $X_x(t)$ and the mean square continuity of $X_x^{(1)}(t)$. Furthermore, according to Remark 9.4, $\mathbf{E}X_x^{(1)}(t) \le e^{K(K+1)t}$, therefore the derivative $\dfrac{\partial}{\partial x}u(t,x)$, $(t,x) \in (0,T) \times \mathbf{R}$, is a bounded function.

Similarly, we can prove that

$$\frac{\partial^2}{\partial x^2} u(t,x) = \mathbf{E}\{\varPhi''(X_x(t)) \big(X_x^{(1)}(t)\big)^2\} + \mathbf{E}\{\varPhi'(X_x(t)\, X_x^{(2)}(t)\}, \tag{13.6}$$

$(t,x) \in (0,T) \times \mathbf{R}$. This derivative is a bounded continuous function.

For any $0 \le v \le t$, the solution $X_x(t)$ of equation (13.1) can be written in the form

$$X_x(t) = X_x(v) + \int_0^{t-v} a(X_x(v+s))\, ds + \int_0^{t-v} b(X_x(v+s))\, d\widetilde{W}_v(s), \tag{13.7}$$

where for a fixed v the process $\widetilde{W}_v(s) = W(s+v) - W(v)$, $s \ge 0$, is a Brownian motion. Note that the process $\widetilde{W}_v$ does not depend on the σ-algebra $\mathcal{G}_0^v$ of events generated by the Brownian motion $W(s)$ for $0 \le s \le v$.

Consider the stochastic differential equation

$$\widetilde{X}_{v,x}(h) = x + \int_0^h a(\widetilde{X}_{v,x}(s))\, ds + \int_0^h b(\widetilde{X}_{v,x}(s))\, d\widetilde{W}_v(s), \tag{13.8}$$

which is similar to equation (13.1). It is clear that the process $\widetilde{X}_{v,x}$ is independent of the σ-algebra $\mathcal{G}_0^v$ and has the same finite-dimensional distributions as the process $X_x(h)$.

In (13.7) we set $t - v = h$. Then by the uniqueness of the solution of the stochastic differential equation, we have (see (9.4)) the equality

$$X_x(h+v) = \widetilde{X}_{v,X_x(v)}(h). \tag{13.9}$$

Let $0 < p < t < q$ and $\delta := q - p$. Further for a fixed t we let $\delta \to 0$. Using the fourth property of the conditional expectations and Lemma 2.1 of Ch. I, we represent $u(q,x)$ as

$$\begin{aligned} u(q,x) &= \mathbf{E}\{\mathbf{E}\{\varPhi(X_x(q))\big|\mathcal{G}_0^\delta\}\} \\ &= \mathbf{E}\{\mathbf{E}\{\varPhi(\widetilde{X}_{\delta,X_x(\delta)}(p))\big|\mathcal{G}_0^\delta\}\} = \mathbf{E}\, u(p, X_x(\delta)). \end{aligned} \tag{13.10}$$

To obtain the last equality we used the fact that the random variable $X_x(\delta)$ is measurable with respect to the σ-algebra $\mathcal{G}_0^\delta$ and $\widetilde{X}_{\delta,z}(h)$, as a random function of the argument z, is independent of $\mathcal{G}_0^\delta$. Therefore, we can use Lemma 2.1 of Ch. I to compute the conditional expectation and to prove the last equality.

Since $X_x(0) = x$, Itô's formula yields

$$u(p, X_x(\delta)) - u(p,x) = \int_0^\delta b(X_x(s)) \frac{\partial}{\partial x} u(p, X_x(s))\, dW(s)$$

$$+\int_0^\delta \Big(a(X_x(s)) \frac{\partial}{\partial x} u(p, X_x(s)) + \frac{1}{2} b^2(X_x(s)) \frac{\partial^2}{\partial x^2} u(p, X_x(s)) \Big) ds.$$

Taking the expectation of both sides of this equality, we get

$$\mathbf{E}u(p, X_x(\delta)) - u(p,x) = \mathbf{E}\int_0^\delta \Big(a(X_x(s)) \frac{\partial}{\partial x} u(p, X_x(s)) + \frac{b^2(X_x(s))}{2} \frac{\partial^2}{\partial x^2} u(p, X_x(s)) \Big) ds. \tag{13.11}$$

By the mean value theorem for integrals, we have

$$\mathbf{E}u(p, X_x(\delta)) - u(p,x) = \mathbf{E}\Big(a(X_x(\tilde{s})) \frac{\partial}{\partial x} u(p, X_x(\tilde{s})) + \frac{1}{2}\, b^2(X_x(\tilde{s})) \frac{\partial^2}{\partial x^2} u(p, X_x(\tilde{s})) \Big) \delta,$$

where $\tilde{s}$ is some, possibly random, point of the interval $(0, \delta)$.

Since the derivatives $\frac{\partial}{\partial x} u(t,x)$, $\frac{\partial^2}{\partial x^2} u(t,x)$, $(t,x) \in [0,T] \times \mathbf{R}$, are continuous and bounded, $X_x(\tilde{s}) \to x$ as $p \to t$, $q \to t$, applying the Lebesgue dominated convergence theorem we obtain

$$\frac{\mathbf{E}u(p, X_x(\delta)) - u(p,x)}{q-p} \to \frac{1}{2}\, b^2(x) \frac{\partial^2}{\partial x^2} u(t,x) + a(x) \frac{\partial}{\partial x} u(t,x). \tag{13.12}$$

Now, by (13.10),

$$\frac{u(q,x) - u(p,x)}{q-p} = \frac{\mathbf{E}u(p, X_x(\delta)) - u(p,x)}{q-p},$$

and, thus it is proved that the function $u(t,x)$, $(t,x) \in (0,t) \times \mathbf{R}$, is the solution of (13.3). □

Theorem 13.2. *Let $\Phi(x)$, $f(x)$, $x \in \mathbf{R}$, be continuous bounded functions with continuous bounded derivatives $\Phi'(x)$, $\Phi''(x)$, $f'(x)$, and $f''(x)$. Assume, in addition, that the function f is nonnegative.*

Then the function

$$u(t,x) := \mathbf{E}\Big\{ \Phi(X_x(t)) \exp\Big(-\int_0^t f(X_x(s))\, ds \Big) \Big\} \tag{13.13}$$

is differentiable with respect to t, twice continuously differentiable with respect to x, and it is the solution of the problem

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\, b^2(x) \frac{\partial^2}{\partial x^2} u(t,x) + a(x) \frac{\partial}{\partial x} u(t,x) - f(x) u(t,x), \tag{13.14}$$

$$u(0,x) = \Phi(x), \tag{13.15}$$

$(t,x) \in (0,T) \times \mathbf{R}$.

Proof. Obviously, the function u is bounded by the same constant as the function Φ.

The initial condition (13.15) is easily verified by passing to the limit under the expectation sign in (13.13) as $t \downarrow 0$.

The existence of continuous bounded derivatives is proved similarly to (13.5). In particular, the function $u(t,x)$ has the continuous bounded first-order partial derivative

$$\frac{\partial}{\partial x}u(t,x) = \mathbf{E}\Big\{\Phi'(X_x(t))\, X_x^{(1)}(t) \exp\Big(-\int_0^t f(X_x(s))\,ds\Big)\Big\}$$

$$-\mathbf{E}\Big\{\Phi(X_x(t)) \exp\Big(-\int_0^t f(X_x(s))\,ds\Big)\int_0^t f'(X_x(s))X_x^{(1)}(s)\,ds\Big\}. \qquad (13.16)$$

In addition, the function $u(t,x)$ has a continuous bounded second-order partial derivative. We use for $0 < p < t < q$, $\delta := q - p$ the relation

$$\exp\Big(-\int_\delta^q f(X_x(s))\,ds\Big) - \exp\Big(-\int_0^q f(X_x(s))\,ds\Big)$$

$$= \int_0^\delta f(X_x(v)) \exp\Big(-\int_v^q f(X_x(s))\,ds\Big)dv. \qquad (13.17)$$

Multiplying this equality by $\Phi(X_x(q))$ and taking into account (13.9), we have

$$\Phi\big(\widetilde{X}_{\delta,X_x(\delta)}(p)\big) \exp\Big(-\int_0^p f\big(\widetilde{X}_{\delta,X_x(\delta)}(s)\big)\,ds\Big) - \Phi(X_x(q)) \exp\Big(-\int_0^q f(X_x(s))\,ds\Big)$$

$$= \int_0^\delta f(X_x(v))\Phi\big(\widetilde{X}_{v,X_x(v)}(q-v)\big) \exp\Big(-\int_0^{q-v} f\big(\widetilde{X}_{v,X_x(v)}(s)\big)\,ds\Big)dv.$$

We take the expectation of both sides of this equality and use the fourth property of conditional expectations. Then we get

$$\mathbf{E}\Big\{\mathbf{E}\Big\{\Phi\big(\widetilde{X}_{\delta,X_x(\delta)}(p)\big) \exp\Big(-\int_0^p f\big(\widetilde{X}_{\delta,X_x(\delta)}(s)\big)\,ds\Big)\Big|\mathcal{G}_0^\delta\Big\}\Big\}$$

$$-\mathbf{E}\Big\{\Phi(X_x(q)) \exp\Big(-\int_0^q f(X_x(s))\,ds\Big)\Big\}$$

$$= \int_0^\delta \mathbf{E}\Big\{f(X_x(v))\mathbf{E}\Big\{\Phi\big(\widetilde{X}_{v,X_x(v)}(q-v)\big) \exp\Big(-\int_0^{q-v} f\big(\widetilde{X}_{v,X_x(v)}(s)\big)\,ds\Big)\Big|\mathcal{G}_0^v\Big\}\Big\}dv.$$

Using for the computation of these conditional expectations Lemma 2.1 Ch. I and the notation (13.13), the above equality can be written in the form

$$\mathbf{E}u(p,X_x(\delta)) - u(q,x) = \int_0^\delta \mathbf{E}\big\{f(X_x(v))\,u(q-v,X_x(v))\big\}dv. \tag{13.18}$$

Since $f(x)$ and $u(v,x)$, $(v,x)\in[0,T]\times\mathbf{R}$, are continuous bounded functions and the process $X_x(v)$ is continuous with respect to v, we have

$$\lim_{p\uparrow t,q\downarrow t}\frac{1}{q-p}\int_0^\delta \mathbf{E}\big\{f(X_x(v))\,u(q-v,X_x(v))\big\}dv = f(x)\,u(t,x).$$

Therefore,

$$\lim_{p\uparrow t,q\downarrow t}\frac{\mathbf{E}u(p,X_x(\delta)) - u(q,x)}{q-p} = f(x)\,u(t,x). \tag{13.19}$$

We now use the equality

$$\frac{u(q,x)-u(p,x)}{q-p} = \frac{\mathbf{E}u(p,X_x(\delta)) - u(p,x)}{q-p} - \frac{\mathbf{E}u(p,X_x(\delta)) - u(q,x)}{q-p}. \tag{13.20}$$

We can apply relation (13.12) to the first term in (13.20). Then from (13.20) and (13.19) we derive that $u(t,x)$, $(t,x)\in(0,T)\times\mathbf{R}$, is the solution of (13.14). □

Exercises.

In the following exercises all functions $u(t,x)$, $(t,x)\in[0,\infty)\times\mathbf{R}$, have the corresponding probabilistic representation (13.13).

13.1. Verify that

$$u(t,x) = \exp\Big(-\gamma xt + \frac{\gamma^2t^3}{6}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv 0$, $f(x)=\gamma x$, and $\varPhi(x)\equiv 1$.

13.2. Verify that

$$u(t,x) = \frac{1}{\sqrt{\operatorname{ch}(t\gamma)}}\exp\Big(-\frac{x^2\gamma\operatorname{sh}(t\gamma)}{2\operatorname{ch}(t\gamma)}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv 0$, $f(x)=\dfrac{\gamma^2}{2}x^2$, and $\varPhi(x)\equiv 1$.

13.3. Verify that

$$u(t,x) = \frac{1}{\sqrt{\operatorname{ch}(t\gamma)+2\beta\gamma^{-1}\operatorname{sh}(t\gamma)}}\exp\Big(-\frac{x^2(\gamma\operatorname{sh}(t\gamma)+2\beta\operatorname{ch}(t\gamma))}{2(\operatorname{ch}(t\gamma)+2\beta\gamma^{-1}\operatorname{sh}(t\gamma))}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv 0$, $f(x)=\dfrac{\gamma^2}{2}x^2$, and $\varPhi(x)=e^{-\beta x^2}$.

13.4. Verify that

$$u(t,x)=\exp\Big(-\gamma xt-\gamma\mu\frac{t^2}{2}+\frac{\gamma^2t^3}{6}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv\mu$, $f(x)=\gamma x$, and $\Phi(x)\equiv 1$.

13.5. Verify that

$$u(t,x)=\frac{1}{\sqrt{\operatorname{ch}(t\gamma)}}\exp\Big(-\mu x-\frac{\mu^2t}{2}-\frac{(x^2\gamma^2-\mu^2)\operatorname{sh}(t\gamma)-2\mu x\gamma}{2\gamma\operatorname{ch}(t\gamma)}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv 1$, $a(x)\equiv\mu$, $f(x)=\dfrac{\gamma^2}{2}x^2$, and $\Phi(x)\equiv 1$.

13.6. Verify that

$$u(t,x)=\exp\Big(-\frac{\gamma x}{\theta}\big(1-e^{-\theta t}\big)+\frac{\gamma^2\sigma^2}{2\theta^2}\Big(2\theta t+1-\big(2-e^{-\theta t}\big)^2\Big)\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv\sigma^2\theta$, $a(x)=-\theta x$, $\theta>0$, $f(x)=\gamma x$, and $\Phi(x)\equiv 1$.

13.7. Verify that

$$u(t,x)=\frac{\sqrt{\gamma}e^{\theta t/2}}{\sqrt{\operatorname{sh}(t\gamma\theta)+\gamma\operatorname{ch}(t\gamma\theta)}}\exp\Big(-\frac{x^2(\gamma^2-1)\operatorname{sh}(t\gamma\theta)}{4\sigma^2(\operatorname{sh}(t\gamma\theta)+\gamma\operatorname{ch}(t\gamma\theta))}\Big)$$

is the solution of (13.14), (13.15) with $b(x)\equiv\sigma^2\theta$, $a(x)=-\theta x$, $\theta>0$, $f(x)=\dfrac{(\gamma^2-1)\theta}{4\sigma^2}x^2$, $\gamma\ge 1$, and $\Phi(x)\equiv 1$.

DISTRIBUTIONS OF FUNCTIONALS OF BROWNIAN MOTION

§1. Distributions of integral functionals of Brownian motion

We consider general methods for computing the joint distributions of integral functionals of Brownian motion and functionals of its infimum and supremum. We begin our study by considering an integral functional of a Brownian motion $W(s)$, $s\ge 0$, since it serves as a starting point for the development of computation methods for others functionals.

We consider first the question how to compute the distribution of the integral functional $\int_0^t f(W(s))\,ds$, where f is a continuous nonnegative function.

To find the distribution function of a nonnegative random variable we can first compute the Laplace transform of the distribution and then invert this Laplace transform. Schematically, this is expressed by the formula

$$F(dy)=\mathcal{L}_\gamma^{-1}\Bigl(\int_0^\infty e^{-\gamma y}F(dy)\Bigr),\qquad \gamma\ge 0,$$

where F is the distribution function of a nonnegative random variable and $\mathcal{L}_\gamma^{-1}$ denotes the operator of the inverse Laplace transform with respect to γ.

For example, if the distribution function of some nonnegative random variable ξ has the form

$$F(y)=c_0\mathrm{1\!I}_{(0,\infty)}(y)+c_q\mathrm{1\!I}_{(q,\infty)}(y)+\int_0^y g(z)\,dz,$$

(there are two mass points 0 and q), then the corresponding distribution is

$$F(dy)=c_0\delta_0(dy)+c_q\delta_q(dy)+g(y)\,dy,$$

where $\delta_x(A)$ is the *Dirac measure*,

$$\delta_x(A):=\begin{cases}1, & \text{if } x\in A,\\ 0, & \text{if } x\notin A.\end{cases}$$

The Laplace transform of ξ with such a distribution function has the following structure:

$$\mathbf{E}e^{-\gamma\xi}=\int_0^\infty e^{-\gamma y}F(dy)=c_0+c_qe^{-\gamma q}+\int_0^\infty e^{-\gamma y}g(y)\,dy. \tag{1.1}$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_3

Since the density g is integrable, the last term tends to zero as $\gamma \to \infty$. Inverting the Laplace transform in (1.1) with respect to γ, we have

$$\mathbf{P}(\xi = 0) = c_0, \qquad \mathbf{P}(\xi = q) = c_q, \qquad \frac{d}{dy}\mathbf{P}(\xi < y) = g(y), \qquad y \neq 0, q.$$

For nonnegative integral functionals of diffusion processes it is easier to compute the Laplace transform of the distribution than the distribution itself. In fact, in many cases distributions can be computed only by means of their Laplace transforms, therefore the inverse transform must be applied. We present now a method of computing distributions of integral functionals step by step.

For brevity, we denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and the expectation with respect to the Brownian motion $W(s)$, $s \geq 0$, under the condition $W(0) = x$.

Set

$$\tilde{u}(t, x) := \mathbf{E}_x \exp\Big(-\gamma \int_0^t f(W(s))\, ds\Big), \qquad \gamma \geq 0. \tag{1.2}$$

Under the assumption that f is a nonnegative, bounded, twice continuously differentiable function with bounded derivatives, Theorem 13.2 Ch. II with $\varPhi(x) \equiv 1$, $b(x) \equiv 1$, $a(x) \equiv 0$ shows that $\tilde{u}$ is the unique solution of the Cauchy problem

$$\frac{\partial}{\partial t}\tilde{u}(t, x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{u}(t, x) - \gamma f(x)\tilde{u}(t, x), \qquad (t, x) \in (0, \infty) \times \mathbf{R},$$
$$\tilde{u}(0, x) = 1.$$

Solving this problem and inverting the Laplace transform with respect to γ, we compute the distribution function of the integral functional of the Brownian motion:

$$\mathbf{P}_x\Big(\int_0^t f(W(s))\, ds \in dy\Big) = \mathcal{L}_\gamma^{-1}(\tilde{u}(t, x)). \tag{1.3}$$

In §11 Ch. II it was shown that by means of the Laplace transform with respect to t, the Cauchy problem for a nonnegative f can be reduced to a problem for an ordinary differential equation. There we gave the natural probabilistic interpretation for the Laplace transform with respect to t in terms of an exponentially distributed with the parameter $\lambda > 0$ random time τ independent of the Brownian motion $W(s)$, $s \geq 0$.

Theorem 12.4 Ch. II implies that the function

$$\widetilde{U}(x) := \lambda \int_0^\infty e^{-\lambda t}\tilde{u}(t, x)\, dt = \mathbf{E}_x \exp\Big(-\gamma \int_0^\tau f(W(s))\, ds\Big)$$

is the unique bounded solution of the equation

$$\frac{1}{2}\widetilde{U}''(x) - (\lambda + \gamma f(x))\widetilde{U}(x) = -\lambda, \qquad x \in \mathbf{R}.$$

Since $\tilde{u}(t,x) = \mathcal{L}_\lambda^{-1}\Big(\frac{1}{\lambda}\widetilde{U}(x)\Big)$, the distribution of the integral functional can be computed as the double inverse Laplace transform of the function $\widetilde{U}(x)$:

$$\mathbf{P}_x\bigg(\int_0^t f(W(s))\,ds \in dy\bigg) = \mathcal{L}_\gamma^{-1}\mathcal{L}_\lambda^{-1}\big(\tfrac{1}{\lambda}\widetilde{U}(x)\big). \tag{1.4}$$

The order of the inverse Laplace transforms can be changed to simplify the computations. Note that the distribution of the integral functional at the random time τ can be computed by the formula

$$\mathbf{P}_x\bigg(\int_0^\tau f(W(s))\,ds \in dy\bigg) = \mathcal{L}_\gamma^{-1}\big(\widetilde{U}(x)\big). \tag{1.5}$$

This is a general approach for computing the distributions of integral functionals of a Brownian motion. The same approach is applicable to a broad class of diffusion processes, which will be considered in Ch. IV. When formulating theorems that enable us to compute the Laplace transforms of distributions of integral functionals, we for brevity exclude the parameter γ, because when needed we can take γf instead of f.

We at once consider the more general function than $\tilde{u}(t,x)$. Set

$$u(t,x) := \mathbf{E}_x\bigg\{\varPhi(W(t))\exp\bigg(-\int_0^t f(W(s))\,ds\bigg)\bigg\}. \tag{1.6}$$

Such a function is important for computing the joint distribution of an integral functional of W and the position of the Brownian motion W at the point t. The Laplace transform of $u(t,x)$, $(t,x)\in[0,\infty)\times\mathbf{R}$, with respect to t is

$$U(x) := \lambda\int_0^\infty e^{-\lambda t}u(t,x)\,dt = \mathbf{E}_x\bigg\{\varPhi(W(\tau))\exp\bigg(-\int_0^\tau f(W(s))\,ds\bigg)\bigg\}. \tag{1.7}$$

As mentioned above, we can give the Laplace transform with respect to t a natural probabilistic interpretation. If we consider the functional at the random time τ, then for computing the distribution of the integral functional, the function U has the same form and the same meaning as the function u for a fixed time t.

We now formulate Theorem 12.4 Ch. II for particular case $\sigma\equiv 1$ and $\mu\equiv 0$. As we explained above, this result is of key importance for computing the distributions of integral functionals of the Brownian motion W, or more exactly, for computing the function U.

Theorem 1.1. *Let $\varPhi(x)$ and $f(x)$, $x\in\mathbf{R}$, be continuous functions. Assume that $\varPhi$ is bounded and f is nonnegative. Then the function $U(x)$, $x\in\mathbf{R}$, is the unique bounded solution of the equation*

$$\frac{1}{2}U''(x) - (\lambda+f(x))U(x) = -\lambda\varPhi(x), \qquad x\in\mathbf{R}. \tag{1.8}$$

Remark 1.1. For a continuous $g(x) \geq 0$, $x \in \mathbf{R}$, which does not vanish identically, the homogeneous equation

$$\frac{1}{2}\phi''(x) - g(x)\phi(x) = 0, \qquad x \in \mathbf{R}, \tag{1.9}$$

has (see Proposition 12.1 Ch. II) two nonnegative convex linearly independent solutions $\psi(x)$ and $\varphi(x)$ such that $\psi(x)$ is increasing, and $\varphi(x)$ is decreasing. The *Wronskian* of these solutions ($w = \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$) is a positive constant.

The functions ψ and φ are called the *fundamental solutions* of (1.9). For a positive solution ϕ the convexity is a consequence of the equality $\phi'' \geq 0$. Being convex, the functions $\psi(x)$ and $\varphi(x)$, $x \in \mathbf{R}$, are unbounded.

It is clear that $u(t,x)$ is bounded by the same constant as Φ, therefore the function U is also bounded by the same constant. Let $g(x) = \lambda + f(x)$. Then the general solution of (1.8) has the form

$$U(x) = U_p(x) + A\psi(x) + B\varphi(x),$$

where $U_p(x)$ is a bounded particular solution of (1.8), and A and B are arbitrary coefficients. Due to this fact there is a unique bounded solution of equation (1.8), because ψ, φ can be included in the general solution of (1.8) only with zero coefficients.

Since continuous functions form a rather narrow class, Theorem 1.1 is applicable only for a restricted class of integral functionals of Brownian motion. For example, the occupation measure of a process (see definition in §5 Ch. II) does not belong to this class. The occupation measure of the interval $[a,b]$ is the integral functional that describes the amount of time spent by a sample path of the Brownian motion W in $[a,b]$ up to the time t:

$$\mu_t([a,b]) = \int_0^t \mathbb{1}_{[a,b]}(W(s))\,ds,$$

where we set $f(x) = \mathbb{1}_{[a,b]}(x)$ to define this integral functional.

However, the indicator function of an interval is a piecewise-continuous function according to the following definition.

A *piecewise-continuous function* is a function $f : \mathbf{R} \to \mathbf{R}$ having at most finitely many points of discontinuity and having left and right limits everywhere.

The following result extends the scope of application of Theorem 1.1.

Theorem 1.2. *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be piecewise-continuous functions. Assume that Φ is bounded and $f \geq 0$. Then the function*

$$U(x) := \mathbf{E}_x\Big\{\Phi(W(\tau))\exp\Big(-\int_0^\tau f(W(s))\,ds\Big)\Big\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the equation

$$\frac{1}{2}U''(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{1.10}$$

Remark 1.2. For piecewise-continuous functions f and Φ equation (1.10) must be interpreted as follows: it holds at all points of continuity of f, Φ and at points of discontinuity of f, Φ its solution is continuous together with its first derivative.

Analogous remarks apply to all subsequent results concerning solutions of differential equations, including piecewise-continuous functions, so we will not repeat them.

Proof of Theorem 1.2. Our aim is to extend Theorem 1.1 to piecewise-continuous functions Φ and f. We do this with the help of the following approach, which is referred to as the *limit approximation method.*

A nonnegative piecewise-continuous function f can be approximated by a sequence of continuous functions $\{f_n\}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad 0 \le f_n(x) \le f(x), \qquad x \in \mathbf{R}.$$

A bounded piecewise-continuous function Φ can be approximated by a sequence of continuous uniformly bounded functions $\{\Phi_n\}$ such that $\sup_{x\in\mathbf{R}} |\Phi_n(x)| \le K$ for all n and $\Phi(x) = \lim_{n\to\infty} \Phi_n(x)$, $x \in \mathbf{R}$.

Set

$$U_n(x) := \mathbf{E}_x\Big\{\Phi_n(W(\tau)) \exp\Big(-\int_0^\tau f_n(W(s))\,ds\Big)\Big\}. \tag{1.11}$$

By the Lebesgue dominated convergence theorem,

$$U_n(x) \to U(x) \qquad \text{as } n \to \infty, \quad \text{for all } x \in \mathbf{R}. \tag{1.12}$$

We assume without loss of generality that the functions f and Φ are continuous at the point zero, otherwise we could choose another point at which they are continuous.

By Theorem 1.1, the function $U_n(x)$, $x \in \mathbf{R}$, satisfies the equation

$$\frac{1}{2}(U_n'(x) - U_n'(0)) - \int_0^x (\lambda + f_n(y))U_n(y)\,dy = -\lambda \int_0^x \Phi_n(y)\,dy, \qquad x \in \mathbf{R}. \tag{1.13}$$

Integrating this equation, we obtain

$$\frac{1}{2}\big(U_n(x) - U_n(0)\big) - \frac{1}{2}U_n'(0)x - \int_0^x \int_0^z (\lambda + f_n(y))U_n(y)\,dydz$$

$$= -\lambda \int_0^x \int_0^z \Phi_n(y)\, dydz. \tag{1.14}$$

The sequence $\{\Phi_n(x)\}$ is uniformly bounded: $|\Phi_n(x)| \le K$ for all n and $x \in \mathbf{R}$. Then, taking into account (1.11), we have $|U_n(x)| \le K$. Since f is a piecewise-continuous function, the estimate $0 \le f_n(x) \le f(x)$, $x \in \mathbf{R}$, shows that the functions f_n are uniformly bounded in any finite interval $[a, b]$. Then using (1.12) and the Lebesgue dominated convergence theorem, we deduce from (1.14) that the limit $\widetilde{U}_0 := \lim_{n\to\infty} U_n'(0)$ exists and

$$\frac{1}{2}\big(U(x) - U(0)\big) - \frac{1}{2}\widetilde{U}_0 x - \int_0^x \int_0^z (\lambda + f(y))U(y)\, dydz = -\lambda \int_0^x \int_0^z \Phi(y)\, dydz.$$

From this it follows that the function $U(x)$ is continuous in $x \in \mathbf{R}$. In addition, it is differentiable for all x including zero, and $U'(0) = \widetilde{U}_0$.

Differentiating the equation above, we get

$$\frac{1}{2}(U'(x) - U'(0)) - \int_0^x (\lambda + f(y))U(y)\, dy = -\lambda \int_0^x \Phi(y)\, dy, \qquad x \in \mathbf{R}. \tag{1.15}$$

This relation is equivalent to the following statement: for the piecewise-continuous functions f and Φ, the function U is the solution of equation (1.10). Indeed, at any point x in which the functions f and Φ are continuous, equality (1.15) can be differentiated with respect to x, which implies (1.10). At the points of discontinuity of f and Φ from (1.15) one can infer only the continuity of the derivative U' and therefore the continuity of U. By Remark 1.2, (1.10) is valid.

The uniqueness of the bounded solution of equation (1.10) on the whole real line can be established in the following way. The real line is decomposed into finitely many intervals in the interiors of which f and Φ are continuous. These intervals can be enumerated by an index $k = 0, 1, 2, \dots, m$. The extreme left and right of these intervals have an infinite length. The general solution of (1.10) in every interval of finite length depends on two arbitrary constants A_k, B_k and has the form

$$U(x) = U_{p,k}(x) + A_k\psi_k(x) + B_k\varphi_k(x), \qquad k = 0, 1, 2, \dots, m,$$

where k is the index of the interval, $U_{p,k}$ is a particular bounded solution of (1.10) in this interval, and ψ_k, φ_k are the fundamental solutions of the corresponding homogeneous equation in the same interval. On the left and right intervals of infinite length the general bounded solution of (1.10) depends only on one unknown constant, because of the unboundedness of the linearly independent solutions ψ and φ of equation (1.9) for $g(x) = \lambda + f(x)$ at plus or minus infinity, respectively. Therefore $B_0 = 0$ and $A_m = 0$. Since there are m boundary points between intervals, the number of unknown constants equals $2m$. At any of these boundary points the function U must be continuous together with its first derivative. Thus, there are $2m$ conditions for the glued together solutions on adjacent intervals to satisfy the condition of continuity of U and its derivative U'. This leads to $2m$ linear algebraic equations for the unknown constants and this system has a unique solution. Theorem 1.2 is proved. □

Proposition 1.1. *Let $f(x)$, $x \in \mathbf{R}$, be a nonnegative piecewise-continuous function which does not vanish identically. Then*

$$\int_0^\infty f(W(s))\,ds = \infty \qquad \text{a.s.} \tag{1.16}$$

Proof. Set

$$U_\lambda(x) := \mathbf{E}_x \exp\Big(-\int_0^\tau f(W(s))\,ds\Big).$$

Then $U_\lambda(x)$ is the unique bounded solution of the equation

$$\frac{1}{2}U''(x) - (\lambda + f(x))U(x) = -\lambda, \qquad x \in \mathbf{R}. \tag{1.17}$$

Since $\mathbf{P}(\tau > t) = e^{-\lambda t}$, we see that $\tau \to \infty$ in probability as $\lambda \to 0$. Therefore,

$$\lim_{\lambda\to 0} U_\lambda(x) = \mathbf{E}_x \exp\Big(-\int_0^\infty f(W(s))\,ds\Big) := U_\infty(x)$$

for every $x \in \mathbf{R}$. Applying for equation (1.17) the limit approximation method described in the proof of Theorem 1.2, we get that $U_\infty(x)$ is the unique bounded solution of the equation

$$\frac{1}{2}U''(x) - f(x)U(x) = 0, \qquad x \in \mathbf{R}.$$

But this solution, in view of Remark 1.1, is zero ($U_\infty(x) \equiv 0$). This is also true for piecewise-continuous f, because the gluing procedure described above implies $U_{p,k}(x) \equiv 0$, $A_k = 0$, $B_k = 0$, $k = 0, 1, 2, \dots, m$. The proposition is proved. □

Example 1.1. We compute the distribution of the total amount of time spent by the Brownian motion W in the interval $[r, \infty)$ up to the time t. That is, we are interested in the distribution of the functional

$$A(t) := \int_0^t \mathbb{1}_{[r,\infty)}(W(s))\,ds.$$

We first solve this problem for the random time τ instead of a fixed time t. Applying Theorem 1.2 with $\Phi(x) \equiv 1$, $f(x) = \gamma\mathbb{1}_{[r,\infty)}(x)$, $\gamma > 0$, we see that the function

$$U(x) = \mathbf{E}_x \exp\Big(-\gamma\int_0^\tau \mathbb{1}_{[r,\infty)}(W(s))\,ds\Big)$$

is the unique bounded continuous solution of the equation

$$\frac{1}{2}U''(x) - (\lambda + \gamma \mathbb{I}_{[r,\infty)}(x))U(x) = -\lambda, \quad x \in (-\infty, \infty). \tag{1.18}$$

A particular bounded solution of (1.18) in the interval $(-\infty, r)$ is equal to 1 and the fundamental solutions of the corresponding homogeneous equation are $e^{-x\sqrt{2\lambda}}$, $e^{x\sqrt{2\lambda}}$. The analogous solutions in the interval (r, ∞) are $\lambda/(\lambda+\gamma)$, $e^{-x\sqrt{2\lambda+2\gamma}}$, and $e^{x\sqrt{2\lambda+2\gamma}}$. Taking into account the boundedness and continuity of the solution of (1.18), we can see that this solution has the following form:

$$U(x) = \begin{cases} 1 + (A-1)e^{-(r-x)\sqrt{2\lambda}}, & x \le r, \\ \dfrac{\lambda}{\lambda+\gamma} + \Big(A - \dfrac{\lambda}{\lambda+\gamma}\Big)e^{-(x-r)\sqrt{2\lambda+2\gamma}}, & r \le x. \end{cases}$$

The unknown constant A is computed from the condition that the derivative of the function U at the point r must also be continuous. Thus,

$$\sqrt{2\lambda}(A-1) = -\sqrt{2\lambda+2\gamma}\Big(A - \frac{\lambda}{\lambda+\gamma}\Big),$$

so $A = \dfrac{\sqrt{\lambda}}{\sqrt{\lambda+\gamma}}$. As a result, we obtain

$$\mathbf{E}_x \exp\Big(-\gamma \int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds\Big)$$
$$= \begin{cases} 1 - \Big(1 - \dfrac{\sqrt{\lambda}}{\sqrt{\lambda+\gamma}}\Big)e^{-(r-x)\sqrt{2\lambda}}, & x \le r, \\ \dfrac{\lambda}{\lambda+\gamma} + \Big(\dfrac{\sqrt{\lambda}}{\sqrt{\lambda+\gamma}} - \dfrac{\lambda}{\lambda+\gamma}\Big)e^{-(x-r)\sqrt{2\lambda+2\gamma}}, & r \le x. \end{cases} \tag{1.19}$$

The structure (see (1.1)) of this Laplace transform with respect to γ is such that the corresponding distribution for $x < r$ has the mass point at zero equal to $1 - e^{-(r-x)\sqrt{2\lambda}}$. Thus,

$$\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds = 0\Big) = \begin{cases} 1 - e^{-(r-x)\sqrt{2\lambda}}, & x \le r, \\ 0, & r \le x. \end{cases} \tag{1.20}$$

The other part of the Laplace transform (1.19) corresponds to the density of the distribution function. Taking the inverse Laplace transform with respect to γ (see formulas a, 5 and 6 of Appendix 3), we obtain

$$\frac{d}{dy}\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))ds < y\Big)$$

$$= \begin{cases} \dfrac{\sqrt{\lambda}}{\sqrt{\pi y}}e^{-\lambda y-(r-x)\sqrt{2\lambda}}, & x \le r, \\ \lambda e^{-\lambda y}\Big(1 - \operatorname{Erfc}\Big(\dfrac{x-r}{\sqrt{2y}}\Big)\Big) + \dfrac{\sqrt{\lambda}}{\sqrt{\pi y}}e^{-\lambda y-(x-r)^2/2y}, & r \le x. \end{cases} \tag{1.21}$$

The error function $\operatorname{Erfc}(x)$, $x \in \mathbf{R}$, is defined in Appendix 2. To obtain the analogous distributions for a fixed time t, it is necessary to divide the right-hand sides of these formulas by λ and then take the inverse Laplace transform with respect to λ. Inverting (1.20), we have

$$\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W(s))ds = 0\Big) = \begin{cases} 1 - \operatorname{Erfc}\Big(\frac{r-x}{\sqrt{2t}}\Big), & x \le r, \\ 0, & r \le x. \end{cases} \tag{1.22}$$

It is clear that $A(t) \le t$, or correspondingly $A(\tau) \le \tau$. For $r < x$ there exists the special case when $\sup_{0\le s\le\tau} W(s) \ge r$, or equivalently $A(\tau) = \tau$ with positive probability.

Therefore,

$$\frac{d}{dy}\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds < y\Big) = \frac{d}{dy}\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds = \tau,\ \tau < y\Big)$$

$$+ \frac{d}{dy}\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds < \tau, \int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds < y\Big), \qquad r < x.$$

Since τ is independent of W, we get

$$\frac{d}{dy}\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W(s))\,ds = \tau, \tau < y\Big)$$

$$= \lambda\frac{d}{dy}\int_0^y e^{-\lambda t}\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W(s))\,ds = t\Big)dt = \lambda e^{-\lambda y}\mathbf{P}_x\Big(\int_0^y \mathbb{I}_{[r,\infty)}(W(s))\,ds = y\Big).$$

Comparing this equality with (1.21) for $r \le x$, we obtain

$$\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W(s))ds = t\Big) = 1 - \operatorname{Erfc}\Big(\frac{x-r}{\sqrt{2t}}\Big), \qquad r \le x. \tag{1.23}$$

The other part of the Laplace transform in (1.21) corresponds to the density. Dividing it by λ and inverting the Laplace transform with respect to λ (see formulas b and 5 of Appendix 3), we get

$$\frac{d}{dy}\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W(s))\,ds < t, \int_0^t \mathbb{I}_{[r,\infty)}(W(s))\,ds < y\Big)$$

$$= \frac{d}{dy}\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W(s))ds < y\Big) = \begin{cases} \dfrac{\mathbb{I}_{(0,t)}(y)}{\pi\sqrt{y(t-y)}}\, e^{-(r-x)^2/2(t-y)}, & x \le r, \\ \dfrac{\mathbb{I}_{(0,t)}(y)}{\pi\sqrt{y(t-y)}}\, e^{-(x-r)^2/2y}, & r \le x. \end{cases} \tag{1.24}$$

The first equality is obvious when considering the cases $0 < y < t$ and $t < y$.

Note that in the special case $x = r$ we have

$$\frac{d}{dy}\mathbf{P}_r\Big(\int_0^t \mathbb{1}_{[r,\infty)}(W(s))\,ds < y\Big) = \frac{\mathbb{1}_{(0,t)}(y)}{\pi\sqrt{y(t-y)}}. \tag{1.25}$$

Integrating with respect to y, we obtain the well-known *arcsine law* of P. Lévy:

$$\mathbf{P}_r\Big(\int_0^t \mathbb{1}_{[r,\infty)}(W(s))\,ds < y\Big) = \frac{2}{\pi}\arcsin\sqrt{y/t}, \qquad 0 \le y \le t. \tag{1.26}$$

Exercises.

1.1. Compute $\mathbf{E}e^{-\alpha\tau+i\beta W(\tau)}$, $\alpha > 0$. Derive from this expression the formula for the density

$$\frac{d}{dz}\mathbf{P}_x(W(\tau) < z).$$

1.2. Compute

$$\mathbf{E}_x\Big\{\exp\Big(-\alpha\tau - \gamma\int_0^\tau \mathbb{1}_{(-\infty,r)}(W(s))ds\Big)\Big\}, \qquad \alpha > 0, \qquad \gamma > 0.$$

1.3. Compute the occupation time distribution of the Brownian motion W in the interval $(-\infty, r)$ up to the time t.

1.4. Compute

$$\mathbf{E}_x \exp\Big(-\int_0^\tau \big(p\mathbb{1}_{(-\infty,r)}(W(s)) + q\mathbb{1}_{[r,\infty)}(W(s))\big)ds\Big), \qquad p > 0, \qquad q > 0.$$

1.5. Compute $\mathbf{E}_x \exp\Big(-\gamma\int_0^\tau \mathbb{1}_{[r,u]}(W(s))ds\Big)$, $\quad \gamma > 0$.

§2. Distributions of integral functionals of Brownian motion and of infimum and supremum functionals

Consider the problem of computation of the joint distribution of the integral functional

$$A(t) = \int_0^t f(W(s))ds,$$

and of the variables $\inf_{0\le s\le t} W(s)$ and $\sup_{0\le s\le t} W(s)$.

In order to simplify formulas, we will use the notation $\mathbf{E}\{\xi; A\} := \mathbf{E}\{\xi \mathbb{1}_A\}$.

The main idea, used for the computation of infimum or supremum type functionals can be illustrated by the following example.

Let $X(s)$, $s \geq 0$, be an arbitrary continuous process. Then

$$\mathbf{P}\Big(\sup_{0\le s\le t} X(s) \le h\Big) = \lim_{\gamma\to\infty} \mathbf{E}\exp\Big(-\gamma\int_0^t \mathbb{1}_{(h,\infty)}(X(s))\,ds\Big).$$

Indeed,

$$\mathbf{E}\exp\Big(-\gamma\int_0^t \mathbb{1}_{(h,\infty)}(X(s))\,ds\Big) = \mathbf{P}\Big(\sup_{0\le s\le t} X(s) \le h\Big)$$

$$+\mathbf{E}\Big\{\exp\Big(-\gamma\int_0^t \mathbb{1}_{(h,\infty)}(X(s))\,ds\Big);\ \sup_{0\le s\le t} X(s) > h\Big\}.$$

In the last expectation we have that $\int_0^t \mathbb{1}_{(h,\infty)}(X(s))\,ds > 0$. Therefore, the exponential function under the expectation sign tends to zero as $\gamma \to \infty$, and hence the expectation, reduced to the set $\sup_{0\le s\le t} X(s) > h$, also tends to zero.

The main result for computing the joint distributions of integral functionals of a Brownian motion and its infimum and supremum is the following theorem.

Theorem 2.1. *Let $\Phi(x)$ and $f(x)$, $x \in [a, b]$, be piecewise-continuous functions. Assume that $f \ge 0$ and Φ is bounded if either $a = -\infty$ or $b = \infty$. Then the function*

$$U(x) := \mathbf{E}_x\Big\{\Phi(W(\tau))\exp\Big(-\int_0^\tau f(W(s))\,ds\Big); a \le \inf_{0\le s\le\tau} W(s),\ \sup_{0\le s\le\tau} W(s) \le b\Big\},$$

$x \in [a, b]$, is the unique solution of the problem

$$\frac{1}{2}U''(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in (a, b), \tag{2.1}$$

$$U(a) = 0, \qquad U(b) = 0. \tag{2.2}$$

Remark 2.1. In the case $a = -\infty$ or $b = \infty$ the corresponding boundary condition in (2.2) must be replaced by the condition that the function $U(x)$ is bounded as x tends to $-\infty$ or ∞.

Proof of Theorem 2.1. The general bounded solution of (2.1) on the intervals of a finite length where the functions Φ and f are continuous depends on two unknown constants. All such constants are uniquely determined by the conditions of continuity of the solution U and its derivative U' at the boundaries of these intervals, as well as by the boundary conditions (2.2). This proves that the solution of the problem (2.1), (2.2) is unique.

Set

$$U_\gamma(x) := \mathbf{E}_x\Big\{\Phi(W(\tau)) \exp\Big(- \int_0^\tau (f(W(s)) + \gamma \mathbb{I}_{\mathbf{R}\setminus[a,b]}(W(s)))\, ds\Big)\Big\}. \tag{2.3}$$

The proof of Theorem 2.1 is based on the following almost obvious relation: for all $x \in (a,b)$,

$$U(x) = \lim_{\gamma\to\infty} U_\gamma(x). \tag{2.4}$$

To see this we represent the expectation in (2.3) as the sum of two terms: the expectation over the set

$$Q = \Big\{a \le \inf_{0\le s\le\tau} W(s),\ \sup_{0\le s\le\tau} W(s) \le b\Big\},$$

and the expectation over its complement $Q^{\rm c}$. The first expectation is exactly equal to $U(x)$, and on the complement $Q^{\rm c}$ we have that $\int_0^\tau \mathbb{I}_{\mathbf{R}\setminus[a,b]}(W(s))\, ds > 0$. Therefore, on $Q^{\rm c}$ the exponential function under the expectation sign tends to zero as $\gamma \to \infty$, and hence the expectation over $Q^{\rm c}$ also tends to zero.

We apply Theorem 1.2 with the function $f(x) + \gamma \mathbb{I}_{\mathbf{R}\setminus[a,b]}(x)$ instead of $f(x)$ $x \in \mathbf{R}$. By this result, the function U_γ is the unique bounded continuous solution of the equation

$$\frac12 U''(x) - (\lambda + f(x) + \gamma \mathbb{I}_{\mathbf{R}\setminus[a,b]}(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{2.5}$$

We will express the solution of equation (2.5) in terms of the fundamental solutions ψ, φ of equation (1.9) with $g(x) = \lambda + f(x)$ and the fundamental solutions ψ_γ, φ_γ of equation (1.9) with $g(x) = \lambda + f(x) + \gamma$. For the piecewise-continuous function f these solutions must be treated according to Remark 1.2. Denote by w and w_γ the Wronskians of these solutions. The bounded solution of (2.5) can be represented as follows:

$$U_\gamma(x) = \begin{cases} A_1(\gamma)\psi_\gamma(x) + \phi_\gamma(x), & x \le a, \\ A_2(\gamma)\psi(x) + B_2(\gamma)\varphi(x) + \phi(x), & a \le x \le b, \\ B_3(\gamma)\varphi_\gamma(x) + \phi_\gamma(x), & b \le x, \end{cases}$$

where

$$\phi_\gamma(x) = \frac{2\lambda}{w_\gamma}\varphi_\gamma(x) \int^x \psi_\gamma(z)\Phi(z)\, dz - \frac{2\lambda}{w_\gamma}\psi_\gamma(x) \int^x \varphi_\gamma(z)\Phi(z)\, dz,$$

$$\phi(x) = \frac{2\lambda}{w}\varphi(x) \int^x \psi(z)\Phi(z)\, dz - \frac{2\lambda}{w}\psi(x) \int^x \varphi(z)\Phi(z)\, dz. \tag{2.6}$$

The coefficients $A_1(\gamma)$, $A_2(\gamma)$, $B_2(\gamma)$ and $B_3(\gamma)$ are uniquely determined from the continuity conditions of the function U_γ and its derivative at a and b.

For a Brownian motion the following fact holds: starting at the boundary of the interval $[a,b]$, the Brownian motion W spends a.s. a positive time outside $[a,b]$ up to the time τ. For the boundary b this is a consequence of (1.20) for $x=r$, because

$$\mathbf{P}_b\biggl(\int_0^\tau \mathbb{I}_{[b,\infty)}(W(s))\,ds>0\biggr)=1-\mathbf{P}_b\biggl(\int_0^\tau \mathbb{I}_{[b,\infty)}(W(s))\,ds=0\biggr)=1.$$

Moreover, the Brownian motion spends a nonzero time outside $[a,b]$ if the initial point of W lies outside this interval. Using this fact and the explicit form of the function U_γ defined in (2.3), we easily get that $U_\gamma(x)\to 0$ for $x\le a$ and $x\ge b$. From (2.4) and the expression for $U_\gamma(x)$, $x\in[a,b]$, in terms of the solutions $\psi(x)$, $\varphi(x)$ we can deduce that there exist the limits $A_2:=\lim_{\gamma\to\infty}A_2(\gamma)$, $B_2:=\lim_{\gamma\to\infty}B_2(\gamma)$. Indeed, we can choose any two points $p>q$ in the interval $[a,b]$ and solve for the unknown constants $A_2(\gamma)$ and $B_2(\gamma)$ the system of algebraic equations

$$\begin{aligned}U_\gamma(p)&=A_2(\gamma)\psi(p)+B_2(\gamma)\varphi(p)+\phi(p),\\ U_\gamma(q)&=A_2(\gamma)\psi(q)+B_2(\gamma)\varphi(q)+\phi(q).\end{aligned}$$

By the monotonicity of the solutions ψ and φ, we have $\psi(p)\varphi(q)-\psi(q)\varphi(p)>0$ and

$$\begin{aligned}A_2(\gamma)&=\frac{(U_\gamma(p)-\phi(p))\varphi(q)-(U_\gamma(q)-\phi(q))\varphi(p)}{\psi(p)\varphi(q)-\psi(q)\varphi(p)},\\ B_2(\gamma)&=\frac{\psi(p)(U_\gamma(q)-\phi(q))-\psi(q)(U_\gamma(p)-\phi(p))}{\psi(p)\varphi(q)-\psi(q)\varphi(p)}.\end{aligned}$$

By (2.4), the sequences $U_\gamma(p)$, $U_\gamma(q)$ converge to the values $U(p)$, $U(q)$. Consequently, the sequences $A_2(\gamma)$, $B_2(\gamma)$ converge to the corresponding limits.

We can choose $p=b$ and $q=a$. Since $U_\gamma(a)\to 0$ and $U_\gamma(b)\to 0$ as $\gamma\to\infty$, we have

$$A_2=\frac{\phi(a)\varphi(b)-\phi(b)\varphi(a)}{\psi(b)\varphi(a)-\psi(a)\varphi(b)},\qquad B_2=\frac{\psi(a)\phi(b)-\psi(b)\phi(a)}{\psi(b)\varphi(a)-\psi(a)\varphi(b)}.\tag{2.7}$$

Therefore, the limiting function $U(x)$, $x\in\mathbf{R}$, has the form

$$U(x)=\begin{cases}0, & x\le a,\\ A_2\psi(x)+B_2\varphi(x)+\phi(x), & a\le x\le b,\\ 0, & b\le x.\end{cases}\tag{2.8}$$

From (2.7) and (2.8) it follows that U is continuous in $x\in\mathbf{R}$ and satisfies the problem (2.1), (2.2). Theorem 2.1 is proved. □

Note that formulas (2.6)–(2.8) give the solution of the problem (2.1), (2.2) in terms of the fundamental solutions ψ and φ of the equation

$$\frac12\phi''(x)-(\lambda+f(x))\phi(x)=0.\tag{2.9}$$

Example 2.1. We compute the distribution of the supremum of Brownian motion, i.e., the variable $\sup_{0\le s\le t} W(s)$, and then the joint distribution of $\inf_{0\le s\le t} W(s)$ and $\sup_{0\le s\le t} W(s)$. Applying Theorem 2.1 with $\Phi\equiv 1$, $f=0$, we see that

$$U(x)=\mathbf{P}_x\Big(a\le \inf_{0\le s\le \tau} W(s),\ \sup_{0\le s\le \tau} W(s)\le b\Big)$$

is the unique solution of the problem

$$\frac{1}{2}U''(x)-\lambda U(x)=-\lambda,\qquad x\in(a,b), \tag{2.10}$$

$$U(a)=0,\qquad U(b)=0. \tag{2.11}$$

The particular solution of equation (2.10) is the function identically equal to 1. We can choose $\operatorname{sh}((b-x)\sqrt{2\lambda})$, $\operatorname{sh}((x-a)\sqrt{2\lambda})$ as the two linearly independent solutions of the corresponding homogeneous equation. These functions are the most suitable for the solutions of the problem (2.10), (2.11), since the first one equals 0 at b and the second one equals 0 at a. Using the boundary conditions (2.11), it is easy to compute the unknown constants figuring in the linear combination of these functions (formula (2.8)). We thus obtain that the solution of the problem (2.10), (2.11) has the form

$$\begin{aligned} U(x)&=\mathbf{P}_x\Big(a\le \inf_{0\le s\le \tau} W(s),\ \sup_{0\le s\le \tau} W(s)\le b\Big)\\ &=1-\frac{\operatorname{sh}((b-x)\sqrt{2\lambda})+\operatorname{sh}((x-a)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})}=1-\frac{\operatorname{ch}((b+a-2x)\sqrt{\lambda/2})}{\operatorname{ch}((b-a)\sqrt{\lambda/2})}. \end{aligned} \tag{2.12}$$

Letting $a\to-\infty$, we can deduce that

$$\mathbf{P}_x\Big(\sup_{0\le s\le \tau} W(s)\le b\Big)=1-e^{-(b-x)\sqrt{2\lambda}},\qquad x\le b.$$

Dividing this equality by λ and inverting the Laplace transform with respect to λ (see formula 6 of Appendix 2), we obtain

$$\mathbf{P}_x\Big(\sup_{0\le s\le t} W(s)\le b\Big)=1-\operatorname{Erfc}\Big(\frac{b-x}{\sqrt{2t}}\Big)=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{(b-x)/\sqrt{t}} e^{-v^2/2}\,dv. \tag{2.13}$$

Hence, using the symmetry property and the spatial homogeneity of a Brownian motion, it is easy to deduce from (2.13) that

$$\mathbf{P}_x\Big(a\le \inf_{0\le s\le t} W(s)\Big)=1-\operatorname{Erfc}\Big(\frac{x-a}{\sqrt{2t}}\Big)=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{(x-a)/\sqrt{t}} e^{-v^2/2}\,dv,\qquad x\ge a. \tag{2.14}$$

These formulas imply the following useful estimate: for any $h > 0$

$$\mathbf{P}_0\Big(\sup_{0\le s\le t}|W(s)| > h\Big) \le \frac{2\sqrt{2}}{\sqrt{\pi}}\int\limits_{h/\sqrt{t}}^{\infty} e^{-v^2/2}\,dv < \frac{2\sqrt{2t}}{h\sqrt{\pi}}\,e^{-h^2/2t}. \tag{2.15}$$

In the last inequality we apply estimate (10.7) of Ch. I.

Now consider the joint distribution of the supremum and infimum of a Brownian motion. Dividing the right-hand side of (2.12) by λ and inverting the Laplace transform with respect to λ (see Section 13 of Appendix 2), we get

$$\mathbf{P}_x\Big(a \le \inf_{0\le s\le t} W(s),\ \sup_{0\le s\le t} W(s) \le b\Big) = 1 - \widetilde{\mathrm{ss}}_t(b-x, b-a) - \widetilde{\mathrm{ss}}_t(x-a, b-a)$$

$$= 1 - \widetilde{\mathrm{cc}}_t((b+a-2x)/2, (b-a)/2).$$

One can get another expression for this probability. It is essentially associated with the similar formula for the Brownian bridge (see (4.44)). One can verify that

$$\mathbf{P}_x\Big(a \le \inf_{0\le s\le \tau} W(s),\ \sup_{0\le s\le \tau} W(s) \le b\Big)$$

$$= \int\limits_a^b \frac{\sqrt{\lambda}\,(\mathrm{ch}((b-a-|z-x|)\sqrt{2\lambda}) - \mathrm{ch}((b+a-z-x)\sqrt{2\lambda})}{\sqrt{2}\,\mathrm{sh}((b-a)\sqrt{2\lambda})}\,dz.$$

Dividing the right-hand side by λ and inverting the Laplace transform with respect to λ (see section 13 of Appendix 2), we obtain

$$\mathbf{P}_x\Big(a \le \inf_{0\le s\le t} W(s),\ \sup_{0\le s\le t} W(s) \le b\Big)$$

$$= \int\limits_a^b \big(\mathrm{cs}_t(b-a-|z-x|, b-a) - \mathrm{cs}_t(b+a-z-x, b-a)\big)\,dz$$

$$= \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\int\limits_a^b \big(e^{-(z-x+2k(b-a))^2/2t} - e^{-(z+x-2a+2k(b-a))^2/2t}\big)\,dz. \tag{2.16}$$

Exercises.

2.1. Compute

$$\mathbf{E}_x\big\{e^{-\alpha\tau};\ a \le \inf_{0\le s\le\tau} W(s)\big\}, \qquad \mathbf{E}_x\big\{e^{-\alpha\tau};\ \sup_{0\le s\le\tau} W(s) \le b\big\}, \qquad \alpha > 0.$$

2.2. Compute

$$\mathbf{E}_x\big\{e^{-\alpha\tau+i\beta W(\tau)}; a \le \inf_{0\le s\le\tau} W(s)\big\}, \qquad \mathbf{E}_x\big\{e^{-\alpha\tau+i\beta W(\tau)};\ \sup_{0\le s\le\tau} W(s) \le b\big\},$$

for $\alpha > 0$ and $\beta \in \mathbf{R}$.

2.3. Compute

$$\mathbf{E}_x\Big\{\cos(W(\tau)); a \le \inf_{0\le s\le\tau} W(s)\Big\}, \qquad \mathbf{E}_x\Big\{\sin(W(\tau)); \sup_{0\le s\le\tau} W(s) \le b\Big\}.$$

2.4. Compute

$$\mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^\tau \mathbb{1}_{(-\infty,r)}(W(s))\,ds\Big); \sup_{0\le s\le\tau} W(s) \le b\Big\}, \qquad \gamma > 0.$$

2.5. Compute

$$\mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^\tau \mathbb{1}_{(r,\infty)}(W(s))\,ds\Big); a \le \inf_{0\le s\le\tau} W(s)\Big\}, \qquad \gamma > 0.$$

§3. Distributions of functionals of Brownian motion and local times

We consider the problem of computing the joint distribution of integral functional of Brownian motion, local time at different levels, and $\inf\limits_{0\le s\le t} W(s)$, $\sup\limits_{0\le s\le t} W(s)$.

We restrict ourselves to additive functionals of the Brownian motion W which can be represented in the form

$$A_{\vec\beta}(t) := \int_0^t f(W(s))\,ds + \sum_{l=1}^m \beta_l\,\ell(t,q_l),$$

where f is a nonnegative piecewise-continuous function, ℓ is the Brownian local time, $\beta_l \ge 0$, $q_l \in \mathbf{R}$, and $m < \infty$.

Following the general approach for computing distributions of functionals, we consider the Laplace transform

$$\widetilde{m}(t,x) := \mathbf{E}_x \exp\Big(-\gamma\int_0^t f(W(s))\,ds - \sum_{l=1}^m \beta_l\,\ell(t,q_l)\Big).$$

Having an explicit expression for this transform, the joint distribution of an integral functional of the Brownian motion and the local time at different levels can be computed as the multiple inverse Laplace transform of the function $\widetilde{m}(t,x)$:

$$\mathbf{P}_x\Big(\int_0^t f(W(s))\,ds \in dy, \ell(t,q_1) \in dy_1, \dots, \ell(t,q_m) \in dy_m\Big)$$

$$= \mathcal{L}_\gamma^{-1}\mathcal{L}_{\beta_1}^{-1}\cdots\mathcal{L}_{\beta_m}^{-1}\big(\widetilde{m}(t,x)\big).$$

Instead of a fixed time t one should take the exponentially distributed with the parameter $\lambda > 0$ random time τ, which corresponds to the Laplace transform with respect to t. For a fixed t the formula is obtained from the appropriate assertion for τ by means of the inverse Laplace transform with respect to λ.

The main result of this section is the following.

Theorem 3.1. *Let $\Phi(x)$, $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \geq 0$. Then the function*

$$M(x) := \mathbf{E}_x \Big\{ \Phi(W(\tau)) \exp(-A_{\vec\beta}(\tau));\ a \leq \inf_{0\leq s\leq\tau} W(s),\ \sup_{0\leq s\leq\tau} W(s) \leq b \Big\},$$

$x \in [a,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}M''(x) - (\lambda + f(x))M(x) = -\lambda\Phi(x), \qquad x \in (a,b) \setminus \{q_1,\dots,q_m\}, \tag{3.1}$$

$$M'(q_k+0) - M'(q_k-0) = 2\beta_k M(q_k), \qquad k = 1,\dots,m, \tag{3.2}$$

$$M(a) = 0, \qquad M(b) = 0. \tag{3.3}$$

Remark 3.1. In the case $a = -\infty$ or $b = \infty$ we in addition assume that Φ is bounded. Then the corresponding boundary condition in (3.3) must be replaced by the condition that the function $M(x)$ is bounded as x tends to $-\infty$ or ∞.

Proof of Theorem 3.1. Arguments similar to those given in the proof of the uniqueness of the bounded solution of equation (1.10) are applicable to the proof of the uniqueness of the solution of the problem (3.1)–(3.3), so the proof of the uniqueness is omitted.

Without loss of generality we assume that only $\beta_1 \neq 0$. Set

$$\chi_\varepsilon(x) := \int_{-\infty}^{x} \frac{1}{\varepsilon} \mathbb{1}_{[q_1,q_1+\varepsilon)}(u)\,du, \qquad x \in \mathbf{R}.$$

By the definition of the Brownian local time (see (5.9) of Ch. II),

$$\ell(t,q_1) = \lim_{\varepsilon\downarrow 0} \int_0^t \frac{1}{\varepsilon} \mathbb{1}_{[q_1,q_1+\varepsilon)}(W(s))\,ds \qquad \text{a.s.} \tag{3.4}$$

uniformly in t from any finite interval and therefore,

$$\frac{1}{\varepsilon}\int_0^\tau \mathbb{1}_{[q_1,q_1+\varepsilon)}(W(s))\,ds \underset{\varepsilon\downarrow 0}{\longrightarrow} \ell(\tau,q_1)$$

in probability.

Set

$$M_\varepsilon(x) := \mathbf{E}_x \Big\{ \Phi(W(\tau)) \exp\Big(-\int_0^\tau \big(f(W(s)) + \frac{\beta_1}{\varepsilon} \mathbb{1}_{[q_1,q_1+\varepsilon)}(W(s)) \big)\,ds \Big);$$

$$a \leq \inf_{0\leq s\leq\tau} W(s),\ \sup_{0\leq s\leq\tau} W(s) \leq b \Big\}.$$

Then by the Lebesgue dominated convergence theorem

$$M(x)=\lim_{\varepsilon\downarrow 0}M_\varepsilon(x)\qquad \text{for every } x\in[a,b]. \tag{3.5}$$

For M_ε we can apply Theorem 2.1. In this case equation (2.1) can be written in the following form: for any $y,x\in(a,b)$

$$\frac{1}{2}(M_\varepsilon'(x)-M_\varepsilon'(y))-\int_y^x(\lambda+f(r))M_\varepsilon(r)\,dr$$

$$-\beta_1\int_y^x M_\varepsilon(r)\,d\chi_\varepsilon(r)=-\lambda\int_y^x \varPhi(r)\,dr. \tag{3.6}$$

In addition, the boundary conditions

$$M_\varepsilon(a)=0,\qquad M_\varepsilon(b)=0 \tag{3.7}$$

hold. Our aim is to pass to the limit as $\varepsilon\downarrow 0$ in the problem (3.6), (3.7), applying (3.5). For this we use the approach, which is the development of the *limit approximation method*, applied in the proof of Theorem 1.2. The main feature here is the presence of the function χ_ε, converging to the indicator function. Furthermore, we must justify the convergence of the boundary conditions (3.7).

Integrating (3.6) with respect to x over the interval (y,v) for some $v>y$, we get

$$\frac{1}{2}\big(M_\varepsilon(v)-M_\varepsilon(y)\big)-\frac{1}{2}M_\varepsilon'(y)(v-y)-\int_y^v\int_y^x(\lambda+f(r))M_\varepsilon(r)\,drdx$$

$$-\beta_1\int_y^v\int_y^x M_\varepsilon(r)\,d\chi_\varepsilon(r)\,dx=-\lambda\int_y^v\int_y^x\varPhi(r)\,drdx. \tag{3.8}$$

Integrating (3.6) with respect to y over the interval (u,x) for some $u<x$, we get

$$-\frac{1}{2}\big(M_\varepsilon(x)-M_\varepsilon(u)\big)+\frac{1}{2}M_\varepsilon'(x)(x-u)-\int_u^x\int_y^x(\lambda+f(r))M_\varepsilon(r)\,dr\,dy$$

$$-\beta_1\int_u^x\int_y^x M_\varepsilon(r)\,d\chi_\varepsilon(r)\,dy=-\lambda\int_u^x\int_y^x\varPhi(r)\,drdy. \tag{3.9}$$

Since $\varPhi$ is bounded by a constant $K_{a,b}^{(1)}$, we have the estimate $\sup_{y\in[a,b]}|M_\varepsilon(y)|\le K_{a,b}^{(1)}$. Let $\rho:=\frac{b-a}{4}$. Then from (3.8) for $v=\frac{b+a}{2}+\rho$ we deduce that

$$\sup_{y\in(a,(b+a)/2]}|M_\varepsilon'(y)|\le\frac{1}{\rho}K_{a,b}^{(2)}$$

for some constant $K^{(2)}_{a,b}$. Similarly, from (3.9) for $u = \frac{b+a}{2} - \rho$, we deduce that

$$\sup_{x\in[(b+a)/2,b)} |M'_\varepsilon(x)| \le \frac{1}{\rho} K^{(2)}_{a,b}.$$

Thus the family of functions $\{M'_\varepsilon(y)\}$ is uniformly bounded on the interval (a,b). Now it follows from (3.8) that the family of functions $\{M_\varepsilon(y)\}_{\varepsilon>0}$ is equicontinuous on the interval $[a+\rho, b]$, and it follows from (3.9) that the family is equicontinuous on the interval $[a, b-\rho]$. Consequently, the family of functions $\{M_\varepsilon(y)\}_{\varepsilon>0}$ is equicontinuous on $[a,b]$. Moreover, it is uniformly bounded.

By the Arzelà–Ascoli theorem, the family of functions $\{M_\varepsilon(x)\}_{\varepsilon>0}$, $[a,b]$, is relatively compact in the uniform norm. This and (3.5) imply that

$$\sup_{x\in[a,b]} \bigl|M_\varepsilon(x) - M(x)\bigr| \to 0 \qquad \text{as } \varepsilon \downarrow 0.$$

A uniform limit of continuous functions is a continuous function, i.e., $M(x)$ is continuous on $[a,b]$. In addition, the boundary conditions for the functions M_ε are transformed to the boundary conditions (3.3).

Now, since $\chi_\varepsilon(x) \to \chi(x) := \mathbb{I}_{[q_1,\infty)}(x)$, we deduce from (3.8), by passage to the limit, that for $y \ne q_1$ the limit $\widetilde{M}(y) = \lim_{\varepsilon\downarrow 0} M'_\varepsilon(y)$ exists and

$$\frac{1}{2}\bigl(M(v) - M(y)\bigr) - \frac{1}{2}\widetilde{M}(y)(v-y) - \int_y^v \int_y^x (\lambda + f(r))M(r)\,drdx$$
$$- \beta_1 \int_y^v \int_y^x M(r)\,d\chi(r)dx = -\lambda \int_y^v \int_y^x \varPhi(r)\,drdx. \tag{3.10}$$

From here and from the limiting analog of (3.9) it follows that $M(v)$ is differentiable for $v \in (a,b)\setminus\{q_1\}$. Differentiating (3.10) with respect to v, we obtain $\widetilde{M}(y) = M'(y)$ and

$$\frac{1}{2}(M'(v) - M'(y)) - \int_y^v (\lambda + f(r))M(r)\,dr - \beta_1 \int_y^v M(r)\,d\chi(r) = -\lambda \int_y^v \varPhi(r)\,dr,$$

$v, y \in (a,b)\setminus\{q_1\}$. This equality, in turn, implies that $M(x)$ satisfies (3.1), (3.2) for $\beta_1 \ne 0$, $\beta_l = 0$, $l = 2,\dots,m$. Indeed, if $q_1 \notin (y,v)$ then the integral with respect to $d\chi$ equals zero and equation (3.1) holds. If $q_1 \in (y,v)$, then the integral with respect to $d\chi$ equals $M(q_1)$. Letting $y \uparrow q_1$, $v \downarrow q_1$, we obtain

$$M'(q_1+0) - M'(q_1-0) = 2\beta_1 M(q_1).$$

Therefore, (3.2) holds. Theorem 3.1 is proved. □

Let us provide an examples of application of Theorem 3.1 to computing distributions.

Example 3.1. We compute the one-dimensional distribution of the Brownian local time $\ell(t,q)$. Applying Theorem 3.1 with $\varPhi \equiv 1$, $f \equiv 0$, $a=-\infty$, $b=\infty$, $\beta_1=\beta$, $q_1=q$ and $\beta_k=0$ for $k\neq 1$, we see that the function $M(x)=\mathbf{E}_x e^{-\beta\ell(\tau,q)}$, $x\in\mathbf{R}$, is the unique bounded continuous solution of the problem

$$\frac{1}{2}M''(x)-\lambda M(x)=-\lambda, \qquad x\in\mathbf{R}\setminus\{q\}, \tag{3.11}$$

$$M'(q+0)-M'(q-0)=2\beta M(q). \tag{3.12}$$

The standard approach for solving this problem is the following. We look for a general bounded solution of (3.11) on each of the intervals $(-\infty,q)$ and (q,∞). Each of these solutions depends on one unknown constant. These constants can be computed from the conditions of continuity of M at q and the condition (3.12) for the jump of the first derivative. However, these computations can be simplified. For example, the jump of the first derivative at q takes place for the function $|x-q|$ and its value equals 2. The fundamental solutions of the homogeneous equation corresponding to (3.11) are $e^{-x\sqrt{2\lambda}}$ and $e^{x\sqrt{2\lambda}}$. They are symmetric and a shift of the argument leads to solutions of the same homogeneous equation, therefore the function $e^{-|x-q|\sqrt{2\lambda}}$ is the bounded continuous solution of the homogeneous equation for $x\neq q$ and its derivative has at the point q a jump equal to $-2\sqrt{2\lambda}$. Since the particular solution of (3.11) equals 1, the continuous bounded solution of the problem (3.11), (3.12) can be represented in the form

$$M(x)=1+Ae^{-|x-q|\sqrt{2\lambda}}, \qquad x\in\mathbf{R}.$$

Condition (3.12) implies the equality $-2A\sqrt{2\lambda}=2\beta(1+A)$, and so the solution of the problem (3.11), (3.12) is

$$M(x)=1-\Big(1-\frac{\sqrt{2\lambda}}{\beta+\sqrt{2\lambda}}\Big)e^{-|x-q|\sqrt{2\lambda}}. \tag{3.13}$$

Inverting the Laplace transform with respect to β, we get

$$\mathbf{P}_x(\ell(\tau,q)=0)=1-e^{-|x-q|\sqrt{2\lambda}}, \tag{3.14}$$

$$\frac{d}{dy}\mathbf{P}_x(\ell(\tau,q)<y)=\sqrt{2\lambda}e^{-(y+|x-q|)\sqrt{2\lambda}}, \quad y>0. \tag{3.15}$$

Dividing these equalities by λ and inverting the Laplace transform with respect to λ (see formulas 6, 5 of Appendix 3), we have

$$\mathbf{P}_x(\ell(t,q)=0)=\frac{\sqrt{2}}{\sqrt{\pi t}}\int_0^{|x-q|} e^{-v^2/2t}dv, \tag{3.16}$$

$$\frac{d}{dy}\mathbf{P}_x(\ell(t,q)<y)=\frac{\sqrt{2}}{\sqrt{\pi t}}e^{-(|x-q|+y)^2/2t}, \qquad y>0. \tag{3.17}$$

Example 3.2. We compute the Laplace transform corresponding to the joint distribution of the Brownian local times $\ell(\tau,r)$ and $\ell(\tau,q)$. Applying Theorem 3.1

with $\Phi \equiv 1$, $f \equiv 0$, $a = -\infty$, $b = \infty$, $\beta_1 = \mu$, $q_1 = r$, $\beta_2 = \eta$, $q_2 = q$ and $\beta_l = 0$ for $l \neq 1, 2$, we see that the function

$$M(x) = \mathbf{E}_x e^{-\mu\ell(\tau,r)-\eta\ell(\tau,q)}, \qquad x \in \mathbf{R},$$

is the unique continuous bounded solution of the problem

$$\frac{1}{2}M''(x) - \lambda M(x) = -\lambda, \qquad x \in \mathbf{R} \setminus \{r, q\}, \tag{3.18}$$
$$M'(r+0) - M'(r-0) = 2\mu M(r), \tag{3.19}$$
$$M'(u+0) - M'(u-0) = 2\eta M(u). \tag{3.20}$$

As in the previous example, the continuous bounded solution of (3.18)–(3.20) can be written as

$$M(x) = 1 - Ae^{-|x-r|\sqrt{2\lambda}} - Be^{-|x-u|\sqrt{2\lambda}}, \qquad x \in \mathbf{R}.$$

Conditions (3.19) and (3.20) imply

$$2A\sqrt{2\lambda} = 2\mu(1 - A - Be^{-|u-r|\sqrt{2\lambda}}), \qquad B2\sqrt{2\lambda} = 2\eta(1 - Ae^{-|u-r|\sqrt{2\lambda}} - B).$$

Solving this system of algebraic equations, we get

$$\mathbf{E}_x e^{-\mu\ell(\tau,r)-\eta\ell(\tau,q)} = 1$$
$$- \frac{\mu\big(\sqrt{2\lambda} + \eta\big(1 - e^{-|u-r|\sqrt{2\lambda}}\big)\big)e^{-|r-x|\sqrt{2\lambda}} + \eta\big(\sqrt{2\lambda} + \mu\big(1 - e^{-|u-r|\sqrt{2\lambda}}\big)\big)e^{-|u-x|\sqrt{2\lambda}}}{(\sqrt{2\lambda} + \mu)(\sqrt{2\lambda} + \eta) - \mu\eta e^{-2|u-r|\sqrt{2\lambda}}}. \tag{3.21}$$

Exercises.

3.1. Compute the expression

$$\mathbf{E}_x\Big\{e^{-\gamma\ell(\tau,r)}; \sup_{0 \le s \le \tau} W(s) \le b\Big\}, \qquad \gamma > 0, \quad x < b, \quad r < b,$$

and compute the joint distribution of the Brownian local time and the supremum of the Brownian motion up to the time τ.

3.2. Compute the expression

$$\mathbf{E}_x\Big\{e^{-\gamma\ell(\tau,r)}; a \le \inf_{0 \le s \le \tau} W(s)\Big\}, \qquad \gamma > 0, \quad a < x, \quad a < r,$$

and compute the joint distribution of the Brownian local time and the infimum of the Brownian motion up to the time τ.

3.3. Compute the expression

$$\mathbf{E}_x\Big\{e^{-\gamma\ell(\tau,r)}; a \le \inf_{0 \le s \le \tau} W(s), \sup_{0 \le s \le \tau} W(s) \le b\Big\}, \qquad \gamma > 0,$$

$a < x < b$, $a < r < b$, and compute the joint distribution of the Brownian local time, the infimum, and the supremum of the Brownian motion up to the time τ.

§4. Distributions of functionals of Brownian bridge

By the definition of the Brownian bridge $W_{x,t,z}(s)$, $s \in [0,t]$ (see §11 Ch. I), for any bounded continuous functional $\wp$ on $C([0,t])$,

$$\mathbf{E}\wp(W_{x,t,z}(s), 0 \le s \le t) = \mathbf{E}_x\{\wp(W(s), 0 \le s \le t) | W(t) = z\} \tag{4.1}$$
$$= \lim_{\delta \downarrow 0} \frac{\mathbf{E}_x\{\wp(W(s), 0 \le s \le t); W(t) \in [z, z+\delta)\}}{\mathbf{P}_x(W(t) \in [z, z+\delta))} = \frac{\frac{d}{dz}\mathbf{E}_x\{\wp(W(s), 0 \le s \le t); W(t) < z\}}{\frac{d}{dz}\mathbf{P}_x(W(t) < z)}.$$

In this section we consider a method for computing the distribution of an integral functional $\int_0^t f(W_{x,t,z}(s))\,ds$ of the Brownian bridge and the infimum and supremum functionals. The approach for the case of the Brownian bridge is the same as for the Brownian motion, namely, it is based on the computation of the Laplace transform of the distribution of an integral functional, i.e., of the function

$$h(t,z) := \mathbf{E}\exp\Big(-\int_0^t f(W_{x,t,z}(s))\,ds\Big)$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\int_0^t f(W(s))\,ds\Big)\Big| W(t) = z\Big\}, \tag{4.2}$$

in which the parameter of the Laplace transform $\gamma > 0$ is included in the function f. It turns out that instead of $h(t,x)$ it is more convenient to compute the function

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^t f(W(s))\,ds\Big); W(t) < z\Big\} = h(t,z)\frac{1}{\sqrt{2\pi t}}e^{-(z-x)^2/2t}. \tag{4.3}$$

This equality is due to (4.1). To avoid equations with partial derivatives it is necessary to consider the Laplace transform of this function with respect to t:

$$G_x(z) := \lambda\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^t f(W(s))\,ds\Big); W(t) < z\Big\}\,dt$$

$$= \frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^\tau f(W(s))\,ds\Big);\ W(\tau) < z\Big\}. \tag{4.4}$$

Here, as before, one uses the probabilistic representation of the Laplace transform with respect to the time parameter, in which τ is an exponentially distributed with the parameter $\lambda > 0$ random variable independent of the Brownian motion.

By the definition of the conditional expectation,

$$G_x(z) = \mathbf{E}_x\Big\{ \exp\Big(-\int_0^\tau f(W(s))\,ds\Big)\Big| W(\tau)=z\Big\}\frac{d}{dz}\mathbf{P}_x(W(\tau)<z), \tag{4.5}$$

and therefore, in view of nonnegativity of f, the function $G_x(z)$ is estimated by the density of the variable $W(\tau)$, which is (see formula 5 of Appendix 3)

$$\frac{d}{dz}\mathbf{P}_x(W(\tau)<z) = \lambda\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{2\pi t}}e^{-(z-x)^2/2t}\,dt = \frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}}. \tag{4.6}$$

The function U defined in Theorem 1.2 can be expressed in terms of the function G_x:

$$\begin{aligned} U(x) &= \mathbf{E}_x\Big\{\Phi(W(\tau))\exp\Big(-\int_0^\tau f(W(v))dv\Big)\Big\} \\ &= \int_{-\infty}^{\infty} \mathbf{E}_x\Big\{\Phi(W(\tau))\exp\Big(-\int_0^\tau f(W(v))dv\Big); W(\tau)\in[z,z+dz)\Big\} \\ &= \int_{-\infty}^{\infty} \Phi(z)\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^\tau f(W(v))dv\Big); W(\tau)<z\Big\}\,dz \\ &= \int_{-\infty}^{\infty} \Phi(z)G_x(z)\,dz. \end{aligned} \tag{4.7}$$

Like $U(x)$, $x\in\mathbf{R}$, the function $G_x(z)$, $z\in\mathbf{R}$, is the solution of a differential problem. In view of (4.7), the function $G_x(z)$, $(z,x)\in\mathbf{R}^2$, is called the *Green function* of equation (1.10).

Theorems 1.2 and 2.1 provide a convenient method for computing the distributions of integral functionals and supremum-type functionals of Brownian motion. This method is essentially based on the variation of the starting point of the Brownian motion. In many problems we are given not only the starting point of a sample path, but also its endpoint. In this case it is useful to have a statement based on variation of the endpoint of a sample path for computing distributions of functionals of a process. The results of this type that include variation of either the starting point or the endpoint can easily be transformed into each other.

To compute the joint distributions of integral, infimum, and supremum functionals of the Brownian bridge, it is necessary to include the variables $\inf_{0\le s\le\tau} W(s)$ and $\sup_{0\le s\le\tau} W(s)$ in the definition of the function $G_x(z)$. The following result is the basic one for the computation of such joint distributions.

Theorem 4.1. *Let $f(x)$, $x \in [a,b]$, be a nonnegative piecewise-continuous function. Then for $a \le x \le b$ the function*

$$G_x(z) := \frac{d}{dz}\mathbf{E}_x\biggl\{\exp\biggl(-\int_0^\tau f(W(s))\,ds\biggr); a \le \inf_{0\le s\le\tau} W(s), \sup_{0\le s\le\tau} W(s) \le b, W(\tau) < z\biggr\},$$

$z \in [a,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}G''(z) - (\lambda + f(z))G(z) = 0, \qquad z \in (a,b)\setminus\{x\}, \tag{4.8}$$

$$G'(x+0) - G'(x-0) = -2\lambda, \tag{4.9}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{4.10}$$

Remark 4.1. The function $G_x(z)$, $(z,x) \in [a,b]\times[a,b]$, is the Green function for the problem (2.1), (2.2), since it follows from the definitions of the functions U and G_x that

$$U(x) = \int_a^b \Phi(z)G_x(z)\,dz. \tag{4.11}$$

This formula is derived similarly to (4.7).

Remark 4.2. In the case $a = -\infty$ or $b = \infty$ the corresponding boundary condition in (4.10) must be replaced by the condition that the function $G_x(z)$ tends to zero as z tends to $-\infty$ or ∞. This is due to the fact that $G_x(z)$ is estimated by the density of the variable $W(\tau)$.

Proof of Theorem 4.1. We deduce Theorem 4.1 from Theorem 2.1. Set

$$U_\Delta(x) := \mathbf{E}_x\biggl\{\frac{\mathbb{1}_{[z,z+\Delta)}(W(\tau))}{\Delta}\exp\biggl(-\int_0^\tau f(W(s))\,ds\biggr); a \le \inf_{0\le s\le\tau} W(s), \sup_{0\le s\le\tau} W(s) \le b\biggr\}.$$

We let $\mathbb{1}_{[y,y+\Delta)}(x) := -\mathbb{1}_{[y+\Delta,y)}(x)$ if $\Delta < 0$. By the definition of the function G_x,

$$G_x(z) = \lim_{\Delta\to 0} U_\Delta(x) =: U_z(x), \qquad (z,x) \in (a,b)\times(a,b). \tag{4.12}$$

Here we rewrite G_x as a function of the variable x, because we derive first the differential problem with respect to x.

By Theorem 2.1, the function U_Δ is the unique solution of the problem

$$\frac{1}{2}U''_\Delta(x) - (\lambda + f(x))U_\Delta(x) = -\lambda\frac{1}{\Delta}\mathbb{1}_{[z,z+\Delta)}(x), \qquad x \in (a,b), \tag{4.13}$$

$$U_\Delta(a) = 0, \qquad U_\Delta(b) = 0. \tag{4.14}$$

We set $\chi_\Delta(x) := \int\limits_{-\infty}^{x} \frac{1}{\Delta} \mathbb{I}_{[z,z+\Delta)}(u)\,du$ for $\Delta \neq 0$. Then (4.13) can be written as follows: for every $y, x \in (a,b)$

$$\frac{1}{2}(U'_\Delta(x) - U'_\Delta(y)) - \int\limits_{y}^{x} (\lambda + f(r))U_\Delta(r)\,dr = -\lambda \int\limits_{y}^{x} d\chi_\Delta(r). \tag{4.15}$$

Our aim is to pass to the limit as $\Delta \to 0$ in (4.15) and (4.14), with the help of (4.12). Here the arguments almost word for word repeat those used to justify the passage to the limit in the problem (3.6), (3.7) and we omit them. We only note that an important role here is played the following uniform estimate, which can be obtained by formula (4.6):

$$|U_\Delta(x)| \le \mathbf{E}_x\Big\{\frac{\mathbb{I}_{[z,z+\Delta)}(W(\tau))}{\Delta}\Big\} = \frac{1}{\Delta}\int\limits_{z}^{z+\Delta} \frac{d}{dy}\mathbf{P}_x(W(\tau) < y)\,dy \le \frac{\sqrt{\lambda}}{\sqrt{2}}. \tag{4.16}$$

By passing to the limit as $\Delta \to 0$ in (4.15), (1.14), we can prove that the function $U_z(x)$, $x \in [a,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}U''_z(x) - (\lambda + f(x))U_z(x) = 0, \qquad x \in (a,b) \setminus \{z\} \tag{4.17}$$

$$U'_z(z+0) - U'_z(z-0) = -2\lambda, \tag{4.18}$$

$$U_z(a) = 0, \qquad U_z(b) = 0. \tag{4.19}$$

Note that, in view of the equality $G_x(z) = U_z(x)$, the problem (4.17)–(4.19), in fact, solves the problem of computation of the function $G_x(z)$. But this is a problem with respect to the starting point x of the process. Now we explain how it can be transformed to a problem with respect to the endpoint z.

We solve the problem (4.17)–(4.19) in terms of fundamental solutions $\psi(x)$ and $\varphi(x)$ of equation (4.17) for $x \in (a,b)$, satisfying the conditions $\psi(a) = 0$ and $\varphi(b) = 0$. The Wronskian $w = \psi'(x)\varphi(x) - \psi(x)\varphi'(x) > 0$ of these solutions is a constant. Taking into account the continuity of $U_z(x)$, $x \in [a,b]$, and condition (4.18), we obtain

$$U_z(x) = \begin{cases} \dfrac{2\lambda}{w}\varphi(z)\psi(x), & a \le x \le z, \\ \dfrac{2\lambda}{w}\psi(z)\varphi(x), & z \le x \le b. \end{cases} \tag{4.20}$$

From this formula it follows that the function $G_x(z) = U_z(x)$, $z \in (a,b)$, can be represented as

$$G_x(z) = \begin{cases} \dfrac{2\lambda}{w}\varphi(x)\psi(z), & a \le z \le x, \\ \dfrac{2\lambda}{w}\psi(x)\varphi(z), & x \le z \le b. \end{cases} \tag{4.21}$$

Hence, it is continuous, and satisfies equation (4.8) for $z \neq x$, because this equation coincides with (4.17). Moreover, at x the derivative has the jump

$$G'_x(x+0) - G'_x(x-0) = \frac{2\lambda}{w}(\psi(x)\varphi'(x) - \psi'(x)\varphi(x)) = -2\lambda. \tag{4.22}$$

The boundary conditions (4.10) also hold. Theorem 4.1 is proved. □

Consider an analogue of Theorem 3.1 for the Brownian bridge. Let's start with an important remark concerning the distribution of the local time at different levels.

Remark 4.3. Based on the proof of Theorem 3.1 we can provide a general rule of formulation of assertions, in which along with an integral functional there is a linear combination of local times. A local time at a level q can be informally considered as an integral functional whose integrand is the Dirac δ-function at the point q, i.e., the function $\delta_q(x)$, $x \in \mathbf{R}$. The function $\delta_q(x)$ can be interpreted as the family of functions $\Big\{\frac{1}{\varepsilon}\mathbb{I}_{[q,q+\varepsilon)}(x)\Big\}_{\varepsilon>0}$. The presence of the Dirac δ-function under the integral sign is characterized by the following procedure: the δ-function is replaced by an element of this family and then the limit is computed as $\varepsilon \downarrow 0$. With this treatment of the Dirac δ-function δ_q equality (3.4) takes the form

$$\ell(t,q_1) = \lim_{\varepsilon\downarrow 0} \int_0^t \frac{1}{\varepsilon}\mathbb{I}_{[q_1,q_1+\varepsilon)}(W(s))\,ds = \int_0^t \delta_{q_1}(W(s))\,ds.$$

For integral functionals we derived Theorem 2.1 and its analogue for the Brownian bridge (Theorem 4.1). Due to the presence of linear combinations of the functions $\delta_{q_k}(x)$, $k = 1,\dots r$, the basic equations of the form (2.1) or (4.8) are not valid at the points q_k. However, at these points their solutions are continuous and the first derivative has a jump. The values of the jumps are computed as follows. We integrate the basic equation, in which along with the usual integrands the linear combination of $\delta_{q_k}(x)$, $k = 1,\dots,r$, appears, over a small interval containing some point q_l. Then letting the boundaries of the interval tend to the point q_l, we obtain an expression for the jump of the first derivative at q_l. This was done in detail in the proof of Theorem 3.1.

Now we illustrate this rule by formulating an analogue of Theorem 3.1 for the Brownian bridge.

The next assertion is stated without proof, because the latter is completely similar to the proofs of Theorems 3.1 and 4.1. This result enables us to compute the joint distribution of an integral functional of the Brownian bridge, its local times at different levels, and the infimum and supremum functionals. Let the functional $A_{\vec\beta}(t)$ be defined as at the beginning of § 3.

Theorem 4.2. *Let $f(x)$, $x \in [a,b]$ be a piecewise-continuous nonnegative function. Then for $a \le x \le b$ the function*

$$G_x(z) = \frac{d}{dz}\mathbf{E}_x\Big\{\exp(-A_{\vec\beta}(\tau)); a \le \inf_{0\le s\le\tau} W(s),\ \sup_{0\le s\le\tau} W(s) \le b,\ W(\tau) < z\Big\},$$

$z \in [a,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}G''(z) - (\lambda + f(z))G(z) = 0, \qquad z \in (a,b)\setminus\{x,q_1,\dots,q_m\}, \tag{4.23}$$

$$G'(x+0) - G'(x-0) = -2\lambda, \tag{4.24}$$

$$G'(q_k+0) - G'(q_k-0) = 2\beta_k G(q_k), \qquad q_k \ne x,\ k = 1,\dots,m, \tag{4.25}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{4.26}$$

Remark 4.4. If $q_k = x$ for some k, then condition (4.24) should be replaced by the condition

$$G'(x+0) - G'(x-0) = 2\beta_k G(x) - 2\lambda.$$

In this case in (4.25) the condition with index k is removed.

Remark 4.5. $G_x(z)$ is the Green function of the problem (3.1)–(3.3), because by the definition of the functions $M(x)$ and $G_x(z)$,

$$M(x) = \int_a^b \varPhi(z) G_x(z)\, dz. \tag{4.27}$$

The function $h(t,z)$, defined by (4.2), is of special interest. As a function of two variables it satisfies an equation in partial derivatives. We will give an original derivation of this equation based on the explicit expression for the Brownian bridge (see (11.7) Ch. I).

Assume that $f(x) = f_+(x) + f_0(x)$, $x \in \mathbf{R}$, where $f_+ \ge 0$ and f_0 is bounded. Let f be twice continuously differentiable function, the derivatives of which obey the estimates $|f'(x)| \le C(1+|x|^m)$ and $|f''(x)| \le C(1+|x|^m)$ for $x \in \mathbf{R}$, and some $C > 0$, $m \ge 0$.

Theorem 4.3. *Under the above conditions, the function $h(t,z)$, $t > 0$, $z \in \mathbf{R}$, is the solution of the problem*

$$\frac{\partial}{\partial t} h(t,z) = \frac{1}{2}\frac{\partial^2}{\partial z^2} h(t,z) - \frac{z-x}{t}\frac{\partial}{\partial z} h(t,z) - f(z)h(t,z), \tag{4.28}$$

$$h(+0,z) = 1. \tag{4.29}$$

Proof. We use (11.7) Ch. I. According to that formula,

$$W_{x,t,z}(s) = W(s) - \frac{s}{t}(W(t) - z), \qquad s \in [0,t],$$

where $W(0) = x$. Therefore,

$$h(t,z) = \mathbf{E} \exp\Bigl(- \int_0^t f\bigl(W(v) - \tfrac{v}{t}(W(t)-z)\bigr)\, dv \Bigr). \tag{4.30}$$

The initial condition (4.29) is easily verified by passage to the limit as $t \downarrow 0$ under the expectation sign in (4.30).

Set

$$V(t,y,z) := \exp\Bigl(- \int_0^t f\bigl(W(v) - \tfrac{v}{t}(y-z)\bigr)\, dv \Bigr).$$

It is clear that $h(t,z) = \mathbf{E} V(t, W(t), z)$. The derivative

$$\frac{\partial}{\partial y} V(t,y,z) = \exp\Bigl(- \int_0^t f\bigl(W(v) - \tfrac{v}{t}(y-z)\bigr)\, dv \Bigr) \int_0^t \frac{v}{t} f'\bigl(W(v) - \tfrac{v}{t}(y-z)\bigr)\, dv \tag{4.31}$$

obeys the estimate

$$\mathbf{E}\Big(\frac{\partial}{\partial x}V(t,W(t),z)\Big)^2 \le C^2e^{C_1t}\mathbf{E}\Big(\int_0^t \tfrac{v}{t}\Big(1+\Big|W(v)-\tfrac{v}{t}(W(t)-z)\Big|^m\Big)dv\Big)^2$$

$$\le C_m e^{C_1t}t^2\big(1+t^m+|z|^{2m}\big), \qquad (4.32)$$

where C_m and C_1 are constants. An analogous estimate holds for the second derivative.

Note that the variables y and z figuring in V with opposite sign, therefore $\frac{\partial}{\partial y}V = -\frac{\partial}{\partial z}V$ and $\frac{\partial^2}{\partial y^2}V = \frac{\partial^2}{\partial z^2}V$. Differentiating the right-hand side of (4.30) with respect to z twice under the sign of expectation, we can easily verify that the derivatives $\frac{\partial}{\partial z}h(t,z)$ and $\frac{\partial^2}{\partial z^2}h(t,z)$ exist, and

$$\frac{\partial}{\partial z}h(t,z) = -\mathbf{E}\frac{\partial}{\partial y}V(t,W(t),z), \quad \frac{\partial^2}{\partial z^2}h(t,z) = \mathbf{E}\frac{\partial^2}{\partial y^2}V(t,W(t),z). \qquad (4.33)$$

Compute the stochastic differential $dV(t,W(t),z)$ by Itô's formula. Note that

$$\frac{\partial}{\partial t}V(t,y,z) = -f(W(t)-y+z)V(t,y,z) - \frac{y-z}{t}\frac{\partial}{\partial y}V(t,y,z),$$

because the variable t is included to the definition of V as the upper limit of integration and through the fraction $\frac{y-z}{t}$ under the integral sign. Then by Itô's formula,

$$dV(t,W(t),z) = -f(z)V(t,W(t),z)\,dt - \frac{W(t)-z}{t}\frac{\partial}{\partial y}V(t,W(t),z)\,dt$$
$$+\frac{\partial}{\partial y}V(t,W(t),z)\,dW(t) + \frac{1}{2}\frac{\partial^2}{\partial y^2}V(t,W(t),z)\,dt.$$

The application of Itô's formula is correct, because the process $V(t,x,z)$, $t \ge 0$, is adapted to the natural filtration of the Brownian motion W. The integral version of this equality has the form

$$V(t,W(t),z)-1 = -f(z)\int_0^t V(s,W(s),z)\,ds + \int_0^t \frac{z-W(s)}{s}\frac{\partial}{\partial y}V(s,W(s),z)\,ds$$
$$+\int_0^t \frac{\partial}{\partial y}V(s,W(s),z)\,dW(s) + \int_0^t \frac{1}{2}\frac{\partial^2}{\partial y^2}V(s,W(s),z)\,ds. \qquad (4.34)$$

Since for every fixed s the variable $W(s)$ is independent of the process $W(v) - \frac{v}{s}(W(s)-z)$, $v \in [0,s]$ (see the reasoning after formula (11.13) Ch.I), we see that $W(s)$ is independent of the variable $\frac{\partial}{\partial y}V(s,W(s),z)$, which, according to (4.31), is completely determined by the sample paths of the process $W(v) - \frac{v}{s}(W(s)-z)$,

$v \in [0, s]$. In view of (4.32), the expectation of the stochastic integral in (4.34) is zero. Therefore, computing the expectation of both sides of (4.34) and using the obvious equality $\mathbf{E}W(s) = x$, we get

$$\mathbf{E}V(t, W(t), z) - 1 = -f(z) \int_0^t \mathbf{E}V(s, W(s), z)\,ds + \int_0^t \frac{z-x}{s}\mathbf{E}\frac{\partial}{\partial y}V(s, W(s), z)\,ds$$

$$+ \int_0^t \frac{1}{2}\mathbf{E}\frac{\partial^2}{\partial y^2}V(s, W(s), z)\,ds.$$

This and (4.33) imply that the function $h(t, z)$ is the solution of the problem (4.28), (4.29). The theorem is proved. □

Remark 4.6. Under the above conditions on the function f, Theorem 4.3 implies Theorem 4.1 for $a = -\infty$, $b = \infty$.

Indeed, set

$$q(t, z) := (h(t, z) - 1)\frac{1}{\sqrt{2\pi t}}e^{-(z-x)^2/2t}.$$

It is easy to verify that the function q is the solution of the equation

$$\frac{\partial}{\partial t}q(t, z) = \frac{1}{2}\frac{\partial^2}{\partial z^2}q(t, z) - f(z)q(t, z) - f(z)\frac{1}{\sqrt{2\pi t}}e^{-(z-x)^2/2t} \tag{4.35}$$

and satisfies the boundary condition $q(+0, z) = 0$. By (4.3), (4.4), and (4.6),

$$Q(z) := \lambda \int_0^\infty e^{-\lambda t}q(t, z)\,dt = G_x(z) + \frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}}.$$

Computing the Laplace transform with respect to t of the left-hand and right-hand sides of (4.35) and using the boundary condition, we deduce that Q is the unique bounded solution of the equation

$$\frac{1}{2}Q''(z) - (\lambda + f(z))Q(z) = f(z)\frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}}, \qquad z \in \mathbf{R}.$$

Now it is easy to see that the function $G_x(z)$, $z \in \mathbf{R}$, is the unique continuous bounded solution of the problem

$$\frac{1}{2}G''(z) - (\lambda + f(z))G(z) = 0, \qquad z \neq x, \tag{4.36}$$

$$G'(x+0) - G'(x-0) = -2\lambda. \tag{4.37}$$

This is precisely the statement of Theorem 4.1 for $a = -\infty$, $b = \infty$. □

Let us give examples of the application of Theorems 4.1 and 4.2.

Example 4.1. According to Theorem 4.1 with $f \equiv 0$ and Remark 4.2 for $a = -\infty$, $b = \infty$, the function $G_x(z) = \frac{d}{dz}\mathbf{P}_x(W(\tau) < z)$ is the unique bounded continuous solution of the problem

$$\frac{1}{2}G''(z) - \lambda G(z) = 0, \qquad z \in (-\infty, x) \cup (x, \infty), \tag{4.38}$$

$$G'(x+0) - G'(x-0) = -2\lambda. \tag{4.39}$$

It is easy to solve this problem, but we already know the solution: it is given by (4.6). Thus the function $G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}} e^{-|z-x|\sqrt{2\lambda}}$, $z \in \mathbf{R}$, satisfies equation (4.38) for $z \neq x$ and its derivative has at the point x the jump -2λ.

Example 4.2. We compute the joint distribution of $\inf_{0 \le s \le t} W(s)$, $\sup_{0 \le s \le t} W(s)$, given the condition $W(t) = z$.

Applying Theorem 4.1 with $f \equiv 0$, we see that

$$G_x(z) = \frac{d}{dz}\mathbf{P}_x\Big(a \le \inf_{0 \le s \le \tau} W(s), \sup_{0 \le s \le \tau} W(s) \le b,\ W(\tau) < z\Big)$$

is the unique continuous solution of the problem

$$\frac{1}{2}G''(z) - \lambda G(z) = 0, \qquad z \in (a, b) \setminus \{x\}, \tag{4.40}$$

$$G'(x+0) - G'(x-0) = -2\lambda, \tag{4.41}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{4.42}$$

The standard way of solving this problem is the following. Find the general solutions of equation (4.40) in the intervals (a, x) and (x, b). Each of these solutions depends on two unknown constants. These four constants can be computed from the continuity condition of the function $G_x(z)$, $z \in (a, b)$, at the point x, condition (4.41) on the jump of the derivative at x, and the boundary conditions (4.42).

However, these computations can be simplified. The fundamental solutions of equation (4.40) satisfying, respectively, the right-hand and the left-hand boundary conditions are $\operatorname{sh}((b-z)\sqrt{2\lambda})$ and $\operatorname{sh}((z-a)\sqrt{2\lambda})$. As it was noticed in Example 4.1, we can choose a function that satisfies (4.40) for $z \neq x$ and whose derivative has at the point x the jump -2λ. In our case it is suitable to take $-\frac{\sqrt{\lambda}}{\sqrt{2}} \operatorname{sh}(|z-x|\sqrt{2\lambda})$ as such a function. Then the solution of the problem (4.40)–(4.42) can be represented as

$$G_x(z) = A \operatorname{sh}((b-z)\sqrt{2\lambda}) + B \operatorname{sh}((z-a)\sqrt{2\lambda}) - \frac{\sqrt{\lambda}}{\sqrt{2}} \operatorname{sh}(|z-x|\sqrt{2\lambda}).$$

The constants A and B can be easily computed from the boundary conditions, and we get

$$G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}} \Big[\frac{\operatorname{sh}((b-z)\sqrt{2\lambda}) \operatorname{sh}((x-a)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})} + \frac{\operatorname{sh}((b-x)\sqrt{2\lambda}) \operatorname{sh}((z-a)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})} - \operatorname{sh}(|z-x|\sqrt{2\lambda})\Big].$$

Using the formulas for products of hyperbolic functions, we have

$$G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}}\Big[\frac{\operatorname{ch}((b+x-z-a)\sqrt{2\lambda}) + \operatorname{ch}((b-x+z-a)\sqrt{2\lambda})}{2\operatorname{sh}((b-a)\sqrt{2\lambda})}$$
$$- \frac{\operatorname{ch}((b+a-z-x)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})} - \operatorname{sh}(|z-x|\sqrt{2\lambda})\Big]$$
$$= \frac{\sqrt{\lambda}}{\sqrt{2}}\Big[\frac{\operatorname{ch}((b-a)\sqrt{2\lambda})\operatorname{ch}((z-x)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})} - \frac{\operatorname{ch}((b+a-z-x)\sqrt{2\lambda})}{\operatorname{sh}((b-a)\sqrt{2\lambda})} - \operatorname{sh}(|z-x|\sqrt{2\lambda})\Big].$$

Finally, the solution of the problem (4.40)–(4.42) is

$$G_x(z) = \frac{d}{dz}\mathbf{P}_x\Big(a \le \inf_{0\le s\le\tau} W(s),\ \sup_{0\le s\le\tau} W(s) \le b,\ W(\tau) < z\Big)$$
$$= \frac{\sqrt{\lambda}\big(\operatorname{ch}((b-a-|z-x|)\sqrt{2\lambda}) - \operatorname{ch}((b+a-z-x)\sqrt{2\lambda})\big)}{\sqrt{2}\operatorname{sh}((b-a)\sqrt{2\lambda})}. \tag{4.43}$$

Dividing (4.43) by λ and inverting the Laplace transform with respect to λ (see Section 13 of Appendix 2), we get

$$\frac{d}{dz}\mathbf{P}_x\Big(a \le \inf_{0\le s\le t} W(s),\ \sup_{0\le s\le t} W(s) \le b, W(t) < z\Big)$$
$$= \operatorname{cs}_t(b-a-|z-x|, b-a) - \operatorname{cs}_t(b+a-z-x, b-a)$$
$$= \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\big(e^{-(z-x+2k(b-a))^2/2t} - e^{-(z+x-2a+2k(b-a))^2/2t}\big). \tag{4.44}$$

Dividing (4.44) by the density

$$\frac{d}{dz}\mathbf{P}_x\big(W(t) < z\big) = \frac{1}{\sqrt{2\pi t}}e^{-(z-x)^2/2t},$$

we obtain the conditional joint distribution of the infimum and supremum of the Brownian motion given the condition $W(t) = z$, i.e., the joint distribution of the infimum and supremum of the Brownian bridge.

Note that (4.44) turns into (2.16) after integration with respect to z over the interval (a, b). The same connection exists between formulas (4.43) and (2.12).

Example 4.3. We compute the distribution of the Brownian local time $\ell(t, q)$ given the condition $W(t) = z$, i.e., the distribution of the local time of the Brownian bridge $W_{x,t,z}(s)$, $s \in [0, t]$. Applying Theorem 4.2 with $a = -\infty$, $b = \infty$, $f \equiv 0$, $\beta_1 = \beta$, $q_1 = q$, and $\beta_k = 0$ for $k \ne 1$, we see that the function

$$G_x(z) = \frac{d}{dz}\mathbf{E}_x\big\{e^{-\beta\ell(\tau,q)}; W(\tau) < z\big\}, \qquad z \in \mathbf{R},$$

is the unique bounded continuous solution of the problem

$$\frac{1}{2}G''(z) - \lambda G(z) = 0, \qquad z \in \mathbf{R}\setminus\{x, q\}, \tag{4.45}$$
$$G'(x+0) - G'(x-0) = -2\lambda, \tag{4.46}$$
$$G'(q+0) - G'(q-0) = 2\beta G(q). \tag{4.47}$$

To find the solution of this problem we use the approach proposed in Example 4.1. The jump of the derivative (4.46) is provided by the term $\frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}}$, which for $z \neq x$ satisfies the equation (4.45). Similarly, the jump of the derivative (4.47) can be provided by the term $e^{-|z-q|\sqrt{2\lambda}}$ with some unknown factor. As a result, the bounded continuous solution of the problem (4.45)–(4.47) can be represented in the form

$$G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}} + Ae^{-|z-q|\sqrt{2\lambda}}.$$

From (4.47) it is easy to deduce that

$$A = -\frac{\beta\sqrt{\lambda}}{\sqrt{2}(\sqrt{2\lambda}+\beta)}e^{-|q-x|\sqrt{2\lambda}}.$$

To invert the Laplace transform with respect to β, it is necessary to decompose A into a sum of two terms. Finally, for G_x we obtain the representation

$$G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}} - \Big(\frac{\sqrt{\lambda}}{\sqrt{2}} - \frac{\lambda}{\sqrt{2\lambda}+\beta}\Big)e^{-(|z-q|+|q-x|)\sqrt{2\lambda}}.$$

Inverting the Laplace transform with respect to β, we have (see formula 1 of Appendix 3)

$$\frac{d}{dz}\mathbf{P}_x(\ell(\tau,q)=0, W(s)<z) = \frac{\sqrt{\lambda}}{\sqrt{2}}\big(e^{-|z-x|\sqrt{2\lambda}} - e^{-(|z-q|+|q-x|)\sqrt{2\lambda}}\big), \tag{4.48}$$

and

$$\frac{\partial}{\partial v}\frac{\partial}{\partial z}\mathbf{P}_x(\ell(\tau,q)<v,\ W(\tau)<z) = \lambda e^{-(v+|z-q|+|q-x|)\sqrt{2\lambda}} \tag{4.49}$$

for $v>0$.

Dividing these equalities by λ, inverting the Laplace transform with respect to λ, and then dividing the obtained expression by the density $\frac{d}{dz}\mathbf{P}_x(W(t)<z)$, we, finally, get (see formulas 5 and 2 of Appendix 3)

$$\mathbf{P}_x(\ell(t,q)=0|W(t)=z) = 1-\exp\Big(-\frac{(|z-q|+|q-x|)^2-(z-x)^2}{2t}\Big), \tag{4.50}$$

$$\frac{d}{dv}\mathbf{P}_x(\ell(t,q)<v|W(t)=z)$$
$$= \frac{v+|z-q|+|q-x|}{t}\exp\Big(-\frac{(v+|z-q|+|q-x|)^2-(z-x)^2}{2t}\Big), \quad v>0. \tag{4.51}$$

Example 4.4. We compute the joint distribution of the Brownian local times $\ell(\tau,r)$ and $\ell(\tau,u)$ given $W(\tau)=z$. In § 2 Ch. V it will be proved that $\ell(\tau,y)$, $y\in\mathbf{R}$, given $W(\tau)=z$ is a Markov process, therefore this example is important for the description of transition probabilities of this process.

Applying Theorem 4.2 with $a=-\infty$, $b=\infty$, $f\equiv 0$, $\beta_1=\gamma$, $\beta_2=\eta$, $q_1=r$, $q_2=u$ and $\beta_k=0$ $k\geq 2$, we see that the function

$$G_x(z) = \frac{d}{dz}\mathbf{E}_x\big\{e^{-\gamma\ell(\tau,r)-\eta\ell(\tau,u)}; W(\tau)<z\big\}, \qquad z\in\mathbf{R},$$

is the unique bounded continuous solution of the problem

$$\frac{1}{2}G''(z) - \lambda G(z) = 0, \qquad z \in \mathbf{R} \setminus \{x, r, u\}, \tag{4.52}$$

$$G'(x+0) - G'(x-0) = -2\lambda, \tag{4.53}$$

$$G'(r+0) - G'(r-0) = 2\gamma G(r), \tag{4.54}$$

$$G'(u+0) - G'(u-0) = 2\eta G(u). \tag{4.55}$$

To find the solution of this problem, we use the approach proposed in Example 4.3. The bounded continuous solution of (4.52)–(4.55) can be written as

$$G_x(z) = \frac{\sqrt{\lambda}}{\sqrt{2}} e^{-|z-x|\sqrt{2\lambda}} + Ae^{-|z-r|\sqrt{2\lambda}} + Be^{-|z-u|\sqrt{2\lambda}}.$$

In this representation we take into account the continuity and boundedness requirements for the solution, condition (4.53) and the conditions on the jumps of the derivative at the points r and u. The coefficients A and B are computed according to (4.54) and (4.55). Solving the system of two algebraic equations with two unknowns A and B, we get

$$\begin{aligned} \frac{d}{dz}\mathbf{E}_x\big\{e^{-\gamma\ell(\tau,r)-\eta\ell(\tau,u)}; W(\tau) < z\big\} &= \frac{\sqrt{\lambda}}{\sqrt{2}} e^{-|z-x|\sqrt{2\lambda}} \\ &- \frac{\sqrt{\lambda}\big(\gamma\sqrt{2\lambda} + \gamma\eta\big(e^{-|r-x|\sqrt{2\lambda}} - e^{-(|u-x|+|u-r|)\sqrt{2\lambda}}\big)\big)}{\sqrt{2}\big(2\lambda + \sqrt{2\lambda}(\gamma+\eta) + \gamma\eta\big(1 - e^{-2|u-r|\sqrt{2\lambda}}\big)\big)} e^{-|z-r|\sqrt{2\lambda}} \\ &- \frac{\sqrt{\lambda}\big(\eta\sqrt{2\lambda} + \gamma\eta\big(e^{-|u-x|\sqrt{2\lambda}} - e^{-(|r-x|+|u-r|)\sqrt{2\lambda}}\big)\big)}{\sqrt{2}\big(2\lambda + \sqrt{2\lambda}(\gamma+\eta) + \gamma\eta\big(1 - e^{-2|u-r|\sqrt{2\lambda}}\big)\big)} e^{-|z-u|\sqrt{2\lambda}}. \end{aligned} \tag{4.56}$$

Note that this formula is invariant under the substitution x to z and (γ, r) to (η, u). This fact also follows from the properties of spatial homogeneity and time reversibility of the Brownian bridge (see § 11 Ch. I).

For brevity set $\Delta := |u - r|$. In order to invert the double Laplace transform with respect to γ and η, we can use formulas (34) and (35) of Appendix 3. For this we transform the right-hand side of (4.56) to the following form:

$$\begin{aligned} \frac{d}{dz}\mathbf{E}_x\big\{e^{-\gamma\ell(\tau,r)-\eta\ell(\tau,u)}; W(\tau) < z\big\} &= \frac{\sqrt{\lambda}}{\sqrt{2}} e^{-|z-x|\sqrt{2\lambda}} - \frac{\sqrt{\lambda}C}{\sqrt{2}\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)} \\ &+ \frac{\lambda D}{\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)^2\Big(\eta + \frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)} + \frac{\lambda F}{\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)^2\Big(\gamma + \frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)} \\ &+ \frac{\lambda\sqrt{2\lambda}}{\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)^3} \frac{\big(C\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big) - D - F\big)}{\Big(\gamma\eta + \frac{\sqrt{2\lambda}(\gamma+\eta)}{1-e^{-2\Delta\sqrt{2\lambda}}} + \frac{2\lambda}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)} \\ &+ \frac{\lambda D}{\big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)^2}\left\{\frac{\gamma + \frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}}{\gamma\eta + \frac{\sqrt{2\lambda}(\gamma+\eta)}{1-e^{-2\Delta\sqrt{2\lambda}}} + \frac{2\lambda}{1-e^{-2\Delta\sqrt{2\lambda}}}} - \frac{1}{\eta + \frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}}\right\} \end{aligned}$$

$$+\frac{\lambda F}{(1-e^{-2\Delta\sqrt{2\lambda}})^2}\Bigg\{\frac{\eta+\frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}}{\gamma\eta+\frac{\sqrt{2\lambda}(\gamma+\eta)}{1-e^{-2\Delta\sqrt{2\lambda}}}+\frac{2\lambda}{1-e^{-2\Delta\sqrt{2\lambda}}}}-\frac{1}{\gamma+\frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}}\Bigg\}, \tag{4.57}$$

where

$$C:=e^{-|z-r|\sqrt{2\lambda}}\big(e^{-|r-x|\sqrt{2\lambda}}-e^{-(|u-x|+\Delta)\sqrt{2\lambda}}\big)$$
$$+e^{-|z-u|\sqrt{2\lambda}}\big(e^{-|u-x|\sqrt{2\lambda}}-e^{-(|r-x|+\Delta)\sqrt{2\lambda}}\big), \tag{4.58}$$

$$D:=\big(e^{-|u-x|\sqrt{2\lambda}}-e^{-(|r-x|+\Delta)\sqrt{2\lambda}}\big)\big(e^{-|z-u|\sqrt{2\lambda}}-e^{-(|z-r|+\Delta)\sqrt{2\lambda}}\big), \tag{4.59}$$

$$F:=\big(e^{-|r-x|\sqrt{2\lambda}}-e^{-(|u-x|+\Delta)\sqrt{2\lambda}}\big)\big(e^{-|z-r|\sqrt{2\lambda}}-e^{-(|z-u|+\Delta)\sqrt{2\lambda}}\big). \tag{4.60}$$

We still need the constant $H:=C\big(1-e^{-2\Delta\sqrt{2\lambda}}\big)-D-F$. It is easy to verify that

$$H=e^{-\Delta\sqrt{2\lambda}}\big(e^{-|r-x|\sqrt{2\lambda}}-e^{-(|u-x|+\Delta)\sqrt{2\lambda}}\big)\big(e^{-|z-u|\sqrt{2\lambda}}-e^{-(|z-r|+\Delta)\sqrt{2\lambda}}\big)$$
$$+e^{-\Delta\sqrt{2\lambda}}\big(e^{-|u-x|\sqrt{2\lambda}}-e^{-(|r-x|+\Delta)\sqrt{2\lambda}}\big)\big(e^{-|z-r|\sqrt{2\lambda}}-e^{-(|z-u|+\Delta)\sqrt{2\lambda}}\big). \tag{4.61}$$

Inverting in (4.57) the Laplace transform with respect to γ and η and dividing the result by the density $\frac{d}{dz}\mathbf{P}_x(W(\tau)<z)=\frac{\sqrt{\lambda}}{\sqrt{2}}e^{-|z-x|\sqrt{2\lambda}}$, we get (see formulas 1, 34, and 35 of Appendix 3)

$$\mathbf{P}_x(\ell(\tau,r)=0,\,\ell(\tau,u)=0|W(\tau)=z)=1-\frac{Ce^{|z-x|\sqrt{2\lambda}}}{\big(1-e^{-2\Delta\sqrt{2\lambda}}\big)}, \tag{4.62}$$

$$\frac{\partial}{\partial g}\mathbf{P}_x(\ell(\tau,r)=0,\,\ell(\tau,u)<g|W(\tau)=z)=\frac{\sqrt{2\lambda}De^{|z-x|\sqrt{2\lambda}}}{(1-e^{-2\Delta\sqrt{2\lambda}})^2}\exp\Big(-\frac{g\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big), \tag{4.63}$$

$$\frac{\partial}{\partial v}\mathbf{P}_x(\ell(\tau,r)<v,\,\ell(\tau,u)=0|W(\tau)=z)=\frac{\sqrt{2\lambda}Fe^{|z-x|\sqrt{2\lambda}}}{(1-e^{-2\Delta\sqrt{2\lambda}})^2}\exp\Big(-\frac{v\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big), \tag{4.64}$$

$$\frac{\partial}{\partial v}\frac{\partial}{\partial g}\mathbf{P}_x(\ell(\tau,r)<v,\,\ell(\tau,u)<g|W(\tau)=z)$$
$$=\frac{\lambda De^{|z-x|\sqrt{2\lambda}}e^{-\Delta\sqrt{2\lambda}}}{(1-e^{-2\Delta\sqrt{2\lambda}})^3}\exp\Big(-\frac{(v+g)\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)\frac{\sqrt{g}}{\sqrt{v}}I_1\Big(\frac{\sqrt{2\lambda gv}}{\operatorname{sh}(\Delta\sqrt{2\lambda})}\Big)$$
$$+\frac{2\lambda Fe^{|z-x|\sqrt{2\lambda}}e^{-\Delta\sqrt{2\lambda}}}{(1-e^{-2\Delta\sqrt{2\lambda}})^3}\exp\Big(-\frac{(v+g)\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)\frac{\sqrt{v}}{\sqrt{g}}I_1\Big(\frac{\sqrt{2\lambda gv}}{\operatorname{sh}(\Delta\sqrt{2\lambda})}\Big)$$
$$+\frac{2\lambda He^{|z-x|\sqrt{2\lambda}}}{(1-e^{-2\Delta\sqrt{2\lambda}})^3}\exp\Big(-\frac{(v+g)\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)I_0\Big(\frac{\sqrt{2\lambda gv}}{\operatorname{sh}(\Delta\sqrt{2\lambda})}\Big). \tag{4.65}$$

Formula (4.65) is invariant under the substitution x to z and (v,r) to (g,u). This fact also follows from the properties of spatial homogeneity and time reversibility of the Brownian bridge.

We fix $x<z$ and assume for definiteness that $r<u$. The coefficients D, F, and H have the simplest expressions if $x<r<u<z$, $r<u<x$ and $z<r<u$. The most complicated expressions arise when $r<x<z<u$. In § 2 Ch. V we will describe the process $\ell(\tau,y)$, $y\in\mathbf{R}$, given $W(\tau)=z$. We will show that it is a Markov process and at the end of the section for $x<z$ we will give formulas for the transition probabilities depending on the intervals $(-\infty,x)$, (x,z), (z,∞).

Exercises.

4.1. Compute the probabilities

$$\frac{d}{dz}\mathbf{P}_x\Big(a \le \inf_{0\le s\le t} W(s),\ W(t) < z\Big), \qquad \frac{d}{dz}\mathbf{P}_x\Big(\sup_{0\le s\le t} W(s) \le b,\ W(t) < z\Big).$$

4.2. Compute the expression

$$\frac{d}{dz}\mathbf{E}_x\Big\{e^{-\gamma\ell(\tau,r)}; \sup_{0\le s\le\tau} W(s) \le b, W(\tau) < z\Big\}, \qquad \gamma > 0,$$

and the joint density

$$\frac{d}{dy}\frac{d}{dz}\mathbf{P}_x\Big(\ell(\tau,r) < y,\ \sup_{0\le s\le\tau} W(s) \le b, W(\tau) < z\Big).$$

4.3. Compute the expression

$$\frac{d}{dz}\mathbf{E}_x\Big\{e^{-\gamma\ell(\tau,r)}; a \le \inf_{0\le s\le\tau} W(s), W(\tau) < z\Big\}, \qquad \gamma > 0,$$

and the joint density

$$\frac{d}{dy}\frac{d}{dz}\mathbf{P}_x\Big(\ell(\tau,r) < y, a \le \inf_{0\le s\le\tau} W(s), W(\tau) < z\Big).$$

4.4. Compute the expression

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^\tau \mathbb{1}_{[r,\infty)}(W(s))\,ds\Big); W(\tau) < z\Big\}, \qquad \gamma > 0.$$

4.5. Compute the expression

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\frac{\gamma^2}{2}\int_0^\tau W^2(s)\,ds\Big); W(\tau) < z\Big\}.$$

Hint: Use the solution of equation 3 of Appendix 4.

§5. Distributions of functionals of Brownian motion stopped at the first exit time

Besides the problem of computation of distributions of functionals of Brownian motion, stopped at a fixed time t or exponential random time τ, of significant interest is the solution of the same problem for the *first exit time* of the Brownian motion W from the interval (a,b), i.e., for the moment

$$H_{a,b} := \min\{s : W(s) \notin (a,b)\}.$$

It is easy to verify (see also Remark 12.3 Ch. II), that $H_{a,b} < \infty$ a.s. Indeed, taking into account (2.12), we have

$$\mathbf{P}(H_{a,b} = \infty) \le \mathbf{P}_x\Big(a \le \inf_{0\le s\le\infty} W(s), \sup_{0\le s\le\infty} W(s) \le b\Big)$$

$$= \lim_{\lambda\downarrow 0} \mathbf{P}_x\Big(a \le \inf_{0\le s\le\tau} W(s), \sup_{0\le s\le\tau} W(s) \le b\Big) = \lim_{\lambda\downarrow 0}\Big(1 - \frac{\operatorname{ch}((b+a-2x)\sqrt{\lambda/2})}{\operatorname{ch}((b-a)\sqrt{\lambda/2})}\Big) = 0.$$

Here we used the fact that $\tau \to \infty$ in probability as $\lambda \downarrow 0$, since $\mathbf{P}(\tau > t) = e^{-\lambda t}$.

As previously, we begin the discussion of this problem with the consideration of a nonnegative integral functional of the Brownian motion. As in the case with an exponential moment, the Laplace transform

$$Q(x) = \mathbf{E}_x \exp\Big(-\gamma\int_0^{H_{a,b}} f(W(s))\,ds\Big) \tag{5.1}$$

plays a key role. To compute the distribution function of the integral functional at the moment $H_{a,b}$ we must invert this Laplace transform with respect to γ. Schematically this can be written as

$$\mathbf{P}_x\Big(\int_0^{H_{a,b}} f(W(s))\,ds < y\Big) = \mathcal{L}_\gamma^{-1}(Q(x)). \tag{5.2}$$

The general approach to computing the distributions of functionals of the Brownian motion stopped at the time $H_{a,b}$ is based on the computation of the function Q for all $\gamma > 0$ and on the subsequent inversion of the Laplace transform with respect to γ.

For the problem of distributions of functionals at the time $H_{a,b}$, the exit across the upper or lower boundary has an important meaning. Thus we must consider the Laplace transform of the distribution of the integral functional reduced to the set $\{W(H_{a,b}) = b\}$ or the set $\{W(H_{a,b}) = a\}$, respectively. The function Q can be expressed as the sum of these functions.

The following result is a consequence of Theorem 12.6 Ch. II, with $\sigma(x) \equiv 1$, $\mu(x) \equiv 0$ and $F(x) \equiv 0$. Nevertheless, we will prove it in a different way to illustrate the variety of probabilistic methods for solving the problems of distribution of functionals. As it was mentioned, the class of integral functionals of a process with only continuous integrands is too narrow. It will be extended to the class of piecewise-continuous functions. For brevity of notations the parameter γ is included in the function f.

Theorem 5.1. *Let $f(x)$, $x \in [a,b]$, be a piecewise-continuous nonnegative function. Then the function*

$$Q_a(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds\Big); W(H_{a,b}) = a\Big\}, \qquad x \in [a,b], \tag{5.3}$$

is the unique solution of the problem

$$\frac{1}{2}Q''(x) - f(x)Q(x) = 0, \qquad x \in (a,b), \tag{5.4}$$

$$Q(a) = 1, \qquad Q(b) = 0. \tag{5.5}$$

Remark 5.1. If the function Q_b is considered, where we set $W(H_{a,b}) = b$, then $Q_b(x)$, $x \in [a,b]$, satisfies (5.4) and

$$Q(a) = 0, \qquad Q(b) = 1. \tag{5.6}$$

This is a consequence of the spatial homogeneity and symmetry properties of the Brownian motion W.

Theorem 5.2. *Let $f(x)$, $x \in [a,b]$, be a piecewise-continuous nonnegative function. Then the function*

$$Q(x) = \mathbf{E}_x \exp\Big(- \int_0^{H_{a,b}} f(W(s))\,ds \Big), \qquad x \in [a,b], \tag{5.7}$$

is the unique solution of the problem

$$\frac{1}{2}Q''(x) - f(x)Q(x) = 0, \qquad x \in (a,b), \tag{5.8}$$

$$Q(a) = 1, \qquad Q(b) = 1. \tag{5.9}$$

Remark 5.2. The function Q defined by (5.7) is the sum of the functions Q_a and Q_b, since the events $\{W(H_{a,b}) = a\}$ and $\{W(H_{a,b}) = b\}$ are complementary. This follows also from the solutions of the differential problems: adding the solutions of the problems (5.4), (5.5) and (5.4), (5.6) we obtain the solution of the problem (5.8), (5.9).

Proof of Theorem 5.1. We first prove two auxiliary results. The first one is the special case of Proposition 12.5 Ch. II, because $S(x) = x$ for a Brownian motion. The second result is a special case of Proposition 12.4 Ch. II for $S(x) = x$, $\Phi(a) = \Phi(b) = 0$, and Theorem 12.6 Ch. II. We, however, give an original derivation of these results for a Brownian motion.

Lemma 5.1. *Let $W(s)$, $s \geq 0$, be a Brownian motion starting at $x \in [a,b]$. Then the probabilities of exit through the boundaries a or b equal*

$$\mathbf{P}_x(W(H_{a,b}) = a) = \frac{b-x}{b-a}, \qquad \mathbf{P}_x(W(H_{a,b}) = b) = \frac{x-a}{b-a}, \tag{5.10}$$

respectively

Proof. We prove, for example, the right-hand side relation in (5.10). Denote $Y_b(x) := \mathbf{P}_x(W(H_{a,b}) = b)$. It is obvious that $Y_b(a) = 0$ and $Y_b(b) = 1$. In view of the strong Markov property and the symmetry of the Brownian motion ($-W(s)$ is also a Brownian motion), we have

$$Y_b(x) = \frac{1}{2} Y_b(x-\delta) + \frac{1}{2} Y_b(x+\delta) \tag{5.11}$$

for any $\delta < \min\{b-x, x-a\}$. Indeed, by the symmetry property, the Brownian motion, starting at x, reaches each of the levels $x-\delta$, $x+\delta$ with probabilities 1/2. By the strong Markov property, the Brownian motion starts anew at the first exit time, regardless of the behavior until the boundary. This implies (5.11). Equality (5.11) can be recast as

$$Y_b(x+\delta) - Y_b(x) = Y_b(x) - Y_b(x-\delta).$$

Thus the function Y has identical increments on the intervals $(x-\delta, x)$ and $(x, x+\delta)$. In view of the boundary conditions, the equalities

$$Y_b(x_k) = \frac{k}{n} = \frac{x_k - a}{b-a} \tag{5.12}$$

hold on the lattice $x_k = a + \dfrac{k}{n}(b-a)$. Using the strong Markov property it is not hard to get that $Y_b(x) = Y_y(x) Y_b(y)$ for $x < y$. This implies the monotonicity of the function $Y_b(x)$, i.e., $Y_b(x) \le Y_b(y)$. Since n is arbitrary, (5.12) is realized on a dense lattice. In view of monotonicity of $Y_b(x)$, it is valid for all x in $[a, b]$. □

Lemma 5.2. *Set*

$$h(x,y) := \begin{cases} \dfrac{2(b-y)(x-a)}{b-a}, & a \le x \le y \le b, \\[2mm] \dfrac{2(b-x)(y-a)}{b-a}, & a \le y \le x \le b. \end{cases}$$

Then for any bounded function $g(y)$,

$$\mathbf{E}_x \int_0^{H_{a,b}} g(W(t))\,dt = \int_a^b g(y) h(x,y)\,dy. \tag{5.13}$$

Proof. Since the event $\{t < H_{a,b}\}$ is equivalent to the event that the Brownian motion lies between the levels a and b up to the time t, we have

$$\mathbf{E}_x \int_0^{H_{a,b}} g(W(t))\,dt = \mathbf{E}_x \int_0^\infty \mathbb{1}_{\{0 \le t < H_{a,b}\}} g(W(t))\,dt$$

$$= \int_0^\infty dt \int_a^b g(y) \mathbf{E}_x \big\{ \mathbb{1}_{\{0 \le t < H_{a,b}\}}, W(t) \in [y, y+dy) \big\} = \int_a^b g(y) \int_0^\infty e^{-\lambda t}$$

$$\times \mathbf{P}_x \Big(a < \inf_{0 \le s \le t} W(s), \sup_{0 \le s \le t} W(s) < b, W(t) \in [y, y+dy) \Big) dt \Big|_{\lambda=0}$$

$$= \int_a^b g(y) \frac{1}{\lambda} \frac{d}{dy} \mathbf{P}_x \Big(a < \inf_{0 \le s \le \tau} W(s), \sup_{0 \le s \le \tau} W(s) < b, W(\tau) < y \Big) \Big|_{\lambda=0} dy.$$

Using (4.43) and the asymptotic expansion $\operatorname{sh} x = x + x^3/3! + \cdots$, we get

$$\frac{1}{\lambda}\frac{d}{dy}\mathbf{P}_x\Big\{a < \inf_{0\le s\le\tau} W(s), \sup_{0\le s\le\tau} W(s) < b, W(\tau) < y\Big\}\Big|_{\lambda=0}$$

$$= \lim_{\lambda\to 0}\frac{1}{\lambda}\frac{\sqrt{\lambda}(\operatorname{ch}((b-a-|y-x|)\sqrt{2\lambda}) - \operatorname{ch}((b+a-y-x)\sqrt{2\lambda})}{\sqrt{2}\operatorname{sh}((b-a)\sqrt{2\lambda}))}$$

$$= \lim_{\lambda\to 0}\frac{\sqrt{2}\operatorname{sh}((2b-y-x-|y-x|)\sqrt{\lambda/2})\operatorname{sh}((y+x-|y-x|-2a)\sqrt{\lambda/2})}{\sqrt{\lambda}\operatorname{sh}((b-a)\sqrt{2\lambda})}$$

$$= \frac{(2b-y-x-|y-x|)(y+x-|y-x|-2a)}{2(b-a)} = h(x,y).$$

Thus the relation(5.13) is proved. □

Now we can pass to the proof of Theorem 5.1. We use the equality

$$Q_a(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds\Big); W(H_{a,b}) = a\Big\}$$

$$= \mathbf{E}_x\Big\{\Big[1 - \int_0^{H_{a,b}} f(W(t))\exp\Big(-\int_t^{H_{a,b}} f(W(s))\,ds\Big)dt\Big]; W(H_{a,b}) = a\Big\}.$$

We set $\widetilde{W}(v) := W(v+t) - W(t) + y$, $v \ge 0$, where $y \in \mathbf{R}$ is the starting point of $\widetilde{W}$. Let $\widetilde{W}(0) = W(t)$ and $\widetilde{H}_{a,b} = \min\{v : \widetilde{W}(v) \notin (a,b)\}$, then $H_{a,b} = t + \widetilde{H}_{a,b}$ for the set $\{t \le H_{a,b}\}$. For a fixed y the Brownian motion $\widetilde{W}$ is independent of $W(t)$. Now, using the 4th and 3rd properties of conditional expectations, and Lemma 2.1 Ch. I with $\mathcal{Q} = \sigma(w(s), 0 \le s \le t)$, we obtain

$$Q_a(x) = \mathbf{P}_x(W(H_{a,b}) = a) - \mathbf{E}_x\Big\{\mathbf{E}\Big\{\int_0^\infty \mathbb{1}_{\{t<H_{a,b}\}} f(W(t))$$

$$\times\exp\Big(-\int_0^{H_{a,b}-t} f(W(v+t) - W(t) + W(t))\,dv\Big)dt\,\mathbb{1}_{\{W(H_{a,b})=a\}}\Big|\mathcal{Q}\Big\}\Big\}$$

$$= \mathbf{P}_x(W(H_{a,b}) = a)$$

$$-\mathbf{E}_x\int_0^\infty \mathbb{1}_{\{t<H_{a,b}\}} f(W(t))\mathbf{E}_{W(t)}\Big\{\exp\Big(-\int_0^{\widetilde{H}_{a,b}} f(\widetilde{W}(v))\,dv\Big); \widetilde{W}(H_{a,b}) = a\Big\}dt$$

$$= \mathbf{P}_x(W(H_{a,b}) = a) - \mathbf{E}_x\int_0^{H_{a,b}} f(W(t))Q_a(W(t))\,dt.$$

By (5.10) and (5.13), we have

$$Q_a(x) = \frac{b-x}{b-a} - \int_a^b f(y)Q_a(y)h(x,y)\,dy.$$

Using the definition of the function $h(x,y)$, we can write

$$Q_a(x) = \frac{b-x}{b-a} - \frac{2(x-a)}{b-a}\int_x^b (b-y)f(y)Q_a(y)\,dy$$
$$- \frac{2(b-x)}{b-a}\int_a^x (y-a)f(y)Q_a(y)\,dy. \tag{5.14}$$

From (5.14) it follows that $Q_a(x)$, $x \in (a,b)$, is continuous and conditions (5.5) hold. Therefore, Q_a is differentiable at the points of continuity of f. Indeed, at those points, we have

$$Q_a'(x) = \frac{1}{b-a} - \frac{2}{b-a}\int_x^b (b-y)f(y)Q_a(y)\,dy + \frac{2}{b-a}\int_a^x (y-a)f(y)Q_a(y)\,dy.$$

This relation implies that $Q_a'(x)$ exists everywhere in (a,b) and it is continuous on (a,b); moreover, at the points of continuity of f the function $Q_a'(x)$, $x \in (a,b)$, is differentiable, and

$$Q_a''(x) = \frac{2(b-x)}{b-a}f(x)Q_a(x) + \frac{2(x-a)}{b-a}f(x)Q_a(x) = 2f(x)Q_a(x).$$

Thus, by Remark 1.2, the function $Q_a(x)$, $x \in (a,b)$, satisfies equation (5.4). Theorem 5.1 is proved. □

The following result enables us to compute the joint distribution of an integral functional of the Brownian motion W and its local times on different levels at the moment $H_{a,b}$. This result is derived from Theorem 5.1 similarly to how Theorem 3.1 was derived from Theorem 2.1. The general idea of the proof was stated in Remark 4.1.

Let the functional $A_{\vec\beta}(t)$ be defined as at the beginning of § 3.

Theorem 5.3. *Let $f(x)$, $x \in [a,b]$ be a piecewise-continuous nonnegative function. Then the function*

$$Q_{a,\vec\beta}(x) := \mathbf{E}_x\{\exp(-A_{\vec\beta}(H_{a,b}); W(H_{a,b}) = a\}, \qquad x \in [a,b],$$

is the unique continuous solution of the problem

$$\frac{1}{2}Q''(x) - f(x)Q(x) = 0, \qquad x \in (a,b) \setminus \{q_1,\dots,q_m\}, \tag{5.15}$$
$$Q'(q_k+0) - Q'(q_k-0) = 2\beta_k Q(q_k), \qquad k = 1,\dots,m, \tag{5.16}$$
$$Q(a) = 1, \qquad Q(b) = 0. \tag{5.17}$$

Remark 5.3. A similar result holds when the expectation is reduced to the set $\{W(H_{a,b}) = b\}$. One only needs to change the boundary conditions (5.17) to the conditions (5.6). If there are no restrictions on the way out across the boundary, conditions (5.17) must be replaced by (5.9).

Let $H_z := \min\{s : W(s) = z\}$ be the *first hitting time* of the level z by the Brownian motion. It is clear that

$$H_z = \begin{cases} \lim\limits_{a\to-\infty} H_{a,z}, & \text{for } x \le z, \\ \lim\limits_{b\to\infty} H_{z,b}, & \text{for } z \le x. \end{cases} \tag{5.18}$$

From Theorem 12.7 Ch. II it follows that for a continuous function $f(x)$, $x \in \mathbf{R}$, the function

$$L_z(x) := \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_z} f(W(s))\,ds\Big)\mathbb{I}_{\{H_z<\infty\}}\Big\} \tag{5.19}$$

can be represented in the form

$$L_z(x) = \begin{cases} \dfrac{\psi(x)}{\psi(z)}, & \text{for } x \le z, \\ \dfrac{\varphi(x)}{\varphi(z)}, & \text{for } z \le x, \end{cases} \tag{5.20}$$

where φ is a convex positive decreasing solution and ψ is a convex positive increasing solution of the equation

$$\frac{1}{2}\phi''(x) - f(x)\phi(x) = 0, \qquad x \in \mathbf{R}. \tag{5.21}$$

Since for $f \equiv 0$ the positive solution of (5.21) on the whole real line is a constant, we see that $L_z(x) \equiv 1$, and hence, $\mathbf{P}_x(H_z < \infty) = 1$. Therefore, the indicator function $\mathbb{I}_{\{H_z<\infty\}}$ in the definition of the function L_z can be removed.

It is clear that (5.20) holds also for piecewise-continuous functions f. This can be proved by using the limit approximation method (see the proof of Theorem 1.2).

Example 5.1. We compute the distribution of the first exit time $H_{a,b}$. Applying Theorem 5.3 with $f \equiv \alpha$, we see that

$$Q_a(x) = \mathbf{E}_x\big\{e^{-\alpha H_{a,b}}; W(H_{a,b}) = a\big\}, \qquad x \in [a,b], \tag{5.22}$$

is the unique solution of the problem

$$\frac{1}{2}Q''(x) - \alpha Q(x) = 0, \qquad x \in (a,b), \tag{5.23}$$

$$Q(a) = 1, \qquad Q(b) = 0. \tag{5.24}$$

Among the linearly independent solutions of equation (5.23) we can choose $\operatorname{sh}((b-x)\sqrt{2\alpha})$ and $\operatorname{sh}((x-a)\sqrt{2\alpha})$. The first solution vanishes at b and the second vanishes at a. Using this property, it is easy to see that

$$M_a(x) = \mathbf{E}_x\Big\{e^{-\alpha H_{a,b}}; W(H_{a,b}) = a\Big\} = \frac{\operatorname{sh}((b-x)\sqrt{2\alpha})}{\operatorname{sh}((b-a)\sqrt{2\alpha})}, \qquad x \in [a,b], \tag{5.25}$$

is the solution of the problem (5.23), (5.24).

Inverting the Laplace transform with respect to α (see Section 13 of Appendix 2), we get

$$\mathbf{P}_x(H_{a,b}\in dt,\ W(H_{a,b})=a)=\mathrm{ss}_t(b-x,b-a)\,dt$$

$$=\sum_{k=-\infty}^{\infty}\frac{x-a+2k(b-a)}{\sqrt{2\pi}t^{3/2}}e^{-(x-a+2k(b-a))^2/2t}\,dt,\qquad x\in[a,b]. \tag{5.26}$$

Similarly, one can show that

$$\mathbf{E}_x\Big\{e^{-\alpha H_{a,b}};W(H_{a,b})=b\Big\}=\frac{\mathrm{sh}((x-a)\sqrt{2\alpha})}{\mathrm{sh}((b-a)\sqrt{2\alpha})},\qquad x\in[a,b], \tag{5.27}$$

and

$$\mathbf{P}_x(H_{a,b}\in dt,\ W(H_{a,b})=b)=\mathrm{ss}_t(x-a,b-a)\,dt$$

$$=\sum_{k=-\infty}^{\infty}\frac{b-x+2k(b-a)}{\sqrt{2\pi}t^{3/2}}e^{-(b-x+2k(b-a))^2/2t}\,dt,\qquad x\in[a,b]. \tag{5.28}$$

Summing (5.25) and (5.27), we have

$$\mathbf{E}_x e^{-\alpha H_{a,b}}=\frac{\mathrm{sh}((b-x)\sqrt{2\alpha})+\mathrm{sh}((x-a)\sqrt{2\alpha})}{\mathrm{sh}((b-a)\sqrt{2\alpha})}$$

$$=\frac{\mathrm{ch}((b+a-2x)\sqrt{\alpha/2})}{\mathrm{ch}((ba)\sqrt{\alpha/2})},\quad x\in[a,b], \tag{5.29}$$

and summing (5.26) and (5.28), we have

$$\mathbf{P}_x(H_{a,b}\in dt)=\mathrm{ss}_t(b-x,b-a)\,dt+\mathrm{ss}_t(x-a,b-a)\,dt,\quad x\in[a,b]. \tag{5.30}$$

Example 5.2. We compute the distribution of the first hitting time H_z. Applying (5.20) and (5.21) with $f\equiv\alpha$, we have

$$L_z(x)=\mathbf{E}_x e^{-\alpha H_z}=e^{-|x-z|\sqrt{2\alpha}},\qquad x\in\mathbf{R}. \tag{5.31}$$

Inverting the Laplace transform with respect to α (see formula 2 of Appendix 3), we obtain

$$\mathbf{P}_x(H_z\in dt)=\frac{|x-z|}{\sqrt{2\pi}t^{3/2}}\exp\Big(-\frac{(x-z)^2}{2t}\Big)\,dt,\qquad x\in\mathbf{R}. \tag{5.32}$$

Example 5.3. We compute the Laplace transform

$$Q_{a,\beta}(x):=\mathbf{E}_x\big\{e^{-\beta\ell(H_{a,b},q)};W(H_{a,b})=a\big\},\qquad \beta>0,$$

and find the joint distribution of the local time $\ell(H_{a,b},q)$ and the variable $W(H_{a,b})$. By Theorem 5.3, the function $Q_{a,\beta}(x)$, $x\in[a,b]$, is the unique continuous solution of the problem

$$\begin{aligned}&\frac{1}{2}Q''(x)=0, \qquad x\in(a,b)\setminus\{q\},\\ &Q'(q+0)-Q'(q-0)=2\beta Q(q),\\ &Q(a)=1, \qquad Q(b)=0.\end{aligned}$$

It is easy to compute that the solution of this problem is

$$\mathbf{E}_x\big\{e^{-\beta\ell(H_{a,b},q)};W(H_{a,b})=a\big\}=\begin{cases}\dfrac{b-x}{b-a+2\beta(b-q)(q-a)}, & q\le x\le b,\\ \dfrac{b-x+2\beta(b-q)(q-x)}{b-a+2\beta(b-q)(q-a)}, & a\le x\le q.\end{cases} \tag{5.33}$$

If we change the boundary conditions, we obtain

$$\mathbf{E}_x\big\{e^{-\beta\ell(H_{a,b},q)};W(H_{a,b})=b\big\}=\begin{cases}\dfrac{x-a+2\beta(x-q)(q-a)}{b-a+2\beta(b-q)(q-a)}, & q\le x\le b,\\ \dfrac{x-a}{b-a+2\beta(b-q)(q-a)}, & a\le x\le q.\end{cases} \tag{5.34}$$

Inverting the Laplace transform in (5.33) with respect to β (see formula 1 of Appendix 3), we get

$$\mathbf{P}_x\big(\ell(H_{a,b},q)\in dy, W(H_{a,b})=a\big)$$

$$=\begin{cases}\dfrac{b-x}{2(b-q)(q-a)}\exp\Big(-\dfrac{(b-a)y}{2(b-q)(q-a)}\Big)dy, & q\le x\le b,\\ \dfrac{x-a}{2(q-a)^2}\exp\Big(-\dfrac{(b-a)y}{2(b-q)(q-a)}\Big)dy, & a\le x\le q,\end{cases} \tag{5.35}$$

$$\mathbf{P}_x\big(\ell(H_{a,b},q)=0, W(H_{a,b})=a\big)=\begin{cases}0, & q\le x\le b,\\ \dfrac{q-x}{q-a}, & a\le x\le q.\end{cases} \tag{5.36}$$

Inverting the Laplace transform in (5.34) with respect to β, we obtain

$$\mathbf{P}_x\big(\ell(H_{a,b},q)\in dy, W(H_{a,b})=b\big)$$

$$=\begin{cases}\dfrac{b-x}{2(b-q)^2}\exp\Big(-\dfrac{(b-a)y}{2(b-q)(q-a)}\Big)dy, & q\le x\le b,\\ \dfrac{x-a}{2(b-q)(q-a)}\exp\Big(-\dfrac{(b-a)y}{2(b-q)(q-a)}\Big)dy, & a\le x\le q,\end{cases} \tag{5.37}$$

$$\mathbf{P}_x\big(\ell(H_{a,b},q)=0, W(H_{a,b})=b\big)=\begin{cases}\dfrac{x-q}{b-q}, & q\le x\le b,\\ 0, & a\le x\le q.\end{cases} \tag{5.38}$$

Exercises.

5.1. Compute the Laplace transform

$$\mathbf{E}_x\bigl\{ \operatorname{ch}(W(H_{a,b}))e^{-\alpha H_{a,b}}\bigr\}, \qquad \alpha > 0.$$

5.2. Compute the expressions

$$\mathbf{E}_x\Bigl\{e^{-\alpha H_{a,b}}; H_{a,b} > \tau\Bigr\}, \qquad \mathbf{E}_x\Bigl\{e^{-\alpha H_{a,b}}; H_{a,b} \le \tau\Bigr\}, \qquad \alpha > 0,$$

where τ is the exponentially distributed with the parameter $\lambda > 0$ random time independent of $H_{a,b}$.

5.3. Compute the Laplace transforms

$$\mathbf{E}_x\biggl\{ \exp\biggl(-\gamma \int_0^{H_{a,b}} \mathbb{1}_{[r,b]}(W(s))\,ds\biggr); W(H_{a,b}) = a\biggr\},$$

$$\mathbf{E}_x\biggl\{ \exp\biggl(-\gamma \int_0^{H_{a,b}} \mathbb{1}_{[r,b]}(W(s))\,ds\biggr); W(H_{a,b}) = b\biggr\}, \qquad \gamma > 0.$$

5.4. Compute the Laplace transform

$$\mathbf{E}_x \exp\biggl(-\gamma \int_0^{H_z} \mathbb{1}_{[r,\infty)}(W(s))\,ds\biggr), \qquad \gamma > 0,$$

and find the distribution of the functional $\int_0^{H_z} \mathbb{1}_{[r,\infty)}(W(s))\,ds$.

5.5. Compute the Laplace transform

$$\mathbf{E}_x \exp\biggl(-\alpha H_z - \frac{\gamma^2}{2} \int_0^{H_z} W^2(s)\,ds\biggr), \qquad \alpha > 0.$$

Hint: Use the solution of equation 3 of Appendix 4.

5.6. Compute the Laplace transform $\mathbf{E}_x e^{-\beta\ell(H_z,q)}$, $\beta > 0$, and find the distribution of the local time $\ell(H_z, q)$.

§6. Distributions of functionals of Brownian motion stopped at the moment inverse of integral functional

We consider the integral functional $\int_0^t g(W(s))\,ds$, $t>0$, where g is a nonnegative piecewise-continuous function, which is not identically equal to zero. For definiteness we assume that at points of discontinuity it takes the values of the right limits, $g(z)=g(z+0)$.

According to Proposition 1.1,

$$\int_0^\infty g(W(s))\,ds=\infty, \qquad \text{a.s.} \tag{6.1}$$

We consider the problem of computing distributions of functionals of Brownian motion stopped at the *moment inverse of integral functional.* This moment is defined by the formula

$$\nu(t):=\min\Big\{s:\int_0^s g(W(v))\,dv=t\Big\}, \qquad t\ge 0.$$

By (6.1), this moment is a.s. finite. It is a stopping time with respect to the natural filtration $\mathcal{G}_0^t=\sigma(W(v),0\le v\le t)$, $t\ge 0$, generated by the Brownian motion. It is possible that $\nu(0+)>0$.

The following result is of key importance for the problem of computing distributions of functionals of Brownian motion stopped at the moment inverse of the integral functional.

Theorem 6.1. *Let $\Phi(x)$, $f(x)$, $x\in\mathbf{R}$, be piecewise-continuous functions. Assume that $f\ge 0$ and Φ is bounded. Then the function*

$$U(x):=\mathbf{E}_x\Big\{\Phi(W(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)} f(W(s))\,ds\Big)\Big\}, \qquad x\in\mathbf{R},$$

is the unique bounded solution of the equation

$$\frac{1}{2}U''(x)-(\lambda g(x)+f(x))U(x)=-\lambda g(x)\Phi(x), \qquad x\in\mathbf{R}. \tag{6.2}$$

Remark 6.1. For $g\equiv 1$ this statement turns into Theorem 1.2.

Remark 6.2. Equation (6.2) has only one bounded solution on $\mathbf{R}$. This follows from the fact that the corresponding homogeneous equation has no bounded nontrivial solutions on the whole real line.

Proof of Theorem 6.1. We assume first that f is a bounded continuous function and Φ is a twice continuously differentiable function with bounded derivatives.

For any $\varrho > 0$ set

$$\eta_\varrho(s) := \Phi(W(s)) \exp\Big(- \int_0^s \big(\varrho + f(W(v))\big)\, dv\Big). \tag{6.3}$$

Applying Itô's formula, we have

$$\eta_\varrho(t) - \eta_\varrho(0) = \int_0^t \exp\Big(- \int_0^s \big(\varrho + f(W(v))\big)\, dv\Big) \Big[\Phi'(W(s))\, dW(s)$$
$$+ \frac{1}{2}\Phi''(W(s))\, ds - \big(\varrho + f(W(s))\big)\Phi(W(s))\, ds\Big].$$

Since g is a nonnegative function, $\int_0^s g(W(v))\, dv$, $s \ge 0$, is an increasing function. This implies that for any $s \ge 0$ and $t \ge 0$

$$\mathbb{1}_{[0,\nu(t))}(s) = \mathbb{1}_{[0,t)}\Big(\int_0^s g(W(v))\, dv\Big).$$

By (3.8) Ch II, we have

$$\eta_\varrho(\nu(t)) - \Phi(x) = \int_0^\infty \mathbb{1}_{[0,t)}\Big(\int_0^s g(W(v))\, dv\Big) \exp\Big(- \int_0^s \big(\varrho + f(W(v))\big)\, dv\Big)$$
$$\times \Big[\Phi'(W(s))\, dW(s) + \frac{1}{2}\Phi''(W(s))\, ds - \big(\varrho + f(W(s))\big)\Phi(W(s))\, ds\Big].$$

Since the expectation of a stochastic integral equals zero,

$$\mathbf{E}_x\eta_\varrho(\nu(t)) - \Phi(x) = \mathbf{E}_x \int_0^\infty \mathbb{1}_{[0,t]}\Big(\int_0^s g(W(v))\, dv\Big) \exp\Big(- \int_0^s \big(\varrho + f(W(v))\big) dv\Big)$$
$$\times \Big[\frac{1}{2}\Phi''(W(s)) - \big(\varrho + f(W(s))\big)\Phi(W(s))\Big]\, ds.$$

Taking the Laplace transform of this equality with respect to t, we obtain

$$\lambda \int_0^\infty e^{-\lambda t}\mathbf{E}_x\eta_\varrho(\nu(t))\, dt - \Phi(x) = \mathbf{E}_x \int_0^\infty e^{-\varrho s} \exp\Big(- \int_0^s \big(\lambda g(W(v)) + f(W(v))\big) dv\Big)$$
$$\times \Big[\frac{1}{2}\Phi''(W(s)) - \big(\varrho + f(W(s))\big)\Phi(W(s))\Big]\, ds.$$

Alongside with the moment τ we consider the independent of the Brownian motion W exponentially distributed random time $\tilde\tau$, for which $\mathbf{P}(\tilde\tau > s) = e^{-\varrho s}$. Then applying Fubini's theorem, we have

$$\mathbf{E}_x\eta_\varrho(\nu(\tau)) - \Phi(x) = \frac{1}{\varrho}\mathbf{E}_x\Big\{ \Big[\frac{1}{2}\Phi''(W(\tilde\tau)) - \big(\varrho + f(W(\tilde\tau))\big)\Phi(W(\tilde\tau))\Big]$$

$$\times \exp\biggl(-\int_0^{\tilde\tau} \bigl(\lambda g(W(v)) + f(W(v))\bigr)dv\biggr)\biggr\}.$$

For the expectation on the right-hand side of this equality we apply Theorem 1.2. Then we see that the function

$$\widetilde U(x) := \mathbf{E}_x \eta_\varrho(\nu(\tau)) - \Phi(x)$$

is the unique bounded solution of the equation

$$\frac12 \widetilde U''(x) - (\varrho + \lambda g(x) + f(x))\widetilde U(x) = -\frac12 \Phi''(x) + (\varrho + f(x))\Phi(x).$$

Hence, the function $V_\varrho(x) := \mathbf{E}_x \eta_\varrho(\nu(\tau)) = \widetilde U(x) + \Phi(x)$ satisfies the equation

$$\frac12 V_\varrho''(x) - (\varrho + \lambda g(x) + f(x))V_\varrho(x) = -\lambda g(x)\Phi(x), \qquad x \in \mathbf{R}, \tag{6.4}$$

Using (6.2), we have

$$U(x) = \lim_{\varrho\downarrow 0} \mathbf{E}_x \eta_\varrho(\nu(\tau)) = \lim_{\varrho\downarrow 0} V_\varrho(x).$$

Passing to the limit in (6.4) as $\varrho \downarrow 0$, we deduce that U satisfies equation (6.2). Thus the theorem is proved for any bounded continuous function f and any twice continuously differentiable function Φ with bounded derivatives.

Every nonnegative piecewise-continuous function f can be approximated by a sequence of continuous bounded functions $\{f_n\}$ such that $0 \le f_n(x) \le f(x)$, $x \in \mathbf{R}$. Every bounded piecewise-continuous function Φ can be approximated by a sequence $\{\Phi_n\}$ of uniformly bounded twice continuously differentiable functions with bounded derivatives. Applying the limit approximation method (see the proof of Theorem 1.2), we can prove that U is the unique bounded solution of (6.2) in the general case. □

Example 6.1. We compute the distribution of the moment inverse of the integral functional that is the time spent by the Brownian motion above the level r, i.e., of the moment

$$\nu(t) = \min\biggl\{s : \int_0^s \mathbb{1}_{[r,\infty)}(W(v))\,dv = t\biggr\}. \tag{6.5}$$

We apply Theorem 6.1 with $\Phi \equiv 1$, $f \equiv \gamma$, $g(x) = \mathbb{1}_{[r,\infty)}(x)$, $x \in \mathbf{R}$. In this case the function $U(x) = \mathbf{E}_x e^{-\gamma\nu(\tau)}$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$\frac12 U''(x) - (\lambda \mathbb{1}_{[r,\infty)}(x) + \gamma)U(x) = -\lambda \mathbb{1}_{[r,\infty)}(x), \qquad x \in \mathbf{R}.$$

It is not hard to verify that the solution has the expression

$$U(x) = \begin{cases} \dfrac{\lambda}{\sqrt{\lambda+\gamma}(\sqrt{\gamma}+\sqrt{\lambda+\gamma})} e^{-(r-x)\sqrt{2\gamma}}, & x \le r, \\ \dfrac{\lambda}{\lambda+\gamma} - \Bigl(\dfrac{\lambda}{\lambda+\gamma} - \dfrac{\lambda}{\sqrt{\lambda+\gamma}(\sqrt{\gamma}+\sqrt{\lambda+\gamma})}\Bigr) e^{-(x-r)\sqrt{2\lambda+2\gamma}}, & r \le x. \end{cases}$$

Dividing this equality by λ and inverting the Laplace transform with respect to λ (see formulas a, 6, 11 of Appendix 3), we get

$$\mathbf{E}_x e^{-\gamma\nu(t)} = \begin{cases} e^{-(r-x)\sqrt{2\gamma}}\operatorname{Erfc}\big(\sqrt{t\gamma}\big), & x \le r, \\ e^{-t\gamma} - e^{-t\gamma}\operatorname{Erfc}\Big(\dfrac{x-r}{\sqrt{2t}}\Big) + e^{(x-r)\sqrt{2\gamma}}\operatorname{Erfc}\Big(\dfrac{x-r}{\sqrt{2t}} + \sqrt{t\gamma}\Big), & r \le x. \end{cases}$$

The structure of this Laplace transform with respect to γ (see (1.1)) is such that the corresponding distribution for $r < x$ has the mass point at t of value $1 - \operatorname{Erfc}\Big(\dfrac{x-r}{\sqrt{2t}}\Big)$. Thus for $r \le x$

$$\mathbf{P}_x(\nu(t) = t) = 1 - \operatorname{Erfc}\Big(\frac{x-r}{\sqrt{2t}}\Big).$$

The other part of the Laplace transform corresponds to the density. Taking the inverse Laplace transform with respect to γ (see formulas a and 12 of Appendix 3), we obtain that for $x \le r$

$$\frac{d}{dy}\mathbf{P}_x(\nu(t) < y) = \Big\{\frac{\sqrt{t}}{\pi y\sqrt{y-t}}\exp\Big(-\frac{(r-x)^2}{2(y-t)}\Big)$$

$$+ \frac{r-x}{\sqrt{2\pi}y^{3/2}}\exp\Big(-\frac{(r-x)^2}{2y}\Big)\operatorname{Erfc}\Big(\frac{(r-x)\sqrt{t}}{\sqrt{2y(y-t)}}\Big)\Big\}\mathbb{I}_{(t,\infty)}(y),$$

and for $r \le x$

$$\frac{d}{dy}\mathbf{P}_x(\nu(t) < y) = \Big\{\frac{\sqrt{t}}{\pi y\sqrt{y-t}}\exp\Big(-\frac{(x-r)^2}{2t}\Big)$$

$$- \frac{x-r}{\sqrt{2\pi}y^{3/2}}\exp\Big(-\frac{(x-r)^2}{2y}\Big)\operatorname{Erfc}\Big(\frac{(x-r)\sqrt{y-t}}{\sqrt{2yt}}\Big)\Big\}\mathbb{I}_{(t,\infty)}(y).$$

Note that for $x \le r$

$$\frac{d}{dy}\mathbf{P}_x(\nu(0+) < y) = \frac{r-x}{\sqrt{2\pi}y^{3/2}}\exp\Big(-\frac{(r-x)^2}{2y}\Big), \qquad 0 < y.$$

It is easy to verify that $\nu(0+) = H_r$ (see (5.32)).

The following theorem enables us to compute the joint distributions of an integral functional, infimum, and supremum of a Brownian motion stopped at the moment inverse of the integral functional.

Theorem 6.2. *Let $\varPhi(x)$, $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \ge 0$. Then the function*

$$U(x) := \mathbf{E}_x\Big\{\varPhi(W(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)} f(W(s))\,ds\Big);$$

$$a \le \inf_{0\le s\le\nu(\tau)} W(s), \sup_{0\le s\le\nu(\tau)} W(s) \le b\Big\}, \qquad x \in [a,b],$$

is the unique solution of the problem

$$\frac{1}{2}U''(x) - (\lambda g(x) + f(x))U(x) = -\lambda g(x)\Phi(x), \quad x \in (a,b), \tag{6.6}$$

$$U(a) = 0, \qquad U(b) = 0. \tag{6.7}$$

Theorem 6.2 can be derived from Theorem 6.1 similarly to the proof of Theorem 2.1.

Remark 6.3. For $g \equiv 1$ this statement turns into Theorem 2.1. Remark 2.1 is also applicable to Theorem 6.2.

Example 6.2. Let us compute the distribution

$$\mathbf{P}_x\Big(\sup_{0\le s\le \nu(t)} W(s) \le b\Big),$$

where the moment ν is defined in (6.5). We apply Theorem 6.2 with $\Phi \equiv 1$, $f \equiv 0$, $g(x) = \mathbb{1}_{[r,\infty)}(x)$, $x \in \mathbf{R}$ and $a = -\infty$. In this case the function

$$U(x) = \mathbf{P}_x\Big(\sup_{0\le s\le \nu(\tau)} W(s) \le b\Big)$$

is the unique bounded solution of the problem

$$\frac{1}{2}U''(x) - \lambda\mathbb{1}_{[r,\infty)}(x)U(x) = -\lambda\mathbb{1}_{[r,\infty)}(x), \qquad x \in (-\infty, b),$$

$$U(b) = 0.$$

Such a solution can be represented as

$$U(x) = \begin{cases} 1 - B, & x \le r, \\ 1 - \dfrac{\operatorname{sh}((x-r)\sqrt{2\lambda})}{\operatorname{sh}((b-r)\sqrt{2\lambda})} - B\dfrac{\operatorname{sh}((b-x)\sqrt{2\lambda})}{\operatorname{sh}((b-r)\sqrt{2\lambda})}, & r \le x \le b. \end{cases}$$

Here in the interval (r,b) we choose linearly independent solutions $\operatorname{sh}((x-r)\sqrt{2\lambda})$ and $\operatorname{sh}((b-x)\sqrt{2\lambda})$ instead of exponentials, because they vanish at the points r and b, respectively. In the expression suggested for U the boundary and continuity conditions were taken into account.

The constant B is computed from the continuity condition for the derivative of U at the point r:

$$B = \frac{1}{\operatorname{ch}((b-r)\sqrt{2\lambda})}.$$

We obtain

$$\mathbf{P}_x\Big(\sup_{0\le s\le \nu(\tau)} W(s) \le b\Big) = \begin{cases} 1 - \dfrac{1}{\operatorname{ch}((b-r)\sqrt{2\lambda})}, & x \le r, \\ 1 - \dfrac{\operatorname{ch}((x-r)\sqrt{2\lambda})}{\operatorname{ch}((b-r)\sqrt{2\lambda})}, & r \le x \le b. \end{cases}$$

Dividing this equality by λ and inverting the Laplace transform with respect to λ (see Section 13 of Appendix 2), we obtain

$$\mathbf{P}_x\Big(\sup_{0\le s\le\nu(t)} W(s)\le b\Big)=\begin{cases}1-\widetilde{\mathrm{cc}}_t(0,b-r), & x\le r,\\ 1-\widetilde{\mathrm{cc}}_t(x-r,b-r), & r\le x\le b.\end{cases}$$

It is useful to have results which enable us to compute the joint distributions of functionals of the Brownian motion stopped at some random moment and the Brownian motion itself at that moment. Such a result for the moment τ was considered in §4. We can compute such a joint distribution for the moment $\nu(\tau)$ with the help of the following theorem.

Theorem 6.3. *Let $f(x)$, $x\in[a,b]$, be a nonnegative piecewise-continuous function. Then for every $z\in(a,b)$ there exists the right derivative*

$$G_{\nu,z}(x):=\frac{d}{dz+}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\nu(\tau)} f(W(s))\,ds\Big);$$
$$a\le\inf_{0\le s\le\nu(\tau)}W(s),\ \sup_{0\le s\le\nu(\tau)}W(s)\le b,\ W(\nu(\tau))<z\Big\},$$

and $G_{\nu,z}(x)$, $x\in[a,b]$, is the unique continuous solution of the problem

$$\frac12 G''(x)-\big(\lambda g(x)+f(x)\big)G(x)=0,\qquad x\in(a,b)\setminus\{z\},\tag{6.8}$$
$$G'(z+0)-G'(z-0)=-2\lambda g(z),\tag{6.9}$$
$$G(a)=0,\qquad G(b)=0.\tag{6.10}$$

Remark 6.4. $G_{\nu,z}(x)$, $(z,x)\in[a,b]\times[a,b]$, is the Green function of the problem (6.6), (6.7), because by the definitions of the functions $U(x)$ and $G_{\nu,z}(x)$,

$$U(x)=\int_a^b \Phi(z)G_{\nu,z}(x)\,dz.$$

Remark 6.5. In contrast to the function $G_x(z)$ from Theorem 4.1, the argument of the function $G_{\nu,z}(x)$ is the starting point of W.

Proof of Theorem 6.3. This result can be proved with the help of Theorem 6.2. Set

$$U_\Delta(x):=\mathbf{E}_x\Big\{\frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(W(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)} f(W(s))\,ds\Big);$$
$$a\le\inf_{0\le s\le\nu(\tau)}W(s),\ \sup_{0\le s\le\nu(\tau)}W(s)\le b\Big\}.\tag{6.11}$$

By Theorem 6.2, with $\Phi(x) = \frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(x)$, $x \in (a,b)$, the function $U_\Delta(x)$ is the unique solution of the problem

$$\frac{1}{2}U''_\Delta(x) - \big(\lambda g(x) + f(x)\big)U_\Delta(x) = -\lambda g(x)\frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(x), \tag{6.12}$$

$$U_\Delta(a) = 0, \qquad U_\Delta(b) = 0. \tag{6.13}$$

Set $\chi_\Delta(x) := \int\limits_{-\infty}^{x} \frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(r)\, dr$ for $0 < \Delta < 1$. Then (6.12) can be written as follows: for every $y, x \in (a,b)$

$$\frac{1}{2}(U'_\Delta(x) - U'_\Delta(y)) - \int\limits_y^x (\lambda g(r) + f(r))U_\Delta(r)\, dr = -\lambda \int\limits_y^x g(r) d\chi_\Delta(r). \tag{6.14}$$

We must justify the passage to the limit in (6.14), (6.13) as $\Delta \downarrow 0$. We carried out such limit transformations for the problems (3.6), (3.7) and (4.15), (4.14). However, there are some aspects that we need to discuss. We have no estimate (4.16) here, because we have no the analogue of (4.6) for the variable $W(\nu(\tau))$. Therefore, we start with the derivation of such an estimate.

We set $U_{0,\Delta}(x) := \frac{1}{\Delta}\mathbf{E}_x \mathbb{I}_{[z,z+\Delta)}(W(\nu(\tau)))$, $0<\Delta<1$, and prove that this family is uniformly bounded on any finite interval. This is important, since $U_\Delta(x) \le U_{0,\Delta}(x)$.

By Theorem 6.1, the function $U_{0,\Delta}(x)$, $x \in \mathbf{R}$, is the unique bounded solution of (6.12) on the whole real line for $f \equiv 0$. According to the theory of differential equations (formula (4.7)), such a solution is expressed in terms of fundamental solutions of the homogeneous equation. Let $\psi_0(x)$ and $\varphi_0(x)$ be the fundamental solutions of the equation

$$\frac{1}{2}\phi''(x) - \lambda g(x)\phi(x) = 0, \tag{6.15}$$

and let $\omega_0 = \psi'_0(x)\varphi_0(x) - \psi_0(x)\varphi'_0(x) > 0$ be their Wronskian, which is a constant. Then by (4.11) and (4.21) with $a = -\infty$ and $b = \infty$,

$$U_{0,\Delta}(x) = \frac{2\lambda\varphi_0(x)}{\Delta}\int\limits_{-\infty}^{x} \mathbb{I}_{[z,z+\Delta)}(r)\frac{g(r)\psi_0(r)}{\omega_0}\, dr + \frac{2\lambda\psi_0(x)}{\Delta}\int\limits_{x}^{\infty} \mathbb{I}_{[z,z+\Delta)}(r)\frac{g(r)\varphi_0(r)}{\omega_0}\, dr.$$

Since the function φ_0 is decreasing and the function ψ_0 is increasing, we have

$$\sup_{x\in\mathbf{R}} U_\Delta(x) \le \sup_{x\in\mathbf{R}} U_{0,\Delta}(x) \le \frac{2\lambda\varphi_0(z)\psi_0(z+1)}{\omega_0} \sup_{z\le r\le z+1} g(r).$$

Hence, the family of functions $\big\{U_\Delta(x)\big\}_{0<\Delta<1}$ is uniformly bounded.

Integrating (6.14) with respect to x over the interval (y,v), we obtain

$$\frac{1}{2}\big(U_\Delta(v) - U_\Delta(y)\big) - \frac{1}{2}U'_\Delta(y)(v-y) - \int\limits_y^v \int\limits_y^x (\lambda g(r) + f(r))U_\Delta(r)\, drdx$$

$$= -\lambda \int\limits_y^v \int\limits_y^x g(r)\, d\chi_\Delta(r)dx. \tag{6.16}$$

Now integrating (6.14) with respect to y over the interval (u,x), we get

$$-\tfrac{1}{2}\big(U_\Delta(x)-U_\Delta(u)\big)+\tfrac{1}{2}U'_\Delta(x)(x-u)-\int_u^x\int_y^x(\lambda+f(r))U_\Delta(r)\,drdy$$
$$=-\lambda\int_u^x\int_y^x g(r)\,d\chi_\Delta(r)dy. \tag{6.17}$$

As in the proof of Theorem 3.1, (6.16) and (6.17) imply that the family of functions $\{U_\Delta(y)\}_{\Delta>0}$ is equicontinuous on the closed interval $[a,b]$. By the Arzelà–Ascoli theorem, the family of functions $\{U_\Delta(y)\}_{\Delta>0}$ is relatively compact in $[a,b]$ in the uniform norm. This implies that from any sequence $\Delta_n\downarrow 0$ one can extract a subsequence Δ_{n_m} such that

$$\sup_{x\in[a,b]}\big|U_{\Delta_{n_m}}(x)-U(x)\big|\to 0, \tag{6.18}$$

where U is a continuous function. In addition, the boundary conditions for the functions U_Δ are transformed to the boundary conditions $U(a)=0$ and $U(b)=0$.

Now, since $\chi_\Delta(x)\to\chi(x):=\mathbb{1}_{[z,\infty)}(x)$, from (6.16), by passage to the limit, we deduce that for $y\neq z$ there exists the limit $\widetilde U(y)=\lim\limits_{\Delta_{n_m}\downarrow 0}U'_{\Delta_{n_m}}(y)$ and

$$\frac{1}{2}\big(U(v)-U(y)\big)-\frac{1}{2}\widetilde U(y)(v-y)-\int_y^v\int_y^x(\lambda g(r)+f(r))U(r)\,drdx$$
$$=-\lambda\int_y^v\int_y^x g(r)\,d\chi(r)dx. \tag{6.19}$$

From (6.19) and from the limiting analog of (6.17) it follows that the function $U(v)$ is differentiable for $v\in(a,b)\setminus\{z\}$. Differentiating (6.19) with respect to v, we obtain that $\widetilde U(y)=U'(y)$ and, consequently,

$$\frac{1}{2}(U'(v)-U'(y))-\int_y^v(\lambda g(r)+f(r))U(r)\,dr=-\lambda\int_y^v g(r)\,d\chi(r).$$

Since g is a right-continuous function, this equality implies that $U(x)$ is the solution of the problem (6.8)–(6.10). Thus the limit in (6.18) does not depend on the choice of a subsequence Δ_{n_m}. This just means that there exists the right derivative $G_{\nu,z}(x):=U(x)$. Theorem 6.3 is proved. □

Example 6.3. We compute the distribution of the Brownian motion W at the moment $\nu(t)$ defined in (6.5). Applying Theorem 6.3 with $f\equiv 0$, $g(x)=\mathbb{1}_{[r,\infty)}(x)$, $x\in\mathbf{R}$, $a=-\infty$ and $b=\infty$, we see that the function

$$G_{\nu,z}(x)=\frac{d}{dz+}\mathbf{P}_x\big(W(\nu(\tau))<z\big)$$

is the unique bounded continuous solution of the problem

$$\frac{1}{2}G''(x) - \lambda 1\!\!\mathrm{I}_{[r,\infty)}(x)G(x) = 0, \qquad x \neq z, \tag{6.20}$$

$$G'(z+0) - G'(z-0) = -2\lambda 1\!\!\mathrm{I}_{[r,\infty)}(z). \tag{6.21}$$

For $z < r$, by the definition of $G_{\nu,z}$, it follows that $G_{\nu,z}(x) = 0$, $x \in \mathbf{R}$, because the Brownian motion at the moment $\nu(\tau)$ can stop only in the interval $[r, \infty)$. The fact that $G_{\nu,z}(x) = 0$ for $z < r$ follows also directly from the solution of the problem (6.20), (6.21), because $1\!\!\mathrm{I}_{[r,\infty)}(z) = 0$. Therefore, the nontrivial case is $r \leq z$. The unique bounded solution of the problem (6.20), (6.21) for $r \leq z$ is the function

$$G_{\nu,z}(x) = \begin{cases} \sqrt{2\lambda}e^{-(z-r)\sqrt{2\lambda}}, & x \leq r, \\ \dfrac{\sqrt{\lambda}}{\sqrt{2}}e^{-|x-z|\sqrt{2\lambda}} + \dfrac{\sqrt{\lambda}}{\sqrt{2}}e^{-(x+z-2r)\sqrt{2\lambda}}, & r \leq x. \end{cases}$$

Dividing this equality by λ and inverting the Laplace transform with respect to λ (see formula 5 of Appendix 3), for $r \leq z$ we obtain

$$\frac{d}{dz+}\mathbf{P}_x\big(W(\nu(t)) < z\big) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\pi t}}e^{-(z-r)^2/2t}, & x \leq r, \\ \dfrac{1}{\sqrt{2\pi t}}e^{-(x-z)^2/2t} + \dfrac{1}{\sqrt{2\pi t}}e^{-(x+z-2r)^2/2t}, & r \leq x. \end{cases}$$

§7. Distributions of functionals of Brownian motion stopped at the inverse local time

Our main goal of this section is to consider the problem of computing distributions of functionals of Brownian motion stopped at the *inverse local time*

$$\varrho(t, z) = \min\{s : \ell(s, z) = t\}.$$

We consider first the more general stopping time. This is the moment defined by the formula

$$\nu(\vec{\alpha}, t) := \min\Big\{s : \int_0^s g(W(v))\,dv + \sum_{k=1}^m \alpha_k \ell(s, q_k) = t\Big\},$$

where g is a nonnegative piecewise-continuous function, which at points of discontinuity takes the right limit values, $\alpha_k \geq 0$, $k = 1, \dots, m$. We generalize the moment ν from §6 by adding a linear combination of the Brownian local times at different levels.

As in §3, set

$$A_{\vec{\beta}}(t) := \int_0^t f(W(s))\,ds + \sum_{k=1}^m \beta_k \ell(t, q_k),$$

where f is a nonnegative piecewise-continuous function, $\beta_k \geq 0$ $k = 1, \dots, m$. Clearly, the points q_k in the definition of the time $\nu(\vec{\alpha}, t)$ and in the functional $A_{\vec{\beta}}(t)$ can be chosen the same, because in the case when these points are different, one can let the corresponding coefficients of α_k, β_k be equal to zero.

On the basis of the following proof of Theorem 7.1, some general remarks concerning results that include linear combinations of local times can be made. We have already done this in Remark 4.3. In view of the special importance of these interesting facts, we return briefly to them.

Remark 7.1. A local time can be considered, informally, as the integral functional of the Brownian motion for the Dirac δ-function. For computing the distributions of integral functionals of Brownian motion different assertions were obtained in § 1–§ 6. Those results are based on the solutions of certain differential problems. Adding to the integrands linear combinations of Dirac δ-functions $\delta_{q_k}(x)$, $x \in \mathbf{R}$, $k = 1, \dots, m$, leads to that the basic equations of these problems (see equation (3.1)) are not satisfied at the points q_k, $k = 1, \dots, m$, whenever at these points the solution is continuous and the first derivative has a corresponding jump.

We illustrate this observation in detail for the proof of the following theorem, which is a generalization of Theorem 6.1.

Theorem 7.1. *Let $\Phi(x)$, $f(x)$, $g(x)$, $x \in \mathbf{R}$, be piecewise-continuous functions. Assume that $f \geq 0$, $g \geq 0$, and Φ is bounded. Then the function*

$$M_\nu(x) := \mathbf{E}_x\big\{\Phi(W(\nu(\vec{\alpha}, \tau))) \exp(-A_{\vec{\beta}}(\nu(\vec{\alpha}, \tau)))\big\}, \qquad x \in \mathbf{R},$$

is the unique bounded continuous solution of the problem

$$\frac{1}{2}M''(x) - (\lambda g(x) + f(x))M(x) = -\lambda g(x)\Phi(x), \quad x \neq q_k, \qquad k = 1, \dots, m, \tag{7.1}$$

$$M'(q_k+0) - M'(q_k-0) = 2(\lambda\alpha_k + \beta_k)M(q_k) - 2\lambda\alpha_k\Phi(q_k), \qquad k = 1, \dots, m. \tag{7.2}$$

Proof. We give a schematic proof of this theorem, illustrating Remark 7.1. Treating the local time as the integral functional of the Brownian motion for the Dirac δ-function, we apply Theorem 6.1, using instead of g the function $g(x) + \sum_{k=1}^{m} \alpha_k \delta_{q_k}(x)$, and instead of f the function $f(x) + \sum_{k=1}^{m} \beta_k \delta_{q_k}(x)$. Then we see that the function

$$M_\nu(x) = \mathbf{E}_x\bigg\{\Phi(W(\nu(\vec{\alpha}, \tau))) \exp\bigg(- \int_0^{\nu(\vec{\alpha},\tau)} \Big(f(W(s)) + \sum_{k=1}^{m} \beta_k \delta_{q_k}(W(s)) \Big)\, ds \bigg) \bigg\}$$

satisfies the equation

$$\frac{1}{2}M''(x) - \Big(\lambda g(x) + \lambda \sum_{k=1}^{m} \alpha_k \delta_{q_k}(x) + f(x) + \sum_{k=1}^{m} \beta_k \delta_{q_k}(x)\Big) M(x)$$

$$= -\lambda \Phi(x)\Big(g(x) + \sum_{k=1}^{m} \alpha_k \delta_{q_k}(x)\Big). \tag{7.3}$$

We must prove that except at the points $\{q_1, \dots, q_m\}$ this equation turns into (7.1). The important fact is that the solution M is continuous at the points $\{q_1, \dots, q_m\}$.

To prove (7.2) we choose the points x_1 and x_2 such that the interval (x_1, x_2) contains only one point q_k for some k. Integrating (7.3) over this interval and using the definition of the Dirac δ-function, we obtain

$$\frac{1}{2}(M'(x_2) - M'(x_1)) - \int_{x_1}^{x_2} (\lambda g(x) + f(x))M(x)\,dx - (\lambda\alpha_k + \beta_k)M(q_k)$$

$$= -\lambda \int_{x_1}^{x_2} \Phi(x)g(x)\,dx - \lambda\alpha_k\,\Phi(q_k). \tag{7.4}$$

Passing in equation (7.4) to the limit as $x_1 \uparrow q_k$ and $x_2 \downarrow q_k$, we see that the condition (7.2) holds.

All the arguments given informally when using the δ-functions, can be replaced by rigorous proof with the corresponding sequences of functions $\{\frac{1}{\varepsilon}\mathbb{I}_{[q_k, q_k+\varepsilon)}(x)\}_{\varepsilon>0}$, $k = 1, \dots, m$, realizing the limit approximation method (see the proof of Theorem 3.1). □

We now turn to the problem of computing distributions of functionals of Brownian motion stopped at the inverse local time. Consider the particular case of the moment $\nu(\vec{\alpha}, t)$. Set $g \equiv 0$, $\alpha_1 = 1$, $q_1 = z$, $\alpha_k = 0$, $k = 2, \dots, m$. Then $\nu(\vec{\alpha}, t) = \varrho(t, z)$ and we formulate the result, which is a particular case of Theorem 7.1 with $\Phi \equiv 1$ if it is assumed that $a = -\infty$ and $b = \infty$. For a finite a or b Theorem 7.2 can be proved analogously to Theorem 2.1, with the help of the result for infinite a and b.

Theorem 7.2. *Let $f(x)$, $x \in [a, b]$, be a nonnegative piecewise-continuous function. Then for $a < z < b$ and $q_1 = z$ the function*

$$D(x) := \mathbf{E}_x\Big\{\exp(-A_{\vec{\beta}}(\varrho(\tau, z))),\ a \le \inf_{0\le s\le\varrho(\tau,z)} W(s),\ \sup_{0\le s\le\varrho(\tau,z)} W(s) \le b\Big\},$$

$x \in [a, b]$, is the unique continuous solution of the problem

$$\frac{1}{2}D''(x) - f(x)D(x) = 0, \qquad x \in (a, b) \setminus \{q_1, \dots, q_m\}, \tag{7.5}$$

$$D'(z+0) - D'(z-0) = 2(\lambda + \beta_1)D(z) - 2\lambda, \tag{7.6}$$

$$D'(q_k+0) - D'(q_k-0) = 2\beta_k D(q_k), \qquad k = 2, \dots, m, \tag{7.7}$$

$$D(a) = 0, \qquad\qquad D(b) = 0. \tag{7.8}$$

Remark 7.2. In the case when $a = -\infty$ or $b = \infty$ the corresponding boundary condition in (7.8) must be replaced by the condition that the function $D(x)$ is bounded as x tends to $-\infty$ or ∞.

Since the parameter λ is involved only in the condition (7.6) on the jump of the derivative, it is possible to formulate the statement for the inverse Laplace transform of the function $\frac{1}{\lambda}D(x)$ with respect to λ.

Let $\varphi(x)$, $x \le b$, and $\psi(x)$, $x \ge a$, be nonnegative continuous linearly independent solutions of the problem

$$\frac{1}{2}\phi''(x) - f(x)\phi(x) = 0, \qquad x \in (a,b) \setminus \{q_1, \dots, q_m\}, \tag{7.9}$$

$$\phi'(q_k+0) - \phi'(q_k-0) = 2\beta_k \phi(q_k), \qquad k = 1, \dots, m, \tag{7.10}$$

with $\varphi(x)$ nonincreasing and $\psi(x)$ nondecreasing, satisfying for $a \ne -\infty$ or $b \ne \infty$ the conditions $\psi(a) = 0$ or $\varphi(b) = 0$. The Wronskian $w = \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$, $x \ne q_k$, $k = 1, \dots, m$, is a constant.

Theorem 7.3. *Let $f(x), x \in [a,b]$, be a nonnegative piecewise-continuous function. Then for $a < z < b$ and $q_1 = z$ the function*

$$d(t,x) := \mathbf{E}_x\Big\{ \exp(-A_{\vec\beta}(\varrho(t,z)),\ a \le \inf_{0 \le s \le \varrho(t,z)} W(s),\ \sup_{0 \le s \le \varrho(t,z)} W(s) \le b \Big\},$$

$t \ge 0$, $x \in [a,b]$, has the form

$$d(t,x) = \begin{cases} \dfrac{\psi(x)}{\psi(z)} \exp\Big(-\dfrac{wt}{2\varphi(z)\psi(z)}\Big), & a \le x \le z, \\ \dfrac{\varphi(x)}{\varphi(z)} \exp\Big(-\dfrac{wt}{2\varphi(z)\psi(z)}\Big), & z \le x \le b, \end{cases} \tag{7.11}$$

where φ and ψ are the fundamental solutions of the problem (7.9), (7.10).

Proof. We take the Laplace transform with respect to the parameter t of the function d, defined by (7.11), and check that this transform satisfies the problem (7.5)–(7.8).

For $a < z < b$ we have

$$D(x) = \lambda \int_0^\infty e^{-\lambda t} d(t,x)\,dt = \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}} \Big(\frac{\psi(x)}{\psi(z)} \mathbb{I}_{[a,z)}(x) + \frac{\varphi(x)}{\varphi(z)} \mathbb{I}_{[z,b]}(x) \Big).$$

We see that (7.5), (7.7), and (7.8) are valid, because φ and ψ satisfy (7.9), (7.10) and the boundary conditions. It is clear that

$$D(z) = \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}},$$

and for $z = q_1$

$$D'(z+0) - D'(z-0) = \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}} \Big(\frac{\varphi'(z+0)}{\varphi(z)} - \frac{\psi'(z-0)}{\psi(z)} \Big)$$

$$= \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}} \Big(\frac{\varphi'(z-0)}{\varphi(z)} - \frac{\psi'(z-0)}{\psi(z)} + 2\beta_1 \Big) = \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}} \Big(-\frac{w}{\varphi(z)\psi(z)} + 2\beta_1 \Big)$$

$$= \frac{\lambda}{\lambda + \frac{w}{2\varphi(z)\psi(z)}} \Big(2(\lambda + \beta_1) - 2\Big(\lambda + \frac{w}{2\varphi(z)\psi(z)}\Big)\Big) = 2(\lambda + \beta_1) D(z) - 2\lambda.$$

Therefore, condition (7.6) holds and Theorem 7.3 is proved. □

Example 7.1. We compute the distribution of the supremum of the Brownian motion over the time $\varrho(t,z)$. We apply Theorem 7.3 with $f \equiv 0$ and $a = -\infty$, $\beta_k = 0$, $k = 1,\dots,m$. The linearly independent nonnegative solutions φ and ψ of the equation

$$\frac{1}{2}\phi''(x) = 0, \qquad x \in (-\infty, b), \tag{7.12}$$

such that $\varphi(b) = 0$ and $\psi(x)$ is nondecreasing, are

$$\varphi(x) = b - x, \qquad \psi(x) \equiv 1, \qquad x \in (-\infty, b).$$

Then by (7.11),

$$\mathbf{P}_x\Big(\sup_{0\le s\le \varrho(t,z)} W(s) \le b\Big) = \begin{cases} \exp\Big(-\frac{t}{2(b-z)}\Big), & x \le z, \\ \frac{b-x}{b-z}\exp\Big(-\frac{t}{2(b-z)}\Big), & z \le x \le b. \end{cases} \tag{7.13}$$

Exercises.

7.1. Compute $\mathbf{E}_x e^{-\alpha\varrho(t,z)}$, $\alpha > 0$, and the distribution of the inverse local time $\varrho(t,z)$.

7.2. Compute $\mathbf{E}_x\Big\{e^{-\alpha\varrho(t,z)}; a \le \inf_{0\le s\le\varrho(t,z)} W(s)\Big\}$, $\alpha > 0$.

7.3. Compute the Laplace transform $\mathbf{E}_x e^{-\gamma\ell(\varrho(t,z),r)}$, $\gamma > 0$, and the distribution of the local time $\ell(\varrho(t,z),r)$.

7.4. Compute $\mathbf{E}_x \exp(-\alpha\varrho(t,z) - \gamma\ell(\varrho(t,z),r))$, $\alpha > 0$, $\gamma > 0$.

7.5. Compute the Laplace transform

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^{\varrho(t,z)} \mathbb{1}_{[r,\infty)}\big(W(s)\big)ds\Big), \qquad \gamma > 0.$$

7.6. Compute the Laplace transform

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^{\varrho(t,z)} \mathbb{1}_{[r,u]}\big(W(s)\big)ds\Big), \qquad \gamma > 0.$$

§8. Distributions of functionals of Brownian motion stopped at the inverse range time

The *range of a process* is the difference between its maximum and minimum values on a finite time interval. This section deals with the method of computation of distributions of functionals of a Brownian motion stopped at the moment inverse of the range of a Brownian motion $W(s)$, $s \in [0, \infty)$.

Let

$$\theta_v = \min\Big\{t : \sup_{0 \le s \le t} W(s) - \inf_{0 \le s \le t} W(s) = v\Big\}$$

be the first moment at which the range of W reaches a given value $v > 0$. The moment θ_v is called an *inverse range time* of the process W.

For results on inverse range time see Feller (1951), Imhof (1986), Vallois (1995), Borodin (1999).

We will prove that the problem of computation of distributions of functionals of the Brownian motion W stopped at the moment θ_v can be transformed into the same problem for the Brownian motion W stopped at the first exit time from some interval, i.e., at the moment $H_{a,b} = \min\{s : W(s) \notin (a, b)\}$. This fact is very important for the proof of many results concerning the distributions of functionals stopped at the moment θ_v.

We consider the problem of computation of distribution of the integral functional

$$A(t) := \int_0^t f(W(s))\, ds$$

stopped the moment θ_v.

The following result is of key importance for computing the conditional distribution of the functional $A(t)$ for the moment θ_v given the condition $W(\theta_v) = z$.

Theorem 8.1. *Let $f(x)$, $x \in \mathbf{R}$, be a piecewise-continuous nonnegative function. Then*

$$\frac{\partial}{\partial z}\mathbf{E}_x\big\{e^{-A(\theta_v)}; W(\theta_v) < z\big\}$$

$$= \begin{cases} \dfrac{\partial}{\partial v}\mathbf{E}_x\big\{e^{-A(H_{z,z+v})}; W(H_{z,z+v}) = z\big\}, & \text{for } x - v < z < x, \\ \dfrac{\partial}{\partial v}\mathbf{E}_x\big\{e^{-A(H_{z-v,z})}; W(H_{z-v,z}) = z\big\}, & \text{for } x < z < x + v. \end{cases} \qquad (8.1)$$

Proof. Set $H_z := \min\{s : W(s) = z\}$. Consider the case $x - v < z < x$. The result for the case $x < z < x + v$ can be derived from the result for the previous case, using the properties of spatial homogeneity and symmetry of a Brownian motion. Let δ be a sufficiently small positive number. Denote by D_δ the set of sample paths of the Brownian motion that start at x, reach the level $z + v$ earlier than the level z, and then start at $z + v$, and reach the level $z + \delta$ earlier than the level $z + v + \delta$. Denote by C_δ the set of sample paths of the Brownian motion that start at x, reach the level $z + v$ earlier than the level $z + \delta$, and start at $z + v$, reach the level $z+\delta$ earlier than the level $z+v+\delta$, and, finally, start at $z+\delta$, reaching the level z earlier than the level $z+v$. It is not hard to see that

$$C_\delta \subset \{W(\theta_v) \in [z, z+\delta)\} \subset D_\delta, \tag{8.2}$$

$$C_\delta \subset \Big\{ \sup_{0\le s\le H_z} W(s) \in [z+v, z+v+\delta)\Big\} \subset D_\delta. \tag{8.3}$$

Applying the strong Markov property of a Brownian motion (see §11 Ch. I) and formula (5.10), we have

$$\mathbf{P}_x(D_\delta \backslash C_\delta) \le 2\mathbf{P}_{z+v}(W(H_{z+\delta,z+v+\delta}) = z+\delta)\mathbf{P}_{z+\delta}(W(H_{z,z+v}) = z+v) = \frac{2\delta^2}{v^2}. \tag{8.4}$$

By $O(x)$ denote any function that satisfies $\sup_x \Big|\frac{O(x)}{x}\Big| < \infty$. By (8.2)–(8.4),

$$\begin{aligned}
&\mathbf{E}_x\big\{e^{-A(\theta_v)}; W(\theta_v) \in [z, z+\delta)\big\}\\
&\quad = \mathbf{E}_x\big\{e^{-A(\theta_v)}; \sup_{0\le s\le H_z} W(s) \in [z+v, z+v+\delta)\big\} + O\Big(\frac{\delta^2}{v^2}\Big)\\
&\quad = \mathbf{E}_x\big\{e^{-A(H_z)}; \sup_{0\le s\le H_z} W(s) \in [z+v, z+v+\delta)\big\} + I_1(\delta) + O\Big(\frac{\delta^2}{v^2}\Big),
\end{aligned} \tag{8.5}$$

where

$$\begin{aligned}
I_1(\delta) &= \mathbf{E}_x\big\{e^{-A(\theta_v)} - e^{-A(H_z)}; \sup_{0\le s\le H_z} W(s) \in [z+v, z+v+\delta)\big\}\\
&= \mathbf{E}_x\big\{e^{-A(\theta_v)} - e^{-A(H_z)}; C_\delta\big\} + O\Big(\frac{\delta^2}{v^2}\Big).
\end{aligned}$$

Since f is a piecewise-continuous function, it is bounded on any finite interval. Therefore,

$$\begin{aligned}
|I_1(\delta)| &\le \mathbf{E}_x\{|A(H_z) - A(\theta_v)|; C_\delta\} + O\Big(\frac{\delta^2}{v^2}\Big)\\
&\le \max_{y\in[z,z+v+\delta]} f(y)\ \mathbf{E}_x\{|H_z - \theta_v|; C_\delta\} + O\Big(\frac{\delta^2}{v^2}\Big).
\end{aligned} \tag{8.6}$$

Differentiating (5.25) with respect to α at the point zero, we obtain

$$\mathbf{E}_x\{H_{a,b}; W(H_{a,b}) = a\} = \frac{(b-x)(x-a)}{3(b-a)}\,(2b-x-a). \tag{8.7}$$

Applying the strong Markov property of Brownian motion and taking into account formulas (5.10) and (8.7), we get

$$\begin{aligned}
\mathbf{E}_x\{|H_z - \theta_v|; C_\delta\} &\le \mathbf{P}_{z+v}(W(H_{z+\delta,z+v+\delta}) = z+\delta)\mathbf{E}_{z+\delta}\{H_{z,z+v}; W(H_{z,z+v}) = z\}\\
&= \frac{\delta}{v}\,\frac{(v-\delta)\delta}{3v}\,(2v-\delta) \le \frac{2}{3}\delta^2.
\end{aligned}$$

Consequently, $I_1(\delta) = O(\delta^2)$. Since

$$\mathbf{E}_x\bigl\{e^{-A(H_z)}; \sup_{0\le s\le H_z} W(s) \in [z+v, z+v+\delta)\bigr\}$$
$$= \mathbf{E}_x\bigl\{e^{-A(H_{z,z+v+\delta})}; W(H_{z,z+v+\delta}) = z\bigr\} - \mathbf{E}_x\bigl\{e^{-A(H_{z,z+v})}; W(H_{z,z+v}) = z\bigr\},$$

we can divide equality (8.5) by δ and pass to the limit as $\delta \downarrow 0$. This proves (8.1). We have only to check that the derivative on the right-hand side of (8.1) exists. Set

$$u_v(x) := \mathbf{E}_x\bigl\{e^{-A(H_{z,z+v})}; W(H_{z,z+v}) = z\bigr\}. \tag{8.8}$$

The function $u_v(x)$ is the solution of the problem (5.4), (5.5) for $a = z$, $b = z + v$. Consequently, this function can be represented in the form

$$u_v(x) = \frac{\psi(z+v)\varphi(x) - \psi(x)\varphi(z+v)}{\psi(z+v)\varphi(z) - \psi(z)\varphi(z+v)}, \quad x \in (z, z+v), \tag{8.9}$$

where ψ and φ are the fundamental solutions of the equation

$$\frac{1}{2}\phi''(x) - f(x)\phi(x) = 0, \qquad x \in \mathbf{R}. \tag{8.10}$$

From formula (8.9) it follows that the function $u_v(x)$ is differentiable with respect to v. Theorem 8.1 is proved. □

We can express the right-hand side of (8.1) in terms of the fundamental solutions ψ and φ of equation (8.10). Let $w := \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$ be the Wronskian. It is a nonnegative constant. Denote

$$\rho(x, y) := \psi(x)\varphi(y) - \psi(y)\varphi(x).$$

It is easy to check that for any a, b and c the equality

$$\rho(b, c)\frac{\partial}{\partial b}\rho(a, b) - \rho(a, b)\frac{\partial}{\partial b}\rho(b, c) = -w\rho(a, c)$$

holds. Using this equality, we can obtain that

$$\frac{\partial}{\partial v}u_v(x) = \frac{\partial}{\partial v}\frac{\rho(z+v, x)}{\rho(z+v, z)} = -\frac{\partial}{\partial v}\frac{\rho(x, z+v)}{\rho(z+v, z)} = \frac{w\rho(x, z)}{\rho^2(z+v, z)}.$$

Similar computations for $x < z < x + v$, yield

$$\frac{\partial}{\partial z}\mathbf{E}_x\{\exp(-A(\theta_v)); W(\theta_v) < z\} = \begin{cases} \dfrac{w\rho(x, z)}{\rho^2(z+v, z)}, & x - v < z \le x, \\[2mm] \dfrac{w\rho(z, x)}{\rho^2(z, z-v)}, & x \le z < x + v. \end{cases} \tag{8.11}$$

Hence, for an arbitrary piecewise-continuous function $\Phi(z)$, we have that

$$\mathbf{E}_x\bigl\{\Phi(W(\theta_v))\exp(-A(\theta_v))\bigr\} = w\int_x^{x+v} \frac{\Phi(z)\rho(z, x) + \Phi(z-v)\rho(x, z-v)}{\rho^2(z, z-v)}\,dz. \tag{8.12}$$

Example 8.1. We compute the distribution of the Brownian motion at the moment θ_v. We apply (8.1) for $f \equiv 0$. The function u_v defined by (8.8) for $f \equiv 0$ takes the form $u_v(x) = \frac{z+v-x}{v}$, $x \in (z, z+v)$ (see 5.10). Then $\frac{\partial}{\partial v}\frac{z+v-x}{v} = \frac{x-z}{v^2}$. Similar computations for $x < z < x+v$, yield

$$\frac{d}{dz}\mathbf{P}_x(W(\theta_v) < z) = \frac{|z-x|}{v^2}, \qquad |z-x| \le v. \tag{8.13}$$

Example 8.2. We compute the distribution of the inverse range time θ_v. We apply (8.1) for $f \equiv \alpha$. In this case $\psi(x) = e^{x\sqrt{2\alpha}}$, $\varphi(x) = e^{-x\sqrt{2\alpha}}$, and

$$\rho(x,y) = 2\,\mathrm{sh}((x-y)\sqrt{2\alpha}).$$

Then from (8.12), $\varPhi \equiv 1$, we have

$$\mathbf{E}_x e^{-\alpha\theta_v} = \sqrt{2\alpha}\int_x^{x+v} \frac{\mathrm{sh}((z-x)\sqrt{2\alpha}) + \mathrm{sh}((x-z+v)\sqrt{2\alpha})}{\mathrm{sh}^2(v\sqrt{2\alpha})}\,dz$$

$$= \frac{2(\mathrm{ch}(v\sqrt{2\alpha})-1)}{\mathrm{sh}^2(v\sqrt{2\alpha})} = \frac{1}{\mathrm{ch}^2(v\sqrt{\alpha/2})}. \tag{8.14}$$

Inverting the Laplace transform with respect to α and using the formula for the binomial series (see Appendix 2), we obtain

$$\frac{d}{dt}\mathbf{P}_x(\theta_v < t) = \frac{4v}{t\sqrt{2\pi t}}\sum_{k=1}^{\infty}(-1)^{k-1}k^2 e^{-k^2v^2/2t}. \tag{8.15}$$

Exercises.

8.1. Compute $\mathbf{E}_x e^{i\beta W(\theta_v)}$.

8.2. Compute $\frac{d}{dz}\mathbf{E}_x\big\{e^{-\alpha\theta_v};\ W(\theta_v) < z\big\}$, $\alpha > 0$, and find the joint distribution of θ_v and $W(\theta_v)$.

8.3. Compute

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^{\theta_v}\mathbb{I}_{[r,\infty)}\big(W(s)\big)ds\Big);\ W(\theta_v) < z\Big\}, \qquad \gamma > 0.$$

8.4. Compute

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\theta_v}\Big(p\mathbb{I}_{(-\infty,r]}\big(W(s)\big) + q\mathbb{I}_{[r,\infty)}\big(W(s)\big)\Big)ds\Big);\ W(\theta_v) < z\Big\}$$

for $p > 0$ and $q > 0$.

§9. Distributions of functionals of Brownian motion with linear drift

Let $W^{(\mu)}(t) := \mu t + W(t)$, $t \in [0,T]$ be a *Brownian motion with linear drift* μ. Analogously to the proof of the existence of the Brownian local time (see §5 Ch. II) one can prove the existence of the *local time* $\ell^{(\mu)}(t,x)$ of the Brownian motion with linear drift μ. The local time $\ell^{(\mu)}(t,x)$ is a.s. a jointly continuous process in $(t,x) \in [0,T] \times \mathbf{R}$ and

$$(W^{(\mu)}(t)-x)^+ - (W^{(\mu)}(0)-x)^+ = \int_0^t \mathbb{1}_{[x,\infty)}(W^{(\mu)}(s))\, dW^{(\mu)}(s) + \frac{1}{2}\ell^{(\mu)}(t,x),$$

where $a^+ = \max\{a,0\}$.

We consider the functional of the Brownian motion with linear drift of the form

$$A^{(\mu)}_{\vec{\beta}}(t) := \int_0^t f(W^{(\mu)}(s))ds + \sum_{l=1}^m \beta_l\, \ell^{(\mu)}(t,q_l),$$

where f is a nonnegative piecewise-continuous function, $\beta_l \ge 0$, and $q_k \in \mathbf{R}$, $m < \infty$.

The next result enables us to compute the joint distributions of the integral functional of the Brownian motion with linear drift, the local time at different levels, and the infimum and supremum functionals.

Theorem 9.1. *Let $\Phi(x)$ and $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \ge 0$. Then the function*

$$M_\mu(x) := e^{\mu x}\mathbf{E}_x\Big\{\Phi(W^{(\mu)}(\tau))\exp(-A^{(\mu)}_{\vec{\beta}}(\tau)); a \le \inf_{0\le s\le\tau} W^{(\mu)}(s), \sup_{0\le s\le\tau} W^{(\mu)}(s) \le b\Big\},$$

$x \in [a,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}M''(x) - \Big(\lambda + \frac{\mu^2}{2} + f(x)\Big)M(x) = -\lambda e^{\mu x}\Phi(x), \quad x \in (a,b)\setminus\{q_1,\dots,q_m\}, \tag{9.1}$$

$$M'(q_k+0) - M'(q_k-0) = 2\beta_k M(q_k), \qquad k = 1,\dots,m, \tag{9.2}$$

$$M(a) = 0, \qquad M(b) = 0. \tag{9.3}$$

Remark 9.1. In the case $a = -\infty$ or $b = \infty$ we, in addition, assume that Φ is bounded. Then the corresponding boundary condition in (9.3) must be replaced by the condition that the function $e^{-\mu x}M(x)$ is bounded as x tends to $-\infty$ or ∞.

Remark 9.2. If $\beta_k = 0$, $k = 1,\dots,m$, i.e.,

$$A^{(\mu)}_0(t) = \int_0^t f(W^{(\mu)}(s))ds,$$

then the conditions (9.2) can be omitted and in (9.1) instead of $(a,b)\setminus\{q_1,\dots,q_m\}$ we can take the interval (a,b).

Proof of Theorem 9.1. By (10.15) Ch. II, for any bounded measurable functional $\wp(W^{(\mu)}(s), 0\le s\le t)$, $W^{(\mu)}(0)=x$,

$$\mathbf{E}\wp\big(W^{(\mu)}(s),0\le s\le t\big)=e^{-\mu x-\mu^2 t/2}\mathbf{E}\big\{e^{\mu W(t)}\wp(W(s),0\le s\le t)\big\}. \tag{9.4}$$

Taking the Laplace transform with respect to t, we have

$$\mathbf{E}\wp\big(W^{(\mu)}(s),0\le s\le \tau\big)=e^{-\mu x}\mathbf{E}\big\{e^{\mu W(\tau)-\mu^2\tau/2}\wp(W(s),0\le s\le \tau)\big\}, \tag{9.5}$$

where τ is the exponentially distributed random time with the parameter $\lambda>0$. It is assumed also that τ is independent of the Brownian motion W. Using (9.5), we can represent the function M_μ in the form

$$M_\mu(x)=\mathbf{E}_x\Big\{e^{\mu W(\tau)}\varPhi(W(\tau))\exp\big(-\tfrac{\mu^2\tau}{2}-A_{\vec\beta}(\tau)\big), a\le \inf_{0\le s\le\tau}W(s), \sup_{0\le s\le\tau}W(s)\le b\Big\},$$

where $A_{\vec\beta}(t)$ is the corresponding functional of the Brownian motion W (see §3). To the right-hand side of this equality we can apply Theorem 3.1 with the function $e^{\mu x}\varPhi(x)$ instead of $\varPhi(x)$ and the function $\frac{\mu^2}{2}+f(x)$ instead of $f(x)$, $x\in[a,b]$. Then we see that the function $M_\mu(x)$, $x\in[a,b]$, is the unique continuous solution of the problem (9.1)–(9.3). □

It is easy to see that from Theorem 9.1 we can derive the analogue of Theorem 4.2. It is necessary to replace the function $\varPhi$ by the Dirac δ-function. What this actually means is that one should take the limit as $\Delta\downarrow 0$ for the family of functions $\Big\{\frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(x)\Big\}_{\Delta>0}$ instead of the function $\varPhi$ (see the proof of Theorem 4.1). The function $G_z^{(\mu)}(x)$, in contrast to the corresponding function $G_x(z)$ of Theorem 4.1, is a function of the starting point of the process $W^{(\mu)}$; otherwise, there appears the conjugate operator. In view of (4.21), the function $G_x(z)$ satisfies the equality $G_x(z)=G_z(x)$.

Theorem 9.2. *Let $f(x)$, $x\in[a,b]$ be a piecewise-continuous nonnegative function. Then for $a<z<b$ the function*

$$G_z^{(\mu)}(x):=e^{\mu x}\frac{d}{dz}\mathbf{E}_x\Big\{\exp(-A_{\vec\beta}^{(\mu)}(\tau)); a\le\inf_{0\le s\le\tau}W^{(\mu)}(s),$$

$$\sup_{0\le s\le\tau}W^{(\mu)}(s)\le b, W^{(\mu)}(\tau)<z\Big\},\qquad x\in[a,b],$$

is the unique continuous solution of the problem

$$\frac12 G''(x)-\Big(\lambda+\frac{\mu^2}{2}+f(x)\Big)G(x)=0,\qquad x\in(a,b)\setminus\{z,q_1,\dots,q_m\}, \tag{9.6}$$

$$G'(z+0)-G'(z-0)=-2\lambda e^{\mu z}, \tag{9.7}$$

$$G'(q_k+0)-G'(q_k-0)=2\beta_k G(q_k),\qquad q_k\ne z,\ k=1,\dots,m, \tag{9.8}$$

$$G(a)=0,\qquad G(b)=0. \tag{9.9}$$

Remark 9.3. The solutions of the problems (9.6)–(9.9) and (4.23)–(4.26) can be expressed in terms of each other. If the solution of (4.23)–(4.26) is denoted by $G_{\lambda,x}(z)$, then

$$G_z^{(\mu)}(x) = \frac{\lambda e^{\mu z}}{\lambda + \mu^2/2} G_{\lambda+\mu^2/2,x}(z). \tag{9.10}$$

It suffices to note that $G_{\lambda+\mu^2/2,x}(z) = G_{\lambda+\mu^2/2,z}(x)$. With the help of this equality, the problem (4.23)–(4.26) can be rewritten as a problem with respect to the variable x.

Equality (9.10) can also be derived, by using the Laplace transform with respect to t, from the equality

$$\frac{d}{dz}\mathbf{E}_x\bigl\{\wp\bigl(W^{(\mu)}(s), 0 \le s \le t\bigr); W^{(\mu)}(t) < z\bigr\}$$

$$= e^{\mu(z-x)-\mu^2 t/2}\frac{d}{dz}\mathbf{E}_x\bigl\{\wp(W(s), 0 \le s \le t); W(t) < z\bigr\}, \tag{9.11}$$

which follows from the coincidence of the bridges of the Brownian motion and the Brownian motion with linear drift (see (11.20) Ch. I).

Example 9.1. We compute the joint distribution of the infimum and supremum of the Brownian motion with linear drift. We apply Theorem 9.1 with $\Phi \equiv 1$, $f \equiv 0$, $\beta_k = 0$, $k = 1, \dots, m$. According to this theorem, the function

$$M_\mu(x) = e^{\mu x}\mathbf{P}_x\Bigl(a \le \inf_{0\le s\le\tau} W^{(\mu)}(s), \sup_{0\le s\le\tau} W^{(\mu)}(s) \le b\Bigr), \qquad x \in [a,b],$$

is the unique solution of the problem

$$\frac{1}{2}M''(x) - \bigl(\lambda + \frac{\mu^2}{2}\bigr)M(x) = -\lambda e^{\mu x}, \qquad x \in (a,b), \tag{9.12}$$

$$M(a) = 0, \qquad M(b) = 0. \tag{9.13}$$

The particular solution of equation (9.12) is $e^{\mu x}$. The two fundamental solutions of the corresponding homogeneous equation are the following: $\operatorname{sh}((b-x)\sqrt{2\lambda+\mu^2})$, $\operatorname{sh}((x-a)\sqrt{2\lambda+\mu^2})$. These solutions are suitable for our problem, because the first one vanishes at b and the second one vanishes at a. Now it is easy to see that the solution of the problem (9.12), (9.13) is

$$M(x) = e^{\mu x} - \frac{e^{\mu a}\operatorname{sh}((b-x)\sqrt{2\lambda+\mu^2}) + e^{\mu b}\operatorname{sh}((x-a)\sqrt{2\lambda+\mu^2})}{\operatorname{sh}((b-a)\sqrt{2\lambda+\mu^2})}.$$

Consequently,

$$\mathbf{P}_x\Bigl(a \le \inf_{0\le s\le\tau} W^{(\mu)}(s), \sup_{0\le s\le\tau} W^{(\mu)}(s) \le b\Bigr)$$

$$= 1 - \frac{e^{\mu(a-x)}\operatorname{sh}((b-x)\sqrt{2\lambda+\mu^2}) + e^{\mu(b-x)}\operatorname{sh}((x-a)\sqrt{2\lambda+\mu^2})}{\operatorname{sh}((b-a)\sqrt{2\lambda+\mu^2})}. \tag{9.14}$$

Proceeding as in the derivation of (2.16), we find that

$$\mathbf{P}_x\Big(a \le \inf_{0\le s\le t} W^{(\mu)}(s),\ \sup_{0\le s\le t} W^{(\mu)}(s) \le b\Big)$$

$$= 1 - e^{-\mu^2 t/2}\big(e^{\mu(a-x)}\widetilde{\mathrm{ss}}_t(b-x, b-a) + e^{\mu(b-x)}\widetilde{\mathrm{ss}}_t(x-a, b-a)\big)$$

$$= \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int_a^b e^{\mu(z-x)}\big(e^{-(z-x+2k(b-a))^2/2t} - e^{-(z+x-2a+2k(b-a))^2/2t}\big)dz.$$

Example 9.2. We compute the Laplace transform of the time spent by the Brownian motion with linear drift $W^{(\mu)}(s)$ in the interval $[r,\infty)$ up to the time τ. We apply Theorem 9.1 with $\Phi(x) \equiv 1$, $f(x) = \gamma \mathbb{I}_{[r,\infty)}(x)$, $\gamma > 0$ and $a = -\infty$, $b = \infty$. According to this theorem and Remark 9.1, the function

$$M_\mu(x) = e^{\mu x}\mathbf{E}_x \exp\Big(-\gamma\int_0^\tau \mathbb{I}_{[r,\infty)}(W^{(\mu)}(s))\,ds\Big), \qquad x \in \mathbf{R},$$

is the unique continuous solution of the equation

$$\frac{1}{2}M''(x) - \Big(\lambda + \frac{\mu^2}{2} + \gamma\mathbb{I}_{[r,\infty)}(x)\Big)U(x) = -\lambda e^{\mu x}, \qquad x \in \mathbf{R}, \tag{9.15}$$

for which the function $e^{-\mu x}M(x)$, $x \in \mathbf{R}$, is bounded.

As in Example 1.1, we get

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^\tau \mathbb{I}_{[r,\infty)}(W^{(\mu)}(s))\,ds\Big) = e^{-\mu x}M_\mu(x)$$

$$= \begin{cases} 1 - \dfrac{\gamma(\sqrt{2\lambda+2\gamma+\mu^2}+\mu)}{(\lambda+\gamma)(\sqrt{2\lambda+2\gamma+\mu^2}+\sqrt{2\lambda+\mu^2})}\, e^{(x-r)(\sqrt{2\lambda+\mu^2}-\mu)}, & x \le r, \\ \dfrac{\lambda}{\lambda+\gamma} + \dfrac{\gamma(\sqrt{2\lambda+\mu^2}-\mu)}{(\lambda+\gamma)(\sqrt{2\lambda+2\gamma+\mu^2}+\sqrt{2\lambda+\mu^2})}\, e^{(r-x)(\sqrt{2\lambda+\mu^2+2\gamma}+\mu)}, & r \le x. \end{cases} \tag{9.16}$$

Since $\tau \to \infty$ as $\lambda \to 0$, passing in (9.16) to the limit, we have

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^\infty \mathbb{I}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds\Big)$$

$$= \mathbb{I}_{(-\infty,0)}(\mu)\begin{cases} 1 - \dfrac{\sqrt{2\gamma+\mu^2}+\mu}{\sqrt{2\gamma+\mu^2}-\mu}e^{-2\mu(x-r)}, & x \le r, \\ -\dfrac{2\mu e^{(r-x)(\sqrt{2\gamma+\mu^2}+\mu)}}{\sqrt{2\gamma+\mu^2}-\mu}, & r \le x. \end{cases} \tag{9.17}$$

Obviously,

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^\infty \mathbb{1}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds\Big)$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^\infty \mathbb{1}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds\Big); \int_0^\infty \mathbb{1}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds < \infty\Big\}.$$

Now passing in (9.17) to the limit as $\gamma \to 0$, we see that for $\mu < 0$

$$\int_0^\infty \mathbb{1}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds < \infty, \qquad \text{a.s.,} \tag{9.18}$$

and for $\mu \geq 0$

$$\int_0^\infty \mathbb{1}_{[r,\infty)}\big(W^{(\mu)}(s)\big)ds = \infty, \qquad \text{a.s.} \tag{9.19}$$

Exercises.

9.1. Compute $\mathbf{E}_x e^{-\alpha\tau + i\beta W^{(\mu)}(\tau)}$, $\alpha > 0$.

9.2. Compute the probabilities

$$\mathbf{P}_x\Big(a \leq \inf_{0\leq s\leq t} W^{(\mu)}(s)\Big), \qquad \mathbf{P}_x\Big(\sup_{0\leq s\leq t} W^{(\mu)}(s) \leq b\Big).$$

9.3. Compute

$$\mathbf{E}_x \exp\Big(-\gamma\int_0^\tau \mathbb{1}_{(-\infty,r)}(W^{(\mu)}(s))\,ds\Big), \qquad \gamma > 0.$$

9.4. Compute

$$\mathbf{E}_x\Big\{e^{i\beta W^{(\mu)}(\tau)}; a \leq \inf_{0\leq s\leq \tau} W^{(\mu)}(s)\Big\}, \qquad \mathbf{E}_x\Big\{e^{i\beta W^{(\mu)}(\tau)}; \sup_{0\leq s\leq \tau} W^{(\mu)}(s) \leq b\Big\}.$$

9.5. Compute

$$\mathbf{E}_x\Big\{e^{-\beta|W^{(\mu)}(\tau)|}; \sup_{0\leq s\leq \tau} |W^{(\mu)}(s)| \leq b\Big\}, \qquad \beta > 0.$$

9.6. Compute the Laplace transform $\mathbf{E}_x e^{-\gamma\ell^{(\mu)}(\tau,r)}$, $\gamma > 0$, and the distribution of the local time $\ell^{(\mu)}(\tau, r)$.

9.7. Compute the Laplace transform $\mathbf{E}_x e^{-\gamma\ell^{(\mu)}(\infty,r)}$, $\gamma > 0$, and the distribution of the local time $\ell^{(\mu)}(\infty, r)$.

§10. Distributions of functionals of reflected Brownian motion

The *reflected Brownian motion* W_+ is the modulus of the Brownian motion W, i.e., $W_+(s) := |W(s)|$, $s \geq 0$. The state space of this process is the nonnegative real half-line.

The *local time* of the reflected Brownian motion W_+ at a level y up to the time t is defined to be the limit

$$\ell_+(t,y) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[y,y+\varepsilon)}(|W(s)|)ds, \quad (t,y) \in [0,\infty)\times[0,\infty).$$

It is clear that

$$\ell_+(t,y) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[y,y+\varepsilon)}(W(s))ds + \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{(-y-\varepsilon,-y]}(W(s))ds$$

$$= \ell(t,y) + \ell(t,-y), \quad (t,y) \in [0,\infty)\times[0,\infty), \tag{10.1}$$

where $\ell(t,y)$ is the Brownian local time.

Therefore, the local time $\ell_+(t,y)$ exists a.s. and is a continuous process in $(t,y) \in [0,\infty)\times[0,\infty)$.

We consider the functional of the reflected Brownian motion of the form

$$A^+_{\vec\beta}(t) := \int_0^t f(W_+(s))ds + \sum_{k=1}^m \beta_k \ell_+(t,q_k),$$

where f is a nonnegative piecewise-continuous function, $\beta_k \geq 0$ and $q_k \in (0,\infty)$, $m < \infty$, $k = 1,\dots,m$. We prove first a basic result which enables us to compute the joint distribution of the functional $A^+_{\vec\beta}(\tau)$ for the case $\beta_k = 0$, $k = 1,\dots,m$, and the supremum of reflected Brownian motion for an exponentially distributed with the parameter $\lambda > 0$ random time τ independent of W_+.

Theorem 10.1. *Let $\Phi(x)$ and $f(x)$, $x \in [0,b]$, be piecewise-continuous functions. Assume that $f \geq 0$. Then the function*

$$M_+(x) := \mathbf{E}_x\Big\{\Phi(W_+(\tau))\exp\Big(-\int_0^\tau f(W_+(s))\,ds\Big);\ \sup_{0\leq s\leq\tau} W_+(s) \leq b\Big\},$$

$x \in [0,b]$, is the unique solution of the problem

$$\frac{1}{2}M''(x) - (\lambda + f(x))M(x) = -\lambda\Phi(x), \qquad x \in (0,b), \tag{10.2}$$

$$M'(0+) = 0, \qquad M(b) = 0. \tag{10.3}$$

Remark 10.1. In the case when $b = \infty$ we assume, in addition, that Φ is bounded. Then the right boundary condition in (10.3) must be replaced by the condition that the function $M_+(x)$ is bounded as x tends to ∞.

Proof of Theorem 10.1. The reflected Brownian motion with starting point $x \ge 0$ can be represented as $W_+(s) = |x + \widetilde{W}(s)|$, $s \ge 0$, where $\widetilde{W}$ is the Brownian motion with $\widetilde{W}(0) = 0$. Then using the symmetry property of $\widetilde{W}$ (see §10 Ch. I), we have

$$M_+(x) = \mathbf{E}_0\Big\{\varPhi(|x + \widetilde{W}(\tau)|) \exp\Big(- \int_0^\tau f(|x + \widetilde{W}(s)|)\, ds\Big);\ \sup_{0\le s\le\tau} |x + \widetilde{W}(s)| \le b\Big\}$$

$$= \mathbf{E}_0\Big\{\varPhi(|x - \widetilde{W}(\tau)|) \exp\Big(- \int_0^\tau f(|x - \widetilde{W}(s)|)\, ds\Big);\ \sup_{0\le s\le\tau} |x - \widetilde{W}(s)| \le b\Big\}$$

$$= \mathbf{E}_0\Big\{\varPhi(|-x + \widetilde{W}(\tau)|) \exp\Big(- \int_0^\tau f(|-x + \widetilde{W}(s)|)\, ds\Big);\ \sup_{0\le s\le\tau} |-x + \widetilde{W}(s)| \le b\Big\}.$$

Considering $M_+(x)$, $x \in \mathbf{R}$, as a function defined on the whole real line, we see that it is even function, i.e., $M_+(x) = M_+(-x)$. Using the equality

$$\Big\{\sup_{0\le s\le\tau} |x + \widetilde{W}(s)| \le b\Big\} = \Big\{ -b \le \inf_{0\le s\le\tau} (x + \widetilde{W}(s)),\ \sup_{0\le s\le\tau} (x + \widetilde{W}(s)) \le b\Big\}$$

and interpreting M_+ as the expectation of the functional of the Brownian motion $x + \widetilde{W}$, we can apply Theorem 2.1. By this theorem, $M_+(x)$, $x \in [-b, b]$, is the unique solution of the problem

$$\frac{1}{2}M''(x) - (\lambda + f(|x|))M(x) = -\lambda\varPhi(|x|), \qquad x \in (-b, b), \tag{10.4}$$

$$M(-b) = 0, \qquad M(b) = 0. \tag{10.5}$$

Since M_+ is an even function, we have $M'_+(0) = 0$ and the problem (10.4), (10.5) transforms into the problem (10.2), (10.3). □

Analogously, using Theorem 3.1 and (10.1) one can prove the following generalization of Theorem 10.1.

Theorem 10.2. *Let $\varPhi(x)$ and $f(x), x \in [0, b]$, be piecewise-continuous functions. Assume that $f \ge 0$. Then the function*

$$M_+(x) := \mathbf{E}_x\Big\{\varPhi(W_+(\tau)) \exp\big(- A^+_{\vec\beta}(\tau)\big);\ \sup_{0\le s\le\tau} W_+(s) \le b\Big\}, \qquad x \in [0, b],$$

is the unique continuous solution of the problem

$$\frac{1}{2}M''(x) - (\lambda + f(x))M(x) = -\lambda\varPhi(x), \qquad x \in (0, b) \setminus \{q_1, \dots, q_m\}, \tag{10.6}$$

$$M'(q_k + 0) - M'(q_k - 0) = 2\beta_k M(q_k), \qquad k = 1, \dots, m, \tag{10.7}$$

$$M'(0+) = 0, \qquad M(b) = 0. \tag{10.8}$$

Example 10.1. We compute the distribution of the supremum of a reflected Brownian motion. Applying Theorem 10.1 with $\Phi \equiv 1$ and $f \equiv 0$, we see that the function $M(x) = \mathbf{P}_x\Big(\sup_{0\le s\le\tau} W_+(s) \le b\Big)$ is the unique solution of the problem

$$\frac{1}{2}M''(x) - \lambda M(x) = -\lambda, \qquad x \in (0,b), \tag{10.9}$$

$$M'(0+) = 0, \qquad M(b) = 0. \tag{10.10}$$

We can choose $\operatorname{ch}(x\sqrt{2\lambda})$, $\operatorname{sh}((b-x)\sqrt{2\lambda})$ as the fundamental solutions of the corresponding homogeneous equation. These solutions satisfy, respectively, the left and the right boundary conditions in (10.10). Then the solution of the problem (10.9), (10.10) can be represented in the form

$$M(x) = 1 + A\operatorname{ch}(x\sqrt{2\lambda}).$$

The second fundamental solution is included with the zero factor, because the derivative at $x = 0$ must be zero. The condition $M(b) = 0$ implies that $A = -1/\operatorname{ch}(b\sqrt{2\lambda})$. Therefore,

$$\mathbf{P}_x\Big(\sup_{0\le s\le\tau} W_+(s) \le b\Big) = 1 - \frac{\operatorname{ch}(x\sqrt{2\lambda})}{\operatorname{ch}(b\sqrt{2\lambda})}, \quad 0 \le x \le b. \tag{10.11}$$

Dividing (10.11) by λ and inverting the Laplace transform with respect to λ (see Section 13 of Appendix 2), we obtain

$$\mathbf{P}_x\Big(\sup_{0\le s\le t} W_+(s) \le b\Big) = 1 - \widetilde{\operatorname{cc}}_t(x,b). \tag{10.12}$$

As a consequence of (2.16) the following formula holds:

$$\mathbf{P}_x\Big(\sup_{0\le s\le t} W_+(s) \le b\Big) = \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\int_{-b}^{b}\big(e^{-(z-x+4kb)^2/2t} - e^{-(2b-z-x+4kb)^2/2t}\big)\,dz.$$

To compute the joint distributions of the integral functional of the reflected Brownian motion and the position of W_+ at the time t we can use the following assertion.

Theorem 10.3. *Let $f(z)$, $z \in [0,b]$, be a nonnegative piecewise-continuous function. Then for $0 < x < b$ the function*

$$G_x(z) := \frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^\tau f(W_+(s))\,ds\Big);\ \sup_{0\le s\le\tau} W_+(s) \le b, W_+(\tau) < z\Big\},$$

$z \in [0,b]$, is the unique continuous solution of the problem

$$\frac{1}{2}G''(z) - (\lambda + f(z))G(z) = 0, \qquad z \in (0,b)\setminus\{x\}, \tag{10.13}$$

$$G'(x+0) - G'(x-0) = -2\lambda, \tag{10.14}$$

$$G'(0+) = 0, \qquad G(b) = 0. \tag{10.15}$$

Example 10.2. We compute the joint distribution of $\sup_{0\le s\le t} W_+(s)$ and the position of the reflected Brownian motion $W_+(t)$ at the time t. We refer to the arguments used in Example 4.2. The solution of the problem (10.13)–(10.15) for $f\equiv 0$ can be represented in the form

$$G_x(z) = A\,\mathrm{ch}(z\sqrt{2\lambda}) + B\,\mathrm{sh}((b-z)\sqrt{2\lambda}) - \frac{\sqrt{\lambda}}{\sqrt{2}}\,\mathrm{sh}(|z-x|\sqrt{2\lambda}).$$

The constants A and B can be easily computed from the boundary conditions and we get

$$G_x(z)=\frac{\sqrt{\lambda}}{\sqrt{2}}\Big[\frac{\mathrm{sh}((b-x)\sqrt{2\lambda})\,\mathrm{ch}(z\sqrt{2\lambda})}{\mathrm{ch}(b\sqrt{2\lambda})}+\frac{\mathrm{sh}((b-z)\sqrt{2\lambda})\,\mathrm{ch}(x\sqrt{2\lambda})}{\mathrm{ch}(b\sqrt{2\lambda})}-\mathrm{sh}(|z-x|\sqrt{2\lambda})\Big].$$

Using the formulas for products of hyperbolic functions, analogously to (4.43) we obtain

$$\frac{d}{dz}\mathbf{P}_x\Big(\sup_{0\le s\le\tau} W_+(s)\le b, W_+(\tau)< z\Big) = \frac{\sqrt{\lambda}\big(\mathrm{sh}((b-|x-z|)\sqrt{2\lambda}) + \mathrm{sh}((b-x-z)\sqrt{2\lambda})\big)}{\sqrt{2}\,\mathrm{ch}(b\sqrt{2\lambda})},$$

Dividing this equality by λ and inverting the Laplace transform with respect to λ (see section 13 of Appendix 2), we get

$$\frac{d}{dz}\mathbf{P}_x\Big(\sup_{0\le s\le t} W_+(s)\le b,\ W_+(t)< z\Big) = \mathrm{sc}_t(b-|x-z|,b) + \mathrm{sc}_t(b-x-z,b)$$

$$= \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}(-1)^k\big(e^{-(z-x+2kb)^2/2t}+e^{-(z+x+2kb)^2/2t}\big), \quad x\vee z<b.$$

Exercises.

10.1. Compute $\mathbf{E}_x e^{-\alpha\tau-\beta|W(\tau)|}$, $\alpha>0$, $\beta>0$, and the distribution of the variable $|W(\tau)|$.

10.2. Compute $\mathbf{E}_x e^{-\beta|W(t)|}$, $\beta>0$, and the distribution of the variable $|W(t)|$.

10.3. Compute the distribution of the local time of the reflected Brownian motion at the time τ, i.e., the distribution of the variable $\ell_+(\tau,q)$.

10.4. Compute

$$\mathbf{E}_x\Big\{|W(\tau)|;\ \sup_{0\le s\le\tau}|W(s)|\le b\Big\} \quad\text{and}\quad \mathbf{E}_x\Big\{W^2(\tau);\ \sup_{0\le s\le\tau}|W(s)|\le b\Big\}.$$

10.5. Compute $\frac{d}{dz}\mathbf{E}_x\{e^{-\alpha\tau}; |W(\tau)|<z\}$, $\alpha>0$.

10.6. Compute

$$\frac{d}{dz}\mathbf{E}_x\Big\{e^{-\alpha\tau};\ \sup_{0\le s\le\tau}|W(s)|\le b, |W(\tau)|<z\Big\}, \qquad \alpha>0.$$

10.7. Compute $\frac{d}{dz}\mathbf{E}_x\{e^{-\gamma\ell_+(\tau,q)};\ |W(\tau)|<z\}$, $\gamma>0$, and the joint distribution of the variables $\ell_+(\tau,q)$ and $|W(\tau)|$.

CHAPTER IV

DIFFUSION PROCESSES

§1. Diffusion processes

Diffusion processes form a very important class of stochastic processes both because of the rich variety of theoretical results concerned with them and their numerous practical applications. They represent a large subclass of Markov processes discussed in § 6 Ch. I.

A Markov process $X(t)$, $t \in [0,T]$, is called a *diffusion* if its transition function $P(t,x,v,\Delta)$ has the following properties:

1) for any $\varepsilon > 0$, all $0 \le t \le T$, and $x \in \mathbf{R}$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int\limits_{|y-x|>\varepsilon} P(t,x,t+\delta,dy) = 0; \tag{1.1}$$

2) there exist functions $a(t,x)$, $b(t,x)$, $(t,x) \in [0,T]\times\mathbf{R}$, such that for some $\varepsilon > 0$, all $0 \le t \le T$, and $x \in \mathbf{R}$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int\limits_{|y-x|\le\varepsilon} (y-x)P(t,x,t+\delta,dy) = a(t,x), \tag{1.2}$$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int\limits_{|y-x|\le\varepsilon} (y-x)^2 P(t,x,t+\delta,dy) = b^2(t,x). \tag{1.3}$$

The functions $a(t,x)$ and $b^2(t,x)$ are called the *drift coefficient* and the *diffusion coefficient* of the process X, respectively.

Supposed that for any fixed s, t, Δ the function $P(s,\cdot,t,\Delta)$ is right continuous. Then using (6.10) and (6.11) of Ch. I, we can give the probabilistic interpretation of conditions 1), 2). They can be reformulated as follows:

1) for any $\varepsilon > 0$, all $0 \le t \le T$, and $x \in \mathbf{R}$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbf{P}(|X(t+\delta)-x| > \varepsilon | X(t)=x) = 0; \tag{1.4}$$

2) there exist functions $a(t,x)$ and $b(t,x)$, $(t,x) \in [0,T]\times\mathbf{R}$, such that for some $\varepsilon > 0$, all $0 \le t \le T$, and $x \in \mathbf{R}$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbf{E}\{(X(t+\delta)-x))\mathbb{1}_{[0,\varepsilon]}(|X(t+\delta)-x|)|X(t)=x\} = a(t,x), \tag{1.5}$$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbf{E}\{(X(t+\delta)-x))^2\mathbb{1}_{[0,\varepsilon]}(|X(t+\delta)-x|)|X(t)=x\} = b^2(t,x). \tag{1.6}$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_4

The first condition states that sample paths of a diffusion cannot undergo big changes for infinitely small times, i.e., they have no jumps. A Markov process X satisfying condition 1) uniformly in (t,x) is continuous (see Gihman and Skorohod (1969)). Therefore, such a diffusion is a continuous process.

The second condition involves the truncation of the increments, because there is no a priori information that the increments of a diffusion have finite first and second moments. For diffusions that are solutions of stochastic differential equations (see § 7 Ch. II) there exist finite second moments of the increments. This is also true for all well-known examples of diffusions.

The meaning of the second condition is the following. The conditional expectation of the truncated increments of a diffusion, starting at a time t from a point x, for infinitely small time period $[t, t+dt]$ has the principal value equal to $a(t,x)\,dt$ and the conditional variance of these increments has the principal value equal to $b^2(t,x)\,dt$.

Let us list some convenient sufficient conditions for a Markov process X to be a diffusion.

A Markov process X is a diffusion, if its transition function $P(t,x,v,\Delta)$ satisfies the following conditions:

1′) for some $\beta>0$, for all $0\le t\le T$, and $x\in\mathbf{R}$

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{-\infty}^{\infty}|y-x|^{2+\beta}P(t,x,t+\delta,dy)=0; \tag{1.7}$$

2′) there exist functions $a(t,x)$ and $b(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}$, such that for all $0\le t\le T$, and $x\in\mathbf{R}$

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{-\infty}^{\infty}(y-x)P(t,x,t+\delta,dy)=a(t,x), \tag{1.8}$$

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_{-\infty}^{\infty}(y-x)^2P(t,x,t+\delta,dy)=b^2(t,x). \tag{1.9}$$

By $o(z)$ denote any function that satisfies $\lim\limits_{z\downarrow 0}\frac{o(z)}{z}=0$.

Condition 1′) implies 1), because

$$\int\limits_{|y-x|>\varepsilon}P(t,x,t+\delta,dy)\le\frac{1}{\varepsilon^{2+\beta}}\int_{-\infty}^{\infty}|y-x|^{2+\beta}P(t,x,t+\delta,dy)=\frac{1}{\varepsilon^{2+\beta}}o(\delta).$$

Analogously, by the estimates

$$\Big|\int\limits_{|y-x|>\varepsilon}(y-x)P(t,x,t+\delta,dy)\Big|\le\frac{1}{\varepsilon^{1+\beta}}\int_{-\infty}^{\infty}|y-x|^{2+\beta}P(t,x,t+\delta,dy)=\frac{1}{\varepsilon^{1+\beta}}o(\delta),$$

$$\int\limits_{|y-x|>\varepsilon} (y-x)^2 P(t,x,t+\delta,dy) \le \frac{1}{\varepsilon^\beta} \int\limits_{-\infty}^{\infty} |y-x|^{2+\beta} P(t,x,t+\delta,dy) = \frac{1}{\varepsilon^\beta} o(\delta),$$

conditions 1′) and 2′) imply 2).

Using (6.11) of Ch. I, we can give the probabilistic interpretation of conditions 1′) and 2′). We have

$$\int\limits_{-\infty}^{\infty} |y-x|^{2+\beta} P(t,x,t+\delta,dy)$$

$$= \mathbf{E}\{|X(t+\delta)-x|^{2+\beta}|X(t)=x\} = \mathbf{E}\{|X(t+\delta)-X(t)|^{2+\beta}|X(t)=x\}.$$

Therefore, condition 1′) can be recast as: for some $\beta > 0$, for all $0 \le t \le T$, and $x \in \mathbf{R}$

$$\mathbf{E}\{|X(t+\delta)-X(t)|^{2+\beta}|X(t)=x\} = o(\delta). \tag{1.10}$$

Analogously, condition 2′) can be recast as: there exist functions $a(t,x)$ and $b(t,x)$, $(t,x) \in [0,T] \times \mathbf{R}$, such that for all $0 \le t \le T$ and $x \in \mathbf{R}$

$$\mathbf{E}\{(X(t+\delta)-X(t))|X(t)=x\} = \delta\, a(t,x) + o(\delta), \tag{1.11}$$

$$\mathbf{E}\{(X(t+\delta)-X(t))^2|X(t)=x\} = \delta\, b^2(t,x) + o(\delta). \tag{1.12}$$

Condition (1.11) means that the principal value of the conditional expectation of increment of the diffusion for a small time period $(t, t+\delta)$ takes the form $\delta\, a(t,x)$. Condition (1.12) means that the principal value of the conditional variance of the increment of the diffusion for a small time interval $(t, t+\delta)$ takes the form $\delta\, b^2(t,x)$.

§ 2. Backward and forward Kolmogorov equations

Let $X(t)$, $t \in [0,T]$, be a diffusion with transition function $P(s,x,t,\Delta)$. For any bounded continuous function $g(y)$, $y \in \mathbf{R}$, and for any fixed $t \in [0,T]$, we set

$$u(s,x) := \int\limits_{-\infty}^{\infty} g(y)\, P(s,x,t,dy), \qquad \text{for } s < t.$$

According to (6.11) Ch. I, if for any fixed s the function $u(s,x)$, $x \in \mathbf{R}$, is continuous, then $u(s,x) = \mathbf{E}\{g(X(t))|X(s)=x\}$. For some processes this function was already considered in § 13 Ch. II.

The following assertion is due to A. N. Kolmogorov.

Theorem 2.1. *Suppose that $u(s,x)$, $(s,x)\in[0,t]\times\mathbf{R}$, has bounded first- and second-orders derivatives with respect to x. Assume that these derivatives and the functions $a(s,x)$, $b(s,x)$ are continuous in (s,x). Then $u(s,x)$ is differentiable with respect to s and satisfies in $(0,t)\times\mathbf{R}$ the equation*

$$-\frac{\partial}{\partial s}u(s,x)=\frac{1}{2}b^2(s,x)\frac{\partial^2}{\partial x^2}u(s,x)+a(s,x)\frac{\partial}{\partial x}u(s,x) \tag{2.1}$$

with the boundary condition

$$\lim_{s\uparrow t}u(s,x)=g(x). \tag{2.2}$$

Equation (2.1) is called the *backward Kolmogorov equation.*

Remark 2.1. If the diffusion coefficient b^2 and the drift coefficient a are such that for every $t\in[0,T]$ the Cauchy problem (2.1), (2.2) in $[0,t]\times\mathbf{R}$ has a unique solution for all bounded continuous functions g, then the transition function $P(s,x,t,\Delta)$ is uniquely determined by the coefficients a and b.

Proof of Theorem 2.1. The boundary condition (2.2) follows from the estimates

$$|u(s,x)-g(x)|=\Bigl|\int_{-\infty}^{\infty}(g(y)-g(x))\,P(s,x,t,dy)\Bigr|\le\int_{|y-x|\le\varepsilon}|g(y)-g(x)|\,P(s,x,t,dy)$$

$$+2\sup_{y\in\mathbf{R}}|g(y)|\int_{|y-x|>\varepsilon}P(s,x,t,dy)\le\max_{|y-x|\le\varepsilon}|g(y)-g(x)|+2\sup_{y\in\mathbf{R}}|g(y)|\,o(|t-s|)$$

true for any $\varepsilon>0$. In the last inequality we used the first property of the transition function.

For any fixed t, for all $0\le r<v<t$, and $x\in\mathbf{R}$ the following integral relation holds:

$$u(r,x)=\int_{-\infty}^{\infty}u(v,z)\,P(r,x,v,dz). \tag{2.3}$$

Indeed, by the Chapman–Kolmogorov equation (see § 6 Ch. I),

$$\int_{-\infty}^{\infty}u(v,z)\,P(r,x,v,dz)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(y)\,P(v,z,t,dy)\,P(r,x,v,dz)$$

$$=\int_{-\infty}^{\infty}g(y)\int_{-\infty}^{\infty}P(r,x,v,dz)P(v,z,t,dy)=\int_{-\infty}^{\infty}g(y)\,P(r,x,t,dy)=u(r,x).$$

For a fixed time r, using (2.3) and the first property of the transition function (see (1.1)), we have that

$$u(r,x)-u(v,x)=\int_{-\infty}^{\infty}(u(v,z)-u(v,x))P(r,x,v,dz)$$

$$= \int\limits_{|z-x|\le\varepsilon} (u(v,z)-u(v,x))P(r,x,v,dz) + 2\sup_{y\in\mathbf{R}}|g(y)|\,o(|v-r|) \tag{2.4}$$

for any small $\varepsilon > 0$, because

$$\sup_{(v,x)\in[0,t]\times\mathbf{R}} |u(v,x)| \le \sup_{y\in\mathbf{R}} |g(y)|.$$

For a small $\varepsilon > 0$, we use the Taylor expansion of the function $u(v,z)$ in the interval $(x-\varepsilon, x+\varepsilon)$:

$$u(v,z)-u(v,x) = \frac{\partial}{\partial x}u(v,x)(z-x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(v,x)(z-x)^2(1+\gamma_\varepsilon(v,z,x)),$$

where

$$|\gamma_\varepsilon(v,z,x)| \le \sup_{|z-x|\le\varepsilon}\Big|\frac{\partial^2}{\partial x^2}u(v,x) - \frac{\partial^2}{\partial z^2}u(v,z)\Big| =: \tilde\gamma_\varepsilon(v,x).$$

Then, using the second property of the transition function (see (1.2) and (1.3)), we have

$$\int\limits_{|z-x|\le\varepsilon} (u(v,z)-u(v,x))P(r,x,v,dz) = \frac{\partial}{\partial x}u(v,x)\int\limits_{|z-x|\le\varepsilon}(z-x)P(r,x,v,dz)$$

$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}u(v,x)\Big(\int\limits_{|z-x|\le\varepsilon}(z-x)^2P(r,x,v,dz) + \int\limits_{|z-x|\le\varepsilon}\gamma_\varepsilon(v,z,x)(z-x)^2P(r,x,v,dz)\Big)$$

$$=\Big(\frac{\partial}{\partial x}u(v,x)a(r,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}u(v,x)b^2(r,x)\Big)(v-r)+\Big(\frac{\partial}{\partial x}u(v,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}u(v,x)\Big)o(|v-r|)$$

$$+O\big(\tilde\gamma_\varepsilon(v,x)\big)(b^2(r,x)(v-r)+o(|v-r|)).$$

Substituting this in (2.4), we get

$$\frac{u(r,x)-u(v,x)}{v-r} = \Big(\frac{\partial}{\partial x}u(v,x)a(r,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}u(v,x)b^2(r,x)\Big)(1+O(\tilde\gamma_\varepsilon(v,x)))+\frac{o(|v-r|)}{v-r}.$$

Letting first $r\uparrow s$, $v\downarrow s$, and then $\varepsilon\to 0$, we see that the function $u(s,x)$, $(0,t)\times\mathbf{R}$, satisfies equation (2.1). □

If there is a transition probability density, i.e., a nonnegative measurable function $p(s,x,t,y)$ such that for all $0\le s<t\le T$, $x\in\mathbf{R}$, and every Borel set Δ

$$P(s,x,t,\Delta) = \int\limits_{\Delta} p(s,x,t,y)\,dy,$$

then the Chapman–Kolmogorov equation can be rewritten in the form

$$p(s,x,t,y) = \int_{-\infty}^{\infty} p(s,x,v,z)\,p(v,z,t,y)\,dz, \tag{2.5}$$

where $s<v<t$.

The following result (equation (2.6)) is called the *forward Kolmogorov equation* or the *Fokker–Planck equation.*

Theorem 2.2. *Suppose the limits* (1.1)–(1.3) *exist uniformly in* $x \in \mathbf{R}$. *Suppose that the partial derivatives*

$$\frac{\partial}{\partial t} p(s,x,t,y), \qquad \frac{\partial}{\partial y}\big(a(t,y)p(s,x,t,y)\big), \qquad \frac{\partial^2}{\partial y^2}\big(b^2(t,y)p(s,x,t,y)\big)$$

exist and are continuous with respect to $y \in \mathbf{R}$. *Then the function* $p(s,x,t,y)$, $t \in (s,T)$, $y \in \mathbf{R}$, *satisfies the equation*

$$\frac{\partial}{\partial t} p(s,x,t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(b^2(t,y)p(s,x,t,y)\big) - \frac{\partial}{\partial y}\big(a(t,y)p(s,x,t,y)\big). \tag{2.6}$$

Proof. We choose an arbitrary twice continuously differentiable function $g(y)$ that vanishes outside a finite interval and we fix t. Arguing in much the same way as in the proof of Theorem 2.1, we obtain that for $h > 0$

$$\int_{-\infty}^{\infty} g(y)p(t,x,t+h,y)\,dy - g(x) = \int_{-\infty}^{\infty} (g(y)-g(x))p(t,x,t+h,y)\,dy$$

$$= \int_{|y-x|\le\varepsilon} \Big[(y-x)g'(x) + \frac{1}{2}(y-x)^2 g''(x)(1+\gamma_\varepsilon(y,x))\Big]\, p(t,x,t+h,y)\,dy$$

$$+O(1)\int_{|y-x|>\varepsilon} p(t,x,t+h,y)\,dy = g'(x)a(t,x)h + \frac{1}{2}g''(x)b^2(t,x)h + o(h) + hO(\tilde\gamma_\varepsilon(x))),$$

where

$$|\gamma_\varepsilon(y,x)| \le \sup_{|y-x|\le\varepsilon} |g''(y) - g''(x)| =: \tilde\gamma_\varepsilon(x)$$

Letting first $h \downarrow 0$ and then $\varepsilon \to 0$, we see that

$$\lim_{h\downarrow 0}\frac{1}{h}\bigg(\int_{-\infty}^{\infty} g(y)p(t,x,t+h,y)\,dy - g(x)\bigg) = g'(x)a(t,x) + \frac{1}{2}g''(x)b^2(t,x), \tag{2.7}$$

uniformly in $x \in \mathbf{R}$. Now using (2.5) and (2.7), we get

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} p(s,x,t,z)\,g(z)\,dz = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} p(s,x,t,z)\,g(z)\,dz$$

$$= \lim_{h\downarrow 0}\frac{1}{h}\int_{-\infty}^{\infty} (p(s,x,t+h,z) - p(s,x,t,z))\,g(z)\,dz$$

$$= \lim_{h\downarrow 0}\frac{1}{h}\bigg(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(s,x,t,z)p(t,z,t+h,y))\,dz\,g(y)\,dy - \int_{-\infty}^{\infty} p(s,x,t,z)\,g(z)\,dz\bigg)$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \int_{-\infty}^{\infty} p(s,x,t,z)\Big(\int_{-\infty}^{\infty} p(t,z,t+h,y))\, g(y)\, dy - g(z)\Big) dz$$

$$= \int_{-\infty}^{\infty} p(s,x,t,z)\Big(g'(z)a(t,z) + \frac{1}{2} g''(z) b^2(t,z)\Big)\, dz$$

$$= \int_{-\infty}^{\infty} \Big(\frac{1}{2}\frac{\partial^2}{\partial z^2}\big(b^2(t,z)p(s,x,t,z)\big) - \frac{\partial}{\partial z}\big(a(t,z)p(s,x,t,z)\big)\Big) g(z)\, dz.$$

Since g is an arbitrary bounded twice continuously differentiable function and the integrand is continuous with respect to z, (2.6) holds. □

§ 3. Diffusions as solutions of stochastic differential equations

In this section we verify that under some conditions on the drift and the diffusion coefficients the diffusion defined in § 1 is a solution of a stochastic differential equation. Let $W(t)$, $t \in [0,T]$, be a Brownian motion.

Theorem 3.1. *Suppose that $X(t)$, $t \in [0,T]$, is a solution of the stochastic differential equation*

$$dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t), \qquad X(0) = \xi, \tag{3.1}$$

where the functions $a(t,x)$, $b(t,x)$, $(t,x) \in [0,T]\times \mathbf{R}$, and the initial value ξ satisfy the conditions of Theorem 7.1 Ch. II.

Then X is a Markov process with the transition function $P(s,x,t,\Delta)$ expressed by

$$P(s,x,t,\Delta) = \mathbf{P}(X_{s,x}(t) \in \Delta), \tag{3.2}$$

where for a fixed s and x the process $X_{s,x}(t)$, $t \in [s,T]$, is the solution of the stochastic differential equation

$$X_{s,x}(t) = x + \int_s^t a(u, X_{s,x}(u))\, du + \int_s^t b(u, X_{s,x}(u))\, dW(u). \tag{3.3}$$

Proof. Equation (3.1) is equivalent to the following one: for any $s \le t \le T$,

$$X(t) = X(s) + \int_s^t a(u, X(u))\, du + \int_s^t b(u, X(u))\, dW(u). \tag{3.4}$$

We fix s. Since the process $X_{s,x}$ is continuous with respect to the initial value x (see § Ch II), the transition function $P(s,x,t,\Delta)$ is also continuous in x, because the distribution of $X_{s,x}(t)$ has no mass points (see Gihman and Skorohod (1972)).

According to Theorem 9.2 Ch. II, a.s. for all $s \in [0,T]$

$$X(t) = X_{s,X(s)}(t) \qquad s \le t \le T. \tag{3.5}$$

Denote by $\mathcal{G}_a^b = \sigma\{X(t), t \in [a,b]\}$ the smallest σ-algebra of events generated by the process X when the time varies from a to b. From Theorem 7.1 Ch. II it follows that

$$\mathcal{G}_a^b \subseteq \sigma\{\xi, W(v), 0 \le v \le s\} \tag{3.6}$$

for $0 \le a \le b \le s$. To prove that X is a Markov process we compute for all $0 \le s < t \le T$ and any Borel set Δ the conditional probabilities

$$\mathbf{P}(X(t) \in \Delta | \mathcal{G}_0^s), \qquad \mathbf{P}(X(t) \in \Delta | \mathcal{G}_s^s),$$

and verify that they are equal.

To compute these probabilities we apply Lemma 2.1 of Ch. I. From equation (3.3) it follows that the process $X_{s,x}(t)$ for $t \ge s$ is determined by the increments $W(u) - W(s)$, $u \in [s,T]$, and hence is independent of the σ-algebras $\mathcal{G}_0^s$ and $\mathcal{G}_s^s$. This is due to (3.6) and the independence of the Brownian motion increments of the initial value ξ.

The random variable $X(s)$ is measurable with respect to the σ-algebras $\mathcal{G}_0^s$ and $\mathcal{G}_s^s$, and the process $H(x,\omega) := \mathbb{1}_\Delta\big(X_{s,x}(t)\big)$ is independent of these σ-algebras, because this is true for the process $X_{s,x}(t)$, $t \in [s,T]$.

We have

$$\mathbf{E}H(x,\omega) = \mathbf{P}\big(X_{s,x}(t) \in \Delta\big) = P(s,x,t,\Delta).$$

Applying (3.5) and Lemma 2.1 of Ch. I with $h(x) = P(s,x,t,\Delta)$, we get

$$\mathbf{P}(X(t) \in \Delta | \mathcal{G}_0^s) = \mathbf{E}\big\{\mathbb{1}_\Delta\big(X_{s,X(s)}(t)\big)\big|\mathcal{G}_0^s\big\} = P(s,X(s),t,\Delta) \qquad \text{a.s.}, \tag{3.7}$$

and, similarly,

$$\mathbf{P}(X(t) \in \Delta | \mathcal{G}_s^s) = \mathbf{E}\big\{\mathbb{1}_\Delta\big(X_{s,X(s)}(t)\big)\big|\mathcal{G}_s^s\big\} = P(s,X(s),t,\Delta) \qquad \text{a.s.} \tag{3.8}$$

Therefore,

$$\mathbf{P}(X(t) \in \Delta | \mathcal{G}_0^s) = \mathbf{P}(X(t) \in \Delta | \mathcal{G}_s^s) = P(s,X(s),t,\Delta) \tag{3.9}$$

and X is a Markov process with the transition function $P(s,x,t,\Delta)$ satisfying the conditions 1)–4) of § 6 Ch. I. Condition 4) follows from (6.8) Ch. I and the continuity of $P(s,x,t,\Delta)$ with respect to x. □

Theorem 3.2. *Let $a(t,x)$, $b(t,x)$, $(t,x) \in [0,T] \times \mathbf{R}$, be continuous functions. Assume that a, b, and the initial value ξ satisfy the conditions of Theorem* 7.1 Ch. II.

Then the solution of the stochastic differential equation (3.1) *is a diffusion process with the drift coefficient $a(t,x)$ and the diffusion coefficient $b^2(t,x)$.*

Proof. It is sufficient, for example, to verify conditions (1.7)–(1.9). In view of (3.2), the sufficient condition for (1.7) is the inequality

$$\mathbf{E}(X_{s,x}(t) - x)^4 \le C(t-s)^2. \tag{3.10}$$

Similarly, the conditions (1.8), (1.9) can be transformed to the following ones:

$$\mathbf{E}(X_{s,x}(t)-x)=(t-s)\,a(s,x)+o(t-s), \tag{3.11}$$

$$\mathbf{E}(X_{s,x}(t)-x)^2=(t-s)\,b^2(s,x)+o(t-s). \tag{3.12}$$

The estimate (3.10) follows from (7.23) Ch. II for $m=2$, applied for the process $X_{s,x}(t)$, $t\in[s,T]$.

To prove (3.11), we start with the equality

$$\mathbf{E}(X_{s,x}(t)-x)=\int_s^t \mathbf{E}a(u,X_{s,x}(u))\,du$$

$$=\int_s^t \mathbf{E}(a(u,X_{s,x}(u))-a(u,x))\,du+\int_s^t a(u,x)\,du.$$

By the mean value theorem for integrals,

$$\int_s^t a(u,x)\,du=(t-s)\,a(s,x)+o(t-s). \tag{3.13}$$

Using for the function $a(u,x)$ the Lipschitz condition (7.4) Ch. II and applying the estimate (3.10), we obtain

$$\int_s^t \mathbf{E}|a(u,X_{s,x}(u))-a(u,x)|\,du\le C_T\int_s^t \mathbf{E}|X_{s,x}(u)-x|\,du$$

$$\le C_T\int_s^t \mathbf{E}^{1/4}|X_{s,x}(u)-x|^4\,du\le C_TC^{1/4}\int_s^t (u-s)^{1/2}\,du=\frac{2}{3}C_TC^{1/4}(t-s)^{3/2}.$$

This estimate together with (3.13) implies (3.11).

To prove (3.12) we use Itô's formula. We have

$$d(X_{s,x}(t)-x)^2=2(X_{s,x}(t)-x)\big(a(t,X_{s,x}(t))\,dt$$

$$+\,b(t,X_{s,x}(t))\,dW(t)\big)+b^2(t,X_{s,x}(t))\,dt. \tag{3.14}$$

Writing this in the integral form and taking the expectation, we obtain

$$\mathbf{E}(X_{s,x}(t)-x)^2=2\int_s^t \mathbf{E}\big((X_{s,x}(u)-x)a(u,X_{s,x}(u))\big)\,du$$

$$+\int_s^t \mathbf{E}\big(b^2(u,X_{s,x}(u))-b^2(u,x)\big)\,du+\int_s^t b^2(u,x)\,du. \tag{3.15}$$

By the mean value theorem for integrals,

$$\int_s^t b^2(u,x)\,du = (t-s)\,b^2(s,x) + o(t-s). \tag{3.16}$$

Using the condition on the linear growth of the function $a(u,x)$ (see (7.5) Ch. II) and the estimates (7.22) Ch. II and (3.10), we obtain

$$\int_s^t \mathbf{E}|(X_{s,x}(u)-x)a(u,X_{s,x}(u))|du \le \int_s^t \mathbf{E}^{1/2}|X_{s,x}(u)-x|^2\mathbf{E}^{1/2}|a(u,X_{s,x}(u))|^2du$$

$$\le C_2\int_s^t \mathbf{E}^{1/4}|X_{s,x}(u)-x|^4\mathbf{E}^{1/2}(1+|X_{s,x}(u)|)^2\,du \le C_3(t-s)^{3/2}. \tag{3.17}$$

Using the Lipschitz condition (see (7.4) Ch. II), the linear growth condition on the function $b(u,x)$ (see (7.5) Ch. II), and the estimate (3.10), we get

$$\begin{aligned}
&\int_s^t \mathbf{E}|b^2(u,X_{s,x}(u))-b^2(u,x)|du\\
&\quad\le \int_s^t \mathbf{E}^{1/2}|b(u,X_{s,x}(u))-b(u,x)|^2\mathbf{E}^{1/2}|b(u,X_{s,x}(u))+b(u,x)|^2\,du\\
&\quad\le C_4\int_s^t \mathbf{E}^{1/2}|X_{s,x}(u)-x|^2\mathbf{E}^{1/2}(2+|x|+|X_{s,x}(u)|)^2du\\
&\quad\le C_5\int_s^t \mathbf{E}^{1/4}|X_{s,x}(u)-x|^4\,du \le C_6(t-s)^{3/2}.
\end{aligned} \tag{3.18}$$

This estimate together with (3.15)–(3.17) imply (3.12). □

Remark 3.1. From Theorem 3.2 it follows that the diffusion with a fixed initial value is uniquely determined by its drift coefficient $a(t,x)$ and diffusion coefficient $b^2(t,x)$.

Remark 3.2. The diffusion X, being the solution of the stochastic differential equation (3.1), is almost surely continuous.

Let the coefficients of a and b of the stochastic differential equation (3.1) be independent of the time parameter. The next result states that in this case the Markov process X is homogeneous, i.e., X is a *homogeneous diffusion.*

Theorem 3.3. *The transition function of a diffusion* $X(t)$, $t\in[0,T]$, *which is a solution of the equation*

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \qquad X(0) = \xi, \tag{3.19}$$

depends on the difference of the time moments, i.e.,

$$P(s,x,t,\Delta) = P(t-s,x,\Delta).$$

Proof. By (3.2),

$$P(s,x,t,\Delta) = \mathbf{P}(X_{s,x}(t)\in\Delta),$$

where $X_{s,x}(t)$, $t\in[s,T]$, is the solution of the stochastic differential equation

$$X_{s,x}(t) = x + \int_s^t a(X_{s,x}(u))\,du + \int_s^t b(X_{s,x}(u))\,dW(u). \tag{3.20}$$

For a fixed s set $h := t-s$, $\widetilde{W}(v) := W(v+s) - W(s)$, $v\ge 0$. The process $\widetilde{W}$ is a Brownian motion. Changing in equation (3.20) the variables by setting $u := v+s$, we have

$$X_{s,x}(h+s) = x + \int_0^h a(X_{s,x}(v+s))\,dv + \int_0^h b(X_{s,x}(v+s))\,d\widetilde{W}(v). \tag{3.21}$$

Consider the same equation in the different notations:

$$\widetilde{X}_x(h) = x + \int_0^h a(\widetilde{X}_x(v))\,dv + \int_0^h b(\widetilde{X}_x(v))\,d\widetilde{W}(v). \tag{3.22}$$

It is clear that the finite-dimensional distributions of the process $\widetilde{X}_x(h)$, $h\ge 0$, are independent of s, because this is true for the process $\widetilde{W}$. By the uniqueness of the solution of the stochastic differential equation,

$$X_{s,x}(h+s) = \widetilde{X}_x(h), \qquad h\in[0,T-s], \qquad \text{a.s.} \tag{3.23}$$

Therefore,

$$P(s,x,t,\Delta) = \mathbf{P}(X_{s,x}(h+s)\in\Delta) = \mathbf{P}(\widetilde{X}_x(h)\in\Delta) =: P(h,x,\Delta)$$

and the transition function depends on the difference $h = t-s$. □

Consider a family of homogeneous diffusions, which are solutions of (3.19) for different nonrandom initial values $X(0) = x\in\mathbf{R}$. Let $T=\infty$. The extension of a solution of the stochastic differential equation (3.19) to the time interval $[0,\infty)$, where the functions $a(x)$, $b(x)$, $x\in\mathbf{R}$, satisfy the conditions of Theorem 7.3 Ch. II, is realized by the standard way, because the solutions on intersecting intervals, containing 0, always coincide. We will prove that this family is strong Markov.

Recall that by $\mathbf{P}_x$ and $\mathbf{E}_x$ we denote the probability and the expectation with respect to a process X with the initial value $X(0)=x$. Such a probability and an expectation will be considered as a functions of the variable $x\in\mathbf{R}$.

Proposition 3.1. *Let $X(t)$, $t \in [0,\infty)$, be a solution of equation* (3.19) *for $\xi = x$. Then the family of stochastic processes*

$$\{X(t), 0 \le t < \infty, X(0) = x\}_{x\in\mathbf{R}}$$

is a strong Markov.

Proof. The following arguments repeat in main features the proof of Proposition 7.3 Ch. I. The continuous diffusion X is progressively measurable with respect to the family of σ-algebras $\mathcal{G}_0^t$. Let τ be a stopping time with respect to the family of σ-algebras $\mathcal{G}_0^t$. It is obvious that $\{\tau < \infty\} \in \mathcal{G}_0^\tau$. Instead of equality (6.13) Ch. I, which defines the strong Markov property, it is sufficient to prove that for any $0 \le t_1 < t_2 < \cdots < t_n$, arbitrary $A \in \mathcal{G}_0^\tau$, and any bounded continuous function $g(\vec{y})$, $\vec{y} \in \mathbf{R}^m$, one has

$$\mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} g(X(t_1+\tau), \dots, X(t_m+\tau))\}$$

$$= \mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} \mathbf{E}_{X(\tau)} g(X(t_1), \dots, X(t_m))\}. \tag{3.24}$$

We now fix arbitrary $s \ge 0$. Let the process $X_{s,x}(t)$, $t \in [s,\infty)$, be the solution of (3.20). This process is independent of σ-algebra $\mathcal{G}_0^s$. For the process $\widetilde{X}_x$, satisfying (3.22), we set

$$h(x) := \mathbf{E} g(\widetilde{X}_x(t_1), \widetilde{X}_x(t_2), \dots, \widetilde{X}_x(t_m))$$
$$= \mathbf{E}_x g(X(t_1), X(t_2), \dots, X(t_m)).$$

By Theorem 9.1 Ch. II, the process $\widetilde{X}_x(t)$ is a.s. continuous with respect to $(t,x) \in [0,\infty) \times \mathbf{R}$. Hence h is a continuous function. By (3.23),

$$h(x) = \mathbf{E} g(X_{s,x}(s+t_1), X_{s,x}(s+t_2), \dots, X_{s,x}(s+t_m)).$$

Applying Lemma 2.1 Ch. I, we get

$$\mathbf{E}_x\{g(X(s+t_1), X(s+t_2), \dots, X(s+t_m)) \big| \mathcal{G}_0^s\}$$

$$= \mathbf{E}_x\{g\big(X_{s,X(s)}(s+t_1), \dots, X_{s,X(s)}(s+t_m)\big) \big| \mathcal{G}_0^s\} = h(X(s))$$

a.s. with respect to the measure $\mathbf{P}_x$.

Consider the stopping times

$$\tau_n = 2^{-n} \mathbb{1}_{\{\tau \le 2^{-n}\}} + \sum_{k=1}^{\infty} k2^{-n} \mathbb{1}_{\{(k-1)2^{-n} < \tau \le k2^{-n}\}}, \qquad n \in \mathbb{N}, \quad k = 1, 2, \dots.$$

It is obvious that $\tau_n \downarrow \tau$ as $n \to \infty$ for all $\omega \in \{\tau < \infty\}$.

By the definition of the σ-algebra $\mathcal{G}_0^\tau$, we have $A \bigcap \{\tau_n = k2^{-n}\} \in \mathcal{G}_0^{k2^{-n}}$. Since $\{\tau < \infty\} = \bigcup\limits_{k=1}^{\infty} \{\tau_n = k2^{-n}\}$, we get

$$\mathbf{E}_x\{\mathbb{1}_A \mathbb{1}_{\{\tau<\infty\}} g(X(t_1+\tau_n), \dots, X(t_m+\tau_n))\}$$

$$= \sum_{k=1}^{\infty} \mathbf{E}_x \{ 1\!\!1_A 1\!\!1_{\{\tau_n = k2^{-n}\}} \, g(X(t_1 + k2^{-n}), \dots, X(t_m + k2^{-n})) \}$$
$$= \sum_{k=1}^{\infty} \mathbf{E}_x \big\{ 1\!\!1_{A \cap \{\tau_n = k2^{-n}\}} \mathbf{E}_x \big\{ g(X(t_1 + k2^{-n}), \dots, X(t_m + k2^{-n})) \big| \mathcal{G}_0^{k2^{-n}} \big\} \big\}$$
$$= \sum_{k=1}^{\infty} \mathbf{E}_x \{ 1\!\!1_{A \cap \{\tau_n = k2^{-n}\}} \, h(X(k2^{-n})) = \mathbf{E}\{ 1\!\!1_A 1\!\!1_{\{\tau < \infty\}} h(X(\tau_n))\}. \qquad (3.25)$$

The process X is continuous, therefore $X(\tau_n + t) \to X(\tau + t)$ a.s. for every $t \ge 0$. Since the function $h(y)$ is continuous and bounded together with $g(\vec{y})$, the Lebesgue dominated convergence theorem yields

$$\mathbf{E}\{ 1\!\!1_A 1\!\!1_{\{\tau < \infty\}} g(X(t_1 + \tau), \dots, X(t_m + \tau))\}$$
$$= \lim_n \mathbf{E}\{ 1\!\!1_A 1\!\!1_{\{\tau < \infty\}} g(X(t_1 + \tau_n), \dots, X(t_m + \tau_n))\}$$

and

$$\mathbf{E}\{ 1\!\!1_A 1\!\!1_{\{\tau < \infty\}} h(X(\tau))\} = \lim_n \mathbf{E}\{ 1\!\!1_A 1\!\!1_{\{\tau < \infty\}} h(X(\tau_n))\}.$$

Now, passing in (3.25) to the limit as $\tau_n \downarrow \tau$, we obtain (3.24). □

Remark 3.3. The process X is continuous. Therefore, (3.24) implies that for any $x \in \mathbf{R}$, any stopping time τ with respect to the family of σ-algebras $\mathcal{G}_0^t$, and any bounded measurable function $\wp(X(t), 0 \le t < \infty)$, defined on $C([0,\infty))$, we have

$$\mathbf{E}_x\{\wp(X(t+\tau), 0 \le t < \infty) | \mathcal{G}_0^\tau\} 1\!\!1_{\{\tau < \infty\}}$$
$$= \mathbf{E}_{X(\tau)} \wp(X(t), 0 \le t < \infty) 1\!\!1_{\{\tau < \infty\}} \qquad (3.26)$$

a.s. with respect to $\mathbf{P}_x$.

§ 4. Distributions of integral functionals of a diffusion and of infimum and supremum functionals

Let X be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))dW(t) + \mu(X(t))\, dt, \qquad X(0) = x, \qquad (4.1)$$

where $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, are continuously differentiable functions, satisfying the linear growth condition:

$$|\mu(x)| + |\sigma(x)| \le C(1 + |x|), \qquad \text{for all } x \in \mathbf{R}.$$

Then by Theorem 7.3 Ch. II, there exists a unique strong solution of equation (4.1). Assume, in addition, that $\sigma^2(x) > 0$ for $x \in \mathbf{R}$ and the derivative $\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'$ is bounded.

Let τ be an exponentially distributed with the parameter $\lambda > 0$ random time independent of the diffusion $X(t)$, $t \ge 0$. It is sufficient that τ is independent of the Brownian motion W, since the Brownian motion determines the diffusion (see the method of successive approximations in § 7 Ch. II).

As in the case of a Brownian motion (see § 1 Ch. III), the following result is of key importance for computing the distributions of integral functionals of a homogeneous diffusion.

Theorem 4.1. *Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be piecewise-continuous functions. Assume that Φ is bounded and f is nonnegative. Then the function*

$$U(x) := \mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau f(X(s))\,ds\Big)\Big\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the equation

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in \mathbf{R}. \tag{4.2}$$

Remark 4.1. Under the above assumptions, according to this theorem and to Proposition 12.2 of Ch. II, the homogeneous equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - (\lambda + f(x))\phi(x) = 0 \tag{4.3}$$

for $f(x) \ge 0$, $x \in \mathbf{R}$, has two linearly independent nonnegative strictly monotone solutions $\psi(x)$ and $\varphi(x)$ such that $\lim_{x\to-\infty} \varphi(x) = \infty$ and $\lim_{x\to\infty} \psi(x) = \infty$.

The monotonicity is strict. This follows from the fact that the derivative $\phi'(x)$ cannot be equal to zero. Indeed, if $\phi'(x) = 0$, then by (4.3), $\phi''(x) > 0$, and there is a local minimum at x, which contradicts the monotonicity of the solution ϕ.

The functions $\psi(x)$ and $\varphi(x)$ are called *fundamental solutions* of the homogeneous equation (4.3).

Remark 4.2. For piecewise-continuous functions f and Φ equation (4.2) must be interpreted as follows: it holds for all points of continuity of f and Φ, and at points of discontinuity of f and Φ its solution is continuous together with the first derivative.

Proof of Theorem 4.1. By Theorem 12.4 Ch. II, the function U is the unique bounded solution of (4.2) for a continuous Φ and f. Our aim is to extend Theorem 12.4 Ch. II up to piecewise-continuous functions Φ and f. We do this with the help of the approach used in the proof of Theorem 1.2 of Ch. III. We refer to this approach as the *limit approximation method*.

A nonnegative piecewise-continuous function f can be approximated by a sequence of continuous functions $\{f_n\}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad 0 \le f_n(x) \le f(x), \qquad x \in \mathbf{R}.$$

A bounded piecewise-continuous function Φ can be approximated by a sequence of continuous uniformly bounded functions $\{\Phi_n\}$ such that $\sup_{x\in\mathbf{R}} |\Phi_n(x)| \le K$ for all n and $\Phi(x) = \lim_{n\to\infty} \Phi_n(x)$, $x \in \mathbf{R}$.

Set

$$U_n(x) := \mathbf{E}_x\Big\{\Phi_n(X(\tau))\exp\Big(-\int_0^\tau f_n(X(s))\,ds\Big)\Big\}. \tag{4.4}$$

By the Lebesgue dominated convergence theorem,

$$U_n(x) \to U(x) \qquad \text{as } n \to \infty, \quad \text{for every } x \in \mathbf{R}. \tag{4.5}$$

By Theorem 12.4 in Ch. II, the function U_n satisfies equation (4.2), i.e.,

$$\frac{1}{2}\sigma^2(x)U_n''(x) + \mu(x)U_n'(x) - (\lambda + f_n(x))U_n(x) = -\lambda\Phi_n(x), \qquad x \in \mathbf{R}. \tag{4.6}$$

We make in (4.6) a change of variable to transform it into an equation without the first derivative. Set

$$p(x) := \exp\Bigl(2\int_0^x \frac{\mu(v)}{\sigma^2(v)}\,dv\Bigr), \qquad y(x) := \int_0^x \exp\Bigl(-2\int_0^u \frac{\mu(v)}{\sigma^2(v)}\,dv\Bigr)\,du, \qquad x \in \mathbf{R}.$$

The function $y(x)$, $x \in \mathbf{R}$, is strictly increasing, $y(0) = 0$, and therefore it has the inverse $y^{(-1)}(y)$, $y \in (l, r)$, i.e., $x = y^{(-1)}(y(x))$, where

$$l := -\int_{-\infty}^0 \exp\Bigl(2\int_u^0 \frac{\mu(v)}{\sigma^2(v)}\,dv\Bigr)du \ge -\infty, \qquad r := \int_0^\infty \exp\Bigl(-2\int_0^u \frac{\mu(v)}{\sigma^2(v)}\,dv\Bigr)du \le \infty.$$

The change of the variable $V_n(y) = U_n(y^{(-1)}(y))$ reduces equation (4.6) (see the proof of Proposition 12.2 Ch. II) to the equation

$$\frac{1}{2}V_n''(y) - \frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}(\lambda + f_n(y^{(-1)}(y)))V_n(y) = -\lambda\Phi_n(y^{(-1)}(y))\frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}. \tag{4.7}$$

By (4.5), $V_n(y) \to V(y) := U(y^{(-1)}(y))$. It was established in the proof of Theorem 1.2 Ch. III (the limit approximation method) that we can pass to the limit in equation (4.7), and that the function $V(y)$, $y \in (l, r)$, is the bounded solution of the equation

$$\frac{1}{2}V''(y) - \frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}(\lambda + f(y^{(-1)}(y)))V(y) = -\lambda\Phi(y^{(-1)}(y))\frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}.$$

Returning in this equation to the original variable, we verify that the function $U(x) = V(y(x))$, $x \in \mathbf{R}$, is the bounded solution of (4.2).

The uniqueness of a bounded solution of (4.2) on the whole real line can be established in the same way as it was done in the proof of Theorem 1.2 Ch. III. Theorem 4.1 is proved. □

Consider the problem of computing the joint distribution of

$$\int_0^t f(X(s))\,ds, \qquad \inf_{0\le s\le t} X(s), \qquad \sup_{0\le s\le t} X(s).$$

The main idea used for computing the infimum or supremum type functionals was illustrated in §2 Ch. III and implemented in the proof of Theorem 2.1 Ch. III. It is applicable to the investigation of homogeneous diffusions. In the same way as it was done in the proof of Theorem 2.1 Ch. III, we can establish the following result, which is of key importance for computing the joint distributions of an integral functional of homogeneous diffusion and its infimum and supremum.

Theorem 4.2. *Let $\Phi(x)$ and $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \ge 0$. Then the function*

$$U(x) := \mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau f(X(s))\,ds\Big); a \le \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) \le b\Big\},$$

$x \in [a,b]$, is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + f(x))U(x) = -\lambda\Phi(x), \qquad x \in (a,b), \tag{4.8}$$

$$U(a) = 0, \qquad U(b) = 0. \tag{4.9}$$

Remark 4.4. In the case $a = -\infty$ or $b = \infty$ we assume, in addition, that Φ is bounded. Then the corresponding boundary condition in (4.9) must be replaced by the condition that the function $U(x)$ is bounded as x tends to $-\infty$ or ∞.

As in the case of Brownian motion, the proof exploits the fact that the function

$$U_\gamma(x) := \mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau (f(X(s)) + \gamma\mathbb{I}_{\mathbf{R}\setminus[a,b]}(X(s)))\,ds\Big)\Big\}$$

converges to $U(x)$ as $\gamma \to \infty$. By Theorem 4.1, the function $U_\gamma(x)$ satisfies the equation (4.2) with $f(x) + \gamma\mathbb{I}_{\mathbf{R}\setminus[a,b]}(x)$ instead of $f(x)$.

It is important that starting at the boundary of the interval $[a,b]$, a diffusion spends a.s. positive time outside $[a,b]$ up to a random moment. This is a consequence of the corresponding fact for a Brownian motion (see § 2 Ch. III) and formula (13.2) of this chapter, which expresses a homogeneous diffusion as the transformation of a Brownian motion. This implies that $U_\gamma(a) \to 0$ and $U_\gamma(b) \to 0$, because the exponential function under the expectation sign tends to zero a.s., and implies the boundary conditions (4.9). For the detailed proof we refer to §2 Ch. III.

Proposition 4.1. *For any finite a and b*

$$p_{[a,b]}(x) := \mathbf{P}_x\Big(a \le \inf_{0\le s<\infty} X(s), \sup_{0\le s<\infty} X(s) \le b\Big) = 0. \tag{4.10}$$

Proof. By Theorem 4.2, the function

$$U_\lambda(x) := \mathbf{P}\Big(a \le \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) \le b\Big)$$

is the solution of the problem

$$\frac{1}{2}\sigma^2(x)U_\lambda''(x) + \mu(x)U_\lambda'(x) - \lambda U_\lambda(x) = -\lambda, \qquad x \in (a,b), \tag{4.11}$$

$$U_\lambda(a) = 0, \qquad U_\lambda(b) = 0. \tag{4.12}$$

Since $\mathbf{P}(\tau > t) = e^{-\lambda t}$, we have $\tau \to \infty$ in probability as $\lambda \to 0$ and, correspondingly, $U_\lambda(x) \downarrow p_{[a,b]}(x)$ as $\lambda \to 0$. In the problem (4.11), (4.12) we can pass to the limit as $\lambda \to 0$. This can be done by the limit approximation method from the proof of Theorem 4.1. Then the limit function $p_{[a,b]}(x)$ is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)p''(x) + \mu(x)p'(x) = 0, \qquad x \in (a,b), \qquad p(a) = 0, \qquad p(b) = 0.$$

But the solution of this problem is $p(x) \equiv 0$. □

Corollary 4.1. *For any finite a and b*

$$p_{(a,b)}(x) := \mathbf{P}_x\Big(a < \inf_{0\le s<\infty} X(s),\ \sup_{0\le s<\infty} X(s) < b\Big) = 0. \tag{4.13}$$

Proof. Let $a_n \downarrow a$ and $b_n \uparrow b$. Then $p_{[a_n,b_n]}(x) \uparrow p_{(a,b)}(x)$. Now (4.13) is the consequence of (4.10) and the countable additivity of the probability measure. □

Note that (4.13) also follows from Remark 12.3 of Ch. II, because

$$\mathbf{P}_x\Big(a < \inf_{0\le s<\infty} X(s),\ \sup_{0\le s<\infty} X(s) < b\Big) = \mathbf{P}_x(H_{a,b} = \infty) = 0,$$

where $H_{a,b} := \min\{s : X(s) \notin (a,b)\}$ is the first exit time from the interval (a,b).

Example 4.1. We compute the distributions of $\inf_{0\le s\le\tau} X(s)$ or $\sup_{0\le s\le\tau} X(s)$ for the Ornstein–Uhlenbeck process $X(t)$, $t \ge 0$, which is the solution of the stochastic differential equation

$$dX(t) = \sigma\, dW(t) - \theta X(t)\, dt, \qquad X(0) = x, \qquad \theta > 0. \tag{4.14}$$

For a more detailed description of the Ornstein–Uhlenbeck process see § 16.

We compute the probability

$$U(x) := \mathbf{P}_x\Big(\sup_{0\le s\le\tau} X(s) \le b\Big).$$

Applying Theorem 4.2 with $\varPhi \equiv 1$, $f = 0$, and $a = -\infty$ we have that the function $U(x)$, $x \in (-\infty, b]$, is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 U''(x) - \theta x U'(x) - \lambda U(x) = -\lambda, \qquad x \in (-\infty, b), \tag{4.15}$$

$$U(b) = 0. \tag{4.16}$$

The particular solution of equation (4.15) is the function, which is identically equal to one. The fundamental solutions of the corresponding homogeneous equation

$$\frac{1}{2}\sigma^2 \phi''(x) - \theta x \phi'(x) - \lambda \phi(x) = 0, \qquad x \in \mathbf{R}, \tag{4.17}$$

are (see Appendix 4, equation 19, $\gamma = 0$)

$$\psi(x) = e^{x^2\theta/2\sigma^2} D_{-\lambda/\theta}\Big(-\frac{x\sqrt{2\theta}}{\sigma}\Big), \qquad \varphi(x) = e^{x^2\theta/2\sigma^2} D_{-\lambda/\theta}\Big(\frac{x\sqrt{2\theta}}{\sigma}\Big), \tag{4.18}$$

where $D_{-\nu}(x)$ is the parabolic cylinder function (see Appendix 2, Section 9). As usual $\psi(x)$, $x \in \mathbf{R}$, denotes the convex increasing solution and $\varphi(x)$, $x \in \mathbf{R}$, denotes the convex decreasing solution.

It is obvious that the bounded solution of the problem (4.15), (4.16) has the form

$$\mathbf{P}_x\Big(\sup_{0\le s\le\tau} X(s) \le b\Big) = U(x) = 1 - \frac{e^{x^2\theta/2\sigma^2} D_{-\lambda/\theta}\big(-x\sqrt{2\theta}/\sigma\big)}{e^{b^2\theta/2\sigma^2} D_{-\lambda/\theta}\big(-b\sqrt{2\theta}/\sigma\big)}.$$

Analogously

$$\mathbf{P}_x\Big(a \le \inf_{0\le s\le\tau} X(s)\Big) = 1 - \frac{e^{x^2\theta/2\sigma^2} D_{-\lambda/\theta}\big(x\sqrt{2\theta}/\sigma\big)}{e^{a^2\theta/2\sigma^2} D_{-\lambda/\theta}\big(a\sqrt{2\theta}/\sigma\big)}.$$

§ 5. Distributions of functionals of a diffusion stopped at the moment inverse of integral functional

Let X be a diffusion satisfying conditions of § 4. We consider the integral functional $\int_0^t g(X(s))\,ds$, $t > 0$, where g is a nonnegative piecewise-continuous function. For definiteness we assume that at points of discontinuity g takes the values of the right limits $(g(z) = g(z+))$.

Consider the problem of computing distributions of functionals of the diffusion X stopped at the *moment inverse of integral functional*. This moment is defined by the formula

$$\nu(t) := \min\Big\{s : \int_0^s g(X(v))\,dv = t\Big\}. \tag{5.1}$$

For a Brownian motion this problem was solved in § 6 Ch. III.

Depending on the relations

$$\int_0^\infty g(X(s))\,ds = \infty \qquad \text{a.s.}, \tag{5.2}$$

or

$$\mathbf{P}\Big(\int_0^\infty g(X(s))\,ds < \infty\Big) > 0,$$

the moment $\nu(t)$ is a.s. finite for all $t \ge 0$, or with positive probability it is infinite for all t exceeding some random value.

For example, for the Brownian motion with negative linear drift and for the function $g(x) = \mathbb{I}_{[r,\infty)}(x)$ (see (9.18) and (9.19) Ch. III), the integral on the left-hand side of (5.2) is a.s. finite. One can verify that for the function g, and for the diffusion with $\sigma(x) \equiv 1$ and with a smooth drift $\mu(x)$, $x \in \mathbf{R}$, satisfying the conditions $\mu(x) = 1$ for $x \ge r + 1$, $\mu(x) = x$ for $-r \le x \le r$, and $\mu(x) = -1$ for $x \le -r - 1$, the integral on the left-hand side of (5.2) takes both finite and infinite values with positive probabilities.

A sufficient condition for the validity of (5.2) is given by Corollary 12.1 of Ch. II:

$$\liminf_{y\to\infty} \frac{1}{y} \int_0^y \frac{g(x)}{\sigma^2(x)}\,dx > 0, \qquad \liminf_{y\to\infty} \frac{1}{y} \int_{-y}^0 \frac{g(x)}{\sigma^2(x)}\,dx > 0. \tag{5.3}$$

The fact that g is a piecewise-continuous function is not a problem, because g is continuous for all sufficiently large arguments.

The following result is of key importance for the problem of computing of distributions of functionals for diffusions stopped at the moment $\nu(t)$. Let τ be an exponentially distributed with the parameter $\lambda > 0$ random time independent of the process $\{X(s),\ s \ge 0\}$.

Theorem 5.1. *Let $\Phi(x)$ and $f(x)$, $x\in[a,b]$, be piecewise-continuous functions. Assume that $f\ge 0$. Then the function*

$$U_\nu(x):=\mathbf{E}_x\biggl\{\Phi(X(\nu(\tau)))\exp\Bigl(-\int\limits_0^{\nu(\tau)} f(X(s))\,ds\Bigr);$$

$$a\le \inf_{0\le s\le\nu(\tau)} X(s),\ \sup_{0\le s\le\nu(\tau)} X(s)\le b\biggr\},\qquad x\in[a,b], \tag{5.4}$$

is the unique continuous solution of the problem

$$\frac12\sigma^2(x)U''(x)+\mu(x)U'(x)-(\lambda g(x)+f(x))U(x)=-\lambda g(x)\Phi(x), \tag{5.5}$$

$$U(a)=0,\qquad U(b)=0. \tag{5.6}$$

Remark 5.1. For $a=-\infty$ and $b=\infty$ we assume, in addition, that Φ is bounded. Then in the definition of $U_\nu(x)$ there appears the set $\nu(\tau)<\infty$ (see (5.8)) and the corresponding boundary condition (5.6) must be replaced by the condition that the function $U(x)$ is bounded as x tends to $-\infty$ or ∞.

Proof of Theorem 5.1. We first prove the result for the case $a=-\infty$ and $b=\infty$. Assume that $\Phi(x)$, $f(x)$ and $g(x)$, $x\in\mathbf{R}$, are bounded continuous functions. Let for some $\rho>0$ the function $V_\rho(x)$, $x\in\mathbf{R}$, be a bounded solution of the equation

$$\frac12\sigma^2(x)V_\rho''(x)+\mu(x)V_\rho'(x)-(\rho+\lambda g(x)+f(x))V_\rho(x)=-\lambda g(x)\Phi(x). \tag{5.7}$$

Such a solution exists due to Theorem 12.4 Ch. II. Set

$$\eta(t):=V_\rho(X(t))\exp\Bigl(-\rho t-\lambda\int\limits_0^t g(X(v))\,dv-\int\limits_0^t f(X(v))\,dv\Bigr).$$

Applying Itô's formula, we obtain that for any $r>0$

$$\eta(r)-\eta(0)=\int\limits_0^r\exp\Bigl(-\int\limits_0^t\bigl(\lambda g(X(v))+\rho+f(X(v))\bigr)\,dv\Bigr)\Bigl[V_\rho'(X(t))\sigma(X(t))\,dW(t)$$

$$+\Bigl(V_\rho'(X(t))\mu(X(t))+\frac12V_\rho''(X(t))\sigma^2(X(t))-\lambda g(X(t))+\rho+f(X(t)))V_\rho(X(t))\Bigr)dt\Bigr].$$

Using equation (5.7) and substituting the time $H_{c,d}:=\min\{s:X(s)\notin(c,d)\}$, we have

$$\eta(r\wedge H_{c,d})-V_\rho(x)=\int\limits_0^r \mathbb{I}_{[0,H_{c,d})}(t)\exp\Bigl(-\int\limits_0^t\bigl(\lambda g(X(v))+\rho+f(X(v))\bigr)\,dv\Bigr)$$

$$\times\big[V_\rho'(X(t))\sigma(X(t))\,dW(t)-\lambda g(X(t))\varPhi(X(t))\,dt\big].$$

Since for every $t\ge 0$ the random variable $1\!\!{\rm I}_{[0,H_{c,d})}(t)$ is $\mathcal{G}_0^t$-measurable and the variables $V_\rho'(X(t))$, $\sigma(X(t))$ are bounded for $t<H_{c,d}$, the expectation of the stochastic integral is zero. Computing the expectation of both sides of the above equation, we get

$$V_\rho(x)=\mathbf{E}_x\eta(r\wedge H_{c,d})$$
$$+\lambda\mathbf{E}_x\int_0^{r\wedge H_{c,d}} g(X(t))\varPhi(X(t))\exp\Big(-\lambda\int_0^t g(X(v))\,dv-\int_0^t(\rho+f(X(v)))\,dv\Big)\,dt.$$

We set $I(t):=\int_0^t g(X(v))\,dv$ and make in the above integral the substitution $I(t)=s$. Then

$$V_\rho(x)=\mathbf{E}_x\eta(r\wedge H_{c,d})+\lambda\mathbf{E}_x\int_0^{I(r\wedge H_{c,d})}\varPhi(X(\nu(s)))\exp\Big(-\lambda s-\int_0^{\nu(s)}(\rho+f(X(v)))\,dv\Big)ds.$$

Note that $H_{c,d}\to\infty$ as $c\to-\infty$ and $d\to\infty$, and the variable $\mathbf{E}_x\eta(r)$ tends to zero as $r\to\infty$. Passing to the limit and using the assumption that τ is independent of X and has the density $\lambda e^{-\lambda s}1\!\!{\rm I}_{[0,\infty)}(s)$, we obtain, by Fubini's theorem, that

$$V_\rho(x)=\mathbf{E}_x\Big\{1\!\!{\rm I}_{[0,I(\infty))}(\tau)\varPhi(W(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)}(\rho+f(W(v)))\,dv\Big)\Big\}$$
$$=\mathbf{E}_x\Big\{\varPhi(W(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)}(\rho+f(W(v)))\,dv\Big);\nu(\tau)<\infty\Big\}.\qquad(5.8)$$

It is obvious that $\lim_{\rho\downarrow 0}V_\rho(x)=U_\nu(x)$. In equation (5.7) we can pass to the limit as $\rho\downarrow 0$ and prove that the limiting function $U_\nu(x)$, $x\in\mathbf{R}$, satisfies equation (5.5). A similar passage to the limit has already been used in the proof of Theorem 4.1. Therefore, Theorem 5.1 is proved for bounded continuous functions $\varPhi$, f, g and $a=-\infty$, $b=\infty$.

For piecewise-continuous functions f, g and $\varPhi$ the theorem is proved by the limit approximation method (see the proof of Theorem 4.1).

For the case $a\ne-\infty$ or $b\ne\infty$, analogously to the case of Brownian motion, we exploit the fact that the function

$$U_{\nu,\gamma}(x):=\mathbf{E}_x\Big\{\varPhi(X(\nu(\tau)))\exp\Big(-\int_0^{\nu(\tau)}(f(X(s))+\gamma1\!\!{\rm I}_{\mathbf{R}\setminus[a,b]}(X(s)))\,ds\Big)\Big\}$$

converges to $U_\nu(x)$ as $\gamma\to\infty$. By the version of Theorem 5.1 proved above, the function $U_{\nu,\gamma}(x)$ satisfies equation (5.5) with $f(x)+\gamma1\!\!{\rm I}_{\mathbf{R}\setminus[a,b]}(x)$ instead of $f(x)$. We must justify the passage to the limit in such an equation as $\gamma\to\infty$. This can be done similarly to the proof of Theorem 2.1 Ch. III. □

§ 6. Distributions of functionals of diffusion bridges

The bridge $X_{x,t,z}(s)$, $s \in [0,t]$, from x to z of a process $X(s)$, $s \ge 0$, with $X(0) = x$ was defined in § 11 Ch. I.

Let $X_{x,t,z}(s)$, $s \in [0,t]$, be the bridge of the diffusion X. We consider the method for computing the joint distribution of the integral functional

$$A(t) := \int_0^t f(X_{x,t,z}(s))\,ds, \qquad f \ge 0,$$

and of the infimum and supremum functionals, $\inf_{0\le s\le t} X_{x,t,z}(s)$ and $\sup_{0\le s\le t} X_{x,t,z}(s)$.

The general approach to the problem of computing distributions of nonnegative integral functionals of bridges of random processes was described in § 4 Ch. III for the Brownian bridge. This approach is valid for other diffusions. As an example, we consider the integral functional $\int_0^t f(X_{x,t,z}(s))\,ds$ for a homogeneous diffusion X with the initial value $X(0) = x$.

If the one-dimensional distribution of the diffusion X has a density (see below (10.6)), then the equality

$$\mathbf{E}\wp(X_{x,t,z}(s), 0 \le s \le t) = \frac{\frac{d}{dz}\mathbf{E}_x\{\wp(X(s), 0 \le s \le t); X(t) < z\}}{\frac{d}{dz}\mathbf{P}_x(X(t) < z)} \tag{6.1}$$

holds for any bounded measurable functional $\wp$ defined on $C([0,t])$, the space of continuous functions (see (11.13) Ch. I). Here the derivative in the numerator of the fraction must be treated in the sense of the density.

The main object for computing the distributions of integral functionals of bridges of X is the function

$$\begin{aligned} G_z^\gamma(x) :&= \lambda \int_0^\infty e^{-\lambda t} \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(-\gamma \int_0^t f(X(s))\,ds\Big); X(t) < z\Big\}\,dt \\ &= \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(-\gamma \int_0^\tau f(X(s))\,ds\Big);\ X(\tau) < z\Big\}. \end{aligned} \tag{6.2}$$

In this formula τ is an exponentially distributed with the parameter $\lambda > 0$ random time independent of the process $\{X(s),\ s \ge 0\}$.

The function $\lambda^{-1}G_z^\gamma$, as the function of the parameters $\lambda > 0$ and $\gamma > 0$, is the double Laplace transform with respect to $t \ge 0$ and $y \ge 0$ of the function

$$\frac{d}{dz}\mathbf{P}_x\Big(\int_0^t f(X(s))\,ds < y,\ X(t) < z\Big).$$

If we want to find for the bridge $X_{x,t,z}$ the distribution of the integral functional $\int_0^t f(X_{x,t,z}(s))\,ds$ at a fixed time t, we must compute the double inverse Laplace transform of the function $\lambda^{-1}G_z^\gamma(x)$ with respect to the parameters λ and γ. Then, following formula (6.1), we must divide the result by the density of the variable $X(t)$. Note that the density itself can be computed in the same way by means of the function $G_z^\gamma(x)$ with $f(x)\equiv 0$.

Under the conditions of § 5 we consider the moment inverse of integral functional:

$$\nu(\tau) := \min\Big\{s : \int_0^s g(X(v))\,dv = \tau\Big\}. \tag{6.3}$$

Suppose that (5.3) holds and that the diffusion X satisfies the conditions of § 4. Then this moment is a.s. finite. It is clear that $\nu(\tau)=\tau$ with $g\equiv 1$.

Our reasoning above shows that the following statement is of key importance for computing the joint distributions of integral functionals and of the infimum and supremum functionals of the diffusion bridge. This assertion, in contrast to Theorem 5.1, contains a condition on the value of the diffusion at the moment $\nu(\tau)$.

Theorem 6.1. *Let $f(x)$, $x\in[a,b]$, be a nonnegative piecewise-continuous function. Then for every $z\in(a,b)$ there exists the right derivative*

$$G_{\nu,z}(x) := \frac{d}{dz+}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\nu(\tau)} f(X(s))\,ds\Big);$$

$$a\le \inf_{0\le s\le\nu(\tau)} X(s),\ \sup_{0\le s\le\nu(\tau)} X(s)\le b, X(\nu(\tau))<z\Big\} \tag{6.4}$$

and $G_{\nu,z}(x)$, $x\in[a,b]$, is the unique continuous solution of the problem

$$\frac12\sigma^2(x)G''(x)+\mu(x)G'(x)-(\lambda g(x)+f(x))G(x)=0,\quad x\in(a,b)\setminus\{z\}, \tag{6.5}$$

$$G'(z+0)-G'(z-0)=-2\lambda g(z)/\sigma^2(z), \tag{6.6}$$

$$G(a)=0,\qquad G(b)=0. \tag{6.7}$$

Remark 6.1. If either $a=-\infty$ or $b=\infty$, then the corresponding boundary condition in (6.7) must be replaced by the condition that the function $G_{\nu,z}(x)$ is bounded as x tends to $-\infty$ or ∞.

Remark 6.2. The function $G_{\nu,z}(x)$, $(z,x)\in[a,b]\times[a,b]$, is the Green function of the problem (5.5), (5.6), because by the definitions of $U_\nu(x)$ and $G_{\nu,z}(x)$,

$$U_\nu(x) = \int_a^b \Phi(z)\,G_{\nu,z}(x)\,dz. \tag{6.8}$$

Proof of Theorem 6.1. Our proof is based on Theorem 5.1. Set

$$U_\Delta(x) := \mathbf{E}_x\biggl\{ \frac{\mathrm{1\!I}_{[z,z+\Delta)}(X(\nu(\tau)))}{\Delta} \exp\biggl(-\int_0^{\nu(\tau)} f(X(s))\,ds\biggr);$$

$$a \le \inf_{0\le s\le \nu(\tau)} X(s), \sup_{0\le s\le \nu(\tau)} X(s) \le b\biggr\}.$$

Clearly, $U_\Delta(x) = 0$ for $x \notin (a,b)$.

By Theorem 5.1, the function $U_\Delta(x)$, $x \in (a,b)$, is the unique bounded solution of the problem

$$\frac{1}{2}\sigma^2(x)U''_\Delta(x) + \mu(x)U'_\Delta(x) - (\lambda g(x) + f(x))U_\Delta(x) = -\frac{\lambda}{\Delta} g(x)\mathrm{1\!I}_{[z,z+\Delta)}(x), \quad (6.9)$$

$$U_\Delta(a) = 0, \qquad U_\Delta(b) = 0. \quad (6.10)$$

We want to prove that the passage to the limit as $\Delta \downarrow 0$ in the problem (6.9), (6.10) implies (6.5)–(6.7). In equation (6.9) we make a change of variables to transform it into an equation without the first derivative. We already did this in the proof of Theorem 4.1. Set $V_\Delta(y) := U_\Delta(y^{(-1)}(y))$. Then (6.9) can be reduced (see the proof of Proposition 12.2 Ch. II) to the form

$$\frac{1}{2}V''_\Delta(y) - \frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}(\lambda g(y^{(-1)}(y)) + f(y^{(-1)}(y)))V_\Delta(y)$$

$$= -\frac{p^2(y^{(-1)}(y))}{\sigma^2(y^{(-1)}(y))}\frac{\lambda}{\Delta} g(y^{(-1)}(y))\mathrm{1\!I}_{[y(z),y(z+\Delta))}(y), \qquad y \in (l,r). \quad (6.11)$$

For this case the boundary conditions transform to the following ones:

$$V_\Delta(y(a)) = 0, \qquad V_\Delta(y(b)) = 0. \quad (6.12)$$

We set $\chi_\Delta(x) := \int_{-\infty}^{x} \frac{1}{\Delta}\mathrm{1\!I}_{[y(z),y(z+\Delta))}(h)\,dh$ for $0 < \Delta < 1$ and remark that

$$\chi_\Delta(x) \to \frac{1}{p(z)}\mathrm{1\!I}_{[y(z),\infty)}(x) := \chi(x) \qquad \text{as } \Delta \downarrow 0,$$

because $\frac{y(z+\Delta)) - y(z)}{\Delta} \sim y'(z) = \frac{1}{p(z)}$. Then (6.11) can be written as follows: for every $y, v \in (a,b)$

$$\frac{1}{2}(V'_\Delta(v) - V'_\Delta(y)) - \int_y^v \frac{p^2(y^{(-1)}(h))}{\sigma^2(y^{(-1)}(h))}(\lambda g(y^{(-1)}(h)) + f(y^{(-1)}(h)))V_\Delta(h)\,dh$$

$$= -\lambda \int_y^v \frac{p^2(y^{(-1)}(h))}{\sigma^2(y^{(-1)}(h))} g(y^{(-1)}(h))d\chi_\Delta(h). \quad (6.13)$$

The passage to the limit as $\Delta \downarrow 0$ in a problem analogous to (6.13) (6.12) was carried out in the proof of Theorem 6.3 Ch. III. It was proved that the functions $V_\Delta(v)$, $v \in [y(a), y(b)]$, converge uniformly in the interval $[y(a), y(b)]$ to a continuous function $V(v)$ that satisfies the equation

$$\frac{1}{2}(V'(v) - V'(y)) - \int_y^v \frac{p^2(y^{(-1)}(h))}{\sigma^2(y^{(-1)}(h))}(\lambda g(y^{(-1)}(h)) + f(y^{(-1)}(h)))V(h)\,dh$$

$$= -\lambda \int_y^v \frac{p^2(y^{(-1)}(h))}{\sigma^2(y^{(-1)}(h))} g(y^{(-1)}(h))\,d\chi(h).$$

This equation is equivalent to the problem

$$\frac{1}{2}V''(v) - \frac{p^2(y^{(-1)}(v))}{\sigma^2(y^{(-1)}(v))}(\lambda g(y^{(-1)}(v)) + f(y^{(-1)}(v)))V(v) = 0, \quad v \in (y(a), y(b)) \setminus \{y(z)\},$$

$$V'(y(z)+) - V'(y(z)-) = -\frac{2\lambda p(z)}{\sigma^2(z)} g(z).$$

By the uniform convergence, the boundary conditions (6.12) are transformed to the conditions $V(y(a)) = 0$, $V(y(b)) = 0$.

Returning in this problem to the initial variable, i.e., to the function $U(x) = V(y(x))$, we see first that the existence of the limit

$$U(x) = \lim_{\Delta \downarrow 0} V_\Delta(y(x)) = \lim_{\Delta \downarrow 0} U_\Delta(x) \tag{6.14}$$

means the existence of the right derivative $G_{\nu,z}(x) := U(x)$, and second that for every $z \in (a, b)$ the function $G_{\nu,z}(x)$, $x \in (a, b)$, is the unique continuous solution of the problem (6.5)–(6.7). □

We formulate a very important consequence of Theorem 6.1 for $g(x) \equiv 1$.

Theorem 6.2. *Let $f(x)$, $x \in [a, b]$, be a nonnegative piecewise-continuous function. Then for every $z \in (a, b)$ there exists the derivative*

$$G_z(x) := \frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\int_0^\tau f(X(s))\,ds\Big); a \le \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) \le b, X(\tau) < z\Big\} \tag{6.15}$$

and $G_z(x)$, $x \in [a, b]$, is the unique continuous solution of the problem

$$\frac{1}{2}\sigma^2(x)G''(x) + \mu(x)G'(x) - (\lambda + f(x))G(x) = 0, \quad x \in (a, b) \setminus \{z\}, \tag{6.16}$$

$$G'(z + 0) - G'(z - 0) = -2\lambda/\sigma^2(z), \tag{6.17}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{6.18}$$

Remark 6.3. We can see from the proof of Theorem 6.1 adapted to the case $g(x) \equiv 1$ that the derivative (6.15) exists. In this case, the parameter Δ can be negative, because the influence of piecewise-continuity of g is eliminated.

Proposition 6.1. *The solution of the problem* (6.16)–(6.18) *has the form*

$$G_z(x) = \begin{cases} \dfrac{2\lambda}{w(z)\sigma^2(z)}\varphi(z)\psi(x), & a \le x \le z, \\ \dfrac{2\lambda}{w(z)\sigma^2(z)}\psi(z)\varphi(x), & z \le x \le b, \end{cases} \tag{6.19}$$

where $\psi(x)$ *and* $\varphi(x)$ *are solutions of equation* (6.16) *for* $x \in (a,b)$, *with* $\psi(x)$ *increasing,* $\varphi(x)$ *decreasing and* $\psi(a) = 0$, $\varphi(b) = 0$. *The function* $w(z) = \psi'(z)\varphi(z) - \psi(z)\varphi'(z) > 0$ *is the Wronskian of these solutions.*

Proof. Indeed, $G_z(x)$, $x \in [a,b]$, is a continuous solution, satisfying for $x \ne z$ equation (6.16) and the boundary conditions (6.18). At the point z the derivative has the jump

$$G_z'(z+0) - G_z'(z-0) = \frac{2\lambda}{w(z)\sigma^2(z)}(\psi(z)\varphi'(z) - \psi'(z)\varphi(z)) = -\frac{2\lambda}{\sigma^2(z)}.$$

□

§ 7. Distributions of integral functionals of a diffusion at the first exit time

Let X be a diffusion satisfying the conditions of § 4, $X(0) = x$. The *first exit time* $H_{a,b} := \min\{s : X(s) \notin (a,b)\}$ is very important in various applications. If the initial value $x \notin (a,b)$, we set $H_{a,b} := 0$. According to Lemma 12.1 and Remark 12.3 Ch. II, $\mathbf{E}_x H_{a,b} < \infty$ and $\mathbf{P}_x(H_{a,b} < \infty) = 1$.

We consider the problem of computing distributions of integral functionals of the diffusion X stopped at the moment $H_{a,b}$. As it was explained for a Brownian motion in §5 Ch. III, the following result is very important for computing of such distributions.

Theorem 7.1. *Let* $f(x)$, $F(x)$, $x \in [a,b]$, *be piecewise-continuous functions and* f *be nonnegative. Let* Φ *be defined only at the points* a *and* b. *Then the function*

$$Q(x) := \mathbf{E}_x\biggl\{\Phi(X(H_{a,b}))\exp\biggl(-\int_0^{H_{a,b}} f(X(s))\,ds\biggr)$$

$$+\int_0^{H_{a,b}} F(X(s))\exp\biggl(-\int_0^s f(X(v))\,dv\biggr)ds\biggr\}, \qquad x \in [a,b], \tag{7.1}$$

is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) - f(x)Q(x) = -F(x), \tag{7.2}$$

$$Q(a) = \Phi(a), \qquad Q(b) = \Phi(b). \tag{7.3}$$

Proof. For continuous functions f and F this result is exactly Theorem 12.6 Ch. II. The nonnegative piecewise-continuous function f can be approximated from below by a sequence of continuous functions $\{f_n\}$, $n = 1, 2, \dots$, such that $0 \le f_n(x) \le f(x)$, $x \in \mathbf{R}$. On the finite interval $[a, b]$ the piecewise-continuous function F can be approximated by a sequence of uniformly bounded continuous functions $\{F_n\}$.

Applying the limit approximation method described in the proof of Theorem 4.1, we can prove that Q is the solution of (7.2), (7.3) for the piecewise-continuous functions f and F. □

We now consider some particular cases of Theorems 7.1 with $F(x) \equiv 0$. For the problem of distribution of functionals of diffusions at the time $H_{a,b}$ the exit across the upper or lower boundary has an important meaning. Thus we must consider the Laplace transform of the distribution of a functional reduced to the set $W(H_{a,b}) = b$ or to the set $W(H_{a,b}) = a$.

The following result actually concerns the Laplace transform of the distribution of a nonnegative integral functional of the diffusion, stopped at the first exit time from an interval across the upper boundary b. To find the distribution it is necessary to apply this result for the product $\gamma f(x)$, $\gamma > 0$, instead of $f(x)$ and then invert the Laplace transform with respect to γ. From Theorem 7.1 with $\varPhi(y) = \mathbb{1}_b(y)$, $F \equiv 0$ we deduce the following assertion.

Theorem 7.2. *Let $f(x)$, $x \in [a, b]$, be a nonnegative piecewise-continuous function. Then the function*

$$Q_b(x) := \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(X(s))\,ds\Big); X(H_{a,b}) = b\Big\}, \qquad x \in [a, b], \tag{7.4}$$

is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) - f(x)Q(x) = 0, \quad x \in (a, b), \tag{7.5}$$

$$Q(a) = 0, \qquad Q(b) = 1, \tag{7.6}$$

Remark 7.1. If we consider the function Q_a (with the restriction $X(H_{a,b}) = a$), i.e., in (7.1) we set $\varPhi(y) := \mathbb{1}_a(y)$, then the function $Q_a(x)$, $x \in [a, b]$, satisfies (7.5) and the boundary conditions

$$Q(a) = 1, \qquad Q(b) = 0. \tag{7.7}$$

From Theorem 7.1 with $\varPhi \equiv 1$, $F \equiv 0$ we deduce the following result.

Theorem 7.3. *Let $f(x)$, $x \in [a, b]$, be a nonnegative piecewise-continuous function. Then the function*

$$Q(x) := \mathbf{E}_x \exp\Big(-\int_0^{H_{a,b}} f(X(s))\,ds\Big), \qquad x \in [a, b],$$

is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) - f(x)Q(x) = 0, \quad x \in (a,b), \tag{7.8}$$

$$Q(a) = 1, \qquad Q(b) = 1. \tag{7.9}$$

Remark 7.2 The function Q is the sum of functions Q_a and Q_b.

Let $H_z := \min\{s : X(s) = z\}$ be the *first hitting time* of the level z by the diffusion X. Using the limit approximation method described in the proof of Theorem 4.1, we can extend Theorem 12.7 Ch. II to piecewise-continuous functions f.

Theorem 7.4. *Let $f(x)$, $x \in \mathbf{R}$, be a nonnegative piecewise-continuous function. Then the function*

$$L_z(x) := \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_z} f(X(s))\,ds\Big); H_z < \infty\Big\}, \qquad x \in \mathbf{R},$$

can be represented in the form

$$L_z(x) = \begin{cases} \dfrac{\psi(x)}{\psi(z)}, & x \le z, \\ \dfrac{\varphi(x)}{\varphi(z)}, & z \le x, \end{cases} \tag{7.10}$$

where φ is a positive decreasing solution and ψ is a positive increasing solution of the equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - f(x)\phi(x) = 0, \qquad x \in \mathbf{R}. \tag{7.11}$$

Example 7.1. We compute the Laplace transform of the distribution of the first hitting time H_z for the Ornstein–Uhlenbeck process $X(t)$, $t \ge 0$, which is the solution of the stochastic differential equation (4.14). By Theorem 7.4,

$$\mathbf{E}_x e^{-\alpha H_z} = \mathbf{E}_x\big\{e^{-\alpha H_z}; H_z < \infty\big\} = \begin{cases} \dfrac{\psi(x)}{\psi(z)}, & x \le z, \\ \dfrac{\varphi(x)}{\varphi(z)}, & z \le x, \end{cases} \qquad \alpha > 0, \tag{7.12}$$

where φ is a positive decreasing solution and ψ is a positive increasing solution of equation (4.17) with $\lambda = \alpha$. These solutions are of the form (4.18). Therefore,

$$\mathbf{E}_x e^{-\alpha H_z} = \begin{cases} \dfrac{e^{x^2\theta/2\sigma^2} D_{-\alpha/\theta}\big(-x\sqrt{2\theta}/\sigma\big)}{e^{b^2\theta/2\sigma^2} D_{-\alpha/\theta}\big(-b\sqrt{2\theta}/\sigma\big)}, & x \le z, \\ \dfrac{e^{x^2\theta/2\sigma^2} D_{-\alpha/\theta}\big(x\sqrt{2\theta}/\sigma\big)}{e^{a^2\theta/2\sigma^2} D_{-\alpha/\theta}\big(a\sqrt{2\theta}/\sigma\big)}, & z \le x. \end{cases} \tag{7.13}$$

It is clear that $\mathbf{P}_x(H_z < \infty) = \lim_{\alpha\downarrow 0} \mathbf{E}_x e^{-\alpha H_z}$. Since $\lim_{\beta\downarrow 0} D_{-\beta}(x) = e^{-x^2/4}$ (see the definition of the function $D_{-\beta}(x)$ in Appendix 2), the moment of the first hitting time H_z for the Ornstein–Uhlenbeck process is finite with probability one.

§ 8. Distributions of functionals of a diffusion connected with location of maximum or minimum

In this section we consider the problem of computing the distributions of integral functionals of a diffusion at the moments in which the maximum or minimum values of the diffusion are reached.

We consider a class of homogeneous diffusions $X(t)$, $t \ge 0$, that are the solutions of the stochastic differential equation (4.1). Assume that the conditions on the coefficients μ and σ given in § 4 are satisfied.

Let $H_b := \min\{s : X(s) = b\}$ be the first hitting time of the level b.

For a nonnegative piecewise-continuous function g we set

$$I(s) := \int_0^s g(X(v))\, dv \tag{8.1}$$

and assume that conditions (5.3) hold.

The variable $\nu(t) := \min\bigl\{s : I(s) = t\bigr\}$ is the moment inverse of the integral functional of the diffusion X. By (5.3), the moment $\nu(t)$ is finite a.s. for all $t \ge 0$.

By the strong Markov property of the process X, we have that given the condition $X(H_b) = b$, the process $\widetilde{X}(s) := X(s + H_b)$, $s \ge 0$, is distributed as the process $X(s)$, $s \ge 0$, with the initial value $X(0) = b$. It is clear that if the event $\{H_b < \nu(t)\}$ is realized, then the following equalities hold:

$$\nu(t) = \min\Biggl\{s \ge H_b : I(H_b) + \int_{H_b}^s g(X(v))\, dv = t\Biggr\}$$

$$= H_b + \min\Biggl\{s : \int_0^s g(\widetilde{X}(v))\, dv = t - I(H_b)\Biggr\} = H_b + \widetilde{\nu}(t - I(H_b)), \tag{8.2}$$

where $\widetilde{\nu}(t)$ is defined as $\nu(t)$ but for the process $\widetilde{X}$.

Let $\wp(X(s), u \le s \le v)$, $0 \le u < v < \infty$, be a bounded measurable functional of the process $X(s)$, $s \in [u, v]$.

If for all $t \ge 0$ and $h > 0$ we have

$$\wp(X(s), t \le s \le t + h) = \wp(X(s + t), 0 \le s \le h),$$

then the functional $\wp(X(s), u \le s \le t)$ is called *homogeneous*. Strictly speaking, the definition is given for a whole family of functionals depending on two parameters that characterize the initial and terminal times of the sample paths.

We assume that $\wp(X(s), 0 \le s \le t)$, $t \ge 0$, is a.s. a piecewise-continuous process with respect to t. Furthermore, we assume that at points of discontinuity it is right continuous.

Let $\mathcal{G}_0^t := \sigma(X(s), 0 \le s \le t)$ be the σ-algebra of events generated by the process X up to the time t, i.e., its natural filtration. The process $\wp(X(s), 0 \le s \le t)$, $t \ge 0$, is adapted to this filtration. Since H_b is a stopping time with respect to the natural filtration, we have, according to Corollary 4.1 of Ch. I, that the random variable $\wp(X(s), 0 \le s \le H_b)$ is $\mathcal{G}_0^{H_b}$-measurable.

Let τ be an exponentially distributed with the parameter $\lambda > 0$ random time independent of the diffusion X.

The next result is, in fact, an expression of the strong Markov property of the diffusion X applied to the stopping time H_b.

Lemma 8.1. *For a homogeneous bounded a.s. piecewise-continuous functionals* $\wp_l$, $l = 1, 2$,

$$\mathbf{E}_x\big\{\wp_1(X(u), 0 \le u \le H_b)\,\wp_2(X(u), H_b \le u \le \nu(\tau)); H_b < \nu(\tau)\big\}$$

$$= \mathbf{E}_x\big\{e^{-\lambda I(H_b)}\wp_1(X(u), 0 \le u \le H_b); H_b < \infty\big\}\mathbf{E}_b\wp_2(X(u), 0 \le u \le \nu(\tau)). \quad (8.3)$$

Proof. Using the independence of τ and the diffusion, and applying the equality

$$\mathbb{I}_{[I(H_b)<t)} = \mathbb{I}_{[H_b<\nu(t))},$$

we transform the left-hand side of (8.3) as follows:

$$\lambda\int_0^\infty e^{-\lambda t}\mathbf{E}_x\Big\{\mathbb{I}_{[0,t)}(I(H_b))\wp_1(X(u), 0 \le u \le H_b)\wp_2(X(u), H_b \le u \le \nu(t))\Big\}dt$$

$$= \mathbf{E}_x\Big\{\wp_1(X(u), 0 \le u \le H_b)\mathbb{I}_{\{H_b<\infty\}}\lambda\int_{I(H_b)}^\infty e^{-\lambda t}\wp_2(X(u), H_b \le u \le \nu(t))\,dt\Big\}$$

$$= \mathbf{E}_x\Big\{e^{-\lambda I(H_b)}\wp_1(X(u), 0 \le u \le H_b)\mathbb{I}_{\{H_b<\infty\}}\lambda\int_0^\infty e^{-\lambda v}\wp_2(\widetilde{X}(s), 0 \le s \le \widetilde{\nu}(v))\,dv\Big\}$$

$$= \mathbf{E}_x\Big\{e^{-\lambda I(H_b)}\wp_1(X(u), 0 \le u \le H_b)\mathbb{I}_{\{H_b<\infty\}}$$

$$\times\mathbf{E}_x\Big\{\lambda\int_0^\infty e^{-\lambda v}\wp_2(\widetilde{X}(s), 0 \le s \le \widetilde{\nu}(v))\,dv\Big|\mathcal{G}_0^{H_b}\Big\}\Big\}.$$

By the strong Markov property of a diffusion (see (3.26)), the right-hand side of this equation equals

$$\mathbf{E}_x\Big\{e^{-\lambda I(H_b)}\wp_1(X(u), 0 \le u \le H_b); H_b < \infty\Big\}\mathbf{E}_b\Big\{\lambda\int_0^\infty e^{-\lambda t}\wp_2(X(u), 0 \le u \le \nu(t))\,dt\Big\}$$

$$= \mathbf{E}_x\big\{e^{-\lambda I(H_b)}\wp_1(X(u), 0 \le u \le H_b); H_b < \infty\big\}\mathbf{E}_b\wp_2(X(u), 0 \le u \le \nu(\tau)).$$

The lemma is proved. $\square$

Let $\psi(x)$, $x \in \mathbf{R}$, be a strictly increasing positive solution, and $\varphi(x)$, $x \in \mathbf{R}$, be a strictly decreasing positive solution, of the equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - \lambda g(x)\phi(x) = 0. \tag{8.4}$$

Let $\omega(z) := \psi'(z)\varphi(z) - \psi(z)\varphi'(z) > 0$ be their Wronskian. Set $\rho(x,y) := \psi(x)\varphi(y) - \psi(y)\varphi(x)$ for $y < x$. The function $\rho(x,y)$ strictly increases in x for a fixed y and strictly decreases in y for a fixed x.

We derive some auxiliary results.

Let $H_{a,b} := \min\{s : X(s) \notin (a,b)\}$ be the first exit time from the interval (a,b).

Lemma 8.2. *The following relation holds*:

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mathbf{E}_a\big\{e^{-\lambda I(H_{a-\delta,b})};\, X(H_{a-\delta,b}) = b\big\} = \frac{\omega(a)}{\rho(b,a)}. \tag{8.5}$$

Remark 8.1. Since $-X(s)$, $s \geq 0$, is also a diffusion, there is the analogous relation

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mathbf{E}_b\big\{e^{-\lambda I(H_{a,b+\delta})};\, X(H_{a,b+\delta}) = a\big\} = \frac{\omega(b)}{\rho(b,a)}. \tag{8.6}$$

Proof of Lemma 8.2. To compute the function

$$Q_b(x) := \mathbf{E}_x\big\{e^{-\lambda I(H_{a-\delta,b})};\, X(H_{a-\delta,b}) = b\big\},$$

we apply Theorem 7.2. Obviously, the problem (7.5), (7.6) with $f(x) = \lambda g(x)$ and $a-\delta$ instead of a has the solution

$$Q_b(x) = \frac{\rho(x, a-\delta)}{\rho(b, a-\delta)}, \qquad x \in (a-\delta, b).$$

Therefore,

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, Q_b(a) = \lim_{\delta\downarrow 0} \frac{1}{\delta}\, \frac{\psi(a)\varphi(a-\delta) - \psi(a-\delta)\varphi(a)}{\psi(b)\varphi(a-\delta) - \psi(a-\delta)\varphi(b)} = \frac{\omega(a)}{\rho(b,a)}.$$

□

Lemma 8.3. *The equality*

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mathbf{P}_b\Big(\sup_{0\le s\le \nu(\tau)} X(s) \le b+\delta\Big) = \frac{2\lambda\omega(b)}{\psi(b)} \int_{-\infty}^{b} \frac{g(z)\psi(z)}{\sigma^2(z)\omega(z)}\, dz \tag{8.7}$$

holds, and for some constant K_b one has the estimate

$$\sup_{x\le b+\delta} \mathbf{P}_x\Big(\sup_{0\le s\le \nu(\tau)} X(s) \in [b, b+\delta)\Big) \le \frac{2\lambda\rho(b+\delta, b)}{\psi(b)} \int_{-\infty}^{b+\delta} \frac{g(z)\psi(z)}{\sigma^2(z)\omega(z)}\, dz \le \delta K_b. \tag{8.8}$$

Proof. Set

$$U_{\nu,b}(x) := \mathbf{P}_x\Big(\sup_{0\le s\le\nu(\tau)} X(s) \le b\Big).$$

To compute this probability we apply (6.8) with $a = -\infty$ and Theorem 6.1 with $f(x) \equiv 0$, $a = -\infty$, taking into account Remark 6.1.

It is not hard to verify that the solution of (6.5)–(6.7) with $f(x) \equiv 0$ $a = -\infty$, has the form

$$G_{b,z}(x) = \begin{cases} \frac{2\lambda g(z)}{\sigma^2(z)} \frac{\rho(b,z)\psi(x)}{\omega(z)\psi(b)}, & x \le z, \\ \frac{2\lambda g(z)}{\sigma^2(z)} \frac{\rho(b,x)\psi(z)}{\omega(z)\psi(b)}, & z \le x \le b. \end{cases} \tag{8.9}$$

Then

$$U_{\nu,b}(x) = \int_{-\infty}^{b} G_{b,z}(x)\,dz. \tag{8.10}$$

For $z \le b$,

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\,G_{b+\delta,z}(b) = \frac{2\lambda g(z)\psi(z)}{\sigma^2(z)\omega(z)}\frac{\omega(b)}{\psi(b)},$$

while for $b \le z \le b+\delta$ we have

$$G_{b+\delta,z}(b) \le L_b\delta$$

for some constant L_b. Now from (8.10) it follows that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\,U_{\nu,b+\delta}(b) = \frac{2\lambda\omega(b)}{\psi(b)}\int_{-\infty}^{b}\frac{g(z)\psi(z)}{\sigma^2(z)\omega(z)}\,dz. \tag{8.11}$$

This proves (8.7).

We now prove (8.8). Using (8.10), we get

$$\mathbf{P}_x\Big(\sup_{0\le s\le\nu(\tau)} X(s) \in [b, b+\delta)\Big) = U_{\nu,b+\delta}(x) - U_{\nu,b}(x)$$

$$= \int_{b}^{b+\delta} G_{b+\delta,z}(x)\,dz + \int_{-\infty}^{b}(G_{b+\delta,z}(x) - G_{b,z}(x))\,dz. \tag{8.12}$$

For $z \le b$, $x \le b$

$$G_{b+\delta,z}(x) - G_{b,z}(x) = \frac{2\lambda g(z)\psi(z)\psi(x)}{\sigma^2(z)\omega(z)}\Big(\frac{\varphi(b)}{\psi(b)} - \frac{\varphi(b+\delta)}{\psi(b+\delta)}\Big),$$

$$\sup_{x\le b}\big(G_{b+\delta,z}(x) - G_{b,z}(x)\big) = \frac{2\lambda g(z)\psi(z)}{\sigma^2(z)\omega(z)}\frac{\rho(b+\delta,b)}{\psi(b+\delta)}. \tag{8.13}$$

In addition,

$$\sup_{x\le b+\delta} G_{b+\delta,z}(x) = G_{b+\delta,z}(z) = \frac{2\lambda g(z)\psi(z)}{\sigma^2(z)}\frac{\rho(b+\delta,z)}{\omega(z)\psi(b)} \tag{8.14}$$

for $z \in (b, b+\delta)$, and

$$\sup_{x\in(b,b+\delta)} G_{b+\delta,z}(x) = \frac{2\lambda g(z)}{\sigma^2(z)} \frac{\rho(b+\delta,b)\psi(z)}{\omega(z)\psi(b)} \tag{8.15}$$

for $z \le b$. From (8.13)–(8.15) it follows that

$$\sup_{x\le b+\delta} \Big(G_{b+\delta,z}(x) - G_{b,z}(x)\Big) \le \frac{2\lambda g(z)\psi(z)}{\sigma^2(z)\omega(z)} \frac{\rho(b+\delta,b)}{\psi(b)}$$

for $z \le b+\delta$. Substituting this estimate in (8.12), we obtain

$$\sup_{x\le b+\delta} \Big(U_{\nu,b+\delta}(x) - U_{\nu,b}(x)\Big) \le \frac{2\lambda\rho(b+\delta,b)}{\psi(b)} \int_{-\infty}^{b+\delta} \frac{g(z)\psi(z)}{\sigma^2(z)\omega(z)} dz.$$

This is the estimate (8.8). □

We will need one more auxiliary estimate. For brevity, we set $\nu := \nu(\tau)$.

Lemma 8.4. *The estimate*

$$\mathbf{P}_a\Big(a-\delta_1 < \inf_{0\le s\le\nu} X(s), \sup_{0\le s\le\nu} X(s) \in [b, b+\delta_2)\Big) \le \delta_1\delta_2 K_{a,b} \tag{8.16}$$

holds, where $K_{a,b}$ is some constant.

Proof. We use the equality analogous to (8.3):

$$\mathbf{P}_a\Big(a-\delta_1 < \inf_{0\le s\le\nu} X(s), \sup_{0\le s\le\nu} X(s) \in [b, b+\delta_2)\Big)$$

$$= \mathbf{E}_a\big\{e^{-\lambda I(H_{a-\delta_1,b})}; X(H_{a-\delta_1,b}) = b\big\}\mathbf{P}_b\Big(a-\delta_1 < \inf_{0\le s\le\nu} X(s), \sup_{0\le s\le\nu} X(s) \le b+\delta_2\Big).$$

The right-hand side of this equality is estimated by the value

$$\mathbf{E}_a\big\{e^{-\lambda I(H_{a-\delta_1,b})}; X(H_{a-\delta_1,b}) = b\big\}\mathbf{P}_b\Big(\sup_{0\le s\le\nu} X(s) \le b+\delta_2\Big),$$

which, by (8.5) and (8.7), does not exceed $\delta_1\delta_2 K_{a,b}$. □

1. Functionals of diffusion at locations of maximum or minimum. The moment

$$\check{H}(t) := \inf\Big\{u \le t : X(u) = \sup_{0\le s\le t} X(s)\Big\}$$

is the *location of the maximum* of the process $X(s)$, $0 \le s \le t$, and the moment

$$\hat{H}(t) := \inf\Big\{u \le t : X(u) = \inf_{0\le s\le t} X(s)\Big\}$$

is the *location of the minimum* of the process $X(s)$, $0 \le s \le t$.

Consider the event $\{H_b < \nu(t)\}$. Then on this set the moments of location of the maximum or minimum have a property analogous to (8.2): if $X(0) < b$, then

$$\check{H}(\nu(t)) = H_b + \check{H}_\circ(\widetilde{\nu}(t - I(H_b))), \tag{8.17}$$

and if $X(0) > b$, then

$$\hat{H}(\nu(t)) = H_b + \hat{H}_\circ(\widetilde{\nu}(t - I(H_b))),$$

where $\check{H}_\circ(t)$ and $\hat{H}_\circ(t)$ are defined similarly to $\check{H}(t)$ and $\hat{H}(t)$ by the process $\widetilde{X}$.

The next result enables us to reduce the problem of computing the joint distributions of functionals of the diffusion X, considered on the intervals $[0, \check{H}(\nu)]$ and $[\check{H}(\nu), \nu]$, to the two previously solved problems, the first of which concerns the distribution of the functional up to the first hitting time of a level and the second one concerns the distribution of the functional up to the moment inverse of the integral functional.

Consider the nonnegative integral functionals of the diffusion X:

$$I_l(s,t) = \int_s^t f_l(X(u))\, dv, \qquad l = 1, 2, 3.$$

It is assumed that f_l, $l = 1, 2, 3$, are piecewise-continuous nonnegative functions.

Theorem 8.1. *Let $\Phi(x)$, $x \le b$, be a piecewise-continuous bounded function. Then*

$$\frac{d}{db}\mathbf{E}_x\Big\{\Phi(X(\nu))\, e^{-I_1(0,\check{H}(\nu))} e^{-I_2(\check{H}(\nu),\nu)};\ \sup_{0\le s\le\nu} X(s) < b\Big\}$$

$$= \mathbf{E}_x\big\{e^{-\lambda I(H_b) - I_1(0,H_b)}; H_b < \infty\big\}$$

$$\times \lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mathbf{E}_b\Big\{\Phi(X(\nu))\, e^{-I_2(0,\nu)};\ \sup_{0\le s\le\nu} X(s) < b + \delta\Big\}, \quad x \le b. \tag{8.18}$$

Remark 8.2. Let $\psi(x)$, $x \in \mathbf{R}$, be an increasing positive solution and $\varphi(x)$, $x \in \mathbf{R}$, be a decreasing positive solution, of the equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - (\lambda g(x) + f_2(x))\phi(x) = 0. \tag{8.19}$$

Let $\omega(z) := \psi'(z)\varphi(z) - \psi(z)\varphi'(z) > 0$ be the Wronskian of these solutions.

Then the limit on the right-hand side of (8.18) exists and has the form

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mathbf{E}_b\Big\{\Phi(X(\nu))\, e^{-I_2(0,\nu)};\ \sup_{0\le s\le\nu} X(s) < b + \delta\Big\} = \frac{2\lambda\omega(b)}{\psi(b)} \int_{-\infty}^b \Phi(z) \frac{g(z)\psi(z)}{\sigma^2(z)\omega(z)}\, dz.$$

Indeed, we set

$$U_{\nu,b}(x) := \mathbf{E}_x\Big\{\varPhi(X(\nu))\exp\Big(-\int_0^\nu f_2(X(s))\,ds\Big);\ \sup_{0\le s\le\nu} X(s) < b\Big\}.$$

Then, applying Theorem 6.1 and (6.8), we can express the function $U_{\nu,b}$ in terms of the Green function $G_{b,z}(x)$ by the formula $U_{\nu,b}(x) = \int_{-\infty}^{b} \varPhi(z)\, G_{b,z}(x)\,dz$. Now, analogously to (8.11), we compute $\lim_{\delta\downarrow 0}\frac{1}{\delta}\, U_{\nu,b+\delta}(b)$, and see that the required statement is valid.

Proof of Theorem 8.1. Here we consider only the right derivative on the left-hand side of (8.18), because the right derivative is a continuous function with respect to b and the total derivative will also exist. The continuity is a consequence of (8.18) and the fact that each factor in the right-hand side of (8.18) is continuous function with respect to b.

The meaning of the arguments given below is to verify that, being restricted to the event $\Big\{\sup_{0\le s\le\nu} X(s) \in [b, b+\delta)\Big\}$, the moments H_b and $\check H(\nu)$ coincide in the limiting case as $\delta\downarrow 0$. We can write

$$\mathbf{E}_x\Big\{\varPhi(X(\nu))e^{-I_1(0,\check H(\nu))}e^{-I_2(\check H(\nu),\nu)};\ \sup_{0\le s\le\nu} X(s) \in [b, b+\delta)\Big\}$$

$$= \mathbf{E}_x\Big\{e^{-I_1(0,H_b)}\,\varPhi(X(\nu))\,e^{-I_1(H_b,\check H(\nu))}$$

$$\times\, e^{-I_2(\check H(\nu),\nu)};\ \sup_{H_b\le s\le\nu} X(s) < b+\delta, H_b < \nu\Big\} =: \Delta_\delta.$$

By (8.2) and (8.17), on the event $\{H_b < \nu(t)\}$ the functional

$$\wp(X(u), H_b \le u \le \nu(t))$$

$$:= \varPhi(X(\nu(t)))\,e^{-I_1(H_b,\check H(\nu(t)))}e^{-I_2(\check H(\nu(t)),\nu(t))}\,\mathbb{1}_{\Big\{\sup\limits_{H_b\le s\le\nu(t)} X(s)<b+\delta\Big\}}$$

satisfies the equality

$$\wp(X(u), H_b \le u \le \nu(t)) = \wp(\widetilde X(s), 0\le s\le \widetilde\nu(t - I(H_b))).$$

Therefore, applying (8.3), we get

$$\Delta_\delta = \mathbf{E}_x\big\{e^{-\lambda I(H_b) - I_1(0,H_b)};\ H_b<\infty\big\}$$

$$\mathbf{E}_b\Big\{\varPhi(X(\nu))\,e^{-I_1(0,\check H(\nu))}e^{-I_2(\check H(\nu),\nu)};\ \sup_{0\le s\le\nu} X(s) < b+\delta\Big\}.$$

In order to prove (8.18) it suffices to verify that for $l=1,2$,

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_b\Big\{\varPhi(X(\nu))\big(1-e^{-I_l(0,\check H(\nu))}\big);\ \sup_{0\le s\le\nu}X(s)<b+\delta\Big\}=0. \tag{8.20}$$

Set

$$\wp_l:=1-e^{-I_l(0,\check H(\nu))},\qquad l=1,2.$$

We choose a sufficiently large number $\tilde\lambda>0$. Let $\tilde\tau$ be independent of X and τ the random variable exponentially distributed with the parameter $\tilde\lambda>0$. This mean that the value $1/\tilde\lambda$ is small. Then for $l=1,2$,

$$\mathbf{E}_b\Big\{\wp_l;\ \sup_{0\le s\le\nu}X(s)<b+\delta\Big\}=\mathbf{E}_b\Big\{\wp_l;\ \sup_{0\le s\le\nu}X(s)<b+\delta,\nu<\tilde\tau\Big\}$$
$$+\mathbf{E}_b\Big\{\wp_l;\ \sup_{0\le s\le\tilde\tau}X(s)<b+\delta,\ \sup_{\tilde\tau\le s\le\nu}X(s)<b,\nu\ge\tilde\tau\Big\}$$
$$+\mathbf{E}_b\Big\{\wp_l;\ \sup_{0\le s\le\tilde\tau}X(s)<b+\delta,\ \sup_{\tilde\tau\le s\le\nu}X(s)\in[b,b+\delta),\nu\ge\tilde\tau\Big\}$$
$$=:\Lambda_{1,l}+\Lambda_{2,l}+\Lambda_{3,l}. \tag{8.21}$$

For piecewise-continuous functions f_l, it holds that $\sup_{a\le x\le b}|f_l(x)|\le L_{a,b}$, where $L_{a,b}$ is some constant dependent on a and b. Then for $k=1,2$, and arbitrary a,

$$\Lambda_{k,l}\le\mathbf{E}_b\Big\{\wp_l;\ \sup_{0\le s\le\nu}X(s)\le b+\delta,\check H(\nu)\le\tilde\tau\Big\}$$
$$\le L_{a,b}\mathbf{E}\tilde\tau\,\mathbf{P}_b\Big(\sup_{0\le s\le\nu}X(s)\le b+\delta\Big)+\mathbf{P}_b\big(X(H_{a,b+\delta})=a,H_{a,b+\delta}<\tilde\tau\big)$$
$$\le\frac{L_{a,b}}{\tilde\lambda}\mathbf{P}_b\Big(\sup_{0\le s\le\nu}X(s)\le b+\delta\Big)+\mathbf{E}_b\Big\{e^{-\tilde\lambda H_{a,b+\delta}};X(H_{a,b+\delta})=a\Big\}$$
$$\le\delta C_b\Big(\frac{L_{a,b}}{\tilde\lambda}+\frac{\omega(b)}{\rho(b,a)}\Big). \tag{8.22}$$

Here we used (8.6) with $g(x)\equiv 1$ and (8.7).

By the Markov property, for the third term in (8.21) ($l=1,2$) we obtain

$$\Lambda_{3,l}\le\mathbf{P}_b\Big(\sup_{0\le s\le\tilde\tau}X(s)<b+\delta\Big)\sup_{y\le b+\delta}\mathbf{P}_y\Big(\sup_{0\le s\le\nu}X(s)\in[b,b+\delta)\Big)\le\delta^2C_b.$$

Here we used (8.7) with $\tilde\tau$ instead of $\nu(\tau)$ and (8.8).

Letting first $\tilde\lambda$ to ∞ and then $a\to-\infty$, we can make $\limsup_{\delta\downarrow 0}\frac{\Lambda_{k,l}}{\delta}$, $k,l=1,2$, smaller than any given positive number. Therefore, (8.20) holds, because $\lim_{\delta\downarrow 0}\frac{\Lambda_{3,l}}{\delta}=0$ and $\varPhi$ is bounded. It follows that $\lim_{\delta\downarrow 0}\frac{1}{\delta}\Delta_\delta$ coincides with the left-hand side of (8.18) and equals the right-hand side of (8.18). □

For the moment of the location of the minimum a similar assertion is valid.

Theorem 8.2. *Let $\Phi(x)$, $x \ge a$, be a piecewise-continuous bounded function. Then*

$$-\frac{d}{da}\mathbf{E}_x\Big\{\Phi(X(\nu))\,e^{-I_1(0,\hat H(\nu))}e^{-I_2(\hat H(\nu),\nu)};\ \inf_{0\le s\le\nu}X(s)>a\Big\}$$

$$=\mathbf{E}_x\big\{e^{-\lambda I(H_a)-I_1(0,H_a)};H_a<\infty\big\}$$

$$\times\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_a\Big\{\Phi(X(\nu))e^{-I_2(0,\nu)};\ \inf_{0\le s\le\nu}X(s)>a-\delta\Big\},\qquad x\ge a. \tag{8.23}$$

2. Relative position of locations of maximum and minimum of a diffusion. Suppose, for example, that the maximum of the diffusion X in the interval $[0,\nu]$ is reached before the minimum, i.e., we consider the event $\{\check H(\nu)<\hat H(\nu)\}$. Analogously to the above results, the following statement enables us to reduce the problem of the joint distribution of functionals of X considered on the intervals $[0,\check H(\nu)]$, $[\check H(\nu),\hat H(\nu))]$ and $[\hat H(\nu)),\nu]$ to three previously solved problems. The first one concerns the distribution of the functional up to the first exit time from an interval through the upper boundary, the second concerns the distribution of the functional up to the first exit time from an interval through the lower boundary, and the third concerns the distribution of the functional up to the moment inverse of integral functional.

Theorem 8.3. *Let $\Phi(x)$, $a\le x\le b$, be a piecewise-continuous function. Then*

$$-\frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathbf{E}_x\Big\{\Phi(X(\nu))\,e^{-I_1(0,\check H(\nu))}e^{-I_2(\check H(\nu),\hat H(\nu))}e^{-I_3(\hat H(\nu),\nu)};$$

$$\check H(\nu)<\hat H(\nu),a<\inf_{0\le s\le\nu}X(s),\ \sup_{0\le s\le\nu}X(s)<b\Big\}$$

$$=\mathbf{E}_x\Big\{e^{-\lambda I(H_{a,b})-I_1(0,H_{a,b})};X(H_{a,b})=b\Big\}$$

$$\times\lim_{\delta\downarrow 0}\frac{1}{\delta}\,\mathbf{E}_b\Big\{e^{-\lambda I(H_{a,b+\delta})-I_2(0,H_{a,b+\delta})};X(H_{a,b+\delta})=a\Big\}$$

$$\times\lim_{\delta\downarrow 0}\frac{1}{\delta}\,\mathbf{E}_a\Big\{\Phi(X(\nu))e^{-I_3(0,\nu)};a-\delta<\inf_{0\le s\le\nu}X(s),\ \sup_{0\le s\le\nu}X(s)<b\Big\} \tag{8.24}$$

for $a\le x\le b$.

Proof. We first prove that the limits on the right-hand side of (8.24) exist. For the first limit, it is sufficient to apply (8.6) in the case when the solutions of equation (8.19) are considered instead of the solutions of equation (8.4).

Consider the second limit. Set

$$U_{\nu,a,b}(x):=\mathbf{E}_x\Big\{\Phi(X(\nu))\,e^{-I_3(0,\nu)};a<\inf_{0\le s\le\nu}X(s),\ \sup_{0\le s\le\nu}X(s)<b\Big\}.$$

Then, by (6.8) and Theorem 6.1,

$$U_{\nu,a,b}(x)=\int_a^b\Phi(z)\,G_{a,b,z}(x)\,dz, \tag{8.25}$$

where the function $G_{a,b,z}$ is the solution of the problem (6.5)–(6.7) with $f = f_3$. In terms of the solutions ψ and φ of equation (8.19) with $f_2 = f_3$ we can write

$$G_{a,b,z}(x) = \begin{cases} \dfrac{2\lambda g(z)}{\sigma^2(z)} \dfrac{\rho(b,z)\rho(x,a)}{\omega(z)\rho(b,a)}, & a \le x \le z, \\ \dfrac{2\lambda g(z)}{\sigma^2(z)} \dfrac{\rho(b,x)\rho(z,a)}{\omega(z)\rho(b,a)}, & z \le x \le b. \end{cases} \tag{8.26}$$

For $a \le z$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, G_{a-\delta,b,z}(a) = \frac{2\lambda g(z)}{\sigma^2(z)} \frac{\rho(b,z)\omega(a)}{\omega(z)\rho(b,a)},$$

while for $a-\delta < z < a$

$$G_{a-\delta,b,z}(a) \le C\delta.$$

Then, by (8.25),

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, U_{\nu,a-\delta,b}(a) = \frac{2\lambda\omega(a)}{\rho(b,a)} \int_a^b \varPhi(z) \frac{g(z)\rho(b,z)}{\sigma^2(z)\omega(z)}\, dz.$$

This proves the existence of the second limit on the right-hand side of (8.24).

The scheme of the proof of (8.24) is similar to that of (8.18). Using a slightly modified version of (8.3), we find that

$$\begin{aligned} &\mathbf{E}_x\Bigl\{\varPhi(X(\nu))\, e^{-I_1(0,\check H(\nu))} e^{-I_2(\check H(\nu),\hat H(\nu))} e^{-I_2(\hat H(\nu),\nu)}; \\ &\qquad \check H(\nu) < \hat H(\nu), \inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) \in [b, b+\delta_2)\Bigr\} \\ &= \Bigl(\mathbf{E}_x\Bigl\{e^{-\lambda I(H_{a,b})-I_1(0,H_{a,b})}; X(H_{a,b}) = b\Bigr\} + O\bigl(\mathbf{P}_a(X(H_{a-\delta_1,b}) = b)\bigr)\Bigr) \\ &\qquad \times \mathbf{E}_b\Bigl\{\varPhi(X(\nu)) e^{-I_1(0,\check H(\nu))} e^{-I_2(\check H(\nu),\hat H(\nu))} e^{-I_3(\hat H(\nu),\nu)}; \\ &\qquad \check H(\nu) < \hat H(\nu), \inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Bigr\}. \end{aligned} \tag{8.27}$$

The probability $\mathbf{P}_a(X(H_{a-\delta_1,b}) = b)$ is estimated analogously to (8.5) by the value $K\delta_1$. The last expectation in (8.27) can be represented in the form

$$\begin{aligned} &\mathbf{E}_b\Bigl\{\varPhi(X(\nu))\, e^{-I_1(0,\check H(\nu))} e^{-I_2(\check H(\nu),\hat H(\nu))} e^{-I_3(\hat H(\nu),\nu)}; \check H(\nu) < \hat H(\nu), \\ &\inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Bigr\} =: \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4, \end{aligned}$$

where

$$\Delta_1 := \mathbf{E}_b\Bigl\{\varPhi(X(\nu))\Bigl(e^{-I_1(0,\check H(\nu))} - 1\Bigr) e^{-I_2(\check H(\nu),\hat H(\nu))} e^{-I_3(\hat H(\nu),\nu)};$$

$$\check{H}(\nu) < \hat{H}(\nu), \inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Big\},$$

$$\Delta_2 := \mathbf{E}_b\Big\{\varPhi(X(\nu))\Big(e^{-I_2(\check{H}(\nu),\hat{H}(\nu))} - e^{-I_2(0,\hat{H}(\nu))}\Big)e^{-I_3(\hat{H}(\nu),\nu)};$$

$$\check{H}(\nu) < \hat{H}(\nu), \inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Big\},$$

$$\Delta_3 := \mathbf{E}_b\Big\{\varPhi(X(\nu))\, e^{-I_2(0,\hat{H}(\nu))} e^{-I_3(\hat{H}(\nu),\nu)};$$

$$\check{H}(\nu) \ge \hat{H}(\nu), \inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Big\},$$

$$\Delta_4 := \mathbf{E}_b\Big\{\varPhi(X(\nu))\, e^{-I_2(0,\hat{H}(\nu))} e^{-I_3(\hat{H}(\nu),\nu)};$$

$$\inf_{0\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{0\le s\le\nu} X(s) < b+\delta_2\Big\}.$$

For $l=1,2$ we have the obvious estimates

$$|\Delta_l| \le L\mathbf{E}_b\Big\{\wp_l; X(H_{a,b+\delta_2}) = a, \inf_{H_a\le s\le\nu} X(s) \in (a-\delta_1, a], \sup_{H_a\le s\le\nu} X(s) < b+\delta_2\Big\}.$$

The variables $\wp_l$ are defined right below the formula (8.20). Analogously to (8.21), (8.22) we can prove that $\limsup\limits_{\delta_1\downarrow 0,\delta_2\downarrow 0} \dfrac{\Delta_l}{\delta_1\delta_2}$, $l=1,2$, is less then any given small number.

Applying (8.3) and taking into account (8.6) and (8.16), we have

$$\Delta_3 \le L\mathbf{P}_b\Big(X(H_{a,b+\delta_2}) = a, H_{a,b+\delta_2} \le \nu, \inf_{H_{a,b+\delta_2}\le s\le\nu} X(s) \in (a-\delta_1, a],$$

$$\sup_{H_{a,b+\delta_2}\le s\le\nu} X(s) \in [b, b+\delta_2)\Big) \le L\mathbf{E}_b\Big\{e^{-\lambda I(H_{a,b+\delta_2})}; X(H_{a,b+\delta_2}) = a\Big\}$$

$$\times\, \mathbf{P}_a\Big(a-\delta_1 < \inf_{0\le s\le\nu} X(s), \sup_{0\le s\le\nu} X(s) \in [b, b+\delta_2)\Big) \le L_{a,b}\delta_1\delta_2^2.$$

Again using the analog of (8.3), we obtain

$$\Delta_4 = \mathbf{E}_b\Big\{e^{-\lambda I(H_{a,b+\delta_2}) - I_2(0,H_{a,b+\delta_2})}; X(H_{a,b+\delta_2}) = a\Big\}$$

$$\times\, \mathbf{E}_a\Big\{\varPhi(X(\nu))\, e^{-I_2(0,\hat{H}(\nu))}\, e^{-I_3(\hat{H}(\nu),\nu)};$$

$$a-\delta_1 < \inf_{0\le s\le\nu} X(s), \sup_{0\le s\le\nu} X(s) \le b+\delta_2\Big\}.$$

Now, using (8.27) and the analog of (8.20) with the boundary b replaced by the boundary a, we see that (8.24) holds. □

The analog of Theorem 8.3 for the opposite order of the locations of the extreme values is the following assertion.

Theorem 8.4. *Let $\Phi(x)$, $a \le x \le b$, be a piecewise-continuous function. Then*

$$-\frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathbf{E}_x\Big\{\Phi(X(\nu))\, e^{-I_1(0,\hat H(\nu))}e^{-I_2(\hat H(\nu),\check H(\nu))}e^{-I_3(\check H(\nu),\nu)};$$

$$\hat H(\nu) < \check H(\nu), a < \inf_{0\le s\le \nu} X(s), \sup_{0\le s\le\nu} X(s) < b\Big\}$$

$$= \mathbf{E}_x\Big\{e^{-\lambda I(H_{a,b})-I_1(0,H_{a,b})}; X(H_{a,b}) = a\Big\}$$

$$\times \lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_a\Big\{e^{-\lambda I(H_{a-\delta,b})-I_2(0,H_{a-\delta,b})}; X(H_{a-\delta,b}) = b\Big\}$$

$$\times \lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_b\Big\{\Phi(X(\nu))e^{-I_3(0,\nu)}; a < \inf_{0\le s\le \nu} X(s), \sup_{0\le s\le\nu} X(s) < b+\delta\Big\}, \qquad (8.28)$$

for $a \le x \le b$.

Example 8.1. We compute the Laplace transform of the joint distribution of the location of extreme points of a diffusion. Instead of $\nu(\tau)$ we consider the time τ, i.e., we set $g(x) \equiv 1$. We are interested in computing of the following expression:

$$L^{\mu,\eta}_{a,b} := -\frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathbf{E}_x\Big\{e^{-\mu\hat H(\nu)-\eta\check H(\nu)}; \check H(\nu) < \hat H(\nu), a < \inf_{0\le s\le \tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big\}$$

for the diffusion X, defined in (4.1).

Obviously,

$$L^{\mu,\eta}_{a,b} = -\frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathbf{E}_x\Big\{\exp\big(-(\mu+\eta)\check H(\nu) - \mu(\hat H(\nu)-\check H(\nu))\big);$$

$$\check H(\nu) < \hat H(\nu), a < \inf_{0\le s\le \tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big\}.$$

We apply (8.24) with $\Phi \equiv 1$, $I_1(0,v) = (\mu+\eta)v$, $I_2(u,v) = \mu(v-u)$, $I_3(0,v) = 0$.

According to this formula,

$$L^{\mu,\eta}_{a,b} = \mathbf{E}_x\Big\{e^{-(\lambda+\mu+\eta)H_{a,b}}; X(H_{a,b}) = b\Big\}\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_b\Big\{e^{-(\lambda+\mu)H_{a,b+\delta}}; X(H_{a,b+\delta}) = a\Big\}$$

$$\times \lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{P}_a\Big(a-\delta < \inf_{0\le s\le \tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big). \qquad (8.29)$$

Let $\psi_\gamma(x)$, $x\in\mathbf{R}$, be a strictly increasing positive solution and $\varphi_\gamma(x)$, $x\in\mathbf{R}$, be a strictly decreasing positive solution of the equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - \gamma\phi(x) = 0,$$

and $\omega_\gamma(z) = \psi'_\gamma(z)\varphi_\gamma(z) - \psi_\gamma(z)\varphi'_\gamma(z) > 0$ be their Wronskian. Let $\rho_\gamma(x,y) := \psi_\gamma(x)\varphi_\gamma(y) - \psi_\gamma(y)\varphi_\gamma(x)$. For $y < x$ we have $\rho_\gamma(x,y) > 0$.

According to Theorem 7.2,

$$\mathbf{E}_x\Big\{e^{-\gamma H_{a,b}}; X(H_{a,b}) = b\Big\} = \frac{\rho_\gamma(x,a)}{\rho_\gamma(b,a)}.$$

Analogously (see Remark 7.1),

$$\mathbf{E}_x\Big\{e^{-\gamma H_{a,b}}; X(H_{a,b}) = a\Big\} = \frac{\rho_\gamma(b,x)}{\rho_\gamma(b,a)}.$$

Consequently,

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{E}_b\Big\{e^{-(\lambda+\mu)H_{a,b+\delta}}; X(H_{a,b+\delta}) = a\Big\} = \lim_{\delta\downarrow 0}\frac{1}{\delta}\frac{\rho_{\lambda+\mu}(b+\delta,b)}{\rho_{\lambda+\mu}(b+\delta,a)} = \frac{\omega_{\lambda+\mu}(b)}{\rho_{\lambda+\mu}(b,a)}.$$

By Theorem 4.2, the function

$$U_{a,b}(x) := \mathbf{P}_x\Big(a < \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big), \quad x \in (a,b),$$

is the solution of the problem

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - \lambda U(x) = -\lambda,$$
$$Q(a) = 0, \qquad Q(b) = 0.$$

It is easy to see that

$$U_{a,b}(x) = 1 - \frac{\rho_\lambda(b,x) + \rho_\lambda(x,a)}{\rho_\lambda(b,a)}.$$

Then

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{P}_a\Big(a-\delta < \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big)$$
$$= \lim_{\delta\downarrow 0}\frac{1}{\delta}\Big(1 - \frac{\rho_\lambda(b,a) + \rho_\lambda(a,a-\delta)}{\rho_\lambda(b,a-\delta)}\Big) = -\frac{\frac{\partial}{\partial a}\rho_\lambda(b,a) + \omega_\lambda(a)}{\rho_\lambda(b,a)}.$$

Substituting the corresponding expressions into (8.29), we obtain

$$L^{\mu,\eta}_{a,b} = -\frac{\rho_{\lambda+\mu+\eta}(x,a)}{\rho_{\lambda+\mu+\eta}(b,a)}\frac{\omega_{\lambda+\mu}(b)}{\rho_{\lambda+\mu}(b,a)}\frac{\big(\frac{\partial}{\partial a}\rho_\lambda(b,a) + \omega_\lambda(a)\big)}{\rho_\lambda(b,a)}. \tag{8.30}$$

The analogue of (8.30) for the opposite order of the location of extreme values of the diffusion X is the following result. Set

$$M^{\mu,\eta}_{a,b} := -\frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathbf{E}_x\Big\{e^{-\mu\hat{H}(\nu)-\eta\check{H}(\nu)}; \hat{H}(\nu) < \check{H}(\nu), a < \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) < b\Big\}.$$

Then

$$M^{\mu,\eta}_{a,b} = \frac{\rho_{\lambda+\mu+\eta}(b,x)}{\rho_{\lambda+\mu+\eta}(b,a)}\frac{\omega_{\lambda+\eta}(a)}{\rho_{\lambda+\eta}(b,a)}\frac{\big(\frac{\partial}{\partial b}\rho_\lambda(b,a) - \omega_\lambda(b)\big)}{\rho_\lambda(b,a)}. \tag{8.31}$$

§9. Semigroup of operators related to diffusion

1. Homogeneous diffusions.

Let $X(t)$, $t \in [0,\infty)$, be a homogeneous diffusion process with the state space $(E, \mathcal{B}(E))$ and with the transition function $P(s,x,t,D) =: P(t-s,x,D)$, $D \in \mathcal{B}(E)$. Usually, E is the interval (l,r) with the endpoints included or not (see §15), and $\mathcal{B}(E)$ is the Borel σ-algebra on E. If the drift and diffusion coefficients satisfy (11.2) and (11.3) Ch. II, and $\sigma(x) > 0$ for $x \in \mathbf{R}$, then $E = \mathbf{R}$. This follows from Corollary 12.3 of Ch. II, because every point of the real line is attained by the diffusion with positive probability. Let $B(E)$ be the space of bounded Borel functions from E into $\mathbf{R}$ with the norm $\|f\| = \sup_E |f(x)|$.

A transition function P is *jointly measurable* if for all $D \in \mathcal{B}(E)$ the mapping $(t,x) \mapsto P(t,x,D)$ is $\mathcal{B}([0,\infty)) \times \mathcal{B}(E)$-measurable. In this case we consider the family of operators $\{\mathbb{T}_t, t \geq 0\}$ connected with the homogeneous Markov process X. These operators map from $B(E)$ into $B(E)$ and act by the formula

$$\mathbb{T}_t f(x) := \int_E f(y) P(t,x,dy) = \mathbf{E}_x f(X(t)), \qquad f \in B(E), \tag{9.1}$$

where the subscript x denotes the condition $X(0) = x$. Then the norm of the operator $\mathbb{T}_t$ obeys the estimate

$$\|\mathbb{T}_t\| := \sup_{\|f\|\leq 1} \|\mathbb{T}_t f\| \leq \sup_{\|f\|\leq 1} \|f\| \sup_{x\in E} P(t,x,E) = 1.$$

Let $\mathbb{T}_0 := \mathbb{I}$ be the identity operator. It is clear that

$$\mathbb{T}_t 1 = 1, \qquad \mathbb{T}_t \mathbb{1}_D(x) = P(t,x,D).$$

By the Chapman–Kolmogorov equation, for all $s,t \in [0,\infty)$, $x \in E$ and any $D \in \mathcal{B}(E)$

$$P(s+t,x,D) = \int_E P(t,x,dy) P(s,y,D).$$

This implies the *semigroup property* for the family of operators:

$$\mathbb{T}_t \mathbb{T}_s f(x) = \mathbb{T}_{t+s} f(x). \tag{9.2}$$

Hence $\{\mathbb{T}_t, t \geq 0\}$ form a family of *commutating operators*

$$\mathbb{T}_t \mathbb{T}_s = \mathbb{T}_s \mathbb{T}_t.$$

The semigroup of operators is *strongly right-continuous* if $\|\mathbb{T}_t f - f\| \to 0$ as $t \downarrow 0$. By the semigroup property, this implies that $\lim_{h\downarrow 0} \mathbb{T}_{t+h} f(x) = \mathbb{T}_t f(x)$.

Let $\mathcal{C}_b(E)$ be the space of bounded continuous functions on E. If for any $t > 0$

$$\mathbb{T}_t(\mathcal{C}_b(E)) \subseteq \mathcal{C}_b(E),$$

then the strongly right-continuous semigroup of operators $\{\mathbb{T}_t, t \geq 0\}$ is called a *Feller semigroup.* A Markov process whose transition function generates a Feller semigroup is called a *Feller process.*

The *resolvent* of the semigroup of operators $\{\mathbb{T}_t, t \geq 0\}$ is the family of operators $\{\mathbb{R}_\lambda : \lambda > 0\}$ from $B(E)$ to $B(E)$ defined by the formula

$$\mathbb{R}_\lambda f(x) := \int_0^\infty e^{-\lambda t}\mathbb{T}_t f(x)\,dt = \frac{1}{\lambda}\mathbf{E}_x f(X(\tau)), \tag{9.3}$$

where τ is an exponentially distributed with the parameter $\lambda > 0$ random time independent of the diffusion X. Since the Laplace transform uniquely determines the transformed function, the resolvent uniquely determines the semigroup of operators.

Properties of the resolvent.

1. Using the semigroup property we have

$$\mathbb{R}_\lambda\mathbb{R}_\mu f(x) := \int_0^\infty\int_0^\infty e^{-\lambda t-\mu s}\mathbb{T}_{t+s} f(x)\,dt\,ds = \int_0^\infty dv\mathbb{T}_v f(x)\int_0^v e^{-\lambda(v-s)-\mu s}\,ds$$

$$= \int_0^\infty dv\mathbb{T}_v f(x) e^{-\lambda v}\frac{e^{(\lambda-\mu)v}-1}{\lambda-\mu} = \frac{1}{\lambda-\mu}(\mathbb{R}_\mu f(x) - \mathbb{R}_\lambda f(x)).$$

Therefore, for all positive λ and μ,

$$\mathbb{R}_\lambda\mathbb{R}_\mu f = \frac{1}{\lambda-\mu}(\mathbb{R}_\mu f - \mathbb{R}_\lambda f). \tag{9.4}$$

2. From (9.4) it follows that

$$\frac{d}{d\lambda}\mathbb{R}_\lambda f = -\mathbb{R}_\lambda^2 f. \tag{9.5}$$

3. By induction with respect to m, we can prove that

$$\frac{d}{d\lambda}\mathbb{R}_\lambda^m f = m\mathbb{R}_\lambda^{m-1}\frac{d}{d\lambda}\mathbb{R}_\lambda f.$$

This in turn implies that

$$\frac{d^n}{d\lambda^n}\mathbb{R}_\lambda f = (-1)^n n!\,\mathbb{R}_\lambda^{n+1} f. \tag{9.6}$$

4. Since

$$|\mathbb{R}_\lambda f(x)| \leq \int_0^\infty e^{-\lambda t}|\mathbb{T}_t f(x)|\,dt \leq \|f\|\int_0^\infty e^{-\lambda t}\,dt = \frac{\|f\|}{\lambda},$$

we have

$$\|R_\lambda\| \leq 1/\lambda. \tag{9.7}$$

5. Since

$$\lambda \mathbb{R}_\lambda f(x) = \int_0^\infty e^{-t} \mathbb{T}_{t/\lambda} f(x)\, dt,$$

we have

$$\lim_{\lambda \to \infty} \lambda \mathbb{R}_\lambda f(x) = f(x).$$

From (9.6) and (9.3) it follows that

$$\lambda^n \mathbb{R}_\lambda^n f(x) = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-t} \mathbb{T}_{t/\lambda} f(x)\, dt,$$

therefore,

$$\lim_{\lambda \to \infty} \lambda^n \mathbb{R}_\lambda^n f(x) = f(x).$$

The (*strong*) *infinitesimal generator* $\mathbb{L}$ of a homogeneous diffusion X (or its transition function P) is defined by

$$\mathbb{L} f := \lim_{t \downarrow 0} \frac{\mathbb{T}_t f - f}{t}$$

for $f \in B(E)$ such that the limit exists in the norm $\|\cdot\|$. The set of all such f is called the *domain* of $\mathbb{L}$ and denoted by $D_{\mathbb{L}}$.

If $f \in D_{\mathbb{L}}$, then $\mathbb{T}_t f \in D_{\mathbb{L}}$ for every t, and

$$\mathbb{L}\mathbb{T}_t f = \mathbb{T}_t \mathbb{L} f = \frac{d}{dt} \mathbb{T}_t f. \tag{9.8}$$

Let $f \in B(E)$. Then

$$\mathbb{T}_h \mathbb{R}_\lambda f - \mathbb{R}_\lambda f = \int_0^\infty e^{-\lambda t} \mathbb{T}_{t+h} f\, dt - \int_0^\infty e^{-\lambda t} \mathbb{T}_t f\, dt$$

$$= (e^{\lambda h} - 1) \mathbb{R}_\lambda f - e^{\lambda h} \int_0^h e^{-\lambda t} \mathbb{T}_t f\, dt.$$

This shows that

$$\|\mathbb{T}_h \mathbb{R}_\lambda f - \mathbb{R}_\lambda f\| \le (e^{\lambda h} - 1) \|\mathbb{R}_\lambda f\| + h e^{\lambda h} \|f\|$$

and that $\mathbb{R}_\lambda f \in D_{\mathbb{L}}$.

Applying the Laplace transform to (9.8), we have the following equalities: for $f \in D_{\mathbb{L}}$

$$\mathbb{R}_\lambda \mathbb{L} f = \lambda \mathbb{R}_\lambda f - f,$$

for $f \in B(E)$

$$\mathbb{L} \mathbb{R}_\lambda f = \lambda \mathbb{R}_\lambda f - f.$$

Consequently, for $f \in B(E)$

$$\mathbb{R}_\lambda f = (\lambda\mathbb{I} - \mathbb{L})^{-1} f,$$

where $\mathbb{I}$ is the identity operator. Therefore, the infinitesimal generator $\mathbb{L}$ uniquely determines the semigroup of operators $\{\mathbb{T}_t, t \geq 0\}$.

Let X be a solution of the stochastic differential equation:

$$dX(t) = \sigma(X(t))dW(t) + \mu(X(t))\,dt, \qquad X(0) = x.$$

Assume that $\sigma(x) > 0$ for $x \in E$ and that σ, μ satisfy the Lipschitz condition. Let $\mathcal{C}_b^2(E)$ be the space of twice continuously differentiable functions with bounded first and second derivatives. According to Itô's formula, the process $X(t)$ has the following infinitesimal generator: for $f \in \mathcal{C}_b^2(E)$

$$\mathbb{L}f := \lim_{h\downarrow 0} \frac{1}{h}(\mathbb{T}_h f - f) = \lim_{h\downarrow 0} \frac{1}{h}(\mathbf{E}_x f(X(h)) - f(x))$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\biggl(\mathbf{E}_x\biggl\{ \int_0^h f'(X(t))(\sigma(X(t))dW(t) + \mu(X(t))dt)$$

$$+ \frac{1}{2}\int_0^h f''(X(t))\sigma^2(X(t))dt\biggr\}\biggr) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x).$$

Thus,

$$\mathbb{L} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}. \tag{9.9}$$

The domain of $\mathbb{L}$ includes the space $\mathcal{C}_b^2(E)$.

2. Nonhomogeneous diffusions.

The case of a nonhomogeneous diffusion $X(t)$, $t \in [0,\infty)$, with the transition function $P(s,x,t,D)$, $D \in \mathcal{B}(\mathbf{R})$, can be reduced to the case of a homogeneous process by extension of the state space.

Let $\mathcal{B}_+$ be the σ-algebra of Borel sets on $[0,\infty)$. For $G = \Delta \times D$, where $\Delta \in \mathcal{B}_+$, $D \in \mathcal{B}(\mathbf{R})$, and for $s \geq 0$, $t \geq 0$, $x \in \mathbf{R}$ we define

$$Q(t,(s,x),G) := \mathbb{I}_\Delta(s+t)P(s,x,s+t,D).$$

The measure $Q(t,(s,x),\cdot)$ can be extended by standard methods to the σ-algebra $\mathcal{B}_+ \times \mathcal{B}(\mathbf{R})$. It can be verified that Q is a homogeneous transition function in the state space $E = [0,\infty) \times \mathbf{R}$, corresponding to the homogeneous diffusion

$$Y(t) = (s+t, X(s+t)), \qquad Y(0) = (s, X(s)).$$

We can now apply the theory of homogeneous diffusions. The semigroup of operators $\{\mathbb{T}_t, t \geq 0\}$ is defined by the formula

$$\mathbb{T}_t f(s,x) = \int_{[0,\infty)\times\mathbf{R}} f(u,y)Q(t,(s,x),du \times dy)$$

$$= \int_{\mathbf{R}} f(s+t,y)P(s,x,s+t,dy) = \mathbf{E}_{s,x} f(s+t, X(s+t)),$$

for f belonging to the space of bounded measurable functions on $[0,\infty)\times\mathbf{R}$ with the norm $\|f\| = \sup_{[0,\infty)\times\mathbf{R}} |f(s,x)|$. Here the subscript s,x at the expectation denotes the condition $X(s)=x$.

The resolvent of the semigroup of operators $\{\mathbb{T}_t, t\ge 0\}$ is defined by the formula

$$\mathbb{R}_\lambda f(s,x) := \int_0^\infty e^{-\lambda t}\mathbb{T}_t f(s,x)\,dt = \frac{1}{\lambda}\mathbf{E}_{s,x} f(s+\tau, X(s+\tau)), \tag{9.10}$$

where τ is an exponentially distributed with the parameter $\lambda>0$ random time independent of the diffusion X.

The infinitesimal generator $\mathbb{L}$ of the nonhomogeneous diffusion X is defined by the limit

$$\mathbb{L}f := \lim_{t\downarrow 0}\frac{\mathbb{T}_t f - f}{t} = \lim_{t\downarrow 0}\frac{1}{t}(\mathbf{E}_{s,x} f(s+t,X(s+t)) - f(s,x)). \tag{9.11}$$

Let X be a solution of the stochastic differential equation

$$dX(t) = \sigma(t,X(t))dW(t) + \mu(t,X(t))\,dt, \qquad X(0)=x. \tag{9.12}$$

Assume that $|\sigma(t,x)|>0$ for $(t,x)\in[0,\infty)\times\mathbf{R}$ and that σ, μ satisfy the Lipschitz condition and the linear growth condition (see (7.17), (7.18) Ch. II). For a certain class of functions, according to Itô's formula, the process $X(t)$ has the infinitesimal generator

$$\mathbb{L}f = \lim_{h\downarrow 0}\frac{1}{h}\,\mathbf{E}_{s,x}\biggl\{\int_s^{s+h}\Bigl(\frac{\partial}{\partial t}f(t,X(t)) + \mu(t,X(t))\frac{\partial}{\partial x}f(t,X(t))\Bigr)\,dt$$

$$+\frac{1}{2}\int_s^{s+h}\sigma^2(t,X(t))\frac{\partial^2}{\partial x^2}f(t,X(t))\,dt\biggr\}.$$

Therefore,

$$\mathbb{L}f(s,x) = \frac{\partial}{\partial s}f(s,x) + \frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}f(s,x) + \mu(s,x)\frac{\partial}{\partial x}f(s,x).$$

It is clear that the main part of this operator concerns the variable x. We use for it the special notation

$$\mathbb{L}^\circ f = \frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}f + \mu(s,x)\frac{\partial}{\partial x}f. \tag{9.13}$$

Consider the following question: how does the generating operator transform with a smooth nonrandom time change and the multiplication of the process by a smooth function? Suppose that $a(t)$ and $b(t)$, $t \geq 0$ are differentiable functions, the function b is increasing, $b(0) = 0$, and the function a does not vanish. Set $Y(t) := a(t)X(b(t))$, $t \geq 0$, where the process X satisfies (9.12). According to the result of the nonrandom time change (see (8.8) Ch. II), the process $V(t) := X(b(t))$ satisfies the stochastic differential equation

$$dV(t) = \mu\big(b(t), V(t)\big)b'(t)\,dt + \sigma\big(b(t), V(t)\big)\sqrt{b'(t)}\,d\widetilde{W}(t), \qquad V(0) = x,$$

for some new Brownian motion $\widetilde{W}(t)$, $t \geq 0$. Now, applying the formula of stochastic differentiation, we get

$$dY(t) = \Big(\frac{a'(t)}{a(t)}Y(t) + a(t)b'(t)\mu\Big(b(t), \frac{Y(t)}{a(t)}\Big)\Big)\,dt + a(t)\sqrt{b'(t)}\sigma\Big(b(t), \frac{Y(t)}{a(t)}\Big)\,d\widetilde{W}(t).$$

Consequently,

$$\mathbb{L}_Y^{\circ} f(s,x) = \frac{1}{2}a^2(s)b'(s)\sigma^2\Big(b(s), \frac{x}{a(s)}\Big)\frac{\partial^2}{\partial x^2}f(s,x)$$

$$+\Big(\frac{a'(s)}{a(s)}x + a(s)b'(s)\mu\Big(b(s), \frac{x}{a(s)}\Big)\Big)\frac{\partial}{\partial x}f(s,x). \tag{9.14}$$

§ 10. Transition density of a homogeneous diffusion

Let $X(t)$, $t \in [0,T]$, be a homogeneous diffusion with transition function $P(t,x,\Delta)$, $\Delta \in \mathcal{B}(\mathbf{R})$.

A nonnegative measurable with respect to $(t,x) \in \mathbf{R}^2$ function $p_X(t,x,z)$ is called a *transition density* of the process X with respect to the Lebesgue measure if for any Borel set Δ

$$P(t,x,\Delta) = \int_{\Delta} p_X(t,x,z)\,dz.$$

In this section we consider the question of the existence of the transition density of a homogeneous diffusion. We also describe how it can be computed. The differential problem for the transition density (forward Kolmogorov equation) is derived. To this end we exhibit a special representation for the transition density. The representation for the transition density of a nonhomogeneous diffusion can be found in Gihman and Skorohod (1972).

Let X be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))\,dW(t) + \mu(X(t))\,dt, \qquad X(0) = x, \tag{10.1}$$

where $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, are continuously differentiable functions with bounded derivatives. Assume, in addition, that σ is twice differentiable, $\inf_{x\in\mathbf{R}} \sigma(x) > 0$, and the functions $\mu(x)\dfrac{\sigma'(x)}{\sigma(x)}$, $\sigma''(x)\sigma(x)$, $x \in \mathbf{R}$, are bounded.

We first consider the process with the diffusion coefficient equal to 1. Let Y be a solution of the stochastic differential equation

$$dY(t) = dW(t) + \widetilde{\mu}(Y(t))\,dt, \qquad Y(0) = x, \tag{10.2}$$

where $\widetilde{\mu}(x)$ is a differentiable function with bounded derivative, i.e., $|\widetilde{\mu}'| \le C$. Let $W(0) = x$.

Set

$$\rho(t) := \exp\Big(\int_0^t \widetilde{\mu}(W(s))\,dW(s) - \frac{1}{2}\int_0^t \widetilde{\mu}^2(W(s))\,ds\Big)$$

$$= \exp\Big(\int_{W(0)}^{W(t)} \widetilde{\mu}(v)\,dv - \frac{1}{2}\int_0^t \widetilde{\mu}'(W(s))\,ds - \frac{1}{2}\int_0^t \widetilde{\mu}^2(W(s))\,ds\Big).$$

Here the second equation is satisfied due to the Itô formula.

Since for some $\delta > 0$

$$\sup_{0\le t\le T} \mathbf{E} e^{\delta\widetilde{\mu}^2(W(t))} \le e^{3\delta(\widetilde{\mu}^2(x)+Cx^2)} \sup_{0\le t\le T} \mathbf{E} e^{3C^2\delta W^2(t)} < \infty$$

(the constant C is taken from the boundedness condition for $|\widetilde{\mu}'|$), we have, by Proposition 6.1 Ch. II, that the stochastic exponent $\rho(t)$, $t \in [0,T]$, is a nonnegative martingale with $\mathbf{E}\rho(t) = 1$ for every $t \in [0,T]$.

According to Girsanov's transformation (10.14) Ch. II, for any Borel set Δ, the equality

$$\mathbf{P}_x(Y(t) \in \Delta) = \mathbf{E}_x\big\{\mathbb{I}_\Delta(W(t))\rho(t)\big\}$$

holds. Here the subscript x indicates that $W(0) = Y(0) = x$. The left-hand side of this equality defines the transition probability of the diffusion Y. In view of this equality, it is absolutely continuous with respect to the transition probability of the Brownian process W, and according to the Radon-Nikodým theorem there exists the transition density

$$p_Y(t,x,y) = \frac{d}{dy}\mathbf{E}_x\big\{\rho(t); W(t) < y\big\} = \mathbf{E}_x\big\{\rho(t)\big|W(t) = y\big\}\frac{1}{\sqrt{2\pi t}}\,e^{-(y-x)^2/2t}.$$

Here we used the definition of the conditional expectation (see (2.18) and (2.19) of Ch. I). According to (11.7) and (11.12) of Ch. I, we have

$$p_Y(t,x,y) = \frac{1}{\sqrt{2\pi t}}\,e^{-(y-x)^2/2t} \exp\Big(\int_x^y \widetilde{\mu}(v)\,dv\Big) \tag{10.3}$$

$$\times\, \mathbf{E}_x \exp\Big(-\frac{1}{2}\int_0^t \Big(\widetilde{\mu}'\big(W(s) - \tfrac{s}{t}(W(t)-y)\big) + \widetilde{\mu}^2\big(W(s) - \tfrac{s}{t}(W(t)-y)\big)\Big)ds\Big).$$

Note that

$$p_Y(t,x,y) \sim \frac{1}{\sqrt{2\pi t}}\, e^{-(y-x)^2/2t} \exp\Big(\int_x^y \widetilde{\mu}(v)\,dv\Big) \qquad \text{as} \quad t\downarrow 0. \tag{10.4}$$

We now consider the process $X(t)$, $t\in[0,T]$, which is the solution of the stochastic differential equation (10.1).

Set $f(x) := \int_0^x \frac{dv}{\sigma(v)}$, $x\in\mathbf{R}$. Then $f'(x) = \frac{1}{\sigma(x)}$ and $f''(x) = -\frac{\sigma'(x)}{\sigma^2(x)}$. Let $f^{(-1)}(x)$, $x\in\mathbf{R}$, be the inverse function of f. According to (8.12)–(8.14) of Ch. II, the process $Y(t) := f(X(t))$, $t\in[0,T]$, is the solution of the stochastic differential equation (10.2) with

$$\widetilde{\mu}(x) = \frac{\mu(f^{(-1)}(x))}{\sigma(f^{(-1)}(x))} - \frac{1}{2}\,\sigma'(f^{(-1)}(x)).$$

Since $\big(f^{(-1)}(x)\big)' = \sigma(f^{(-1)}(x))$, we have

$$\widetilde{\mu}'(x) = \mu'(f^{(-1)}(x)) - \mu(f^{(-1)}(x))\frac{\sigma'(f^{(-1)}(x))}{\sigma(f^{(-1)}(x))} - \frac{1}{2}\,\sigma''(f^{(-1)}(x))\sigma(f^{(-1)}(x)).$$

By assumption, the derivative $\widetilde{\mu}'$ is bounded. Therefore, the process Y has the density given by (10.3). According to formula (1.2) of Ch. I, applied to the inverse function instead of the original one, the random variable $X(t) = f^{(-1)}(Y(t))$ has a density of the form $p_X(y) = p_Y(f(y))/\sigma(y)$. Note that if $X(0) = x$, then $Y(0) = f(x)$. Therefore, the transition density of the process X is expressed via the transition density of the process Y by the formula

$$p_X(t,x,y) = p_Y(t,f(x),f(y))/\sigma(y). \tag{10.5}$$

Changing the variable in the integral, we have

$$\int_{f(x)}^{f(y)} \widetilde{\mu}(v)\,dv = \int_x^y \Big(\frac{\mu(u)}{\sigma(u)} - \frac{1}{2}\sigma'(u)\Big)\frac{1}{\sigma(u)}du = \int_x^y \frac{\mu(u)}{\sigma^2(u)}du - \ln\Big(\frac{\sigma^{1/2}(y)}{\sigma^{1/2}(x)}\Big).$$

Set $b(x) := \widetilde{\mu}'(x) + \widetilde{\mu}^2(x)$. From (10.3) and (10.5) it follows that

$$p_X(t,x,y) = \frac{\sigma^{1/2}(x)}{\sqrt{2\pi t}\,\sigma^{3/2}(y)} \exp\bigg(\int_x^y \frac{\mu(u)}{\sigma^2(u)}\,du - \frac{1}{2t}\Big(\int_x^y \frac{du}{\sigma(u)}\Big)^2\bigg)$$

$$\times\, \mathbf{E}_{f(x)} \exp\bigg(-\frac{1}{2}\int_0^t b\Big(W(s) - \frac{s}{t}(W(t)-f(y))\Big)ds\bigg). \tag{10.6}$$

Note that

$$p_X(t,x,y) \sim \frac{\sigma^{1/2}(x)}{\sqrt{2\pi t}\,\sigma^{3/2}(y)} \exp\biggl(\int_x^y \frac{\mu(u)}{\sigma^2(u)}du - \frac{1}{2t}\biggl(\int_x^y \frac{du}{\sigma(u)}\biggr)^2\biggr) \quad \text{as } t\downarrow 0. \qquad (10.7)$$

In the expression for the density $p_X(t,x,y)$ we can choose the symmetric part, i.e., the function, which does not change under the permutation of the variables x and y. We set

$$m(y) := \frac{2}{\sigma^2(y)} \exp\biggl(\int_0^y \frac{2\mu(u)}{\sigma^2(u)}\,du\biggr), \qquad (10.8)$$

and

$$p_X^{\circ}(t,x,y) = \frac{\sqrt{\sigma(x)\sigma(y)}}{2\sqrt{2\pi t}} \exp\biggl(-\int_0^x \frac{\mu(u)}{\sigma^2(u)}\,du - \int_0^y \frac{\mu(u)}{\sigma^2(u)}\,du - \frac{1}{2t}\biggl(\int_x^y \frac{du}{\sigma(u)}\biggr)^2\biggr)$$

$$\times\, \mathbf{E}_{f(x)} \exp\biggl(-\frac{1}{2}\int_0^t b\Bigl(W(s) - \frac{s}{t}(W(t) - f(y))\Bigr)ds\biggr). \qquad (10.9)$$

Then

$$p_X(t,x,y) = p_X^{\circ}(t,x,y)\,m(y). \qquad (10.10)$$

The function $p_X^{\circ}(t,x,y)$ is symmetric with respect to the variables x and y, because the Brownian bridge $W_{x,t,y}(s) = W(s) - \frac{s}{t}(W(t) - y)$, $s \in [0,t]$, is time-reversible process, i.e., the finite-dimensional distributions of the processes $W_{x,t,y}(s)$ and $W_{y,t,x}(t-s)$ are coincide (see § 11 Ch.I). The function $p_X^{\circ}(t,x,y)$ is called the *transition density with respect to the speed measure*, because the function $m(y)$, $y \in \mathbf{R}$, is called the *density of the speed measure*, which will be considered in the next section.

Let us now derive an equation for the transition density. We start with an equation for the transition density $p_Y(t,x,y)$ of the process Y. Set

$$h(t,y) := \mathbf{E}_x \exp\biggl(-\frac{1}{2}\int_0^t \Bigl(\widetilde{\mu}'(W(s) - \frac{s}{t}(W(t)-y)) + \widetilde{\mu}^2(W(s) - \frac{s}{t}(W(t)-y))\Bigr)ds\biggr).$$

Suppose that the function $\widetilde{\mu}'(x) + \widetilde{\mu}^2(x)$ satisfies the conditions of Theorem 4.3 Ch. III. For this it is sufficient that the drift coefficient $\mu(x)$, $x \in \mathbf{R}$, has three bounded derivatives, and the diffusion coefficient $\sigma(x)$, $x \in \mathbf{R}$, has four bounded derivatives. We apply Theorem 4.3 Ch. III. According to this result, the function $h(t,y)$, $(t,y) \in (0,\infty) \times \mathbf{R}$, is the solution of the problem

$$\frac{\partial}{\partial t}h(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}h(t,y) - \frac{y-x}{t}\frac{\partial}{\partial y}h(t,y) - \frac{1}{2}\bigl(\widetilde{\mu}'(y) + \widetilde{\mu}^2(y)\bigr)h(t,y), \qquad (10.11)$$

$$h(+0,y) = 1. \qquad (10.12)$$

Then similarly to (4.35) Ch. III, the function

$$q(t,y) := h(t,y)\frac{1}{\sqrt{2\pi t}}\,e^{-(y-x)^2/2t}$$

satisfies the equation

$$\frac{\partial}{\partial t}q(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}q(t,y) - \frac{1}{2}\big(\widetilde{\mu}'(y) + \widetilde{\mu}^2(y)\big)q(t,y). \qquad (10.13)$$

Finally, the substitution $p(t,y) := \exp\Big(\int\limits_x^y \widetilde{\mu}(v)\,dv\Big)q(t,y)$ leads to the equation

$$\frac{\partial}{\partial t}p(t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}p(t,y) - \widetilde{\mu}(y)\frac{\partial}{\partial y}p(t,y) - \widetilde{\mu}'(y)p(t,y). \qquad (10.14)$$

As a result, we see that the transition density $p_Y(t,x,y)$, expressed by the formula (10.3), is the solution of the equation

$$\frac{\partial}{\partial t}p_Y(t,x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}p_Y(t,x,y) - \frac{\partial}{\partial y}\big(\widetilde{\mu}(y)p_Y(t,x,y)\big) \qquad (10.15)$$

with the boundary condition (10.4).

Consider the process X. The substitution (10.5) leads to the equation

$$\frac{\partial}{\partial t}p_X(t,x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(y)p_X(t,x,y)\big) - \frac{\partial}{\partial y}\big(\mu(y)p_X(t,x,y)\big) \qquad (10.16)$$

with the boundary condition (10.7).

Equation (10.16) is called (see (2.6)) the *forward Kolmogorov equation.*

Since

$$p^{\circ}_X(t,x,y) = \frac{\sigma(y)}{2}\exp\bigg(-\int\limits_0^y \frac{2\mu(u)}{\sigma^2(u)}du\bigg)p_Y(t,f(x),f(y)),$$

using the symmetry property of this function and (10.15), it is not hard to prove that for any fixed y the function $p^{\circ}_X(t,x,y)$ is the solution of the differential problem

$$\frac{\partial}{\partial t}p^{\circ}_X(t,x,y) = \frac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2}p^{\circ}_X(t,x,y) + \mu(x)\,\frac{\partial}{\partial x}p^{\circ}_X(t,x,y), \quad t>0,\ x\in\mathbf{R}, \qquad (10.17)$$

$$p^{\circ}_X(t,x,y) \sim \frac{1}{\sqrt{2\pi t\sigma(x)\sigma(y)m(x)m(y)}}\exp\bigg(-\frac{1}{2t}\bigg(\int\limits_x^y \frac{du}{\sigma(u)}\bigg)^2\bigg) \quad \text{as} \quad t\downarrow 0. \qquad (10.18)$$

For the diffusion X, equation (10.17) is the so-called *backward Kolmogorov equation*, obtained by Kolmogorov (1931), and subsequently studied by Feller (1936).

§ 11. Main characteristics of a homogeneous diffusion

We consider a homogeneous diffusion $X(t)$, $t \in [0,T]$, that is the solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))\,dW(t) + \mu(X(t))\,dt, \qquad X(0) = x. \tag{11.1}$$

where the coefficients $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, satisfy the conditions of § 4. Let $P(t,x,\Delta)$, $\Delta \in \mathcal{B}(\mathbf{R})$, be the transition function of X.

Set $B(x) := \int^x \frac{2\mu(z)}{\sigma^2(z)}\,dz$. The function $S(x) := \int^x e^{-B(z)}\,dz$ is called the *scale function* of the diffusion X. The corresponding measure $S[a,b] := S(b) - S(a)$ is referred to as the *scale measure*, and the function $s(x) := e^{-B(x)}$ as the *density* of the scale measure.

The lack of the lower integration limit in the integral means that one considers the antiderivative.

The function $m(x) := \frac{2e^{B(x)}}{\sigma^2(x)}$ is called the *speed density* and the measure induced by $m(x)$, i.e., $M[c,x] := M(x) - M(c) = \int_c^x m(y)\,dy$, is referred to as the *speed measure* of the diffusion X.

The scale measure and the speed measure are not uniquely defined. They are defined up to an arbitrary factor. However, in formulas this lack of uniqueness is eliminated, because the factors are reduced.

In §9 we have shown that the *infinitesimal generator of the diffusion X* with drift coefficient μ and diffusion coefficient σ^2 has the form

$$\mathbb{L} := \frac{1}{2}\sigma^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx}, \qquad x \in \mathbf{R}. \tag{11.2}$$

This differential operator can be written in the form

$$\mathbb{L} = \frac{\sigma^2(x)}{2}e^{-B(x)}\frac{d}{dx}\Big(e^{B(x)}\frac{d}{dx}\Big) = \frac{d}{dM(x)}\frac{d}{dS(x)}, \tag{11.3}$$

because

$$\frac{d}{dx}\Big(e^{B(x)}\frac{d}{dx}f(x)\Big) = \frac{2\mu(x)}{\sigma^2(x)}e^{B(x)}f'(x) + e^{B(x)}f''(x).$$

This form is very convenient for the detailed investigations of the differential equations related to this operator.

The domain of $\mathbb{L}$ consists of all continuous bounded functions in $\mathcal{C}_b(\mathbf{R})$ such that $\mathbb{L}f \in \mathcal{C}_b(\mathbf{R})$.

Sometimes the function σ^2 is called the *infinitesimal variance* (*diffusion coefficient*), μ is called the *infinitesimal mean* (*drift coefficient*), and in view of (3.11), (3.12)

$$\lim_{t\to 0}\frac{1}{t}\,\mathbf{E}_x(X(t) - x) = \mu(x),$$

$$\lim_{t\to 0}\frac{1}{t}\,\mathbf{E}_x(X(t)-x)^2=\sigma^2(x).$$

Notice that the scale function S is a solution of the differential equation

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}S(x)+\mu(x)\frac{d}{dx}S(x)=0, \tag{11.4}$$

and the density of the speed measure is a solution of the adjoint differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}(\sigma^2(x)m(x))-\frac{d}{dx}(\mu(x)m(x))=0.$$

Let $H_z:=\inf\{t:\ X(t)=z\}$ be the first hitting time of $z\in\mathbf{R}$. Then by (7.12), for $\lambda>0$

$$\mathbf{E}_x e^{-\lambda H_z}=\begin{cases}\dfrac{\psi_\lambda(x)}{\psi_\lambda(z)}, & x\le z,\\[2mm] \dfrac{\varphi_\lambda(x)}{\varphi_\lambda(z)}, & x\ge z,\end{cases}$$

where φ_λ and ψ_λ are nonnegative linearly independent solutions of the differential equation

$$\mathbb{L}\phi(x)-\lambda\phi(x)=0, \tag{11.5}$$

with φ_λ decreasing and ψ_λ increasing. By Proposition 6.1 with $f\equiv 0$, $a=-\infty$, $b=\infty$, the function

$$G_z(x)=\frac{d}{dz}\mathbf{P}_x(X(\tau)<z)=\lambda\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(X(t)<z)\,dt$$

has the form

$$G_z(x)=\begin{cases}\dfrac{2\lambda}{w_\lambda(z)\sigma^2(z)}\psi_\lambda(x)\varphi_\lambda(z), & x\le z,\\[2mm] \dfrac{2\lambda}{w_\lambda(z)\sigma^2(z)}\psi_\lambda(z)\varphi_\lambda(x), & z\le x,\end{cases} \tag{11.6}$$

where $w_\lambda(z)=\psi'_\lambda(z)\varphi_\lambda(z)-\psi_\lambda(z)\varphi'_\lambda(z)$ is the Wronskian of these solutions.

The function $p(t,x,z):=\frac{d}{dz}\mathbf{P}_x(X(t)<z)$ is the *transition density with respect to the Lebesgue measure*, because for any Borel set Δ we have

$$P(t,x,\Delta)=\int_\Delta p(t,x,z)\,dz,$$

where $P(t,x,\Delta)$ is the transition function of the diffusion X.

The function

$$w_\lambda^\circ:=\frac{d\psi_\lambda(x)}{dS(x)}\varphi_\lambda(x)-\psi_\lambda(x)\frac{d\varphi_\lambda(x)}{dS(x)}$$

is a constant independent of x. Indeed, by (11.3),

$$\frac{d}{dM(x)}w_\lambda^\circ=\mathbb{L}\psi_\lambda(x)\,\varphi_\lambda(x)+\frac{d\psi_\lambda(x)}{dS(x)}\frac{d\varphi_\lambda(x)}{dM(x)}$$

$$-\frac{d\psi_\lambda(x)}{dM(x)}\frac{d\varphi_\lambda(x)}{dS(x)} - \psi_\lambda(x)\,\mathbb{L}\varphi_\lambda(x) = 0.$$

It is clear that $w_\lambda^\circ = e^{B(x)}w_\lambda(x)$. Therefore, $\frac{2}{w_\lambda(z)\sigma^2(z)} = \frac{m(z)}{w_\lambda^\circ}$. We introduce the *Green function* of the diffusion X as

$$G_\lambda^\circ(x,z) := \frac{1}{\lambda m(z)}G_z(x) = \begin{cases} \frac{1}{w_\lambda^\circ}\psi_\lambda(x)\varphi_\lambda(z), & x \le z, \\ \frac{1}{w_\lambda^\circ}\psi_\lambda(z)\varphi_\lambda(x), & z \le z. \end{cases} \tag{11.7}$$

This function is symmetric with respect to x, z. If we consider $p^\circ(t,x,z)$, the *transition density with respect to the speed measure*, i.e., $p(t,x,z) = p^\circ(t,x,z)m(z)$, then

$$G_\lambda^\circ(x,z) := \int_0^\infty e^{-\lambda t}p^\circ(t,x,z)dt.$$

This implies that $p^\circ(t,x,z)$ is a symmetric function with respect to x, z.

The finite speed measure M is easily seen to be a *stationary* (or *equilibrium* or *invariant*) *measure*, because it satisfies for all t and $\Delta \in \mathcal{B}(\mathbf{R})$ the equality

$$\int_{\mathbf{R}} P(t,x,\Delta)\,M(dx) = M(\Delta).$$

Indeed, using the symmetry property of p°, we have

$$\int_{\mathbf{R}} P(t,x,\Delta)\,M(dx) = \int_{\mathbf{R}}\int_{\Delta} p^\circ(t,x,z)\,M(dz)\,M(dx)$$

$$= \int_{\Delta}\int_{\mathbf{R}} p^\circ(t,z,x)\,M(dx)\,M(dz) = \int_{\Delta} M(dz) = M(\Delta).$$

§ 12. Recurrence and explosion criteria for a homogeneous diffusion

1. Recurrence criterion.

Let X be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))dW(t) + \mu(X(t))\,dt, \qquad X(0) = x, \tag{12.1}$$

where $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, satisfy the following conditions.

For each $N > 0$, there exists a constant K_N such that for all $x, y \in [-N, N]$

$$|\sigma(x) - \sigma(y)| + |\mu(x) - \mu(y)| \le K_N|x - y|, \tag{12.2}$$

and there exists a constant K, such that for all $x \in \mathbf{R}$

$$|\sigma(x)| + |\mu(x)| \le K(1 + |x|). \tag{12.3}$$

We also assume that $\sigma(x) > 0$ for $x \in \mathbf{R}$.

Let $H_z = \min\{s : X(s) = z\}$ be the first hitting time of the level z. A diffusion X is called *regular* if $\mathbf{P}_x(H_z < \infty) > 0$ for every $x, z \in \mathbf{R}$. A diffusion X is said to be *recurrent* if $\mathbf{P}_x(H_z < \infty) = 1$ for every $x, z \in \mathbf{R}$. As already mentioned,

$$\mathbf{P}_x(H_z < \infty) = \lim_{\lambda \to 0} \mathbf{E}_x(e^{-\lambda H_z}).$$

A diffusion that is not recurrent is called *transient.*

A recurrent diffusion is called *null recurrent* if $\mathbf{E}_x(H_z) = \infty$ for all $x, z \in \mathbf{R}$, and *positively recurrent* if $\mathbf{E}_x(H_z) < \infty$ for all $x, z \in \mathbf{R}$.

Consider the scale function of the diffusion X,

$$S(x) = \int_0^x \exp\Bigl(- \int_0^y \frac{2\mu(v)}{\sigma^2(v)}\, dv \Bigr)\, dy, \qquad x \in \mathbf{R},$$

and the density of the speed measure

$$m(x) = \frac{2}{\sigma^2(x)} \exp\Bigl(\int_0^x \frac{2\mu(v)}{\sigma^2(v)}\, dv \Bigr).$$

Proposition 12.1. *The condition*

$$S(x) \to \pm\infty \qquad \text{as} \quad x \to \pm\infty \tag{12.4}$$

is necessary and sufficient for the diffusion X to be recurrent.

Proof. Indeed, according to (12.38) Ch. II, the exit probabilities of the diffusion X across the boundaries of the interval $[a, b]$ have the form

$$\mathbf{P}_x(X(H_{a,b}) = a) = \frac{S(b) - S(x)}{S(b) - S(a)}, \qquad \mathbf{P}_x(X(H_{a,b}) = b) = \frac{S(x) - S(a)}{S(b) - S(a)}.$$

If $x < z$, then

$$\mathbf{P}_x(H_z < \infty) = \lim_{a \to -\infty} \mathbf{P}_x(X(H_{a,z}) = z) = \lim_{a \to -\infty} \frac{S(x) - S(a)}{S(z) - S(a)} = 1.$$

Analogously, if $z < x$, then

$$\mathbf{P}_x(H_z < \infty) = \lim_{b \to \infty} \mathbf{P}_x(X(H_{z,b}) = z) = \lim_{b \to \infty} \frac{S(b) - S(x)}{S(b) - S(z)} = 1.$$

□

Proposition 12.2. *Let m be the density of the speed measure of a diffusion. If*

$$\int_{-\infty}^{\infty} m(y)\,dy < \infty, \tag{12.5}$$

then the recurrent diffusion is positively recurrent.

If

$$\int_{-\infty}^{0} m(y)\,dy = \infty \qquad \text{and} \qquad \int_{0}^{\infty} m(y)\,dy = \infty, \tag{12.6}$$

then the recurrent diffusion is null recurrent.

Proof. From (12.39) Ch. II and (12.4) it follows that for $x \le z$

$$\mathbf{E}_x H_z = \int_x^z (S(z) - S(y))\, m(y)\,dy + (S(z) - S(x)) \lim_{a\to-\infty} \int_a^x \frac{S(y) - S(a)}{S(z) - S(a)}\, m(y)\,dy,$$

and for $x \ge z$

$$\mathbf{E}_x H_z = \int_z^x (S(y) - S(z))\, m(y)\,dy + (S(x) - S(z)) \lim_{b\to\infty} \int_x^b \frac{S(b) - S(y)}{S(b) - S(z)}\, m(y)\,dy.$$

If (12.5) holds, then for any given $\varepsilon > 0$ we can choose $C > 0$ such that

$$\int_{-\infty}^{-C} m(y)\,dy < \varepsilon, \qquad \int_{C}^{\infty} m(y)\,dy < \varepsilon.$$

Representing the integrals under the limit signs as sums of the integrals over the intervals $(a, -C)$, $[-C, x]$ and $[x, C]$, (C, b), respectively, and taking into account that the integrands are uniformly bounded and converge to 1, we obtain

$$\mathbf{E}_x H_z = \begin{cases} \int_x^z (S(z) - S(y))\, m(y)\,dy + (S(z) - S(x)) \int_{-\infty}^x m(y)\,dy, & \text{for } x \le z, \\ \int_z^x (S(y) - S(z))\, m(y)\,dy + (S(x) - S(z)) \int_x^{\infty} m(y)\,dy, & \text{for } z \le x. \end{cases}$$

Let (12.6) holds. Then for a fixed x and an arbitrarily large $B > 0$ we can choose $C > 0$ such that

$$\int_{-C}^{x} m(y)\,dy > B, \qquad \int_{x}^{C} m(y)\,dy > B. \tag{12.7}$$

Therefore,

$$\mathbf{E}_x H_z > \begin{cases} \int\limits_x^z (S(z)-S(y))\, m(y)\, dy + (S(z)-S(x))\, B, & \text{for } x \le z, \\ \int\limits_z^x (S(y)-S(z))\, m(y)\, dy + (S(x)-S(z))\, B, & \text{for } x \le z. \end{cases}$$

Consequently, $\mathbf{E}_x(H_z) = \infty$ for all $x, z \in \mathbf{R}$. □

For the Brownian motion $W(t)$, $t \ge 0$, we have $S(x) = x$, $m(x) \equiv 2$, $x \in \mathbf{R}$. Therefore, W is a null recurrent diffusion.

For the Ornstein-Uhlenbeck process $U(t) := \sigma e^{-\gamma t} W\Big(\frac{e^{2\gamma t}-1}{2\gamma}\Big)$, $t \ge 0$, $\gamma > 0$ (see Subsection 4 of § 16), we have $S(x) = \int\limits_0^x e^{\gamma y^2/\sigma^2}\, dy$, $m(x) = \frac{2}{\sigma^2} e^{-\gamma x^2/\sigma^2}$, $x \in \mathbf{R}$. Therefore, U is a positively recurrent diffusion.

2. Explosion criterion.

For a diffusion $X(t)$, $t \ge 0$, a random moment $\mathfrak{e}$ is an *explosion time* if $\lim\limits_{t\uparrow \mathfrak{e}} X(t) = \infty$ or $\lim\limits_{t\uparrow \mathfrak{e}} X(t) = -\infty$ and $\mathbf{P}(\mathfrak{e} < \infty) > 0$. In this case we say that *explosion* occurs.

Suppose the restrictions (12.3) on the linear growth of the coefficients of the stochastic differential equation (12.1) hold. Then according to Theorem 7.3 Ch. II, there is a unique solution of (12.1) defined for all time moments. Therefore there is no explosion.

Now suppose that condition (12.3) is not satisfied, but nevertheless equation (12.1) has a solution.

Example 12.1. Let W be a Brownian motion process, $W(0) = x$, and let $H_{-\pi/2,\pi/2}$ be the first exit time through the boundary of the interval $[-\pi/2, \pi/2]$. This moment is finite with probability one. Then the diffusion $X(t) := \tan(W(t))$, $t \ge 0$, obviously explodes as $t \to H_{-\pi/2,\pi/2}$. By Itô's formula, the process X is the solution of the equation

$$dX(t) = (1 + X^2(t))\, dW(t) + X(t)(1 + X^2(t))\, dt, \qquad X(0) = \tan x. \tag{12.8}$$

For this equation, condition (12.3) fails.

Consider the diffusion in *natural scale*, when the drift coefficient equals zero, i.e., the diffusion $Y(t) = S(X(t))$. The function $S(x)$ is the solution of equation (11.4). Assume it satisfies the boundary conditions

$$S(0) = 0, \qquad S'(0) = 1. \tag{12.9}$$

Set

$$g(x) := \sigma(S^{(-1)}(x)) S'(S^{(-1)}(x)), \tag{12.10}$$

where $S^{(-1)}$ is the inverse function. Then according to Itô's formula, the process $Y(t) = S(X(t))$, $t \ge 0$, has the stochastic differential

$$\begin{aligned} dY(t) &= S'(X(t))\big(\sigma(X(t))\, dW(t) + \mu(X(t))\, dt\big) + \frac{1}{2} S''(X(t))\sigma^2(X(t))\, dt \\ &= S'(X(t))\sigma(X(t))\, dW(t) = g(Y(t)) dW(t). \end{aligned} \tag{12.11}$$

Therefore it is a diffusion with mean zero and diffusion coefficient $g^2(x)$.

The following result of W. Feller is a test for explosion.

Theorem 12.1. *Let X be a solution of* (12.1) *and let the function g be defined by* (12.10). *Then explosion is impossible* ($\mathbf{P}(\mathfrak{e} = \infty) = 1$) *if*

$$\lim_{n\to\infty} \int_0^{S(n)} dy \int_0^y \frac{dv}{g^2(v)} = \infty \tag{12.12}$$

and

$$\lim_{n\to\infty} \int_{S(-n)}^0 dy \int_y^0 \frac{dv}{g^2(v)} = \infty \tag{12.13}$$

simultaneously.

If one of these limits is finite, then there is the explosion, i.e., $\mathbf{P}(\mathfrak{e} < \infty) > 0$.

Corollary 12.1. If $S(x) \to \pm\infty$ as $x \to \pm\infty$, i.e., the diffusion is recurrent, then there is no explosion.

Proof of Theorem 12.1. Let $\phi(x)$, $x \in \mathbf{R}$, be a positive even convex solution of the differential equation

$$\frac{1}{2}\, g^2(x)\phi''(x) - \phi(x) = 0, \qquad \phi(0) = 1. \tag{12.14}$$

For $x \ge 0$ the function ϕ can be represented as

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x), \qquad \phi_0(x) \equiv 1,$$

where the functions ϕ_n satisfy the relations

$$\phi''_{n+1}(x) = \frac{2\phi_n(x)}{g^2(x)}, \qquad \phi_{n+1}(0) = 0, \qquad \phi'_{n+1}(0) = 0, \qquad n \ge 0.$$

Then

$$\phi_{n+1}(x) = 2\int_0^x dy \int_0^y \frac{\phi_n(v)}{g^2(v)}\, dv = 2\int_0^x \frac{x-v}{g^2(v)}\phi_n(v)\, dv.$$

The inequalities

$$1 + \phi_1(x) \le \phi(x) \le e^{\phi_1(x)} \tag{12.15}$$

hold, where

$$\phi_1(x) = 2\int_0^x dy \int_0^y \frac{dv}{g^2(v)} = 2\int_0^x \frac{x-v}{g^2(v)}\, dv.$$

In (12.15) the left inequality is obvious. To verify the right one, we will prove the estimate $\phi_n(x) \le \dfrac{(\phi_1(x))^n}{n!}$ by induction on n. Indeed, assuming that this estimate holds for n, we have

$$\phi_{n+1}(x) \le \frac{2^{n+1}}{n!} \int_0^x \frac{x-v}{g^2(v)} \biggl(\int_0^v \frac{v-u}{g^2(u)}\, du\biggr)^n dv \le \frac{2^{n+1}}{n!} \int_0^x \biggl(\int_0^v \frac{x-u}{g^2(u)}\, du\biggr)^n d\biggl(\int_0^v \frac{x-u}{g^2(u)}\, du\biggr)$$

$$= \frac{2^{n+1}}{(n+1)!}\biggl(\int_0^x \frac{x-u}{g^2(u)}\,du\biggr)^{n+1} = \frac{1}{(n+1)!}\,\phi_1^{n+1}(x).$$

This implies (12.15).

The remaining part of the proof of Theorem 12.1 uses a special technique. Applying Itô's formula and (12.11), we obtain

$$e^{-r}\phi(Y(r)) - \phi(S(x)) = -\int_0^r e^{-t}\phi(Y(t))\,dt + \int_0^r e^{-t}\phi'(Y(t))\,dY(t)$$

$$+\int_0^r \frac{e^{-t}}{2} g^2(Y(t))\phi''(Y(t))\,dt = \int_0^r e^{-t}\phi'(Y(t))g(Y(t))\,dW(t). \tag{12.16}$$

Let $H_{-n,n} := \inf\{t : |X(t)| = n\}$. Since for $t \le H_{-n,n}$ the random functions $\bigl(\phi'(S(X(t)))\bigr)^2$ and $g^2(S(X(t)))$ are bounded, we have

$$\mathbf{E}_x \int_0^{H_{-n,n}} e^{-2t}\bigl(\phi'(S(X(t)))g(S(X(t)))\bigr)^2 dt < \infty.$$

Therefore, we can compute the expectation of the stochastic integral in (12.16) with r replaced by the stopping time $H_{-n,n}$. This yields

$$\mathbf{E}_x\bigl\{e^{-H_{-n,n}}\phi(S(X(H_{-n,n})))\bigr\} - \phi(S(x))$$

$$= \mathbf{E}_x \int_0^\infty 1\!\!1_{\{t<H_{-n,n}\}} e^{-t}\phi'(S(X(t)))g(S(X(t)))\,dW(t) = 0,$$

and so

$$\mathbf{E}_x\bigl\{e^{-H_{-n,n}}\phi(S(X(H_{-n,n})))\bigr\} = \phi(S(x)). \tag{12.17}$$

Since $\mathbf{P}(X(H_{-n,n}) = n) + \mathbf{P}(X(H_{-n,n}) = -n) = 1$, equation (12.17) can be rewritten in the form

$$\phi(S(n))\mathbf{E}_x\bigl\{e^{-H_{-n,n}}; X(H_{-n,n}) = n\bigr\}$$

$$+\,\phi(S(-n))\mathbf{E}_x\bigl\{e^{-H_{-n,n}}; X(H_{-n,n}) = -n\bigr\} = \phi(S(x)). \tag{12.18}$$

Conditions (12.12), (12.13) mean that $\phi_1(S(n)) \to \infty$, $\phi_1(S(-n)) \to \infty$. Inequalities (12.15) imply that $\phi(S(n)) \to \infty$, $\phi(S(-n)) \to \infty$. For every fixed x from (12.18), it follows that $\mathbf{P}_x(\lim\limits_{n\to\infty} H_{-n,n} = \infty) = 1$ and, consequently, there is no explosion.

Assume now that one of the limits (12.12), (12.13) is finite. Let, for example, $\lim\limits_{n\to\infty} \phi_1(S(n)) < \infty$. Then by (12.15), $\lim\limits_{n\to\infty} \phi(S(n)) < \infty$. Set $H_{0,n} = \inf\{t : X(t) \notin [0,n]\}$. Similarly to (12.17), for $0 < x < n$ we have

$$\mathbf{E}_x\bigl\{e^{-H_{0,n}}\phi(S(X(H_{0,n})))\bigr\} = \phi(S(x)).$$

Set $\mathfrak{e} := \lim_{n\to\infty} H_{0,n}$. Since $\phi(S(0)) = \phi(0) = 1$ and $\phi(x) > 1$, we have

$$1 < \phi(S(x)) = \lim_{n\to\infty} \mathbf{E}_x\big\{e^{-H_{0,n}}\phi(S(X(H_{0,n})))\big\}$$

$$= \lim_{n\to\infty}\Big(\phi(S(n))\mathbf{E}_x\big\{e^{-H_{0,n}}; X(H_{0,n}) = n\big\} + \phi(S(0))\mathbf{E}_x\big\{e^{-H_{0,n}}; X(H_{0,n}) = 0\big\}\Big)$$

$$\le \lim_{n\to\infty}\phi(S(n))\,\mathbf{E}_x\big\{e^{-\mathfrak{e}}; \lim_{n\to\infty} X(H_{0,n}) = \infty\big\} + 1.$$

Therefore, $\mathbf{E}_x\big\{e^{-\mathfrak{e}}; \lim_{n\to\infty} X(H_{0,n}) = \infty\big\} > 0$ or $\mathbf{P}_x(\mathfrak{e} = \infty) < 1$. Thus $\mathfrak{e}$ is an explosion time. □

Example 12.2. Consider the Brownian motion with linear drift, i.e., the process $W^{(\mu)}(t) = \mu t + W(t)$, $\mu \ne 0$. It is a homogeneous diffusion with drift coefficient μ and diffusion coefficient 1.

The scale function of this process is $S(x) = \dfrac{1 - e^{-2\mu x}}{2\mu}$. Consider, for example, the positive drift $\mu > 0$. Obviously, $\lim_{x\to\infty} S(x) = \dfrac{1}{2\mu}$. Therefore, $W^{(\mu)}$ is nonrecurrent.

The Brownian motion with linear drift $W^{(\mu)}$ is a diffusion without explosion. Indeed, $S^{(-1)}(x) = -\dfrac{\ln(1-2\mu x)}{2\mu}$, $S'(S^{(-1)}(x)) = 1 - 2\mu x$ for $x < 1/2\mu$, and hence,

$$\frac{1}{2}\phi_1(S(n)) = \int_0^{(1-e^{-2\mu n})/2\mu} dy \int_0^y \frac{dv}{(1-2\mu v)^2}$$

$$= \int_0^{(1-e^{-2\mu n})/2\mu} \frac{1}{2\mu}\Big(\frac{1}{1-2\mu v} - 1\Big)dy = \frac{1}{2\mu}\Big(n - \frac{1-e^{-2\mu n}}{2\mu}\Big) \to \infty.$$

Analogously,

$$\frac{1}{2}\phi_1(S(-n)) = \int_{(1-e^{2\mu n})/2\mu}^0 dy \int_y^0 \frac{dv}{(1-2\mu v)^2} = \frac{1}{2\mu}\Big(\frac{e^{2\mu n}-1}{2\mu} - n\Big) \to \infty.$$

By Theorem 12.1, there is no explosion.

§ 13. Random time change

Let $X(t)$, $t \ge 0$, be a solution of the stochastic differential equation

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dW(t), \quad X(0) = x, \tag{13.1}$$

with differentiable coefficients μ and σ. Suppose that $\sigma(x) > 0$, $x \in \mathbf{R}$. Let $g(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function with $g'(x) \ne 0$ for all $x \in \mathbf{R}$.

Then the function g has an inverse $g^{(-1)}(x)$, $x \in g(\mathbf{R})$. We also assume that there exists the bounded derivative $\left(\frac{\mu(x)}{\sigma^2(x)}\right)'$, $x \in \mathbf{R}$, and

$$\liminf_{y\to\infty} \frac{1}{y} \int_0^y (g'(x))^2 \, dx > 0, \qquad \liminf_{y\to\infty} \frac{1}{y} \int_{-y}^0 (g'(x))^2 \, dx > 0.$$

Under these conditions, according to Corollary 12.1 Ch. II,

$$\int_0^\infty \bigl(g'(X(s))\sigma(X(s))\bigr)^2 \, ds = \infty \qquad \text{a.s.}$$

Consider the integral functional

$$A(t) := \int_0^t \bigl(g'(X(s))\sigma(X(s))\bigr)^2 \, ds, \qquad t \in [0,\infty),$$

and define the inverse process:

$$a_t := \min\bigl\{s : A(s) = t\bigr\}, \quad t \in [0,\infty).$$

Under the above conditions, a_t is a.s. finite for any $t \ge 0$. Since $A(t)$, $t \ge 0$, is a strictly increasing function, $\alpha_{0+} = 0$.

Theorem 13.1. *The process*

$$\widetilde{X}(t) := g\bigl(X(a_t)\bigr), \quad t \in [0,\infty), \tag{13.2}$$

is a diffusion with $\sigma(x) = 1$ *and* $\widetilde{\mu}(x) = D(g^{(-1)}(x))$, *where*

$$D(x) := \frac{g''(x)}{2(g'(x))^2} + \frac{\mu(x)}{g'(x)\sigma^2(x)}. \tag{13.3}$$

Remark 13.1. For $g(x) = S(x)$, we obtain (see (11.4)) that $D(x) \equiv 0$ and that $\widetilde{X}(t)$ is a Brownian motion.

Proof of Theorem 13.1. Indeed, by Itô's formula, for any $r > 0$

$$g(X(r)) - g(x) = \int_0^r g'(X(s))\sigma(X(s)) \, dW(s) + \int_0^r g'(X(s))\mu\bigl(X(s)\bigr) \, ds$$

$$+ \frac{1}{2} \int_0^r g''(X(s))\sigma^2(X(s)) \, ds.$$

Replacing r by a_t, we get

$$\widetilde{X}(t)-\widetilde{X}(0)=\int_0^{a_t} g'(X(s))\sigma(X(s))\,dW(s)+\int_0^{a_t}\big(g'(X(s))\sigma(X(s))\big)^2 D(X(s))\,ds.$$

Since a_t is the inverse of $A(t)$, $t\ge 0$, and $A'(t)=\big(g'(X(t))\sigma(X(t))\big)^2$, we have

$$a'_t=\frac{1}{A'(a_t)}=\frac{1}{\big(g'(X(a_t))\sigma(X(a_t))\big)^2}.$$

By Lévy's theorem (see Theorem 8.1 Ch. II), the process

$$\widetilde{W}(t):=\int_0^{a_t} g'(X(s))\sigma(X(s))\,dW(s),\qquad t\in[0,\infty),$$

is a Brownian motion. Consequently,

$$\widetilde{X}(t)-\widetilde{X}(0)=\widetilde{W}(t)+\int_0^t\big(g'(X(a_s))\sigma(X(a_s))\big)^2 D(X(a_s))\,da_s$$

$$=\widetilde{W}(t)+\int_0^t\widetilde{\mu}(\widetilde{X}(s))\,ds.$$

□

We can prove Theorem 13.1 in a different way, with the help of Theorem 5.1.

Let $\varPhi(x)$ and $f(x)$, $x\in\mathbf{R}$, be continuous functions. Assume that $\varPhi$ is bounded and f is nonnegative. Then, by Theorem 5.1, the function

$$V(x):=\mathbf{E}_x\Big\{\varPhi(X(a_\tau))\exp\Big(-\int_0^{a_\tau} f(X(s))\,ds\Big)\Big\},\qquad x\in\mathbf{R},$$

is the unique bounded solution of the equation

$$\frac{1}{2}V''(x)+\frac{\mu(x)}{\sigma^2(x)}V'(x)-\Big(\lambda\big(g'(x)\big)^2+\frac{f(x)}{\sigma^2(x)}\Big)V(x)=-\lambda\big(g'(x)\big)^2\varPhi(x).\tag{13.4}$$

Changing the variable by

$$Q(x):=V(g^{(-1)}(x)),\tag{13.5}$$

we see that the function $Q(x)$, $x\in\mathbf{R}$, satisfies the equation

$$\frac{1}{2}Q''(x)+\widetilde{\mu}(x)Q'(x)-\Big(\lambda+\frac{f(g^{(-1)}(x))}{\big(g'(g^{(-1)}(x))\sigma(g^{(-1)}(x))\big)^2}\Big)Q(x)=-\lambda\varPhi(g^{(-1)}(x)).\tag{13.6}$$

According to Theorem 4.1, the solution of (13.6) has the probabilistic expression

$$Q(x) = \mathbf{E}_x\Big\{\varPhi(g^{(-1)}(\widetilde X(\tau)))\exp\Big(-\int_0^\tau \frac{f(g^{(-1)}(\widetilde X(s)))}{(g'(g^{(-1)}(\widetilde X(s))\sigma(g^{(-1)}(\widetilde X(s)))^2}\,ds\Big)\Big\},$$

where the diffusion $\widetilde X(t)$ is determined by the coefficients $\widetilde\mu(x)$ and $\sigma(x) = 1$. To clarify the connection between the processes $X(t)$ and $\widetilde X(t)$, we transform the function $V(g^{(-1)}(x))$ as follows:

$$V(g^{(-1)}(x)) := \mathbf{E}_{g^{(-1)}(x)}\Big\{\varPhi(X(a_\tau))\exp\Big(-\int_0^\tau f(X(a_s))\,da_s\Big)\Big\}$$
$$= \mathbf{E}_{g^{(-1)}(x)}\Big\{\varPhi(X(a_\tau))\exp\Big(-\int_0^\tau \frac{f(X(a_s))}{(g'(X(a_s)\sigma(X(a_s)))^2}\,ds\Big)\Big\}.$$

We compare the expressions obtained for the two sides of equality (13.5) in terms of the expectations of the corresponding variables. Since the functions $\varPhi$ and f are arbitrary, we conclude that the processes $X(a_t)$ and $g^{(-1)}(\widetilde X(t))$ are identical in law. We note that in the proof of Theorem 13.1, the application of Lévy's theorem gave us the equality $X(a_t) = g^{(-1)}(\widetilde X(t))$, which does not follow from the arguments given above. However, in the study of distributions of functionals, this equality is not important, because the coincidence of the finite-dimensional distributions of the processes is sufficient.

§ 14. Diffusion local time

Let X be a solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))dW(t) + \mu(X(t))\,dt, \qquad X(0) = x, \tag{14.1}$$

where $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, are continuous functions satisfying the Lipschitz condition, and $|\sigma(x)| > 0$.

Let $Y(t) := S(X(t))$ be the *process in natural scale*, where $S(x)$, $x \in \mathbf{R}$, is the scale of the diffusion X. We use this name for the process Y, because its scale is identically equal to x. Indeed, this process satisfies the stochastic differential equation (12.11) and its drift coefficient is equal to zero.

The definition of the local time of a random process was given in § 5 Ch. II. In order to identify the local time with the corresponding process, we append the process as a subscript to the local time. If the process $Y(t)$, $t \ge 0$, has the local time, i.e., a.s. for all $(t, y) \in [0, \infty) \times \mathbf{R}$ there exists the limit (see (5.3) Ch. II)

$$\ell_Y(t, y) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(Y(v))\,dv,$$

then the process $X(t)$, $t \ge 0$, also has a local time, namely

$$\ell_X(t,y) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(X(v))\,dv = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(S^{(-1)}(Y(v)))\,dv$$

$$= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[S(y),S(y)+S'(y)\varepsilon)}(Y(v))\,dv = S'(y)\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^t \mathbb{I}_{[S(y),S(y)+\delta)}(Y(v))\,dv$$

$$= S'(y)\ell_Y(t,S(y)). \tag{14.2}$$

If there exists the bounded derivative $\Bigl(\frac{\mu(x)}{\sigma^2(x)}\Bigr)'$, $x \in \mathbf{R}$, and

$$\liminf_{y\to\infty}\frac{1}{y}\int_0^y \exp\biggl(-4\int_0^z \frac{\mu(v)}{\sigma^2(v)}\,dv\biggr)dz > 0, \quad \liminf_{y\to\infty}\frac{1}{y}\int_{-y}^0 \exp\biggl(4\int_z^0 \frac{\mu(v)}{\sigma^2(v)}\,dv\biggr)dz > 0,$$

then for the integral functional

$$A(t) := \int_0^t \sigma^2(X(s))\bigl(S'(X(s))\bigr)^2\,ds, \qquad t \in [0,\infty),$$

we have (see (5.3)) the equality $A(\infty) = \infty$ a.s. Under these assumptions, the random moment

$$a_t := \min\bigl\{s : A(s) = t\bigr\}, \quad t \in [0,\infty),$$

is a.s. finite and, according to Remark 13.1, the process $\widetilde{W}(t) := S(X(\alpha_t))$, $t \ge 0$, is a Brownian motion.

The process Y (in the natural scale) can now be written as $Y(t) = \widetilde{W}(A(t))$, $t \ge 0$, and hence

$$X(t) = S^{(-1)}(\widetilde{W}(A(t))). \tag{14.3}$$

Let the function $g(x)$, $x \in \mathbf{R}$, be defined by (12.10). Then $A'(t) = g^2(\widetilde{W}(A(t)))$. Therefore,

$$\ell_Y(t,y) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(Y(v))\,dv = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(\widetilde{W}(A(v)))\,dv$$

$$= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{A(t)} \frac{1}{A'(\alpha_u)}\mathbb{I}_{[y,y+\varepsilon)}(\widetilde{W}(u))\,du = \int_0^{A(t)} \frac{1}{A'(\alpha_u)}\,\ell_{\widetilde{W}}(du,y)$$

$$= \int_0^{A(t)} \frac{1}{g^2(\widetilde{W}(u))}\ell_{\widetilde{W}}(du,y) = \frac{1}{g^2(y)}\ell_{\widetilde{W}}(A(t),y).$$

The Brownian local time exists (see § 5 Ch. II) and consequently the local time $\ell_Y(t,y)$ also exists. In the last equality we used the fact that for any fixed y the local time $\ell_{\widetilde W}(u,y)$, $u \ge 0$, increases only on the set $\{u : \widetilde W(u) = y\}$.

Now, taking into account (14.2), we finally get

$$\ell_X(t,y) = S'(y)\ell_Y(t,S(y)) = \frac{S'(y)}{g^2(S(y))}\ell_{\widetilde W}(A(t),S(y))$$

$$= \frac{m(y)}{2}\ell_{\widetilde W}(A(t),S(y)), \tag{14.4}$$

where m is the density of the speed measure of the diffusion X.

Consider the *local time with respect to the speed measure* M. To distinguish it from the local time with respect to the Lebesgue measure, we denote it by L. Then

$$L_X(t,y) := \lim_{\varepsilon\downarrow 0}\frac{1}{M[y,y+\varepsilon)}\int_0^t \mathbb{1}_{[y,y+\varepsilon)}(X(s))\,ds$$

$$= \frac{1}{m(y)}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[y,y+\varepsilon)}(X(s))\,ds = \frac{1}{m(y)}\ell_X(t,y). \tag{14.5}$$

Since the density of the speed measure for a Brownian motion equals 2, (14.4) and (14.5) imply that

$$L_X(t,y) = L_{\widetilde W}(A(t),S(y)). \tag{14.6}$$

§ 15. Boundary classification for regular diffusion

In this section we consider a homogeneous diffusion $X(t)$, $t \ge 0$. Let the state space of the process X be a finite or infinite interval I of the form $[l,r]$, $(l,r]$, $[l,r)$ or (l,r), where $-\infty \le l < r \le \infty$.

Let the drift coefficient $\mu(x)$ and the diffusion coefficient $\sigma^2(x)$, $x \in I$, be continuous functions, and $\inf_{x\in(l,r)} \sigma(x) > 0$. Assume that for any subinterval $[l_n,r_n] \in (l,r)$ there exists a constant K_n such that

$$|\mu(x)-\mu(y)| + |\sigma(x)-\sigma(y)| \le K_n|x-y|, \qquad x,y \in [l_n,r_n].$$

Let $l_n \downarrow l$ and $r_n \uparrow r$. Consider two sequences of functions $\mu_n(x)$ and $\sigma_n(x)$, $x \in \mathbf{R}$, satisfying Theorem 7.3 of Ch. II such that $\mu_n(x) = \mu(x)$, $\sigma_n(x) = \sigma(x)$ for $x \in [l_n,r_n]$. By Theorem 7.3 of Ch. II, there exist a unique strong solution of the stochastic differential equation

$$dX_n(t) = \mu_n(X_n(t))\,dt + \sigma_n(X_n(t))\,dW(t), \qquad X_n(0) = X(0).$$

Let $H_n = \inf\{s : X_n(s) \notin [l_n,r_n]\}$ if the set $\{s : X_n(s) \notin [l_n,r_n]\}$ is not empty, and let $H_n = \infty$ otherwise. By Theorem 7.2 of Ch. II, for $m > n$ the processes $X_m(t)$ and $X_n(t)$ coincide for $t \in [0,H_n]$ and $H_m \ge H_n$. It is clear that for $t < H_n$,

$$\mu_n(X_n(t)) = \mu(X_n(t)), \qquad \sigma_n(X_n(t)) = \sigma(X_n(t)).$$

Set $H := \sup_n H_n$. Since for $m > n$, the process X_m is the continuation of the process X_n, we can set $X(t) := X_n(t)$ for $t \le H_n$, $n = 1, 2, \dots$. It is clear that $X(t)$ for $t < H$ is the solution of the stochastic differential equation

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dW(t). \tag{15.1}$$

We refer to the process $X(t)$, $t < H$, as the solution of (15.1) in the interval (l, r). The moment H is the first exit time of this process through the boundary of the interval. If $H = \infty$, then $X(t)$ is determined for all $t > 0$ and the boundaries do not influence the behavior of the process. Otherwise, this behavior depends on the infinitesimal parameters $\mu(x)$ and $\sigma^2(x)$ as functions of $x \in (l, r)$. Our aim is to describe the behavior of the diffusion X at the boundaries l and r.

Let $H_z := \inf\{s : X(s) = z\}$ be the first hitting time of the level z by the process X. If the level z is not attained, we set $H_z = \infty$.

Similarly to processes whose state space is the whole real line, the diffusion X is called *regular* if for all $l < x < r$, $l < y < r$

$$\mathbf{P}_x(H_y < \infty) > 0.$$

In other words, every point from interior of the interval I can be attained with some positive probability from an arbitrary different point from the interior of I.

For simplicity we can assume that $l < 0 < r$; otherwise, in place of 0 one can take an arbitrary interior point of (l, r). Then we can set

$$S(x) := \int_0^x \exp\Big(- \int_0^y \frac{2\mu(v)}{\sigma^2(v)}\,dv \Big)\,dy, \qquad x \in (l, r),$$

for the scale function and set

$$m(x) := \frac{2}{\sigma^2(x)} \exp\Big(\int_0^x \frac{2\mu(v)}{\sigma^2(v)}\,dv \Big), \qquad x \in (l, r),$$

for the density of the speed measure $(dM(x) = m(x)\,dx)$.

A very important role is played (see § 11) by the differential operator

$$\mathbb{L} := \tfrac{1}{2}\sigma^2(x)\,\frac{d^2}{dx^2} + \mu(x)\,\frac{d}{dx} = \frac{d}{dM(x)}\,\frac{d}{dS(x)}.$$

This operator is called the *infinitesimal generator* of the diffusion X with drift coefficient μ and diffusion coefficient σ^2.

For a bounded function F, we consider the differential equation

$$\tfrac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) = -F(x), \qquad \Big(\frac{d}{dM(x)}\,\frac{d}{dS(x)}U(x) = -F(x)\Big). \tag{15.2}$$

Using the right-hand side expression, we have

$$\frac{d}{dS(x)}U(x) = c_1 - \int_a^x F(z)\,dM(z),$$

where c_1 and $a \in (l,r)$ are some constants. Integrating with respect to $dS(x)$, we obtain

$$U(x) = c_1(S(x) - S(a)) - \int_a^x dS(y)\int_a^y F(z)\,dM(z) + c_2. \tag{15.3}$$

For $[a,b] \subseteq I$ denote

$$H_{a,b} := \inf\{s : X(s) \notin (a,b)\} = \min\{H_a, H_b\},$$

which is the first exit time from the interval (a,b). We compute the expectation of the first exit time, i.e., $\mathbf{E}_x H_{a,b}$. We have already discussed this formula without proof (see (12.39) Ch. II). We next verify that $\mathbf{E}_x H_{a,b}$ is the solution of the corresponding differential problem. To do this we apply Theorem 7.1 with the functions $f \equiv 0$, $F \equiv 1$, and $\Phi \equiv 0$. By this theorem, the function $Q(x) := \mathbf{E}_x H_{a,b}$, $x \in [a,b]$, is the solution of the problem

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) = -1, \tag{15.4}$$

$$Q(a) = 0, \qquad Q(b) = 0. \tag{15.5}$$

From (15.3) with $F \equiv 1$ it follows that

$$Q(x) = \frac{S(x) - S(a)}{S(b) - S(a)}\int_a^b M[a,y]\,dS(y) - \int_a^x M[a,y]\,dS(y). \tag{15.6}$$

One can write (15.6) in a symmetric form. Splitting the first integral into two integrals and using the integration by parts formula, we have

$$\mathbf{E}_x H_{a,b} = \frac{S(x) - S(a)}{S(b) - S(a)}\int_x^b M[a,y]\,dS(y) - \frac{S(b) - S(x)}{S(b) - S(a)}\int_a^x M[a,y]\,dS(y)$$

$$= \frac{S(b) - S(x)}{S(b) - S(a)}\int_a^x (S(y) - S(a))\,dM(y) + \frac{S(x) - S(a)}{S(b) - S(a)}\int_x^b (S(b) - S(y))\,dM(y)$$

$$= \frac{S(b) - S(x)}{S(b) - S(a)}\Sigma[a,x] + \frac{S(x) - S(a)}{S(b) - S(a)}N[x,b], \tag{15.7}$$

where we did set

$$\Sigma[a,x] := \int_a^x (S(y) - S(a))\,dM(y), \qquad N[x,b] := \int_x^b (S(b) - S(y))\,dM(y).$$

We focus our attention on the lower boundary l, the treatment of the upper boundary being completely similar.

Since $S(a)$ is a monotone function of a, one can set $S(l+) := \lim_{a\downarrow l} S(a)$. Also, we can set $H_{l+} := \lim_{a\downarrow l} H_a$, where the limit is less than or equal to infinity. Let us check that $H_{l+} = H_l$. Since $H_a \le H_l$ for $x \in (a,b)$, we have $H_{l+} \le H_l$. If $H_{l+} = \infty$, then $H_l = \infty$. Assume that $H_{l+} < \infty$. By the continuity of X, $X(H_{l+}) = \lim_{a\downarrow l} X(H_a) = \lim_{a\downarrow l} a = l > -\infty$. Hence, $H_{l+} \ge H_l = \inf\{s \ge 0 : X(s) = l\}$.

Note that $H_l = \infty$ if l is not a state point of X.

The following assertion is a direct consequence of the formula for the exit probability of the diffusion through the upper boundary of an interval (see (12.38) Ch. II).

Proposition 15.1. *Suppose that* $S(l+) > -\infty$. *Then* $\mathbf{P}_x(H_{l+} < H_b) > 0$ *for all* $l < x < b < r$.

Suppose that $S(l+) = -\infty$. *Then* $\mathbf{P}_x(H_{l+} < H_b) = 0$ *for all* $l < x < b < r$.

The boundary l is said to be *attractive* if $S(l+) > -\infty$.

Consider an attractive boundary l, i.e., the case $S(l+) > -\infty$. The following question arises: when is the boundary attainable in a finite expected time? According to (15.7), for arbitrary $b \in (l,r)$ the inequality

$$\lim_{a\downarrow l} \mathbf{E}_x H_{a,b} < \infty \tag{15.8}$$

holds if and only if

$$\Sigma(l) := \lim_{a\downarrow l} \Sigma[a,x] < \infty. \tag{15.9}$$

In the notation for $\Sigma(l)$ we omit the argument x, because this quantity is finite or infinite for all $x \in (l,r)$ simultaneously and for us only this fact will be important.

It is clear that

$$\Sigma(l) = \int_l^x (S(y) - S(l+))\,dM(y) = \int\limits_{l<z<y<x} dS(z)\,dM(y). \tag{15.10}$$

The boundary l is said to be *attainable* if $\Sigma(l) < \infty$, and *unattainable* if $\Sigma(l) = \infty$.

Proposition 15.2. *Let l be an attractive boundary. Then it is attainable iff* $\mathbf{P}_x(H_l < \infty) > 0$.

Proof. We prove only the direct implication. We have

$$\mathbf{E}_x \min\{H_l, H_b\} = \mathbf{E}_x \min\{H_{l+}, H_b\} = \mathbf{E}_x H_{l+,b}.$$

Since l is attainable, by (15.7), $\mathbf{E}_x H_{l+,b} < \infty$ and consequently $\min\{H_l, H_b\} < \infty$ a.s. Then $H_l \ne H_b$, because one of this moments is finite and the diffusion at this moment takes the corresponding value. Since l is attractive, $\mathbf{P}_x(H_l < H_b) > 0$ and

$$\mathbf{P}_x(H_l < \infty) \ge \mathbf{P}_x(H_l < H_b) > 0.$$

□

Next we introduce the quantities $M(l,b] := \lim_{a\downarrow l} M[a,b]$ and

$$N(l) := \int_l^x M(l,y]\,dS(y) = \int_{l<z<y<x} dM(z)\,dS(y).$$

The quantity $M(l,b]$ measures the speed of the diffusion near the boundary l.

For the lower boundary l, the modern classification of boundary behavior of the diffusion is based on four quantities $S(l+)$, $\Sigma(l)$, $N(l)$ and $M(l,b]$, specifically, on whether they are finite or infinite. We give here a description of the *boundary classification* without rigorous justification (see, for example, Itô (1963)).

For a complete characterization of the diffusion X, the behavior at the boundary must be specified and this can be done by assigning a meaning to the speed measure $M\{l\}$ at the boundary itself. The behavior can be range from absorbtion ($M\{l\} = \infty$) to reflection ($M\{l\} = 0$).

If the diffusion can both enter and leave the boundary, then the boundary is called *regular*. The boundary l is regular iff $\Sigma(l) < \infty$ and $N(l) < \infty$.

To establish that the boundary l is regular, it suffices to check that $S(l+) > -\infty$ and $M(l,b] < \infty$ for some $b \in (l,r)$.

The boundary l is an *exit* boundary if for all $t > 0$

$$\lim_{b\downarrow l} \lim_{x\downarrow l} \mathbf{P}_x(H_b < t) = 0.$$

This means that starting at l, or starting at an initial point that approaches l, the diffusion cannot attains any interior state b no matter how near b is to l. The boundary l is exit iff $\Sigma(l) < \infty$ and $N(l) = \infty$.

To establish that l is an exit boundary, it suffices to check that $\Sigma(l) < \infty$ and $M(l,b] = \infty$.

The *entrance* boundary cannot be attained from a point of the interior of the state space. At the same time, one can consider the diffusion starting from this boundary. Such a diffusion moves to the interior and never returns to the entrance boundary. The boundary l is an entrance boundary iff $\Sigma(l) = \infty$ and $N(l) < \infty$.

To establish that l is an entrance boundary, it suffices to check that $S(l+) = -\infty$ and $N(l) < \infty$.

A boundary which is neither an entrance one, nor an exit one is called *natural*. A natural boundary cannot be attained in finite time and the diffusion does not start from this boundary. Therefore, this boundary does not belong to the state space of the diffusion. The boundary l is natural iff $\Sigma(l) = \infty$ and $N(l) = \infty$.

To establish that the boundary l is natural, it suffices to check that $S(l+) = -\infty$ and $M(l,b] = \infty$.

The analogous criteria for the upper boundary r are based on the quantities

$$S(r-) := \lim_{b\uparrow r} S(b), \qquad M[a,r) := \lim_{b\uparrow r} M[a,b],$$

$$\Sigma(r) := \int_x^r (S(r-) - S(y))\, dM(y) = \int_{x<y<z<r} dS(z)\, dM(y),$$

and

$$N(r) := \int_x^r M[y,r)\, dS(y) = \int_{x<y<z<r} dM(z)\, dS(y).$$

Exercises.

15.1. Consider a diffusion $X(t)$, $t \geq 0$, with the state space $(0, \infty)$, the drift coefficient $\mu(x) = \alpha x^{p-1}$, and the diffusion coefficient $\sigma^2(x) = \beta x^p$, where $1 \leq p$, $\beta > 0$, $-\infty < \alpha < \infty$.

Classify the boundaries 0 and ∞ in terms of p, α, β.

15.2. A diffusion $X(t)$, $t \geq 0$, with the state space $(0, \infty)$ used in a population growth model is characterized by the drift $\mu(x) = \alpha x$ and the diffusion coefficient $\sigma^2(x) = \beta x + \gamma x^2$. Classify the boundary 0 under the various assumptions on $\beta \geq 0$, $\gamma \geq 0$ with $\beta + \gamma > 0$ and $-\infty < \alpha < \infty$.

§ 16. Special diffusions

1. Brownian motion with linear drift.

A *Brownian motion with linear drift* $W^{(\mu)}(t) = \mu t + W(t)$, $t \geq 0$, is a homogeneous diffusion with the drift coefficient μ and the diffusion coefficient 1.

The generator of this diffusion is

$$\mathbb{L} = \frac{1}{2}\frac{d^2}{dx^2} + \mu \frac{d}{dx}, \qquad x \in \mathbf{R}.$$

The domain $\mathcal{D}$, where this operator is defined, consists of bounded twice continuously differentiable functions on $\mathbf{R}$ such that $\mathbb{L}f$ is a bounded continuous function ($\mathcal{D} = \{f : f, \mathbb{L}f \in C_b(\mathbf{R})\}$).

The scale function is $S(x) = \dfrac{1 - e^{-2\mu x}}{2\mu}$, $x \in \mathbf{R}$, if $\mu \neq 0$, and $S(x) = x$, if $\mu = 0$.

The speed density is $m(x) = 2e^{2\mu x}$, $x \in \mathbf{R}$.

For the negative drift $\mu < 0$, since $S(-\infty) = 1/2\mu > -\infty$ and

$$\Sigma(-\infty) = \int_{-\infty}^x \frac{-e^{-2\mu y}}{2\mu}\, 2\, e^{2\mu y}\, dy = \infty,$$

the boundary $-\infty$ is attractive, but is not attainable.

If $\mu = 0$ (the case of Brownian motion), the boundary $l = -\infty$ is not attractive. Since $\Sigma(l) = \infty$, $N(l) = \infty$, the boundary $l = -\infty$ is natural.

The transition density with respect to the Lebesgue measure has (see (11.17) Ch. I) the form

$$p(t,x,y) = \frac{d}{dy}\mathbf{P}(W^{(\mu)}(t) < y | W^{(\mu)}(0) = x) = \frac{1}{\sqrt{2\pi t}} e^{\mu(y-x) - \mu^2 t/2 - (y-x)^2/2t}.$$

2. Reflected Brownian motion.

Let W be a Brownian motion. Then $W_+(t) := |W(t)|$, $t \geq 0$, is the Brownian motion on $[0,\infty)$ reflected at 0, or a *reflected Brownian motion.*

Proposition 16.1. *For a symmetric Markov process $X(t)$, $t \geq 0$, ($-X(t)$ has the same finite-dimensional distributions) and for an even function f, the composition $f(X(t))$, $t \geq 0$, is a Markov process.*

Proof. It is sufficient to prove this statement for $f(x) = |x|$. We can consider a Brownian motion W, because only the symmetry and the Markov property are used. For $s < t$ and arbitrary sets $\Delta \in \mathcal{B}[0,\infty)$, $B \in \sigma(|W(v)|, 0 \leq v < s)$, it is sufficient to check that

$$\mathbf{P}\big(|W(t)| \in \Delta \big| B \textstyle\bigcap \{|W(s)| = y\}\big) = \mathbf{P}\big(|W(t)| \in \Delta \big| |W(s)| = y\big). \tag{16.1}$$

By the definition of the conditional probability,

$$\mathbf{P}\big(|W(t)| \in \Delta \big| B \textstyle\bigcap \{|W(s)| = y\}\big) = \frac{\mathbf{P}(|W(t)| \in \Delta, B, |W(s)| \in dy)}{\mathbf{P}(B, |W(s)| \in dy)}$$

$$= \frac{\mathbf{P}(|W(t)| \in \Delta, B, W(s) \in dy) + \mathbf{P}(|W(t)| \in \Delta, B, W(s) \in -dy)}{\mathbf{P}(B, W(s) \in dy) + \mathbf{P}(B, W(s) \in -dy)}.$$

Using the symmetry, we have

$$\mathbf{P}(|W(t)| \in \Delta, B, W(s) \in dy) = \mathbf{P}(|W(t)| \in \Delta, B, W(s) \in -dy).$$

Therefore,

$$\mathbf{P}\big(|W(t)| \in \Delta \big| B \textstyle\bigcap \{|W(s)| = y\}\big) = \frac{\mathbf{P}(|W(t)| \in \Delta, B, W(s) \in dy)}{\mathbf{P}(B, W(s) \in dy)}$$

$$= \mathbf{P}\big(|W(t)| \in \Delta \big| B \textstyle\bigcap \{W(s) = y\}\big) = \mathbf{P}\big(|W(t)| \in \Delta \big| W(s) = y\big).$$

In the last equality we apply the Markov property of the Brownian motion. This equality holds also for $B = \Omega$. Since the right-hand side is independent of B, the equality (16.1) holds. □

The reflected Brownian motion W_+ is a diffusion with the state space $[0,\infty)$. The point 0 is the reflecting boundary.

The generator has the form

$$\mathbb{L} = \frac{1}{2}\frac{d^2}{dx^2}, \qquad x > 0,$$

with domain $\mathcal{D} = \{f : f, \mathbb{L}f \in C_b([0,\infty)), f'(0+) = 0\}$.

The transition density with respect to the Lebesgue measure is

$$p(t,x,y) = \frac{d}{dy}\mathbf{P}(|W(t)| < y | W(0) = x) = \frac{d}{dy}\mathbf{P}(-y < W(t) < y | W(0) = x)$$

$$= \frac{d}{dy}\int_{-y}^{y} \frac{1}{\sqrt{2\pi t}} e^{-(z-x)^2/2t}\,dz = \frac{1}{\sqrt{2\pi t}}\big(e^{-(y-x)^2/2t} + e^{-(y+x)^2/2t}\big), \quad x,y \in [0,\infty).$$

3. Geometric (or exponential) Brownian motion.

Let $W(t)$, $t \geq 0$, be a Brownian motion with $W(0) = 0$. For $x > 0$ the process

$$V(t) := x\,\exp((\mu - \sigma^2/2)t + \sigma\,W(t)), \qquad t \geq 0.$$

is the *geometric* or (*exponential*) *Brownian motion* starting at x and depending on the parameters $\mu \in \mathbf{R}$ and $\sigma > 0$. It is a nonnegative process often used to describe interest rates in financial mathematics.

The stochastic differential of the process V is

$$dV(t) = \big(\mu - \frac{\sigma^2}{2}\big)V(t)\,dt + \sigma V(t)\,dW(t) + \frac{\sigma^2}{2}V(t)\,dt = \mu V(t)\,dt + \sigma V(t)\,dW(t).$$

Therefore, the process V satisfies in $(0, \infty)$ the linear stochastic differential equation

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dW(t), \qquad V(0) = x.$$

Then the generator has the form

$$\mathbb{L} = \frac{1}{2}\,\sigma^2 x^2 \frac{d^2}{dx^2} + \mu\,x \frac{d}{dx}, \qquad x > 0,$$

with domain $\mathcal{D} = \{f : f, \mathbb{L}f \in C_b((0, \infty))\}$.

Since $B(x) = \int^x \frac{2\mu z}{\sigma^2 z^2}\,dz = \frac{2\mu}{\sigma^2}\ln x$, the scale function is

$$S(x) = \int^x z^{-2\mu/\sigma^2}\,dz = \begin{cases} \dfrac{x^{1-2\mu/\sigma^2}}{1 - 2\mu/\sigma^2}, & \text{if} \quad 2\mu/\sigma^2 \neq 1, \\ \ln x, & \text{if} \quad 2\mu/\sigma^2 = 1, \end{cases}$$

and the speed density is $m(x) = \frac{2}{\sigma^2}x^{2\mu/\sigma^2 - 2}$, $x \in (0, \infty)$.

Both boundaries 0 and ∞ are natural.

The transition density of the geometric Brownian motion with respect to the Lebesgue measure is computed as follows

$$p(t, x, y) = \frac{d}{dy}\mathbf{P}(V(t) < y | V(0) = x) = \frac{d}{dy}\mathbf{P}\big((\mu - \sigma^2/2)t + \sigma W(t) < \ln(y/x)\big)$$

$$= \frac{1}{y|\sigma|}\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{(\ln(y/x) - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t}\Big).$$

4. Ornstein–Uhlenbeck process.

Let $W(t)$, $t \geq 0$, be a Brownian motion with $W(0) = x/\sigma$, $x \in \mathbf{R}$, $\sigma > 0$. Set for $\gamma > 0$

$$U(t) := \sigma e^{-\gamma t} W\Big(\frac{e^{2\gamma t} - 1}{2\gamma}\Big), \qquad t \geq 0. \tag{16.2}$$

The process U is called an *Ornstein–Uhlenbeck process* with the initial value x.

One often considers the analogous process with a random initial value that is independent of the process U and having a Gaussian distribution with mean zero and variance σ^2. This is a strictly stationary process $U_s(t) := \sigma e^{-\gamma t} W(e^{2\gamma t})$, $t \geq 0$, $W(0) = 0$.

The process U is a Gaussian process with the mean $\mathbf{E}U(t) = xe^{-\gamma t}$, $t \geq 0$, and the covariance

$$\mathrm{Cov}(U(s), U(t)) = \sigma^2 e^{-\gamma t} \frac{\mathrm{sh}(\gamma s)}{\gamma}, \qquad \text{for} \quad s \leq t.$$

Using the nonrandom time change (see § 8 Ch. II), we have

$$U(t) = xe^{-\gamma t} + \sigma e^{-\gamma t} \Big(W\Big(\int_0^t e^{2\gamma s}\, ds \Big) - W(0) \Big) = xe^{-\gamma t} + \sigma e^{-\gamma t} \int_0^t e^{\gamma s}\, d\widetilde{W}(s),$$

where $\widetilde{W}$ is another Brownian motion.

The stochastic differential of the process U is

$$dU(t) = -\gamma x e^{-\gamma t}\, dt - \gamma \sigma e^{-\gamma t} \int_0^t e^{\gamma s}\, d\widetilde{W}(s)\, dt + \sigma d\widetilde{W}(t) = -\gamma U(t)\, dt + \sigma d\widetilde{W}(t).$$

Therefore, the process U satisfies in $\mathbf{R}$ the linear stochastic differential equation

$$dU(t) = -\gamma U(t)\, dt + \sigma d\widetilde{W}(t), \qquad U(0) = x. \tag{16.3}$$

Equation (16.3) has a physical interpretation. Let $U(t)$, $t \geq 0$, be the velocity of a particle of mass $1/\sigma$ suspended in a liquid. Then $\sigma^{-1} dU(t)$ is the change of momentum of the particle during the time dt. Let $-\gamma\sigma^{-1}U(t)$ be the viscous resistance force proportional to the velocity and, accordingly, $-\gamma\sigma^{-1}U(t)\, dt$ be the loss of momentum during dt due to the viscous force. Let $d\widetilde{W}(t)$ be the momentum transferred to the particle by molecular collisions during the time dt. Then the following equality holds:

$$\sigma^{-1} dU(t) = -\gamma \sigma^{-1} U(t)\, dt + d\widetilde{W}(t),$$

which is exactly (16.3).

From (16.3) it follows that the generator of the Ornstein–Uhlenbeck process has the form

$$\mathbb{L} = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} - \gamma x \frac{d}{dx}, \qquad x \in \mathbf{R},$$

with domain $\mathcal{D} = \{f : f, \mathbb{L}f \in C_b(\mathbf{R})\}$.

The fundamental solutions of equation (11.5) have (see Appendix 4, equation 19, $\gamma = 0$) the form

$$\psi_\lambda(x) = e^{\gamma x^2/2\sigma^2} D_{-\lambda/\gamma}\Big(-\frac{x\sqrt{2\gamma}}{\sigma} \Big), \qquad \varphi_\lambda(x) = e^{\gamma x^2/2\sigma^2} D_{-\lambda/\gamma}\Big(\frac{x\sqrt{2\gamma}}{\sigma} \Big).$$

The corresponding Green function has the form

$$G^{\circ}_{\lambda}(x,z)=\frac{1}{w^{\circ}_{\lambda}}\,e^{\gamma x^2/2\sigma^2}D_{-\lambda/\gamma}\Big(-\frac{x\sqrt{2\gamma}}{\sigma}\Big)e^{\gamma z^2/2\sigma^2}D_{-\lambda/\gamma}\Big(\frac{z\sqrt{2\gamma}}{\sigma}\Big),\qquad x\le z,$$

where the Wronskian is $w^{\circ}_{\lambda}=\dfrac{2\sigma\sqrt{\gamma\pi}}{\Gamma(\lambda/\gamma)}$.

Since $B(x)=\int^{x}\frac{-2\gamma z}{\sigma^2}\,dz=-\frac{\gamma x^2}{\sigma^2}$, the scale function is $S(x)=\int_0^{x}e^{\gamma y^2/\sigma^2}\,dy$.

The density of the speed measure is $m(x)=\dfrac{2}{\sigma^2}e^{-\gamma x^2/\sigma^2}$, $x\in\mathbf{R}$.

One can show that $\Sigma(\infty)=\infty$, $N(\infty)=\infty$. Therefore, both boundaries $-\infty$ and ∞ are natural like for the Brownian motion.

The transition density of the process U with respect to the Lebesgue measure is expressed by the formula

$$\begin{aligned}p(t,x,y)&=\frac{d}{dy}\mathbf{P}\Big(\sigma e^{-\gamma t}W\Big(\frac{e^{2\gamma t}-1}{2\gamma}\Big)<y\Big|W(0)=\frac{x}{\sigma}\Big)\\&=\frac{d}{dy}\mathbf{P}\Big(W\Big(\frac{e^{2\gamma t}-1}{2\gamma}\Big)-W(0)<\frac{e^{\gamma t}y-x}{\sigma}\Big)\\&=\frac{\sqrt{\gamma}e^{\gamma t}}{\sigma\sqrt{\pi(e^{2\gamma t}-1)}}\exp\Big(-\frac{\gamma(e^{\gamma t}y-x)^2}{\sigma^2(e^{2\gamma t}-1)}\Big).\end{aligned}$$

The bridge of the Ornstein–Uhlenbeck process (see Proposition 11.5 Ch. I) is represented in the form

$$U_{x,t,z}(s)=U(s)-\frac{\mathrm{Cov}(U(s),U(t))}{\mathrm{Var}\,U(t)}(U(t)-z)=U(s)-\frac{\mathrm{sh}(\gamma s)}{\mathrm{sh}(\gamma t)}(U(t)-z),\qquad s\in[0,t].$$

5. Bessel processes.

Let $\{W_l(s),s\ge0\}$, $l=1,2,\dots,n$, be a family of independent Brownian motions, $n\ge2$. The process $R^{(n)}$ defined by the formula

$$R^{(n)}(t):=\sqrt{W_1^2(t)+W_2^2(t)+\dots+W_n^2(t)},\qquad t\ge0,$$

is called an *n-dimensional Bessel process* or a *Bessel process of order $n/2-1$*. It is clear that $R^{(n)}$ is the *radial part* of the n-dimensional Brownian motion $\vec{W}(t)=(W_1(t),W_2(t),\dots,W_n(t))$.

Proposition 16.2. *Let $\mathcal{G}^k_t=\sigma(W_k(s),0\le s\le t)$ be the σ-algebra of events generated by the Brownian motion W_k, $k=1,2,\dots,n$. Let $f_k(t)$, $t\ge0$, be a progressively measurable process with respect to the filtration $\mathcal{G}^k_t$, $k=1,2,\dots,n$.*

Then there exists a Brownian motion $W(t)$, $t\ge0$, such that for any $t>0$ the variable $W(t)$ is measurable with respect to $\mathcal{G}_t:=\sigma\Big(\bigcup_{k=1}^{n}\mathcal{G}^k_t\Big)$, and

$$\sum_{k=1}^{n}\int_0^t f_k(s)\,dW_k(s)=\int_0^t\Big(\sum_{k=1}^{n}f_k^2(s)\Big)^{1/2}dW(s).\tag{16.4}$$

Proof. Set

$$W(t) := \sum_{k=1}^{n} \int_0^t \frac{f_k(v)}{\Big(\sum_{l=1}^{n} f_l^2(v)\Big)^{1/2}}\, dW_k(v).$$

Since every Brownian motion W_k is adapted to the filtration $\mathcal{G}_t$ and for $v > t$ the increments $W_k(v) - W_k(t)$ are independent of the σ-algebra $\mathcal{G}_t$, we have, by property (2.3) Ch. II, that for $s < t$

$$\mathbf{E}\{W(t) - W(s)|\mathcal{G}_s\} = 0 \qquad \text{a.s.}$$

Using the joint independence of W_k, $k = 1, 2, \dots, n$, and the property (2.4) Ch. II, we have

$$\mathbf{E}\{(W(t) - W(s))^2|\mathcal{G}_s\} = \sum_{k=1}^{n} \mathbf{E}\bigg\{\bigg(\int_s^t \frac{f_k(v)}{\Big(\sum_{l=1}^{n} f_l^2(v)\Big)^{1/2}} dW_k(v)\bigg)^2 \bigg|\mathcal{G}_s\bigg\}$$

$$= \sum_{k=1}^{n} \int_s^t \mathbf{E}\bigg\{\frac{f_k^2(v)}{\sum_{l=1}^{n} f_l^2(v)}\bigg|\mathcal{G}_s\bigg\}\, dv = t - s \qquad \text{a.s.}$$

Being a sum of stochastic integrals, W is a continuous process. By Lévy's characterization theorem (see § 10 Ch. I), the process W is a Brownian motion with respect to the filtration $\mathcal{G}_t$. The equality (16.4) becomes obvious upon substituting W. □

Consider the *squared Bessel process* $Y_n(t) := (R^{(n)}(t))^2$, $t \geq 0$. Using Itô's formula and (16.4) with $f_k(s) = 2W_k(s)$, we have

$$dY_n(t) = \sum_{k=1}^{n} 2W_k(t)\, dW_k(t) + ndt$$

$$= ndt + 2\Big(\sum_{k=1}^{n} W_k^2(t)\Big)^{1/2} dW(t) = ndt + 2\sqrt{Y_n(t)}\, dW(t).$$

Therefore, the process Y_n satisfies the stochastic differential equation

$$dY_n(t) = ndt + 2\sqrt{Y_n(t)}\, dW(t). \tag{16.5}$$

The generator of the n-dimensional squared Bessel process has the form

$$\mathbb{L}_2^{(n)} = 2x\frac{d^2}{dx^2} + n\frac{d}{dx}, \qquad x > 0. \tag{16.6}$$

Applying Itô's formula, we have

$$d\sqrt{Y_n(t)} = \Big(\frac{n}{2\sqrt{Y_n(t)}} - \frac{1}{2}\frac{Y_n(t)}{Y_n^{3/2}(t)}\Big)dt + dW(t) = \frac{n-1}{2\sqrt{Y_n(t)}}\, dt + dW(t).$$

Consequently, $R^{(n)}(t) = \sqrt{Y_n(t)}$, $t \geq 0$, satisfies the stochastic differential equation

$$dR^{(n)}(t) = \frac{n-1}{2R^{(n)}(t)}\,dt + dW(t).$$

This implies that for $n \geq 2$ the generator of the n-dimensional Bessel process has the form

$$\mathbb{L}^{(n)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{n-1}{2x}\frac{d}{dx}, \qquad x > 0,$$

with domain $\mathcal{D} = \Big\{f : f, \mathbb{L}f \in C_b([0,\infty)), \frac{d}{dS(x)}f(0+) = 0\Big\}$, where S is the scale function of the process $R^{(n)}(t)$, $t \geq 0$. The fundamental solutions of equation (11.5) are (see Appendix 4, equation 12, $\nu = n/2 - 1$)

$$\psi_\lambda(x) = x^{-n/2+1} I_{n/2-1}(x\sqrt{2\lambda}), \qquad \varphi_\lambda(x) = x^{-n/2+1} K_{n/2-1}(x\sqrt{2\lambda}).$$

The Green function has (see (11.7)) the form

$$G^\circ_\lambda(x,z) = x^{-n/2+1} z^{-n/2+1} I_{n/2-1}(x\sqrt{2\lambda}) K_{n/2-1}(z\sqrt{2\lambda}), \qquad x \leq z.$$

Since $B(x) = \int^x \frac{n-1}{z}\,dz = (n-1)\ln x$, for the scale function S we have

$$S(x) = \int^x e^{-(n-1)\ln y}\,dy = \int^x y^{1-n}\,dy = \begin{cases} \dfrac{x^{2-n}}{2-n}, & \text{if} \quad n \neq 2, \\ \ln x, & \text{if} \quad n = 2. \end{cases}$$

The density of the speed measure is $m(x) = 2x^{n-1}$, $x \geq 0$.

Since for $n > 2$

$$\Sigma(0) = \int_0^1\int_z^1 m(y)\,dy\,dS(z) = \frac{2}{n}\int_0^1 (1-z^n)z^{1-n}\,dz = \frac{2}{n}\Big(\int_0^1 z^{1-n}\,dz - \frac{1}{2}\Big) = \infty,$$

$$N(0) = \int_0^1\int_z^1 dS(y)m(z)\,dz = \int_0^1 (S(1)-S(z))m(z)\,dz = \frac{2}{2-n}\int_0^1 (1-z^{2-n})z^{n-1}\,dz$$

$$= \frac{2}{2-n}\int_0^1 z^{n-1}\,dz - \frac{1}{2-n} < \infty,$$

we have that 0 is an entrance boundary. For $n = 2$ we have the same situation.

Applying the inverse Laplace transform with respect to λ to the Green function, we get (see formula 28 of Appendix 3) that the transition density of the Bessel process $R^{(n)}$ with respect to the Lebesgue measure has the expression

$$p^{(n)}(t,x,y) = \frac{x}{t}\frac{y^{n/2}}{x^{n/2}}\exp\Big(-\frac{x^2+y^2}{2t}\Big) I_{n/2-1}\Big(\frac{xy}{t}\Big),$$

where $I_\nu(z) := \sum_{k=0}^\infty \frac{(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}$ is the modified Bessel function of order ν.

6. Radial Ornstein–Uhlenbeck processes.

Let $\{U_l(s), s\ge 0\}$, $l=1,2,\dots,n$, be a family of independent Ornstein–Uhlenbeck processes. The process $Q^{(n)}$ defined by the formula

$$Q^{(n)}(t) := \sqrt{U_1^2(t)+U_2^2(t)+\cdots+U_n^2(t)}, \qquad t\ge 0,$$

is called a *radial Ornstein–Uhlenbeck process* of order $n/2-1$.

It is clear that $Q^{(n)}$ is the *radial part* of n-dimensional Ornstein–Uhlenbeck process $\vec{U}(t) = (U_1(t), U_2(t),\dots,U_n(t))$.

Using the expression (16.2) for the Ornstein–Uhlenbeck process and the definition of the n-dimensional Bessel process $R^{(n)}$, we have

$$Q^{(n)}(t) = \sigma e^{-\gamma t} R^{(n)}\Big(\frac{e^{2\gamma t}-1}{2\gamma}\Big), \qquad t\ge 0, \quad \gamma>0, \quad \sigma>0. \tag{16.7}$$

Note that for $\sigma=1$ and $\gamma\to 0$ the process $Q^{(n)}(t)$, $t\ge 0$, is transformed to the Bessel process $R^{(n)}(t)$.

Consider the *squared radial Ornstein–Uhlenbeck process* $Z_n(t) := \big(Q^{(n)}(t)\big)^2$, $t\ge 0$. Using (16.3) and applying Itô's formula, we have

$$dU_k^2(t) = \big(\sigma^2 - 2\gamma U_k^2(t)\big)dt + 2\sigma U_k(t)\,dW_k(t), \qquad k=1,2,\dots,n,$$

for some independent Brownian motions W_k, and, consequently,

$$dZ_n(t) = \Big(n\sigma^2 - 2\gamma\sum_{k=1}^n U_k^2(t)\Big)dt + 2\sigma\sum_{k=1}^n U_k(t)\,dW_k(t).$$

Applying Proposition 16.2 for $f_k(s) = 2U_k(s)$, $s\ge 0$, we obtain that the process Z_n satisfies the stochastic differential equation

$$dZ_n(t) = \big(n\sigma^2 - 2\gamma Z_n(t)\big)dt + 2\sigma\sqrt{Z_n(t)}\,dW(t).$$

Thus the generator of the squared radial Ornstein–Uhlenbeck process is

$$\mathbb{L}_2^{(n)} = 2x\sigma^2\frac{d^2}{dx^2} + (n\sigma^2 - 2\gamma x)\frac{d}{dx}. \tag{16.8}$$

Applying Itô's formula, we have

$$dQ^{(n)}(t) = d\sqrt{Z_n(t)} = \Big(\frac{n\sigma^2 - 2\gamma Z_n(t) - \sigma^2}{2\sqrt{Z_n(t)}}\Big)dt + \sigma dW(t).$$

Consequently, the process $Q^{(n)}(t) = \sqrt{Z_n(t)}$, $t\ge 0$, satisfies the stochastic differential equation

$$dQ^{(n)}(t) = \Big(\frac{\sigma^2(n-1)}{2Q^{(n)}(t)} - \gamma Q^{(n)}(t)\Big)\,dt + \sigma dW(t).$$

This implies that the generator of the radial Ornstein–Uhlenbeck process for $n \geq 2$ has the form

$$\mathbb{L}^{(n)} = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + \Big(\frac{\sigma^2(n-1)}{2x} - \gamma x\Big)\frac{d}{dx}, \tag{16.9}$$

with domain $\mathcal{D} = \Big\{f : f, \mathbb{L}f \in C_b([0,\infty)), \frac{d}{dS(x)}f(0+) = 0\Big\}$, where S is the scale function of the process $Q^{(n)}(t)$, $t \geq 0$.

We can consider the generator $\mathbb{L}$ with an arbitrary parameter $\nu \geq 0$ instead of $\nu = n/2 - 1$, $n \in \mathbb{N}$, $n \geq 2$.

The fundamental solutions of equation (11.5) have (see Appendix 4, equation 22, $\nu = n/2 - 1$) the form

$$\psi_\lambda(x) = M\Big(\frac{\lambda}{2\gamma}, \nu+1, \frac{\gamma x^2}{\sigma^2}\Big), \qquad \varphi_\lambda(x) = U\Big(\frac{\lambda}{2\gamma}, \nu+1, \frac{\gamma x^2}{\sigma^2}\Big).$$

The Wronskian of these solutions is

$$w(\psi_\lambda(z), \varphi_\lambda(z)) = \frac{2\Gamma(\nu+1)\, e^{\gamma z^2}}{\Gamma(\lambda/2\gamma)\, z^{2\nu+1}\gamma^\nu}.$$

According to (11.6), the Laplace transform of the transition density of the radial Ornstein–Uhlenbeck process $Q^{(n)}$ with respect to the Lebesgue measure is given by the formula

$$\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(Q^{(n)}(t) < z)\, dt \tag{16.10}$$

$$= \begin{cases} \dfrac{\Gamma(\lambda/2\gamma)\, z^{2\nu+1}\gamma^\nu}{\Gamma(\nu+1)\sigma^{2\nu+2}\, e^{\gamma z^2/\sigma^2}} M\Big(\dfrac{\lambda}{2\gamma}, \nu+1, \dfrac{\gamma x^2}{\sigma^2}\Big)U\Big(\dfrac{\lambda}{2\gamma}, \nu+1, \dfrac{\gamma z^2}{\sigma^2}\Big), & 0 \leq x \leq z, \\ \dfrac{\Gamma(\lambda/2\gamma)\, z^{2\nu+1}\gamma^\nu}{\Gamma(\nu+1)\sigma^{2\nu+2}\, e^{\gamma z^2/\sigma^2}} U\Big(\dfrac{\lambda}{2\gamma}, \nu+1, \dfrac{\gamma x^2}{\sigma^2}\Big)M\Big(\dfrac{\lambda}{2\gamma}, \nu+1, \dfrac{\gamma z^2}{\sigma^2}\Big), & z \leq x. \end{cases}$$

One can compute (see formula 33 of Appendix 3) the inverse Laplace transform with respect to λ, and thus obtain the following expression for the transition density of the process $Q^{(n)}$ with respect to the Lebesgue measure:

$$p^{(n)}(t,x,z) = \frac{\gamma z^{\nu+1} e^{\gamma t(\nu+1)}}{\sigma^2 x^\nu \operatorname{sh}(\gamma t)} \exp\Big(\frac{\gamma(x^2 - z^2)}{2\sigma^2} - \frac{\gamma(x^2+z^2)\operatorname{ch}(\gamma t)}{2\sigma^2\operatorname{sh}(\gamma t)}\Big) I_\nu\Big(\frac{\gamma x z}{\sigma^2 \operatorname{sh}(\gamma t)}\Big).$$

Since $B(x) = \int^x \Big(\frac{2\nu+1}{z} - \frac{2\gamma z}{\sigma^2}\Big)\, dz = (2\nu+1)\ln x - \gamma x^2/\sigma^2$, the scale function S equals

$$S(x) = \int^x y^{-2\nu-1}\exp(\gamma y^2/\sigma^2)\, dy.$$

The density of the speed measure is $m(x) = \frac{2}{\sigma^2}x^{2\nu+1}e^{-\gamma x^2/\sigma^2}$, $x \geq 0$.

7. Hyperbolic Bessel process.

The *hyperbolic Bessel process* (we denote it by $H_\theta^{(\nu)}(t)$, $t \ge 0$) is the special case ($\gamma = 0$) of the hypergeometric diffusion, defined in Subsection 9. The state space of this process is the nonnegative half-line. Let $\nu > -1$ and $\theta > 0$. The generator of the hyperbolic Bessel process has the form

$$\mathbb{L}f(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) + \big(\nu + \frac{1}{2}\big)\theta \operatorname{cth}(\theta x)\frac{d}{dx}f(x), \quad x > 0. \tag{16.11}$$

Let $\mu := \sqrt{(\nu + \frac{1}{2})^2 + \frac{2\lambda}{\theta^2}}$. Then, according to equation (23) of Appendix 4, the fundamental solutions of equation (11.5) are

$$\psi_\lambda(x) = \frac{P^{-\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)}, \qquad \varphi_\lambda(x) = \frac{\widetilde{Q}^{\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)}.$$

The Wronskian of these solutions has the form

$$w\Big(\frac{P^{-\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)}, \frac{\widetilde{Q}^{\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)}\Big) = \frac{\theta}{\operatorname{sh}^{2\nu+1}(\theta x)}.$$

The Legendre functions $P^{-\nu}_{\mu-1/2}$ and $\widetilde{Q}^{\nu}_{\mu-1/2}$ are defined in Appendix 2, section 12.

Let us examine these fundamental solutions in details. Since $(\mu - \nu - 1/2)(\mu + \nu + 1/2) = 2\lambda/\theta^2$, using the formula (see Appendix 2 section 12)

$$\frac{d}{dx}\big((x^2-1)^{q/2}P^q_p(x)\big) = (p+q)(p-q+1)(x^2-1)^{(q-1)/2}P^{q-1}_p(x),$$

we get

$$\frac{d}{dx}\Big(\frac{P^{-\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)}\Big) = \frac{2\lambda}{\theta}\operatorname{sh}(\theta x)\frac{P^{-(\nu+1)}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^{\nu+1}(\theta x)}. \tag{16.12}$$

Therefore, ψ_λ is an increasing solution. Moreover, for $\nu > -1$

$$\lim_{x\downarrow 0}\psi_\lambda(x) = \lim_{x\downarrow 0}\frac{P^{-\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))}{\operatorname{sh}^\nu(\theta x)} = \lim_{y\downarrow 1}(y^2-1)^{-\nu/2}P^{-\nu}_{\mu-1/2}(y) = \frac{2^{-\nu}}{\Gamma(1+\nu)} \tag{16.13}$$

(see Bateman and Erdélyi (1954), formulas 3.4 (3) and 3.4 (4)).

According to the formula (see Appendix 2 Section 12)

$$\frac{d}{dx}\Big(\frac{\widetilde{Q}^q_p(x)}{(x^2-1)^{q/2}}\Big) = -\frac{\widetilde{Q}^{q+1}_p(x)}{(x^2-1)^{(q+1)/2}},$$

we have

$$\frac{d}{dz}\Big(\frac{\widetilde{Q}^{\nu}_{\mu-1/2}(\operatorname{ch}(\theta z))}{\operatorname{sh}^\nu(\theta z)}\Big) = -\theta\operatorname{sh}(\theta z)\frac{\widetilde{Q}^{\nu+1}_{\mu-1/2}(\operatorname{ch}(\theta z))}{\operatorname{sh}^{\nu+1}(\theta z)}. \tag{16.14}$$

Therefore, φ_λ is a decreasing solution. Using the definition of the function $\widetilde{Q}^{\nu}_{\mu-1/2}$, we have that for $-1 < \nu < 0$ there exists the limit

$$\lim_{x\downarrow 0}\varphi_\lambda(x) \to \frac{\sqrt{\pi}\Gamma(\nu+\mu+1/2)}{2^{\mu+1/2}\Gamma(\mu+1)}F\Big(\frac{\nu+\mu+1/2}{2}, \frac{\nu+\mu+3/2}{2}, \mu+1, 1\Big) = \frac{\Gamma(\nu+\mu+1/2)\Gamma(-\nu)}{2^{\nu+1}\Gamma(\mu-\nu+1/2)}.$$

For $\nu \ge 0$ there is no finite limit: $\lim_{x\downarrow 0}\varphi_\lambda(x) \to \infty$; and being bounded, the Green function is defined by (11.6) for $0 < x < z$ uniquely. On the contrary, for $-1 < \nu < 0$ both fundamental solutions are bounded at the point zero and to define the Green function uniquely we must impose some boundary condition at this point.

For the Green function $G^\nu_\mu(x,z)$, $0 < x < z$, $-1 < \nu < 0$, we impose the reflecting condition $\frac{d}{dS(x)}G^\nu_\mu(0+,z) = 0$, where $S(x)$, $x > 0$, is the scale function:

$$S(x) = \theta^{2\nu+1}\int^x \mathrm{sh}^{-2\nu-1}(\theta y)\,dy.$$

Note that the density of the speed measure has the form $m(z) = \frac{2\,\mathrm{sh}^{2\nu+1}(\theta z)}{\theta^{2\nu+1}}$.

Using (16.12), we have

$$\frac{d}{dS(x)}\psi_\lambda(x) = \frac{\mathrm{sh}^{2\nu+1}(\theta x)}{\theta^{2\nu+1}}\frac{d}{dx}\Big(\frac{P^{-\nu}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)}\Big) = \frac{2\lambda\,\mathrm{sh}^{2\nu+2}(\theta x)}{\theta^{2\nu+2}}\frac{P^{-(\nu+1)}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^{\nu+1}(\theta x)}.$$

In view of (16.13), we have $\frac{d}{dS(x)}\psi_\lambda(0+) = 0$ if $-1 < \nu < 0$, and the solution ψ_λ satisfies the reflecting boundary condition.

Finally for $-1 < \nu$, by (11.6), we have

$$G^\nu_\mu(x,z)\frac{2\,\mathrm{sh}^{2\nu+1}(\theta z)}{\theta^{2\nu+1}} := \int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(H^{(\nu)}_\theta(t) < z)\,dt$$

$$= \begin{cases} \dfrac{2\,\mathrm{sh}^{\nu+1}(\theta z)\,\widetilde{Q}^\nu_{\mu-1/2}(\mathrm{ch}(\theta z))\,P^{-\nu}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\theta\ \mathrm{sh}^\nu(\theta x)}, & \text{if}\quad 0 \le x \le z,\\[2ex] \dfrac{2\,\mathrm{sh}^{\nu+1}(\theta z)\,\widetilde{Q}^\nu_{\mu-1/2}(\mathrm{ch}(\theta x))\,P^{-\nu}_{\mu-1/2}(\mathrm{ch}(\theta z))}{\theta\ \mathrm{sh}^\nu(\theta x)}, & \text{if}\quad 0 < z \le x.\end{cases} \tag{16.15}$$

Here $G^\nu_\mu(x,z)$ is (see (11.7)) the Green function of the transition density of the hyperbolic Bessel process with respect to the speed measure. We denote by $p^\circ_\nu(t,x,z)$ the corresponding transition density with respect to the speed measure. A nice property here is that the Green function and the transition density are symmetric in the variables x and z.

The following integral representations hold (see Bateman and Erdélyi (1954), formulas 3.7(8) and 3.7(4)):

$$\frac{P^{-\nu}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)} = \frac{\sqrt{2}}{\sqrt{\pi}\Gamma(\nu+1/2)\,\mathrm{sh}^{2\nu}(\theta x)}\int_0^{\theta x}\frac{\mathrm{ch}(\mu y)}{(\mathrm{ch}(\theta x)-\mathrm{ch}\,y)^{1/2-\nu}}\,dy, \qquad \nu > -\frac12,$$

and

$$\frac{\widetilde{Q}^\nu_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)} = \frac{\sqrt{\pi}}{\sqrt{2}\Gamma(1/2-\nu)}\int_{\theta x}^\infty \frac{e^{-\mu y}}{(\mathrm{ch}\,y-\mathrm{ch}(\theta x))^{\nu+1/2}}\,dy, \quad \mu+\nu+\frac12 > 0, \qquad \nu < 1/2.$$

In view of (16.13), from (16.15) and the integral representation for $\widetilde{Q}^{\nu}_{\mu-1/2}(\operatorname{ch}(\theta x))$ we obtain

$$\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_0(H^{(\nu)}_\theta(t)<z)\,dt=\frac{2^{1-\nu}\sqrt{\pi}\operatorname{sh}^{2\nu+1}(\theta z)}{\theta\sqrt{2}\Gamma(\nu+1)\Gamma(1/2-\nu)}\int_{\theta z}^\infty\frac{e^{-\mu y}}{(\operatorname{ch}y-\operatorname{ch}(\theta z))^{\nu+1/2}}\,dy.$$

Computing the inverse Laplace transform with respect to λ of the function

$$e^{-\mu y}=\exp\Big(-y\sqrt{(\nu+\tfrac{1}{2})^2+\tfrac{2\lambda}{\theta^2}}\Big)$$

(see Appendix 3, formulas a and 2), we conclude that for $-1<\nu<1/2$

$$p^\circ_\nu(t,0,z)\frac{2\operatorname{sh}^{2\nu+1}(\theta z)}{\theta^{2\nu+1}}=\frac{d}{dz}\mathbf{P}_0(H^{(\nu)}_\theta(t)<z)$$

$$=\frac{\operatorname{sh}^{2\nu+1}(\theta z)\,e^{-(\nu+1/2)^2\theta^2t/2}}{2^\nu\Gamma(\nu+1)\Gamma(1/2-\nu)t^{3/2}}\int_z^\infty\frac{y\,e^{-y^2/2t}}{(\operatorname{ch}(\theta y)-\operatorname{ch}(\theta z))^{\nu+1/2}}\,dy. \tag{16.16}$$

About this formula see also Gruet (1997).

Similarly, using the integral representations, we deduce from (16.15) that for $-1/2<\nu<1/2$ and $x\le z$

$$p^\circ_\nu(t,x,z)\frac{2\operatorname{sh}^{2\nu+1}(\theta z)}{\theta^{2\nu+1}}=\frac{d}{dz}\mathbf{P}_x(H^{(\nu)}_\theta(t)<z)$$

$$=\frac{\sqrt{2}\theta\operatorname{sh}^{2\nu+1}(\theta z)\,e^{-(\nu+1/2)^2\theta^2t/2}}{\sqrt{\pi}\operatorname{sh}^{2\nu}(\theta x)\Gamma(\nu+1/2)\Gamma(1/2-\nu)t^{3/2}} \tag{16.17}$$

$$\times\int_0^x dy_1\int_z^\infty dy_2\frac{e^{-(y_1^2+y_2^2)/2t}(\operatorname{ch}(\theta x)-\operatorname{ch}(\theta y_1))^{\nu-1/2}\Big(y_2\operatorname{ch}\big(\frac{y_1y_2}{t}\big)-y_1\operatorname{sh}\big(\frac{y_1y_2}{t}\big)\Big)}{(\operatorname{ch}(\theta y_2)-\operatorname{ch}(\theta z))^{\nu+1/2}}.$$

The values $\nu=-1/2$ and $\nu=1/2$ are critical parameters. The value $\nu=-1/2$ corresponds to the reflected Brownian motion, and $\nu=1/2$ corresponds to the hyperbolic Bessel process of dimension 3, for which

$$\frac{d}{dz}\mathbf{P}_x(H^{(1/2)}_\theta(t)<z)=\frac{\operatorname{sh}(\theta z)\,e^{-\theta^2t/2}}{\operatorname{sh}(\theta x)\sqrt{2\pi t}}\Big(e^{-(z-x)^2/2t}-e^{-(z+x)^2/2t}\Big). \tag{16.18}$$

For $\nu\in(k+1/2,k+3/2]$, $k=0,1,\dots$, an expression for the transition density can be derived as follows. From (16.15) and (16.13) it follows that

$$G^\nu_{\mu_\nu}(0,z)=\frac{\theta^{2\nu}}{2^\nu}\frac{\widetilde{Q}^\nu_{\mu_\nu-1/2}(\operatorname{ch}(\theta z))}{\Gamma(1+\nu)\operatorname{sh}^\nu(\theta z)}.$$

The index ν in the notation μ_ν indicates that the parameter ν is included in the expression for μ. From (16.14) one can deduce that

$$\frac{d}{dz}G^\nu_{\mu_\nu}(0,z)=-\frac{2(1+\nu)\operatorname{sh}(\theta z)}{\theta}G^{\nu+1}_{\mu_\nu}(0,z). \tag{16.19}$$

Since $\mu_{\nu+1} = \sqrt{(\nu+\frac{1}{2})^2 + \frac{2}{\theta^2}(\lambda + (\nu+1)\theta^2)}$, the inverse Laplace transform of the function $G^{\nu+1}_{\mu_{\nu+1}}(0,z)$ with respect to λ differs from that of the function $G^{\nu+1}_{\mu_\nu}(0,z)$ by the factor $e^{-(\nu+1)\theta^2 t}$. Inverting in (16.19) the Laplace transform with respect to λ, we derive (see Appendix 3, formula a) that

$$p^\circ_{\nu+1}(t,0,z) = -\frac{\theta e^{-(\nu+1)\theta^2 t}}{2(1+\nu)\,\mathrm{sh}(\theta z)}\frac{d}{dz}p^\circ_\nu(t,0,z). \tag{16.20}$$

From (16.15) it follows that

$$G^\nu_{\mu_\nu}(x,z) = \theta^{2\nu}\frac{P^{-\nu}_{\mu_\nu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)}\frac{\widetilde{Q}^\nu_{\mu_\nu-1/2}(\mathrm{ch}(\theta z))}{\mathrm{sh}^\nu(\theta z)}.$$

Further, (16.12) and (16.14) yield

$$\frac{d}{dx}\frac{d}{dz}G^\nu_{\mu_\nu}(x,z) = -\frac{2\lambda}{\theta^2}\,\mathrm{sh}(\theta x)\,\mathrm{sh}(\theta z)G^{\nu+1}_{\mu_\nu}(x,z). \tag{16.21}$$

Inverting in this equality the Laplace transform with respect to λ, we have

$$p^\circ_{\nu+1}(t,x,z) = -\frac{\theta^2 e^{-(\nu+1)\theta^2 t}}{2\,\mathrm{sh}(\theta x)\,\mathrm{sh}(\theta z)}\frac{d}{dx}\frac{d}{dz}\int_0^t p^\circ_\nu(s,x,z)\,ds. \tag{16.22}$$

The recurrence formulas (16.20) and (16.22) enable us to compute expressions for the transition density of the hyperbolic Bessel process for ν from the interval $(k+1/2, k+3/2]$, $k=0,1,\dots$, using (16.16)–(16.18) for the interval $(-1/2,1/2]$.

8. Hyperbolic Ornstein–Uhlenbeck process.

The *hyperbolic Ornstein–Uhlenbeck process* $U^{(\rho)}_\theta(t)$, $t \ge 0$, $\rho > -1$, and $\theta > 0$, has the generator

$$\mathbb{L}f(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) - \rho\theta\,\mathrm{th}(\theta x)\frac{d}{dx}f(x), \quad x \in \mathbf{R}. \tag{16.23}$$

The state space of this process is the whole real line. For $\rho = \gamma/\theta^2$ in the limiting case as $\theta \downarrow 0$ this process becomes the Ornstein–Uhlenbeck process (see Subsection 4, $\sigma \equiv 1$).

Let $\mu := -\sqrt{\rho^2 + 2\lambda/\theta^2}$. Then $-\mu+\rho > 0$, $\mu+\rho < 0$, and according to equation (24) of Appendix 4, and the properties of the Legendre functions (see Appendix 2), the fundamental solutions of equation (11.5) are

$$\psi_\rho(x) = \mathrm{ch}^\rho(\theta x)\widetilde{P}^\mu_\rho(-\mathrm{th}(\theta x)), \qquad \varphi_\rho(x) = \mathrm{ch}^\rho(\theta x)\widetilde{P}^\mu_\rho(\mathrm{th}(\theta x)).$$

The Wronskian of these solutions is

$$w\big(\psi_\rho(x), \varphi_\rho(x)\big) = \frac{2\theta\,\mathrm{ch}^{2\rho}(\theta x)}{\Gamma(-\mu-\rho)\Gamma(1-\mu+\rho)}.$$

By (11.6), we have

$$G^\mu_\rho(x,z):=\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(U^{(\rho)}_\theta(t)<z)\,dt$$

$$=\frac{\Gamma(1-\mu+\rho)\Gamma(-\mu-\rho)}{\theta\operatorname{ch}^\rho(\theta z)\operatorname{ch}^{-\rho}(\theta x)}\begin{cases}\widetilde P^\mu_\rho(\operatorname{th}(\theta z))\,\widetilde P^\mu_\rho(-\operatorname{th}(\theta x)), & \text{if}\quad x\le z,\\ \widetilde P^\mu_\rho(\operatorname{th}(\theta x))\,\widetilde P^\mu_\rho(-\operatorname{th}(\theta z)), & \text{if}\quad z\le x.\end{cases} \tag{16.24}$$

Here $G^\mu_\rho(x,z)$ is the Green function of the hyperbolic Ornstein–Uhlenbeck process with respect to the Lebesgue measure. Let $p^\theta_\rho(t,x,z):=\frac{d}{dz}\mathbf{P}_x(U^{(\rho)}_\theta(t)<z)$ be the corresponding transition density with respect to the Lebesgue measure. It is easy to compute that $m_\rho(z)=\dfrac{2}{\operatorname{ch}^{2\rho}(\theta z)}$ is the speed measure of this process. The function $\frac{1}{2}p^\theta_\rho(t,x,z)\operatorname{ch}^{2\rho}(\theta z)$ is the transition density with respect to the speed measure; it is a symmetric function with respect to x,z (see (11.7)).

For the critical parameters $\rho=0$ and $\rho=-1$, the Green function has the expressions $G^\mu_0(x,z)=\frac{1}{\sqrt{2\lambda}}e^{-|x-z|\sqrt{2\lambda}}$ and

$$G^\mu_{-1}(x,z)=\frac{\operatorname{ch}(\theta z)}{\operatorname{ch}(\theta x)\sqrt{\theta^2+2\lambda}}\,e^{-|x-z|\sqrt{\theta^2+2\lambda}}.$$

The parameter value $\rho=0$ corresponds to the Brownian motion, while $\rho=-1$ corresponds to the hyperbolic Ornstein–Uhlenbeck process with the transition density

$$\frac{d}{dz}\mathbf{P}_x(U^{(-1)}_\theta(t)<z)=\frac{\operatorname{ch}(\theta z)}{\operatorname{ch}(\theta x)\sqrt{2\pi t}}\,e^{-\theta^2t/2-(z-x)^2/2t}. \tag{16.25}$$

To compute the transition density for $-1<\rho<0$ we can use the following approach. From the integral representation for the Legendre functions (see Appendix 2) it follows with the help of integration by substitution that for $-1<x<1$, $\mu+\rho<0$, and $\rho>-1$

$$\Gamma(-\rho-\mu)\widetilde P^\mu_\rho(\operatorname{th}x)=\frac{1}{\Gamma(1+\rho)\operatorname{ch}^\rho x}\int_x^\infty e^{u\mu}(\operatorname{sh}u-\operatorname{sh}x)^\rho du. \tag{16.26}$$

Then using (16.24) and the equality $\widetilde P^\mu_\rho(x)=\widetilde P^\mu_{-\rho-1}(x)$, we derive that for $x\le z$

$$\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(U^{(\rho)}_\theta(t)<z)\,dt=\frac{\Gamma(-\mu-\rho)\widetilde P^\mu_\rho(-\operatorname{th}(\theta x))}{\theta\operatorname{ch}^{-\rho}(\theta x)}\,\frac{\Gamma(1-\mu+\rho)\widetilde P^\mu_{-\rho-1}(\operatorname{th}(\theta z))}{\operatorname{ch}^\rho(\theta z)}. \tag{16.27}$$

Computing the inverse Laplace transform with respect to λ of the function $e^{\mu\theta r}=\exp\big(-r\sqrt{\rho^2\theta^2+2\lambda}\big)$ (see Appendix 3, formulas a and 2), we get

$$\mathcal{L}^{-1}_\lambda\big(e^{\mu\theta r}\big)=\frac{1}{\sqrt{2\pi}t^{3/2}}r\,e^{-\rho^2\theta^2t/2}\,e^{-r^2/2t}.$$

Using (16.27), the integral representation (16.26), and the relation

$$\Gamma(1+\rho)\Gamma(-\rho) = \frac{\pi}{\sin(-\pi\rho)},$$

we conclude that for $-1 < \rho < 0$ and $x \le z$

$$p_\rho^\theta(t,x,z) := \frac{d}{dz}\mathbf{P}_x(U_\theta^{(\rho)}(t) < z) \tag{16.28}$$

$$= \frac{\theta\sin(-\pi\rho)\operatorname{ch}(\theta z)\, e^{-\rho^2\theta^2 t/2}}{\sqrt{2}(\pi t)^{3/2}} \int_{-x}^{\infty} du \int_{z}^{\infty} dv (u+v)\, e^{-(u+v)^2)/2t} \frac{(\operatorname{sh}(\theta u)+\operatorname{sh}(\theta x))^\rho}{(\operatorname{sh}(\theta v)-\operatorname{sh}(\theta z))^{\rho+1}}.$$

For $\rho \in (k, k+1]$, $k = 0,1,\dots$, an expression for the transition density can be computed in the following way. From (16.24) it follows that

$$\frac{d}{dz}\frac{d}{dx}\frac{G_\rho^\mu(x,z)}{\operatorname{ch}(\theta z)\operatorname{ch}^{2\rho+1}(\theta x)} = \frac{\Gamma(1-\mu+\rho)\Gamma(-\mu-\rho)}{\theta}\frac{d}{dz}\frac{\widetilde{P}_\rho^\mu(\operatorname{th}(\theta z))}{\operatorname{ch}^{\rho+1}(\theta z)}\frac{d}{dx}\frac{\widetilde{P}_\rho^\mu(-\operatorname{th}(\theta x))}{\operatorname{ch}^{\rho+1}(\theta x)}.$$

From the corresponding property of the Legendre functions (see Appendix 2) we derive that

$$\frac{d}{dx}\frac{\widetilde{P}_\rho^\mu(\operatorname{th} x)}{\operatorname{ch}^{\rho+1} x} = (\mu-\rho-1)\frac{\widetilde{P}_{\rho+1}^\mu(\operatorname{th} x)}{\operatorname{ch}^{\rho+1} x}.$$

Applying this formula, we have

$$\frac{d}{dz}\frac{d}{dx}\frac{G_\rho^{\mu_\rho}(x,z)}{\operatorname{ch}(\theta z)\operatorname{ch}^{2\rho+1}(\theta x)} = -\frac{\theta^2(\mu_\rho^2-(\rho+1)^2)}{\operatorname{ch}^{2\rho+2}(\theta x)}G_{\rho+1}^{\mu_\rho}(x,z). \tag{16.29}$$

The index ρ in the notation μ_ρ indicates that the parameter ρ is included in the expression for μ. Since $\rho^2 = (\rho+1)^2 - 2\rho - 1$, the inverse Laplace transform with respect to λ of an arbitrary function of the argument $\mu_{\rho+1}$ differs from that of the same function of the argument μ_ρ by the factor $e^{(2\rho+1)\theta^2 t/2}$ (see Appendix 3, formula a). Thus, inverting in (16.29) the Laplace transform, we obtain

$$\frac{d}{dz}\frac{d}{dx}\Big(\frac{p_\rho^\theta(t,x,z)}{\operatorname{ch}(\theta z)\operatorname{ch}^{2\rho+1}(\theta x)}\Big) = \frac{d}{dz}\frac{d}{dx}\Big(\frac{1}{\operatorname{ch}(\theta z)\operatorname{ch}^{2\rho+1}(\theta x)}\mathcal{L}_\lambda^{-1}\big(G_\rho^{\mu_\rho}(x,z)\big)\Big)$$

$$= -\frac{\theta^2}{\operatorname{ch}^{2\rho+2}(\theta x)}\mathcal{L}_\lambda^{-1}\big((\mu_\rho^2-(\rho+1)^2)G_{\rho+1}^{\mu_\rho}(x,z)\big)$$

$$= -\frac{\theta^2 e^{(2\rho+1)\theta^2 t/2}}{\operatorname{ch}^{2\rho+2}(\theta x)}\mathcal{L}_\lambda^{-1}\big((\mu_{\rho+1}^2-(\rho+1)^2)G_{\rho+1}^{\mu_{\rho+1}}(x,z)\big)$$

$$= -\frac{e^{(2\rho+1)\theta^2 t/2}}{\operatorname{ch}^{2\rho+2}(\theta x)}\mathcal{L}_\lambda^{-1}\big(2\lambda G_{\rho+1}^{\mu_{\rho+1}}(x,z)\big)$$

$$= -\frac{2e^{(2\rho+1)\theta^2 t/2}}{\operatorname{ch}^{2\rho+2}(\theta x)}\Big(\frac{d}{dt}p_{\rho+1}^\theta(t,x,z) + p_{\rho+1}^\theta(0+,x,z)\delta_0(t)\Big),$$

where $\delta_0(t)$ is the Dirac δ-function. Integrating the left-hand and the right-hand sides of this relation with respect to t, we finally get

$$\frac{p^\theta_{\rho+1}(t,x,z)}{\operatorname{ch}^{2\rho+2}(\theta x)} = -\frac{1}{2}\int\limits_0^t e^{-(2\rho+1)\theta^2 s/2}\frac{d}{dz}\frac{d}{dx}\frac{p^\theta_\rho(s,x,z)}{\operatorname{ch}(\theta z)\operatorname{ch}^{2\rho+1}(\theta x)}\,ds. \tag{16.30}$$

This recurrence formula enables us to compute expressions for the transition density of the hyperbolic Ornstein–Uhlenbeck process for ρ from the interval $(k, k+1]$, $k = 0, 1, \dots,$ using expression (16.28) for the interval $(-1, 0)$ and the Brownian density for $\rho = 0$.

The *reflected hyperbolic Ornstein–Uhlenbeck process* $|U^{(\rho)}_\theta(t)|$, $t \ge 0$, is the special case $(\nu = -1/2, \rho = \gamma/\theta^2)$ of the hypergeometric diffusion defined in Subsection 9. The state space of this process is the nonnegative half-line.

9. Hypergeometric diffusion.

The *hypergeometric diffusion* $X^{(\nu)}_{\gamma,\theta}$, whose state space is the nonnegative real half-line, is determined by the generating operator

$$\mathbb{L}f(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) + \Big(\big(\nu+\tfrac{1}{2}\big)\theta\operatorname{cth}(\theta x) - \frac{\gamma}{\theta}\operatorname{th}(\theta x)\Big)\frac{d}{dx}f(x), \quad x > 0. \tag{16.31}$$

Equation (11.5) for this operator has the form

$$\tfrac{1}{2}\phi''(x) + \Big(\big(\nu+\tfrac{1}{2}\big)\theta\operatorname{cth}(\theta x) - \frac{\gamma}{\theta}\operatorname{th}(\theta x)\Big)\phi'(x) - \lambda\phi(x) = 0, \qquad x > 0. \tag{16.32}$$

For $\nu > -1$, $\gamma \ge 0$, $\theta > 0$, $\lambda > 0$ we consider the fundamental solutions of this equation (see Appendix 4, equation 26),

$$\psi_\lambda(x) = \frac{1}{\operatorname{ch}^{2\alpha}(\theta x)}F\big(\alpha,\beta,\nu+1,\operatorname{th}^2(\theta x)\big) \tag{16.33}$$

and

$$\varphi_\lambda(x) = \frac{\Gamma(\beta-\nu)}{\operatorname{ch}^{2\alpha}(\theta x)}G\big(\alpha,\beta,\nu+1,\operatorname{th}^2(\theta x)\big), \tag{16.34}$$

where

$$\alpha := \frac{1}{2}\sqrt{\Big(\frac{\gamma}{\theta^2}-\nu-\frac{1}{2}\Big)^2 + \frac{2\lambda}{\theta^2}} - \frac{1}{2}\Big(\frac{\gamma}{\theta^2}-\nu-\frac{1}{2}\Big) \qquad \text{and} \quad \beta := \alpha + \frac{1}{2} + \frac{\gamma}{\theta^2}.$$

It is easy to check that $\alpha > 0$, $\beta > \nu + 1$.

The Wronskian of these solutions is

$$w(\psi_\lambda(x), \varphi_\lambda(x)) = \frac{2\theta\Gamma(\beta-\nu)\Gamma(\nu+1)\operatorname{ch}^{2\gamma/\theta^2}(\theta x)}{\Gamma(\alpha)\Gamma(\beta)\operatorname{sh}^{2\nu+1}(\theta x)}. \tag{16.35}$$

The hypergeometric functions $F(\alpha,\beta,\nu+1,x)$ and $G(\alpha,\beta,\nu+1,x)$, $0 < x < 1$ are defined in Appendix 2, Section 11.

For $\beta > \nu + 1$ the formula (see Appendix 2, Section 11)

$$\frac{d}{dx}\big((1-x)^\alpha F(\alpha,\beta,\nu+1,x)\big) = \frac{\alpha(\beta-\nu-1)}{\nu+1}(1-x)^{\alpha-1}F(\alpha+1,\beta,\nu+2,x)$$

implies that

$$\psi'_\lambda(x) = \frac{2\theta\alpha(\beta-\nu-1)\operatorname{th}(\theta x)}{(\nu+1)\operatorname{ch}^{2\alpha}(\theta x)} F\big(\alpha+1,\beta,\nu+2,\operatorname{th}^2(\theta x)\big) > 0. \tag{16.36}$$

Therefore, ψ_λ is a strictly increasing solution of (16.32). Moreover, $\lim_{x\downarrow 0}\psi_\lambda(x) = 1$ and for $\nu > -1$

$$\frac{d}{dS(x)}\psi_\lambda(x) = \frac{2\operatorname{sh}^{2\nu+2}(\theta x)}{\theta^{2\nu}\operatorname{ch}^{2\alpha+1+2\gamma/\theta^2}(\theta x)} F\big(\alpha+1,\beta,\nu+2,\operatorname{th}^2(\theta x)\big) \underset{x\downarrow 0}{\longrightarrow} 0, \tag{16.37}$$

where $S(x)$, $x > 0$, is the scale function:

$$S(x) = \theta^{2\nu+1}\int^x \frac{\operatorname{ch}^{2\gamma/\theta^2}(\theta y)}{\operatorname{sh}^{2\nu+1}(\theta y)}\,dy.$$

This expression follows from the equality

$$2\int^x \Big((\nu+\tfrac12)\theta\operatorname{cth}(\theta y) - \frac{\gamma}{\theta}\operatorname{th}(\theta y)\Big)\,dy = \ln\left(\frac{\operatorname{sh}^{2\nu+1}(\theta x)}{\operatorname{ch}^{2\gamma/\theta^2}(\theta x)}\right) + \text{const}, \qquad x > 0.$$

The factor $\theta^{2\nu+1}$ is introduced to guarantee the existence of the limit as $\theta \downarrow 0$.

Note that the density of the speed measure has the form $m(x) = \dfrac{2\operatorname{sh}^{2\nu+1}(\theta x)}{\theta^{2\nu+1}\operatorname{ch}^{2\gamma/\theta^2}(\theta x)}$.

From the formula (see Appendix 2, Section 11)

$$\frac{d}{dx}\big((1-x)^\alpha G(\alpha,\beta,\nu+1,x)\big) = -\alpha(1-x)^{\alpha-1}G(\alpha+1,\beta,\nu+2,x)$$

it follows that

$$\varphi'_\lambda(x) = -\frac{2\theta\alpha\Gamma(\beta-\nu)\operatorname{th}(\theta x)}{\operatorname{ch}^{2\alpha}(\theta x)} G\big(\alpha+1,\beta,\nu+2,\operatorname{th}^2(\theta x)\big) < 0.$$

Therefore, φ_λ is a strictly decreasing solution of equation (16.32). Moreover, from the definition of the function G (see Appendix 2, Section 11) it follows that for $-1 < \nu < 0$

$$\lim_{x\downarrow 0}\varphi_\lambda(x) = \Gamma(\beta-\nu)G(\alpha,\beta,\nu+1,0) = \frac{\Gamma(-\nu)}{\Gamma(\alpha-\nu)}.$$

It is clear that $\lim_{x\downarrow 0}\varphi_\lambda(x) = \infty$ for $\nu \ge 0$.

Hence, for $-1 < \nu < 0$ both fundamental solutions are bounded at the point zero and to define the Green function uniquely we should impose some boundary condition at this point. For the Green function $G^\nu_\mu(x,z)$, $0 \le x < z$, we impose the reflecting condition $\frac{d}{dS(x)}G^{\nu}_{\mu}(0+,z)=0$. Since, by (16.37), the solution ψ_λ satisfies this condition, from (11.6) we have

$$G^{\nu}_{\mu}(x,z)\,m(z):=\int_0^\infty e^{-\lambda t}\frac{d}{dz}\mathbf{P}_x(X^{(\nu)}_{\gamma,\theta}(t)<z)\,dt \tag{16.38}$$

$$=\begin{cases}\dfrac{\Gamma(\alpha)\Gamma(\beta)\,\mathrm{sh}^{2\nu+1}(\theta z)F\big(\alpha,\beta,\nu+1,\mathrm{th}^2(\theta x)\big)G\big(\alpha,\beta,\nu+1,\mathrm{th}^2(\theta z)\big)}{\theta\Gamma(\nu+1)\,\mathrm{ch}^{2\alpha+2\gamma/\theta^2}(\theta z)\,\mathrm{ch}^{2\alpha}(\theta x)}, & \text{if} \quad \begin{array}{l}0\le x,\\ x\le z,\end{array}\\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)\,\mathrm{sh}^{2\nu+1}(\theta z)F\big(\alpha,\beta,\nu+1,\mathrm{th}^2(\theta z)\big)G\big(\alpha,\beta,\nu+1,\mathrm{th}^2(\theta x)\big)}{\theta\Gamma(\nu+1)\,\mathrm{ch}^{2\alpha+2\gamma/\theta^2}(\theta z)\,\mathrm{ch}^{2\alpha}(\theta x)}, & \text{if} \quad z\le x.\end{cases}$$

Let $H_z:=\inf\{t: X^{(\nu)}_{\gamma,\theta}(t)=z\}$. According to (12.38) Ch. II,

$$\mathbf{P}_x(H_a<H_b)=\frac{\displaystyle\int_x^b\frac{\mathrm{ch}^{2\gamma/\theta^2}(\theta y)}{\mathrm{sh}^{2\nu+1}(\theta y)}\,dy}{\displaystyle\int_a^b\frac{\mathrm{ch}^{2\gamma/\theta^2}(\theta y)}{\mathrm{sh}^{2\nu+1}(\theta y)}\,dy},\qquad a\le x\le b. \tag{16.39}$$

Let us dwell on some particular cases. For $\gamma=0$ we have $\alpha=\frac{\mu+\nu}{2}+\frac{1}{4}$, $\beta=\frac{\mu+\nu}{2}+\frac{3}{4}$, where μ is defined in Subsection 7. In this case we have the fundamental solutions, corresponding to the hyperbolic Bessel process (see Appendix 2, Section 12):

$$\psi_\lambda(x)=\frac{1}{\mathrm{ch}^{\mu+\nu+1/2}(\theta x)}F\Big(\frac{\mu+\nu}{2}+\frac{1}{4},\frac{\mu+\nu}{2}+\frac{3}{4},\nu+1,1-\frac{1}{\mathrm{ch}^2(\theta x)}\Big)$$

$$=\frac{2^\nu\Gamma(\nu+1)P^{-\nu}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)},$$

$$\varphi_\lambda(x)=\frac{\Gamma(\frac{\mu-\nu}{2}+\frac{3}{4})}{\mathrm{ch}^{\mu+\nu+1/2}(\theta x)\Gamma(\mu+1)}F\Big(\frac{\mu+\nu}{2}+\frac{1}{4},\frac{\mu+\nu}{2}+\frac{3}{4},\mu+1,\frac{1}{\mathrm{ch}^2(\theta x)}\Big)$$

$$=\frac{2^{1-\nu}\Gamma(\frac{\mu-\nu}{2}+\frac{3}{4})}{\Gamma(\frac{\mu+\nu}{2}+\frac{1}{4})\Gamma(\frac{\mu+\nu}{2}+\frac{3}{4})}\frac{\widetilde{Q}^{\nu}_{\mu-1/2}(\mathrm{ch}(\theta x))}{\mathrm{sh}^\nu(\theta x)}.$$

As the next case we consider the limiting behavior of the hypergeometric diffusion as $\theta\downarrow 0$. In this case, the operator (16.31) is transformed to the operator of the radial Ornstein–Uhlenbeck process determined by (16.9) with $\sigma\equiv 1$, $\nu=\frac{n}{2}-1$. We verify that the transition density (16.38) is transformed to that given by formula (16.10) with $\sigma\equiv 1$. It is easily seen that the following relations hold as $\theta\downarrow 0$: $\alpha\sim\frac{\lambda}{2\gamma}$, $\beta\sim\frac{\gamma}{\theta^2}$,

$$\mathrm{ch}^{2\beta}(\theta z)\sim\Big(1+\frac{\theta^2z^2}{2}\Big)^{2\gamma/\theta^2}\sim e^{\gamma z^2},\qquad\text{and}\qquad\frac{\Gamma(\beta)}{\Gamma(\beta-\nu)}\sim\Big(\frac{\gamma}{\theta^2}\Big)^\nu.$$

Due to the asymptotic behavior of the hypergeometric functions (see Appendix 2, Section 11), we have

$$\psi_\lambda(x) = \frac{1}{\mathrm{ch}^{2\alpha}(\theta x)} F\big(\alpha, \beta, \nu+1, \mathrm{th}^2(\theta x)\big) \to M\Big(\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big)$$

and

$$\varphi_\lambda(x) = \frac{\Gamma(\beta-\nu)}{\mathrm{ch}^{2\alpha}(\theta x)} G\big(\alpha, \beta, \nu+1, \mathrm{th}^2(\theta x)\big) \to U\Big(\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\Big)$$

as $\theta \downarrow 0$.

Using these formulas, it is easy to check that the limit as $\theta \downarrow 0$ on the right-hand side of (16.38) coincides with (16.10) for $\sigma \equiv 1$. Therefore, the transition density of the hypergeometric diffusion converges as $\theta \downarrow 0$ to the transition density of the radial Ornstein–Uhlenbeck process. The densities of the speed measure and the scale function of the hypergeometric diffusion also converge as $\theta \downarrow 0$ to the corresponding characteristics of the radial Ornstein–Uhlenbeck process.

Exercises.

16.1. Let $X(t)$, $t \in [0, \infty)$, be a diffusion in the natural scale $S(x) = x$ with a given density of the speed measure $m(x)$, which is continuous and positive. Compute the diffusion coefficients $\mu(x)$ and $\sigma^2(x)$.

16.2. Let X be a regular diffusion with the state space $[0, 1]$ and the diffusion coefficient $\sigma^2(x) = x^2(1-x)^2$. Prove that the process

$$Y(t) := \ln \frac{X(t)}{1 - X(t)}$$

has a constant diffusion coefficient.

16.3. Let $X(t)$, $t \in [0, \infty)$, be a diffusion with the state space $(0, \infty)$, with the drift coefficient $\mu(x) = \alpha x + r$, $r > 0$, and with the diffusion coefficient $\sigma^2(x) = 4x$. What is the form of the drift and the diffusion coefficients for the process $\sqrt{X(t)}$?

16.4. Let $X(t)$, $t \in [0, \infty)$, be a diffusion with the state space $(-\infty, \infty)$, with the drift coefficient $\mu(x) = \operatorname{sign} x$, and with the diffusion coefficient $\sigma^2(x) = 1$. Compute the scale function $S(x)$ and the density of the speed measure $m(x)$, $x \in \mathbf{R}$.

16.5. Let $V(t)$, $t \in [0, \infty)$, be a geometric Brownian motion, i.e.,

$$dV(t) = \mu V(t)\, dt + \sigma V(t)\, dW(t), \qquad V(0) = x_0.$$

Compute the generator of the diffusion $V^\gamma(t)$, $\gamma \in \mathbf{R}$.

16.6. Let $V(t)$, $t \in [0, \infty)$, be a geometric Brownian motion with $\mu < \sigma^2/2$. Since $V(t) = x \exp((\mu - \sigma^2/2)\, t + \sigma\, W(t))$, we have $V(t) \to 0$ as $t \to \infty$. What is the probability that the process V with $V(0) = x < b$ ever attains b?

Hint:

$$\mathbf{P}_x(H_b < \infty) = \lim_{a\downarrow 0} \mathbf{P}_x(H_b < H_a).$$

16.7. Let $U(t)$, $t \in [0,\infty)$, be the Ornstein–Uhlenbeck process with the infinitesimal parameters $\mu(x) = -\gamma x$ and $\sigma^2(x) = \sigma^2$. Compute the infinitesimal parameters of the process $Z(t) = \exp(U(t))$.

16.8. Derive the expression of the transition density of the process $Q^{(n)}$ from the expression of the transition density of the Bessel process $R^{(n)}$ and the representation (16.7).

§ 17. Homogeneous diffusions with identical bridges

In this section we consider sufficient conditions for two homogeneous diffusions to have the same bridges if they have identical diffusion coefficients, but different drift coefficients. The definition of the bridge of a process was given at the beginning of § 11 Ch. I.

We use the notations of § 15. Consider the processes $X(t)$ and $Y(t)$, $t \geq 0$, whose state space is the interval (l,r), where $-\infty \leq l < r \leq \infty$. Assume that the boundaries cannot be attained by each of these processes in a finite time. So for the process X that is a solution of the stochastic differential equation (15.1), this means that $H_n \to \infty$ a.s. or, equivalently, for every $x \in (l,r)$

$$Q_n(x) := \mathbf{E}_x e^{-\alpha H_n} \to 0 \qquad \text{for any} \quad \alpha > 0. \tag{17.1}$$

Formally, this means that the boundaries can be entrance boundaries for the processes, but not exit ones. In this case we do not consider the process starting from the boundary.

Let us determine when (17.1) holds. We apply Theorem 7.3. According to Remark 4.1 of this chapter, for $\sigma(x) > 0$, $x \in (l,r)$ and $\alpha > 0$ the homogeneous equation

$$\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - \alpha\phi(x) = 0, \qquad x \in (l,r), \tag{17.2}$$

has two linearly independent nonnegative strictly monotone solutions ψ and φ such that ψ increases and φ decreases. Suppose that

$$\lim_{l_n\downarrow l} \varphi(l_n) = \infty \qquad \text{and} \qquad \lim_{r_n\uparrow r} \psi(r_n) = \infty. \tag{17.3}$$

By Theorem 7.3, the function Q_n is the solution of (7.8), (7.9) with $f \equiv \alpha$, $a = l_n$ and $b = r_n$. This solution has the form

$$Q_n(x) = \frac{\psi(r_n)\varphi(x) - \psi(x)\varphi(r_n)}{\psi(r_n)\varphi(l_n) - \psi(l_n)\varphi(r_n)} + \frac{\psi(x)\varphi(l_n) - \psi(l_n)\varphi(x)}{\psi(r_n)\varphi(l_n) - \psi(l_n)\varphi(r_n)}. \tag{17.4}$$

Under the assumption (17.3)

$$\lim_{n\to\infty} Q_n(x) = \lim_{n\to\infty} \left(\frac{\varphi(x) - \psi(x)\varphi(r_n)/\psi(r_n)}{\varphi(l_n) - \psi(l_n)\varphi(r_n)/\psi(r_n)} + \frac{\psi(x) - \varphi(x)\psi(l_n)/\varphi(l_n)}{\psi(r_n) - \varphi(r_n)\psi(l_n)/\varphi(l_n)} \right) = 0.$$

Thus the condition (17.3) is sufficient for the diffusion X to fail reaching the boundary of the interval (l,r) in a finite time.

Let $\phi(x)$, $x \in (l,r)$, be a nonnegative solution of equation (17.2). For example, ϕ is a linear combination of solutions ψ and φ with nonnegative coefficients.

For our purposes the following condition is essential: for all $t \geq 0$ and $x \in (l,r)$

$$\mathbf{E}_x \phi(X(t)) < \infty. \tag{17.5}$$

Under certain assumptions it may happen that

$$\mathbf{E}_x \phi(X(t)) = e^{\alpha t}\phi(x). \tag{17.6}$$

The function ϕ satisfying this equality is called *α-invariant* for the process X. Equality (17.6) holds, for example, under the assumptions of Theorem 13.1 Ch. II (the backward Kolmogorov equation). Indeed, in view of (13.3), (13.4) Ch. II, the function $u(t,x) := \mathbf{E}_x \phi(X(t))$ is the unique solution of the problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\,\sigma^2(x)\frac{\partial^2}{\partial x^2}u(t,x) + \mu(x)\frac{\partial}{\partial x}u(t,x), \tag{17.7}$$

$$u(0,x) = \phi(x). \tag{17.8}$$

Now, taking into account equation (17.2), it is easy to verify that $u(t,x) = e^{\alpha t}\phi(x)$ satisfies the problem (17.7), (17.8). This proves (17.6).

In the proof of Theorem 17.1, we will verify that (17.5) implies (17.6), whenever ϕ is a solution of (17.2).

Let $X(t)$ and $Y(t)$, $t \geq 0$, be the solutions of the following homogeneous stochastic differential equations

$$dX(t) = \sigma(X(t))\,dW(t) + \mu(X(t))\,dt, \tag{17.9}$$

$$dY(t) = \sigma(Y(t))\,dW(t) + \eta(Y(t))\,dt \tag{17.10}$$

with the same initial values $X(0) = Y(0) = x$. Suppose that the coefficients σ, μ, η satisfy the conditions of § 15. Assume also that the processes X and Y have continuous transition densities. For the homogeneous diffusion the existence of the transition density was proved in § 10.

Theorem 17.1. *Suppose that ϕ is a nonnegative solution of the homogeneous equation* (17.2) *for some $\alpha > 0$. Assume also that for this solution* (17.5) *holds. Let*

$$\frac{\eta(x) - \mu(x)}{\sigma^2(x)} = \frac{\phi'(x)}{\phi(x)} = (\ln(\phi(x)))', \qquad x \in (l,r). \tag{17.11}$$

Then for any x, t, and z the processes X and Y have the same bridges $X_{x,t,z}(s)$ and $Y_{x,t,z}(s)$, $s \in [0,t]$, i.e., the finite-dimensional distributions of these bridges coincide.

Proof. We use Girsanov's transformation (Theorem 10.3 Ch. II). According to this theorem, for every $t \geq 0$ and for any bounded measurable functional $\wp(Z(s), 0 \leq s \leq t)$ on $C([0,t])$ we have

$$\mathbf{E}_x \wp(Y(s), 0 \leq s \leq t) = \mathbf{E}_x\big\{\wp\big(X(s), 0 \leq s \leq t\big)\rho(t)\big\}, \tag{17.12}$$

where

$$\rho(t) := \exp\Big(\int_0^t (\ln(\phi(X(s))))'\sigma(X(s))\,dW(s) - \frac{1}{2}\int_0^t \Big(\frac{\phi'(X(s))}{\phi(X(s))}\Big)^2\sigma^2(X(s))\,ds\Big).$$

We transform this stochastic exponent. By Itô's formula,

$$\ln(\phi(X(t))) - \ln(\phi(x)) = \int_0^t (\ln(\phi(X(s))))'\sigma(X(s))\,dW(s)$$

$$+\int_0^t \frac{\phi'(X(s))}{\phi(X(s))}\mu(X(s))\,ds + \frac{1}{2}\int_0^t \Big(\frac{\phi'(X(s))}{\phi(X(s))}\Big)'\sigma^2(X(s))\,ds.$$

Expressing the stochastic integral from this equality and substituting it into $\rho(t)$, we get

$$\rho(t) := \frac{\phi(X(t))}{\phi(x)}\exp\Big(-\int_0^t \Big(\frac{\phi'(X(s))}{\phi(X(s))}\mu(X(s)) + \frac{1}{2}\frac{\phi''(X(s))}{\phi(X(s))}\sigma^2(X(s))\Big)ds\Big).$$

Now, taking into account equation (17.2), we have

$$\rho(t) = e^{-\alpha t}\frac{\phi(X(t))}{\phi(x)}. \tag{17.13}$$

Since $\mathbf{E}\phi(X(t)) < \infty$ for all $t > 0$, then the stochastic exponent $\rho(t)$, $t \ge 0$, is a martingale, and application of the Girsanov transformation is correct without any additional assumptions of Theorem 10.3 Ch. II.

Since $\rho(t)$, $t \ge 0$, is a martingale, $\mathbf{E}_x\rho(t) = \mathbf{E}_x\rho(0) = 1$. This and (17.13) imply (17.6).

We choose arbitrary bounded piecewise-continuous function $\varPhi(x)$, $x \in (l, r)$. Taking in (17.12) instead of the functional $\wp(Z(s), 0 \le s \le t)$ the functional $\varPhi(Z(t))\wp(Z(s), 0 \le s \le t)$, we have

$$\mathbf{E}_x\big\{\varPhi(Y(t))\wp(Y(s), 0\le s\le t)\big\} = \frac{e^{-\alpha t}}{\phi(x)}\mathbf{E}_x\big\{\phi(X(t))\varPhi(X(t))\wp\big(X(s), 0\le s\le t\big)\big\}. \tag{17.14}$$

From this equality it is easy to deduce that for every $z \in (l, r)$

$$\frac{d}{dz}\mathbf{E}_x\big\{\wp(Y(s), 0\le s\le t); Y(t) < z\big\} = \frac{e^{-\alpha t}\phi(z)}{\phi(x)}\frac{d}{dz}\mathbf{E}_x\big\{\wp\big(X(s), 0\le s\le t\big); X(t) < z\big\}. \tag{17.15}$$

Indeed, choosing in (17.14) the family of functions $\varPhi_\Delta(x) := \frac{1}{\Delta}\mathbb{I}_{[z,z+\Delta)}(x)$, $0 < \Delta < 1$, and using the fact that the processes X and Y have continuous transition densities, we can pass in (17.14) to the limit as $\Delta \downarrow 0$ and prove (17.15).

Letting in (17.15) $\wp \equiv 1$, we obtain for the transition densities the equality

$$\frac{d}{dz}\mathbf{P}_x\big(Y(t)<z\big) = \frac{e^{-\alpha t}\phi(z)}{\phi(x)}\frac{d}{dz}\mathbf{P}_x\big(X(t)<z\big). \tag{17.16}$$

Dividing equation (17.15) by (17.16), we get

$$\mathbf{E}_x\big\{\wp(Y(s),0\le s\le t)\big|Y(t)=z\big\} = \mathbf{E}_x\big\{\wp\big(X(s),0\le s\le t\big)\big|X(t)=z\big\}. \tag{17.17}$$

Thus the processes X and Y have the same bridges, i.e., their bridges are identical in law. □

We consider some examples of processes that have the same bridges.

1. Processes with bridges identical to the Brownian bridge. For $l=-\infty$, $r=\infty$, $\mu(x)\equiv 0$ and $\sigma(x)\equiv 1$ the process $X(t)$, defined by (17.9), is the Brownian motion W with the initial value $W(0)=x$. We describe the class of processes whose bridges coincide with the bridge of the Brownian motion. For different $c>0$ and $\delta\in\mathbf{R}$, a nonnegative solution of equation (17.2) is $\phi(x) = c\,\mathrm{ch}((x-\delta)\sqrt{2\alpha})$, $x\in\mathbf{R}$. This formula includes the solutions $e^{\pm x\sqrt{2\alpha}}$ if we take into account the limiting case $c=e^{\pm\delta\sqrt{2\alpha}}$ as $\delta\to\infty$.

Obviously, $\dfrac{\phi'(x)}{\phi(x)} = \sqrt{2\alpha}\,\mathrm{th}((x-\delta)\sqrt{2\alpha})$. It is natural to extend the definition of this function for $\delta=\pm\infty$ by the equality $\dfrac{\phi'(x)}{\phi(x)} = \mp\sqrt{2\alpha}$.

Let $Y_{\delta,\alpha}$ be a solution of the stochastic differential equation

$$dY(t) = dW(t) + \sqrt{2\alpha}\,\mathrm{th}\big((Y(t)-\delta)\sqrt{2\alpha}\big)\,dt, \qquad Y(0)=x, \tag{17.18}$$

From Theorem 17.1 it follows that for $\delta\in\mathbf{R}\bigcup\{\pm\infty\}$ and $\alpha>0$ the bridges of the homogeneous diffusions $Y_{\delta,\alpha}$ coincide with the Brownian bridge. The Brownian motion with linear drift $\mp\sqrt{2\alpha}$ corresponds to the values $\delta=\pm\infty$.

Formula (17.16) in this example has the form

$$\frac{d}{dz}\mathbf{P}_x(Y(t)<z) = \frac{\mathrm{ch}((z-\delta)\sqrt{2\alpha})}{\sqrt{2\pi t}\,\mathrm{ch}((x-\delta)\sqrt{2\alpha})}e^{-\alpha t}e^{-(z-x)^2/2t}. \tag{17.19}$$

The corresponding Green function is

$$\frac{d}{dz}\mathbf{P}_x(Y(\tau)<z) = \frac{\lambda\,\mathrm{ch}((z-\delta)\sqrt{2\alpha})}{\sqrt{2\lambda+2\alpha}\,\mathrm{ch}((x-\delta)\sqrt{2\alpha})}e^{-|z-x|\sqrt{2\lambda+2\alpha}}dz. \tag{17.20}$$

2. Processes with bridges identical to the bridge of an n-dimensional Bessel process. The Bessel process $R^{(n)}(t)$, $t\ge 0$, $n\ge 2$, satisfies (see Subsection 5 § 16) the stochastic differential equation

$$dR^{(n)}(t) = dW(t) + \frac{n-1}{2R^{(n)}(t)}\,dt.$$

The drift coefficient $\mu(x) = \dfrac{n-1}{2x}$, $x>0$, has a singularity at the origin. For $n\ge 2$ the Bessel process, starting from a nonnegative point, never hits zero.

Equation (17.2) with $\mu(x) = \dfrac{n-1}{2x}$, has (see Appendix 4, equation 12) the two linearly independent solutions

$$\psi(x) = x^{1-n/2} I_{n/2-1}(x\sqrt{2\alpha}) \qquad \text{and} \qquad \varphi(x) = x^{1-n/2} K_{n/2-1}(x\sqrt{2\alpha}),$$

where I_ν and K_ν are modified Bessel functions. Since it is necessary that the singularity of the function $\eta(x)$ in (17.10) be the same as for the function $\mu(x)$, $x > 0$, only the first fundamental solution is suitable. Therefore, the drift coefficient

$$\eta(x) = \frac{n-1}{2x} + \frac{\sqrt{2\alpha} I_{n/2}(x\sqrt{2\alpha})}{I_{n/2-1}(x\sqrt{2\alpha})}, \qquad x > 0, \qquad \alpha > 0, \tag{17.21}$$

determines the class of processes Y, whose bridges coincide with the bridges of an n-dimensional Bessel process, $n \geq 2$.

Remark 17.1. In the important special case $n = 3$ the drift coefficient is given by $\eta(x) = \sqrt{2\alpha}\,\mathrm{cth}(x\sqrt{2\alpha})$, $x > 0$.

For a three-dimensional Bessel process the relation (17.16) transforms into

$$\frac{d}{dz}\mathbf{P}_x\big(Y(t) < z\big) = \frac{x\,\mathrm{sh}(z\sqrt{2\alpha})}{z\,\mathrm{sh}(x\sqrt{2\alpha})} e^{-\alpha t} \frac{d}{dz}\mathbf{E}_x\big(R^{(3)}(t) < z\big)$$

$$= \frac{\mathrm{sh}(z\sqrt{2\alpha}) e^{-\alpha t}}{\sqrt{2\pi t}\,\mathrm{sh}(x\sqrt{2\alpha})} \Big(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\Big). \tag{17.22}$$

BROWNIAN LOCAL TIME

§ 1. Brownian local time

The local time of a Brownian motion occupies a special place in the theory of local times of stochastic processes. Among the reasons for this are, first, the fact that it allows the construction of a theory very rich in content and, second, that in its example we can see features of the behavior of local times of more general processes, in particular, stable processes and diffusions. The study of different properties of local time is often based on knowledge of distributions of various functionals of its sample paths. Therefore, the development of methods for computing distributions of functionals forms the main part of the investigation.

The definition of the local time of a measurable stochastic process $X(s)$, $s \geq 0$, with values in the Euclidean space $\mathbf{R}$ was given in § 5 Ch. II. Using Itô's stochastic differentiation formula we proved that for a Brownian motion W there exists the local time $\ell(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$. Moreover, we derived Tanaka's formula (see (5.6) Ch. II).

The process $\ell(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$, is called the local time (in other words, the time at a point), because for each x it characterizes the amount of time the process $W(s)$, $s \geq 0$, spends at x up to the time t. For many stochastic processes the Lebesgue measure of the time the process spends at an individual point up to some moment is equal to zero, and the most natural nonzero scale for such a time turned out to be the derivative of the sojourn measure of the process with respect to the Lebesgue measure. For an individual sample path of the process W and for a specific moment of time t, the local time $\ell(t,x)$ is not uniquely determined, since it can be assigned different values on sets of the Lebesgue measure zero. In this connection it is natural to choose a version of the local time $\ell(t,x)$ that has the nicest properties, for example, $\ell(t,x)$ is a continuous process of two parameters t and x.

In § 5 Ch. II we used the theory of stochastic integration to prove the existence of the Brownian local time. This is a rather special technique. Here we consider a more natural approach to the problem of existence of the local time of the Brownian motion $W(s)$, $s \geq 0$, $W(0) = 0$.

Theorem 1.1. *The process W has an* a.s. *continuous local time $\ell(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$.*

Proof. It is obvious that the support of the random measure

$$\mu_t(A) := \int_0^t \mathbb{I}_A(W(s))ds, \qquad A \in \mathcal{B}(\mathbf{R}), \qquad 0 \leq t \leq T,$$

is contained in the set

$$\Big\{x : \inf_{0\leq s\leq t} W(s) \leq x \leq \sup_{0\leq s\leq t} W(s)\Big\}.$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_5

Here $\mathbb{I}_A(\cdot)$ is the indicator function of the Borel set A.

Applying the theorem of integration by substitution we obtain that for any bounded measurable function f there holds (see (5.4) Ch. II) the equality

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)\mu_t(dx). \tag{1.1}$$

In particular, if there exists a local time $\ell(t,x)$, then

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)\ell(t,x)\,dx. \tag{1.2}$$

Taking for f the function $e^{i\lambda x}$, $x \in \mathbf{R}$, we have

$$\int_{-\infty}^{\infty} e^{i\lambda x}\ell(t,x)\,dx = \int_0^t e^{i\lambda W(s)}\,ds.$$

The left-hand side of this equality is the Fourier transform of the density $\ell(t,x)$. Inverting this transform, we get

$$\ell(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_0^t e^{i\lambda W(s)}\,dsd\lambda.$$

This inversion formula can be given a rigorous mathematical meaning in the setting of the L_2-theory and we essentially do this below. Thus as an approximate value for $\ell(t,x)$ we can take

$$\ell^{(n)}(t,x) := \frac{1}{2\pi}\int_{-n}^{n} e^{-i\lambda x}\int_0^t e^{i\lambda W(s)}\,ds\,d\lambda = \frac{1}{\pi}\int_0^t\int_0^n \cos(\lambda(W(s)-x))\,d\lambda\,ds,$$

where $n = 1,2,\dots$. We remark that $\ell_n(t,x)$ is a real-valued process.

Lemma 1.1. *There exists a stochastic process $\ell(t,x)$, $(t,x) \in [0,T]\times\mathbf{R}$, such that for any $T > 0$*

$$\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbf{R}} \mathbf{E}|\ell^{(n)}(t,x)-\ell(t,x)|^2 = 0. \tag{1.3}$$

Proof. We verify that $\ell^{(n)}(t,x)$, $n=1,2,\dots$, is a Cauchy sequence in the norm $\|\cdot\| := \sup_{(t,x)\in[0,T]\times\mathbf{R}} \mathbf{E}^{1/2}|\cdot|^2$. Indeed, if $m>n$, then

$$\mathbf{E}|\ell^{(m)}(t,x)-\ell^{(n)}(t,x)|^2 \le \frac{1}{\pi^2}\int_n^m\int_n^m d\lambda d\zeta \int_0^t\int_0^t dudv\big|\mathbf{E}e^{i\lambda W(u)}e^{i\zeta W(v)}\big|$$

$$\le \frac{2}{\pi^2}\int_n^m\int_n^m d\lambda d\zeta \int_0^t du\int_0^u dv\big|\mathbf{E}e^{i(\lambda+\zeta)W(v)}\mathbf{E}e^{i\lambda(W(u)-W(v))}\big|$$

$$= \frac{2}{\pi^2}\int_n^m\int_n^m d\lambda d\zeta \int_0^t du\int_0^u dv\, e^{-v(\lambda+\zeta)^2/2}e^{-(u-v)\lambda^2/2}$$

$$\le \frac{8}{\pi^2}\int_n^m d\lambda\int_n^m d\zeta \frac{(1-e^{-t\lambda^2/2})}{\lambda^2}\frac{(1-e^{-t(\lambda+\zeta)^2/2})}{(\lambda+\zeta)^2}$$

$$\le \frac{8t}{\pi^2}\int_{-\infty}^{\infty}\frac{(1-e^{-\zeta^2/2})}{\zeta^2}d\zeta\int_{n\sqrt{t}}^{m\sqrt{t}}\frac{(1-e^{-\lambda^2/2})}{\lambda^2}d\lambda \le \frac{32t}{\pi^2}\int_{-\infty}^{\infty}\frac{d\zeta}{1+\zeta^2}\int_{n\sqrt{t}}^{m\sqrt{t}}\frac{d\lambda}{1+\lambda^2}. \tag{1.4}$$

Thus $\ell^{(n)}(t,x)$ is a Cauchy sequence in the norm $\|\cdot\|$, and hence Lemma 1.1 is proved. □

Remark 1.1. The process $\ell(t,x), (t,x)\in[0,\infty)\times\mathbf{R}$, has the following *scaling property*: for any fixed $c>0$ the finite-dimensional distributions of the process $\sqrt{c}\ell(t/c,x/\sqrt{c})$ coincide with those of $\ell(t,x)$.

Indeed, using the scaling property of the process W (the finite-dimensional distributions of the processes $\sqrt{c}W(s/c)$ and $W(s)$, $s\in[0,T]$, coincide), it is not hard to see that the finite-dimensional distributions of the processes $\sqrt{c}\ell^{(n\sqrt{c})}(t/c,x/\sqrt{c})$ and $\ell^{(n)}(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}$, coincide, and, in view of Lemma 1.1, this yields the required statement.

Lemma 1.2. *For any* $0\le s\le t$, $x,y\in\mathbf{R}$, *and for any integer* k,

$$\mathbf{E}|\ell(t,x)-\ell(s,x)|^{2k} \le 2^{2k}(2k)!\,|t-s|^k, \tag{1.5}$$

$$\mathbf{E}|\ell(t,x)-\ell(t,y)|^{2k} \le (2k)!\,2^{6k+1}|x-y|^k t^{k/2}. \tag{1.6}$$

Remark 1.2. The estimates (1.5) and (1.6) can be combined into a single estimate: for $s,t\in[0,T]$,

$$\mathbf{E}|\ell(t,x)-\ell(s,y)|^{2k} \le (2k)!2^{4k}\big(|t-s|^k + 2^{4k+1}|x-y|^k s^{k/2}\big). \tag{1.7}$$

Choosing $k = 3$ and applying Theorem 3.3 Ch. I for a two-parameter process, we get that the process $\ell(t,x)$ has an a.s. continuous in $[0,\infty)\times\mathbf{R}$ modification, which can be also denoted by $\ell(t,x)$. The relations (1.3), (1.5), and (1.6) are obviously valid for this modification. Moreover, the sample paths of the process $\ell(t,x)$ satisfy a.s. the Hölder condition of any order $\gamma < 1/2$:

$$\sup_{\substack{t,s\in[0,T]\\|t-s|\le h}} \sup_{\substack{x,y\in[-N,N]\\|x-y|\le h}} |\ell(t,x)-\ell(s,y)| \le C_{T,N,\gamma} h^{\gamma}, \tag{1.8}$$

where $C_{T,N,\gamma}$ is a random constant depending on T, N, and γ. This follows from Theorem 3.3 Ch. I, because k in (1.7) can be chosen large enough to ensure that $(k-2)/2k > \gamma$.

Proof of Lemma 1.2. Set $\vec{\lambda} := (\lambda_1,\dots,\lambda_{2k})$. Using Fatou's lemma, putting $u_{2k+1} = 0$, and carrying out computations analogous to (1.4), we get

$$\mathbf{E}|\ell(t,x)-\ell(s,x)|^{2k} \le \liminf_{n\to\infty} \mathbf{E}|\ell^{(n)}(t,x)-\ell^{(n)}(s,x)|^{2k}$$

$$\le \frac{1}{(2\pi)^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \int_s^t dv_1 \cdots \int_s^t dv_{2k} \Big|\mathbf{E}\exp\Big(i\sum_{l=1}^{2k}\lambda_l W(v_l)\Big)\Big|$$

$$\le \frac{(2k)!}{(2\pi)^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \int_s^t du_1 \cdots \int_s^{u_{2k-1}} du_{2k} \Big|\mathbf{E}\prod_{l=1}^{2k}\exp\Big(i\Big(\sum_{j=1}^{l}\lambda_j\Big)(W(u_l)-W(u_{l+1})\Big)\Big|$$

$$\le \frac{(2k)!}{(2\pi)^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \int_s^t du_1 \cdots \int_s^{u_{2k-1}} du_{2k}\, \mathbf{E}\prod_{l=1}^{2k}\exp\Big(-\frac{1}{2}\Big(\sum_{j=1}^{l}\lambda_j\Big)^2(u_l-u_{l+1})\Big)$$

$$\le \frac{(2k)!}{\pi^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \prod_{l=1}^{2k} \frac{1-\exp\Big(-(t-s)\Big|\sum\limits_{j=1}^{l}\lambda_j\Big|^2/2\Big)}{\Big|\sum\limits_{j=1}^{l}\lambda_j\Big|^2}$$

$$\le \frac{(2k)!2^{2k}|t-s|^k}{\pi^{2k}}\Big(\int_{-\infty}^{\infty}\frac{d\lambda}{1+\lambda^2}\Big)^{2k} = 2^{2k}(2k)!\,|t-s|^k.$$

We now prove (1.6). It suffices to prove (1.6) for $|x-y|/\sqrt{t} \le 1/16$. Indeed, by (1.5), $s = 0$, for $|x-y|/\sqrt{t} > 1/16$ the following inequality holds:

$$\mathbf{E}|\ell(t,x)-\ell(t,y)|^{2k} \le 2^{2k+1}(2k)!\,t^k \le 2^{2k+1}(2k)!\,16^k|x-y|^k t^{k/2}.$$

By Remark 1.1, it suffice to prove (1.6) for $t = 1$. We set $\Delta := |x-y|$ and

$$D_n(s) := \frac{1}{2\pi}\int_{-n}^{n}(e^{-i\lambda x}-e^{-i\lambda y})e^{i\lambda W(s)}\,d\lambda.$$

By Fatou's lemma,

$$\mathbf{E}|\ell(1,x)-\ell(1,y)|^{2k} \le \liminf_{n\to\infty} \mathbf{E}|\ell^{(n)}(1,x)-\ell^{(n)}(1,y)|^{2k}$$
$$= \liminf_{n\to\infty} \mathbf{E}\Bigl(\int_0^1 D_n(s)\,ds\Bigr)^{2k}. \tag{1.9}$$

Set $\vec{s} := (s_1,\dots,s_{2k})$ and $s_0 := 0$. Then

$$\mathbf{E}\Bigl(\int_0^1 D_n(s)\,ds\Bigr)^{2k} = (2k)! \int\cdots\int_{0\le s_1<\cdots<s_{2k}\le 1} \prod_{l=1}^{2k} D_n(s_l)\,d\vec{s}$$
$$= \frac{(2k)!}{(2\pi)^{2k}} \int_{[-n,n]^{2k}} d\vec{\lambda} \prod_{l=1}^{2k} (e^{-i\lambda_l x} - e^{-i\lambda_l y})$$
$$\times \int\cdots\int_{0\le s_1<\cdots<s_{2k}\le 1} \prod_{l=1}^{2k} \mathbf{E}\exp\Bigl(i\Bigl(\sum_{m=l}^{2k}\lambda_m\Bigr)(W(s_l)-W(s_{l-1}))\Bigr)\,d\vec{s}$$
$$\le \frac{(2k)!}{\pi^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \prod_{l=1}^{2k} |e^{i\lambda_l\Delta}-1| \prod_{l=1}^{2k} \frac{1-\exp\Bigl(-\Bigl|\sum_{j=1}^{l}\lambda_j\Bigr|^2/2\Bigr)}{\Bigl|\sum_{j=1}^{l}\lambda_j\Bigr|^2}$$
$$\le \frac{2^{2k}(2k)!}{\pi^{2k}} \int_{\mathbf{R}^{2k}} d\vec{\lambda} \prod_{l=1}^{2k} \frac{1\wedge|\lambda_l|\Delta}{1+\Bigl|\sum_{j=1}^{l}\lambda_j\Bigr|^2}. \tag{1.10}$$

Denote $I_0(\zeta)\equiv 1$,

$$I_p(\zeta) := \int_{\mathbf{R}^p} d\vec{\lambda} \prod_{l=1}^{p} \frac{1\wedge|\lambda_l|\Delta}{1+\Bigl|\zeta+\sum_{j=1}^{l}\lambda_j\Bigr|^2}.$$

This yields the recurrence relation:

$$I_p(\zeta) = \int_{-\infty}^{\infty} d\lambda\,\frac{1\wedge|\lambda|\Delta}{1+|\zeta+\lambda|^2} I_{p-1}(\zeta+\lambda). \tag{1.11}$$

Using induction on $k=0,1,2,\dots$, we prove that for all ζ and k

$$I_{2k}(\zeta) \le (4\pi)^{2k}\Delta^k\bigl(1+(1\wedge|\zeta|\Delta)\ln(1+1/\Delta^2)\bigr). \tag{1.12}$$

For $k=0$ this estimate is obvious. Assume that (1.12) holds for $k=p-1$ and let us prove that it holds for $k=p$. We deduce two auxiliary estimates. We have

$$\int_{-\infty}^{\infty} \frac{1\wedge|\lambda|\Delta}{1+(\zeta+\lambda)^2}\,d\lambda = \int_{-\infty}^{\infty} \frac{1\wedge|\eta-\zeta|\Delta}{1+\eta^2}\,d\eta \le \int_{-\infty}^{\infty} \frac{(1\wedge|\eta|\Delta)+(1\wedge|\zeta|\Delta)}{1+\eta^2}\,d\eta$$
$$= (1\wedge|\zeta|\Delta)\int_{-\infty}^{\infty}\frac{d\eta}{1+\eta^2} + \Delta\int_0^{1/\Delta}\frac{d(1+\eta^2)}{1+\eta^2} + 2\int_{1/\Delta}^{\infty}\frac{d\eta}{1+\eta^2}$$
$$\le \pi(1\wedge|\zeta|\Delta) + \Delta(2+\ln(1+1/\Delta^2))$$

and

$$\int_{-\infty}^{\infty} \frac{(1\wedge|\lambda|\Delta)(1\wedge|\zeta+\lambda|\Delta)}{1+(\zeta+\lambda)^2}\,d\lambda = \int_{-\infty}^{\infty} \frac{(1\wedge|\eta-\zeta|\Delta)(1\wedge|\eta|\Delta)}{1+\eta^2}\,d\eta$$

$$\le \int_{-\infty}^{\infty} \frac{(1\wedge|\eta|\Delta)^2}{1+\eta^2}\,d\eta + (1\wedge|\zeta|\Delta)\int_{-\infty}^{\infty} \frac{1\wedge|\eta|\Delta}{1+\eta^2}\,d\eta$$

$$\le \Delta(6+(1\wedge|\zeta|\Delta)\ln(1+1/\Delta^2)).$$

Using the induction hypothesis, these estimates, and the double inequality

$$4 \le \ln(1+1/\Delta^2) \le \sqrt{2/\Delta} \qquad \text{for} \qquad 0<\Delta\le 1/16,$$

we deduce from (1.11) and (1.12) for $k=p-1$ that

$$|I_{2p-1}(\zeta)| \le (4\pi)^{2(p-1)}\Delta^{p-1}\int_{-\infty}^{\infty} d\lambda\,\frac{1\wedge|\lambda|\Delta}{1+(\zeta+\lambda)^2}\big(1+(1\wedge|\zeta+\lambda|\Delta)\ln(1+1/\Delta^2)\big)$$

$$\le (4\pi)^{2(p-1)}\Delta^{p-1}\Bigg[\pi(1\wedge|\zeta|\Delta)+\Delta(2+\ln(1+1/\Delta^2))+\ln(1+1/\Delta^2)$$

$$\times\int_{-\infty}^{\infty} \frac{(1\wedge|\lambda|\Delta)(1\wedge|\zeta+\lambda|\Delta)}{1+(\zeta+\lambda)^2}\,d\lambda\Bigg] \le (4\pi)^{2p-1}\Delta^{p-1}\big[(1\wedge|\zeta|\Delta)+\Delta\ln(1+1/\Delta^2)\big].$$

Again using (1.11) and the estimates obtained above, we find that

$$|I_{2p}(\zeta)| \le (4\pi)^{2p-1}\Delta^{p-1}\int_{-\infty}^{\infty} d\lambda\frac{(1\wedge|\lambda|\Delta)}{1+(\zeta+\lambda)^2}\big((1\wedge|\zeta+\lambda|\Delta)+\Delta\ln(1+1/\Delta^2)\big)$$

$$\le (4\pi)^{2p-1}\Delta^{p-1}\big[\Delta\big(6+(1\wedge|\zeta|\Delta)\ln(1+1/\Delta^2)\big)+\Delta\ln(1+1/\Delta^2)\big(\pi(1\wedge|\zeta|\Delta)$$

$$+\Delta(2+\ln(1+1/\Delta^2))\big)\big] \le (4\pi)^{2p}\Delta^{p}\big[1+(1\wedge|\zeta|\Delta)\ln(1+1/\Delta^2)\big].$$

We derived the estimate (1.12) for $k=p$; this concludes the induction proof of the estimate.

The right-hand side of (1.10) is equal to $\frac{2^{2k}(2k)!}{\pi^{2k}}I_{2k}(0)$, and by (1.12), it does not exceed $(2k)!2^{6k}\Delta^k$, In view of (1.9), this proves (1.6) for $t=1$. As already mentioned, for arbitrary t (1.6) follows from (1.6) for $t=1$ by the scaling property of the process $\ell(t,x)$. □

Lemma 1.3. *A.s. for any $a<b$ and $t>0$*

$$\mu_t([a,b)) = \int_a^b \ell(t,x)\,dx. \tag{1.13}$$

Proof. Associating the increasing function $F_t(x) := \mu_t((-\infty, x))$, $x \in \mathbf{R}$, with the sojourn measure $\mu_t(A)$, we can rewrite (1.1) as

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)\,dF_t(x).$$

Using this formula for $f(x) = e^{i\lambda x}$, $x \in \mathbf{R}$, and the definition of $\ell^{(n)}(t,x)$, we find that

$$\int_a^b \ell^{(n)}(t,x)\,dx = \frac{1}{2\pi}\int_{-n}^{n} \frac{e^{-i\lambda b} - e^{-i\lambda a}}{-i\lambda} \int_{-\infty}^{\infty} e^{i\lambda x} dF_t(x)\,d\lambda.$$

By the inversion formula for characteristic functions (see (1.9) Ch. I),

$$\int_a^b \ell^{(n)}(t,x)\,dx \underset{n\to\infty}{\longrightarrow} F_t(b) - F_t(a) \tag{1.14}$$

at all points of continuity of F_t. For any $y \in \mathbf{R}$, the set Ω_y of the sample paths of the process W such that $F_t(x)$ is continuous at y has probability 1. Indeed,

$$\left\{\omega : \int_0^t \mathbb{1}_{\{y\}}(W(s))\,ds = 0\right\} \subset \Omega_y$$

and

$$\mathbf{E}\int_0^t \mathbb{1}_{\{y\}}(W(s))\,ds = \int_0^t \mathbf{P}(W(s) = y)\,ds = 0.$$

This proves the required statement.

Let us verify that for any pair $a < b$ of points of continuity of the function F_t and any $t > 0$ the relation (1.13) holds a.s. We have

$$\left|\mu_t([a,b)) - \int_a^b \ell(t,x)\,dx\right| \le \left|F_t(b) - F_t(a) - \int_a^b \ell^{(n)}(t,x)\,dx\right| + \int_a^b |\ell^{(n)}(t,x) - \ell(t,x)|dx.$$

The first term on the right-hand side of this inequality tends to zero a.s. in view of (1.14) and the second one tends to zero in probability by (1.3) and the estimate

$$\mathbf{E}\int_a^b |\ell^{(n)}(t,x) - \ell(t,x)|dx \le (b-a)\int_a^b \mathbf{E}|\ell^{(n)}(t,x) - \ell(t,x)|^2 dx.$$

Consequently, for some sequence n_k the second term also tends to zero a.s. (see Proposition 1.1 Ch. I). Thus the set of the sample paths of the process W for which (1.13) holds has probability 1, but it can depend on t, a, and b. It now remains to verify that this set can be chosen to be independent of t, a and b. By the countable additivity of the probability measure this set can be chosen to be independent of rational values of t, a, b with probability 1. Since $F_t(x)$ is left continuous for a fixed t, this set does not depend on a, b. Further, since $\ell(t,x)$ is an a.s. uniformly continuous increasing process with respect to t, while $\mu_t([a,b))$ is equicontinuous with respect to t for all $a < b$, this set does not depend on t. □

Lemma 1.3 concludes the proof of the existence of a continuous local time for the Brownian motion W. □

The next relation, which follows from the existence of a continuous local time (see (1.2)), often proves useful: a.s. for any $t \geq 0$ and $x \in \mathbf{R}$

$$\ell(t,x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{[x,x+\varepsilon)}(W(s))\, ds, \tag{1.15}$$

where $\mathbb{I}_A(\cdot)$ is the indicator function of a Borel set $A \subseteq \mathbf{R}$. For example, (1.15) implies that $\ell(0,x) = 0$.

According to Theorem 1.1 there exists an a.s. continuous two-parameter process $\ell(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$, such that for all $t > 0$ and for any Borel set $A \in \mathbf{R}$,

$$\int_0^t \mathbb{I}_A(W(s))\, ds = \int_A \ell(t,x)\, dx. \tag{1.16}$$

The process $\ell(t,x)$ is referred to as the *Brownian local time.*

We now summarize the properties of the Brownian local time that follow directly from its definition.

Relation (1.15) implies that for every x, the process $\ell(t,x)$ is nondecreasing as a process with respect to t and increases only on the set $Z^x = \{s : W(s) = x\}$. It is also obvious that

$$\{(t,x) : \ell(t,x) > 0\} \subseteq \Big\{(t,x) : \inf_{0 \leq s \leq t} W(s) < x < \sup_{0 \leq s < t} W(s)\Big\}. \tag{1.17}$$

For any function $f(x)$, $x \in \mathbf{R}$, that is integrable on any finite interval the equality (1.2) holds a.s. for any $t \geq 0$.

The Brownian local time possesses the *scaling property* (see Remark 1.1). By (1.15), this property can be expressed as follows: for any fixed $c > 0$, the process $\sqrt{c}\ell(t/c, x/\sqrt{c})$ is the local time of the Brownian motion $\sqrt{c}W(t/c)$.

§ 2. Markov property of Brownian local time stopped at an exponential moment

In this section we describe the Brownian local time $\ell(t,y)$ as a process with respect to the parameter y. It turns out that this is a Markov process if some random time is taken instead of t. As examples of such random times one can take τ, which is exponentially distributed and independent of the Brownian motion W, the first exit time $H_{a,b}$, the moment θ_v inverse of the range of the process, and some others. This phenomenon has the following explanation. Consider first a simple random walk with independent steps, taking the values ± 1 with probability $1/2$. An *excursion of the random walk* is a part of the trajectory between subsequent visits to some selected level. The *random walk local time* is the number of times the random walk hits a selected level up to a certain moment of time. By the strong Markov property for random walks, under the condition that the random walk hits a selected level a given number of times, the corresponding excursions are independent. Due to this fact, for a fixed number of visits to the selected level, the numbers of hits of different levels above and below the selected one are independent. This exactly means that the random walk local time with respect to the parameter that characterizes the level is a Markov process. Properly normalized random walk converges to the Brownian motion and normalized random walk local time converges (see § 6 Ch. VII) to the Brownian local time. The Markov property is usually preserved during the passage to the limit. As a result, the Markov property is valid for the Brownian local time as a process of the space variable, if at the end of the path of Brownian motion certain conditions are imposed. The rigorous justification of the Markov property for the Brownian local time when the Brownian motion is stopped at the first hitting moment of a level, was given by F. Knight (1963). D. Ray (1963) proposed a purely analytic proof of this property without using the limiting approximation. We also follow the analytic approach based on the solutions of differential problems instead of that based on integral equations used by Ray. In using such an approach we are forced to give different proofs for different stopping times. Nevertheless the scheme of these proofs is the same.

An exponentially distributed with parameter $\lambda > 0$ stopping time τ that is independent of the Brownian motion W has a special significance and has been already discussed repeatedly. If we know the distributions of the process $\ell(\tau, y)$, $y \in \mathbf{R}$, or the distribution of a functional of this process, then by inverting the Laplace transform with respect to λ we can compute the corresponding distributions of the process $\ell(t, y)$, $y \in \mathbf{R}$, or the distribution of the corresponding functional for any fixed t.

We assume that $W(0) = x$. The probability measure and the expectation corresponding to the Brownian motion W with this starting point are usually denoted by $\mathbf{P}_x$ and $\mathbf{E}_x$.

Theorem 2.1. *Given $W(\tau) = z$, the process $\ell(\tau, y)$, $y \in \mathbf{R}$, is a Markov process that can be represented for $z \geq x$ in the form*

$$\ell(\tau, y) = \begin{cases} V_1(y - z), & \text{for } z \leq y, \\ V_2(z - y), & \text{for } x \leq y \leq z, \\ V_3(x - y), & \text{for } y \leq x, \end{cases}$$

where $V_k(h)$, $h \geq 0$, $k = 1, 2, 3$, are independent homogeneous diffusions under fixed starting points. The generating operators of the processes V_k, $k = 1, 2, 3$, *have the form*

$$\mathbb{L}_1 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda}\frac{d}{dv}\Big), \quad \mathbb{L}_2 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda}\frac{d}{dv}\Big) + 2\frac{d}{dv}, \quad \mathbb{L}_3 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda}\frac{d}{dv}\Big),$$

respectively. The initial values satisfy the equalities $V_1(0) = V_2(0)$, $V_3(0) = V_2(z)$, *and their distributions are determined by the formula*

$$\mathbf{P}_x\{\ell(\tau, y) \geq v | W(\tau) = z\} = \exp(-(|y-x| + |y-z| - |z-x| + v)\sqrt{2\lambda}). \quad (2.1)$$

Remark 2.1. For $z \leq x$ an analogous description holds in view of the spatial homogeneity and the time reversal property of the Brownian bridge.

Remark 2.2. It follows from (2.1) that the processes V_1, V_2, and V_3 have the same initial exponential distribution, with the density $\sqrt{2\lambda}\, e^{-v\sqrt{2\lambda}}\, \mathbb{1}_{[0,\infty)}(v)$.

Remark 2.3. In Theorem 2.1, a special description of the process $\ell(\tau, y)$, $y \in \mathbf{R}$, is given. In this specification the time y of the process V_1 varies in the natural direction, while for the processes V_2 and V_3 it varies in the opposite direction, i.e., for the processes of V_2 and V_3 the time parameter y is taken with the minus sign. There is a certain convenience in such a description. The process V_1 degenerates to zero, because $\ell(\tau, y) = 0$ for $y \geq \sup_{0 \leq s \leq \tau} W(s)$, and the same happens with the process V_3, because $\ell(\tau, y) = 0$ for $y \leq \inf_{0 \leq s \leq \tau} W(s)$. If we define the process V_3 in the direct time, it would start from zero at the random moment $y_0 = \inf_{0 \leq s \leq \tau} W(s)$, which is not convenient. By the reversibility in time property of the Brownian bridge (see § 11 Ch. I), the process V_2 is the same in either direction of time. By the same reason, the infinitesimal characteristics of the processes V_1 and V_3 coincide.

Proof of Theorem 2.1. The distribution (2.1) follows from formula (4.49) of Ch. III. Since the Brownian motion is spatially homogeneous (see § 10 Ch. I), one can assume that $W(0) = 0$, i.e., $x = 0$. We consider the new probability space generated by the conditional measure $\mathbf{P}_0^z(B) = \mathbf{P}_0\{B | W(\tau) = z\}$, $B \in \mathcal{F}$. The symbols for the probability and expectation relative to this space will have the superscript z and subscript 0.

Let $f(x)$, $x \in \mathbf{R}$, be an arbitrary bounded continuous function and q be an arbitrary real number. Set

$$f_+(x) := f(x)\mathbb{1}_{(q,\infty)}(x), \qquad f_-(x) := f(x)\mathbb{1}_{(-\infty,q]}(x).$$

Let $\mathcal{G}_u^v := \sigma(\ell(\tau, y), u \leq y \leq v)$ be the σ-algebra of events generated by the Brownian local time $\ell(\tau, y)$ on the interval $[u, v]$. To prove that $\ell(\tau, y)$, $y \in \mathbf{R}$, is a Markov process given $W(\tau) = z$, it suffices to verify (see Proposition 6.3 of

Ch. I and (2.17)–(2.19) Ch. I) that for any $q \in \mathbf{R}$ and $v \in [0, \infty)$,

$$\begin{aligned} \mathbf{E}_0^z\Big\{ \exp\Big(-\int_{-\infty}^{\infty} f(y)\ell(\tau,y)\,dy\Big)\Big|\ell(\tau,q)=v\Big\} \\ = \mathbf{E}_0^z\Big\{ \exp\Big(-\int_{-\infty}^{\infty} f_+(y)\ell(\tau,y)\,dy\Big)\Big|\ell(\tau,q)=v\Big\} \\ \times \mathbf{E}_0^z\Big\{ \exp\Big(-\int_{-\infty}^{\infty} f_-(y)\ell(\tau,y)\,dy\Big)\Big|\ell(\tau,q)=v\Big\}. \end{aligned} \tag{2.2}$$

Indeed, since $f(y) = f(y)_- + f_+(y)$, $y \in \mathbf{R}$, and the exponent on the left-hand side of this equality equals the product of exponents on the right-hand side, the equality (2.2) means that in the conditional probability space (with the condition $W(\tau) = z$), the future of the process $\ell(\tau, y)$ (the σ-algebra $\mathcal{G}_q^\infty$) for a fixed present state ($\ell(\tau, q) = v$) does not depend on the past (the σ-algebra $\mathcal{G}_{-\infty}^q$).

We remark that, according to (1.2),

$$\int_{-\infty}^{\infty} f(y)\ell(\tau,y)\,dy = \int_0^{\tau} f(W(s))\,ds,$$

and hence (2.2) can be rewritten as

$$\begin{aligned} \mathbf{E}_0^z\Big\{ \exp\Big(-\int_0^{\tau} f(W(s))\,ds\Big)\Big|\ell(\tau,q)=v\Big\} \\ = \mathbf{E}_0^z\Big\{ \exp\Big(-\int_0^{\tau} f_+(W(s))\,ds\Big)\Big|\ell(\tau,q)=v\Big\} \\ \times \mathbf{E}_0^z\Big\{ \exp\Big(-\int_0^{\tau} f_-(W(s))\,ds\Big)\Big|\ell(\tau,q)=v\Big\}. \end{aligned} \tag{2.3}$$

We prove (2.3) by computing for the expectations figuring in it explicit formulas in terms of fundamental solutions of the equation

$$\frac{1}{2}\phi''(y) - (\lambda + f(y))\phi(y) = 0, \qquad y \in \mathbf{R}. \tag{2.4}$$

For piecewise-continuous functions f the solutions of (2.4) must be understood according to Remark 1.2 Ch. III.

For definiteness we take $q > 0$. Let $\gamma > 0$. Set

$$G(z) := \frac{d}{dz}\mathbf{E}_0\Big\{ \exp\Big(-\int_0^{\tau} f(W(s))\,ds - \gamma\ell(\tau,q)\Big); W(\tau) < z\Big\}, \qquad z \in \mathbf{R}.$$

We compute G, using Theorem 4.2 Ch. III, with $a=-\infty$, $b=\infty$, $\beta_1=\gamma$, $q_1=q$ and $\beta_l=0$, $l\neq 1$. Let $\psi(y)$ and $\varphi(y)$, $y\in\mathbf{R}$, be two fundamental solutions (see Remark 1.1 Ch. III) of equation (2.4), satisfying the condition $\psi(q)=\varphi(q)=1$ and w be their Wronskian. By Theorem 4.2 Ch. III, the function G is the unique bounded continuous solution of the problem

$$\frac{1}{2}G''(z)-(\lambda+f(z))G(z)=0, \qquad z\in\mathbf{R}\setminus\{0,q\}, \tag{2.5}$$

$$G'(+0)-G'(-0)=-2\lambda, \tag{2.6}$$

$$G'(q+0)-G'(q-0)=2\gamma G(q). \tag{2.7}$$

The solution can be represented in the form

$$G(z)=\begin{cases} C_1\psi(z), & \text{for} \quad z\le 0,\\ B_1\psi(z)+B_2\varphi(z), & \text{for} \quad 0\le z\le q,\\ C_2\varphi(z), & \text{for} \quad q\le z.\end{cases}$$

The continuity conditions and (2.6), (2.7) give the following values for the constants C_k, B_k, $k=1,2$:

$$C_1=\frac{2\lambda\psi(0)}{w+2\gamma}+\frac{2\lambda}{w}(\varphi(0)-\psi(0)), \qquad C_2=\frac{2\lambda\psi(0)}{w+2\gamma},$$

$$B_1=\frac{2\lambda\psi(0)}{w+2\gamma}-\frac{2\lambda}{w}\psi(0), \qquad B_2=\frac{2\lambda}{w}\psi(0),$$

where $w=\psi'(x)\varphi(x)-\varphi'(x)\psi(x)>0$ is a constant quantity. Consequently,

$$G(z)=\begin{cases} \dfrac{2\lambda\psi(0)\psi(z)}{w+2\gamma}+\dfrac{2\lambda}{w}(\varphi(0)-\psi(0))\psi(z), & \text{for} \quad z\le 0,\\ \dfrac{2\lambda\psi(0)\psi(z)}{w+2\gamma}+\dfrac{2\lambda}{w}\psi(0)(\varphi(z)-\psi(z)), & \text{for} \quad 0\le z\le q,\\ \dfrac{2\lambda\psi(0)\varphi(z)}{w+2\gamma}, & \text{for} \quad q\le z.\end{cases}$$

Inverting the Laplace transform with respect to γ, we get

$$\frac{\partial}{\partial z}\frac{\partial}{\partial v}\mathbf{E}_0\Big\{\exp\Big(-\int_0^\tau f(W(s))\,ds\Big);\ell(\tau,q)\le v, W(\tau)<z\Big\}$$

$$=\begin{cases}\lambda\psi(0)\psi(z)e^{-vw/2}, & \text{for} \quad z\le q,\\ \lambda\psi(0)\varphi(z)e^{-vw/2}, & \text{for} \quad q\le z,\end{cases} \qquad v>0, \tag{2.8}$$

and

$$\frac{\partial}{\partial z}\mathbf{E}_0\Big\{\exp\Big(-\int_0^\tau f(W(s))\,ds\Big);\ell(\tau,q)=0, W(\tau)<z\Big\}$$

$$=\begin{cases}\dfrac{2\lambda}{w}(\varphi(0)-\psi(0))\psi(z), & \text{for} \quad z\le 0,\\ \dfrac{2\lambda}{w}\psi(0)(\varphi(z)-\psi(z)), & \text{for} \quad 0\le z\le q.\end{cases} \tag{2.9}$$

For $f \equiv 0$ the fundamental solutions of (2.4) with the property $\psi_0(q) = \varphi_0(q) = 1$ have the form

$$\psi_0(y) = e^{(y-q)\sqrt{2\lambda}}, \qquad \varphi_0(y) = e^{(q-y)\sqrt{2\lambda}},$$

and have the Wronskian $w_0 = 2\sqrt{2\lambda}$. Then from (2.8), (2.9) it follows that for $v > 0$, $q > 0$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial v}\mathbf{P}_0(\ell(\tau,q) \le v, W(\tau) < z) = \begin{cases} \lambda e^{-(2q+v-z)\sqrt{2\lambda}}, & \text{for} \quad z \le q, \\ \lambda e^{-(v+z)\sqrt{2\lambda}}, & \text{for} \quad q \le z, \end{cases} \tag{2.10}$$

$$\frac{\partial}{\partial z}\mathbf{P}_0(\ell(\tau,q) = 0, W(\tau) < z) = \begin{cases} \frac{\sqrt{\lambda}}{\sqrt{2}}(e^{z\sqrt{2\lambda}} - e^{(z-2q)\sqrt{2\lambda}}), & \text{for } z \le 0, \\ \frac{\sqrt{\lambda}}{\sqrt{2}}(e^{-z\sqrt{2\lambda}} - e^{(z-2q)\sqrt{2\lambda}}), & \text{for } 0 \le z \le q. \end{cases} \tag{2.11}$$

Since for $q = \infty$ the left-hand side of (2.11) becomes $\frac{d}{dz}\mathbf{P}_0(W(\tau) < z)$, we can easily establish (2.1) for $y \ge 0$, $x = 0$, by dividing (2.10) and (2.11) by (2.10) and (2.11) with $q = \infty$, respectively. For $y \le 0$ the equality (2.1) can be verified with the help of the symmetry property of Brownian motion.

Further, dividing (2.8) and (2.9) by (2.10) and (2.11), respectively, we get

$$\mathbf{E}_0^z\Big\{\exp\Big(-\int_0^\tau f(W(s))\,ds\Big)\Big|\ell(\tau,q) = v\Big\}$$
$$= \begin{cases} e^{(2q-z+v)\sqrt{2\lambda}}\psi(0)\psi(z)e^{-tw/2}, & \text{for} \quad z \le q, \\ e^{(z+v)\sqrt{2\lambda}}\psi(0)\varphi(z)e^{-tw/2}, & \text{for} \quad q \le z, \end{cases} \qquad v > 0, \tag{2.12}$$

and

$$\mathbf{E}_0^z\Big\{\exp\Big(-\int_0^\tau f(W(s))\,ds\Big)\Big|\ell(\tau,q) = 0\Big\}$$
$$= \begin{cases} \frac{2\sqrt{2\lambda}(\varphi(0) - \psi(0))\psi(z)}{w(e^{z\sqrt{2\lambda}} - e^{(z-2q)\sqrt{2\lambda}})}, & \text{for} \quad z \le 0, \\ \frac{2\sqrt{2\lambda}\psi(0)(\varphi(z) - \psi(z))}{w(e^{-z\sqrt{2\lambda}} - e^{(z-2q)\sqrt{2\lambda}})}, & \text{for} \quad 0 \le z \le q. \end{cases} \tag{2.13}$$

Let us compute the analogous expressions for the function $f_+(x)$, $x \in \mathbf{R}$, in terms of functions φ and ψ. Since formulas (2.12) and (2.13) were obtained for an arbitrary nonnegative piecewise continuous function f in terms of fundamental solutions of equation (2.4), one can use this formulas for the function f_+. To do this we express $\psi_+(y)$, $\varphi_+(y)$, the fundamental solutions of equation (2.4) with the function $f_+(y)$ instead of $f(y)$, in terms of the fundamental solutions $\psi(y)$, $\varphi(y)$. We get

$$\psi_+(y) = \begin{cases} e^{(y-q)\sqrt{2\lambda}}, & \text{for} \quad y \le q, \\ A\psi(y) + (1-A)\varphi(y), & \text{for} \quad q \le y, \end{cases}$$

$$\varphi_+(y) = \begin{cases} Be^{(y-q)\sqrt{2\lambda}} + (1-B)e^{(q-y)\sqrt{2\lambda}}, & \text{for} \quad y \le q, \\ \varphi(y), & \text{for} \quad q \le y. \end{cases}$$

In this representation we have taken into account that ψ_+ is an increasing function, φ_+ is a decreasing one, and $\psi_+(q) = \varphi_+(q) = 1$. The continuity of the derivative at q enables us to compute the constants A and B, obtaining that

$$\psi_+(y) = \begin{cases} e^{(y-q)\sqrt{2\lambda}}, & \text{for} \quad y \le q, \\ \dfrac{\sqrt{2\lambda} - \varphi'(q)}{w}\psi(y) + \dfrac{\psi'(q) - \sqrt{2\lambda}}{w}\varphi(y), & \text{for} \quad q \le y, \end{cases}$$

$$\varphi_+(y) = \begin{cases} \Big(\frac{1}{2} + \dfrac{\varphi'(q)}{2\sqrt{2\lambda}}\Big)e^{(y-q)\sqrt{2\lambda}} + \Big(\frac{1}{2} - \dfrac{\varphi'(q)}{2\sqrt{2\lambda}}\Big)e^{(q-y)\sqrt{2\lambda}}, & \text{for} \quad y \le q, \\ \varphi(y), & \text{for} \quad q \le y. \end{cases}$$

The Wronskian of ψ_+ and φ_+ is

$$w_+ = \psi_+'(q) - \varphi_+'(q) = \sqrt{2\lambda} - \varphi'(q).$$

Substituting ψ_+ and φ_+ in place of ψ and φ in (2.12) and (2.13), we get

$$\mathbf{E}_0^z\Big\{ \exp\Big(-\int_0^\tau f_+(W(s))\,ds \Big) \Big| \ell(\tau,q) = v \Big\}$$
$$= \begin{cases} e^{v(\sqrt{2\lambda}+\varphi'(q))/2}, & \text{for} \quad z \le q, \\ \varphi(z)\, e^{(z-q)\sqrt{2\lambda}}\, e^{v(\sqrt{2\lambda}+\varphi'(q))/2}, & \text{for} \quad q \le z, \end{cases} \qquad v > 0, \tag{2.14}$$

and

$$\mathbf{E}_0^z\Big\{ \exp\Big(-\int_0^\tau f_+(W(s))\,ds \Big) \Big| \ell(\tau,q) = 0 \Big\} = 1 \qquad \text{for } z \le q. \tag{2.15}$$

We carry out the computations for $f_-(x)$, $x \in \mathbf{R}$. We represent the fundamental solutions of equation (2.4) with f^- instead of f in the form

$$\psi_-(y) = \begin{cases} \psi(y), & \text{for} \quad y \le q, \\ Ce^{(y-q)\sqrt{2\lambda}} + (1-C)e^{(q-y)\sqrt{2\lambda}}, & \text{for} \quad q \le y, \end{cases}$$

$$\varphi_-(y) = \begin{cases} D\psi(y) + (1-D)\varphi(y), & \text{for} \quad y \le q, \\ e^{(q-y)\sqrt{2\lambda}}, & \text{for} \quad q \le y. \end{cases}$$

By the continuity of the derivative at q,

$$\psi_-(y) = \begin{cases} \psi(y), & \text{for} \quad y \le q, \\ \Big(\frac{1}{2} + \dfrac{\psi'(q)}{2\sqrt{2\lambda}}\Big)e^{(y-q)\sqrt{2\lambda}} + \Big(\frac{1}{2} - \dfrac{\psi'(q)}{2\sqrt{2\lambda}}\Big)e^{(q-y)\sqrt{2\lambda}}, & \text{for} \quad q \le y, \end{cases}$$

$$\varphi_-(y)=\begin{cases}\dfrac{\sqrt{2\lambda}+\psi'(q)}{w}\varphi(y)-\dfrac{\sqrt{2\lambda}+\varphi'(q)}{w}\psi(y), & \text{for}\quad y\le q,\\ e^{(q-y)\sqrt{2\lambda}}, & \text{for}\quad q\le y.\end{cases}$$

The Wronskian of these solutions is

$$w_-=\psi_-'(q)-\varphi_-'(q)=\psi'(q)+\sqrt{2\lambda}.$$

Substituting ψ_- and φ_- in place of ψ and φ in (2.12) and (2.13), we have

$$\mathbf{E}_0^z\Big\{\exp\Big(-\int_0^\tau f_-(W(s))\,ds\Big)\Big|\ell(\tau,q)=v\Big\}$$
$$=\begin{cases}\psi(0)\,\psi(z)\,e^{(2q-z)\sqrt{2\lambda}}\,e^{v(\sqrt{2\lambda}-\psi'(q))/2}, & \text{for}\quad z\le q,\\ \psi(0)\,e^{q\sqrt{2\lambda}}\,e^{v(\sqrt{2\lambda}-\psi'(q))/2}, & \text{for}\quad q\le z,\end{cases}\qquad v>0, \tag{2.16}$$

and

$$\mathbf{E}_0^z\Big\{\exp\Big(-\int_0^\tau f_-(W(s))\,ds\Big)\Big|\ell(\tau,q)=0\Big\}$$
$$=\begin{cases}\dfrac{2\sqrt{2\lambda}(\varphi(0)-\psi(0))\psi(z)}{w(e^{z\sqrt{2\lambda}}-e^{(z-2q)\sqrt{2\lambda}})}, & \text{for}\quad z\le 0,\\ \dfrac{2\sqrt{2\lambda}\psi(0)(\varphi(z)-\psi(z))}{w(e^{-z\sqrt{2\lambda}}-e^{(z-2q)\sqrt{2\lambda}})}, & \text{for}\quad 0\le z\le q.\end{cases} \tag{2.17}$$

Since (2.12) is equal to the product of (2.14) and (2.16), while (2.13) is equal to the product of (2.15) and (2.17), this proves (2.3).

We have assumed that $q>0$. The cases $q=0$ and $q<0$ can be dealt with similarly. But this is not necessary, since the case $q<0$ can be reduced to the case $q>0$ with the help of the symmetry property of a Brownian motion, and the case $q=0$ can be reduced to the case $q=z$ due to the fact that a Brownian bridge is spatially homogeneous and time reversible (see § 11 Ch. I). Thus we have established that $\ell(\tau,y)$, $y\in\mathbf{R}$, is a Markov process given $W(\tau)=z$.

Next derive a number of characteristics of this Markov process. We compute the generating operators of the processes V_1, V_2, and V_3. It is not hard to see that thanks to the time reversibility property of the Brownian bridge the generating operator of V_3 coincides with the generating operator of V_1. To compute the generating operators of V_1 and V_2 we first compute for $h\ge 0$, $v>0$ and $\eta>0$ the expressions

$$u_1(h,v):=\mathbf{E}_0^z\big\{e^{-\eta\ell(\tau,q+h)}\big|\ell(\tau,q)=v\big\},\qquad \text{for}\quad 0\le z\le q,$$
$$u_2(h,v):=\mathbf{E}_0^z\big\{e^{-\eta\ell(\tau,q-h)}\big|\ell(\tau,q)=v\big\},\qquad \text{for}\quad 0\le q-h\le q\le z.$$

These functions are the Laplace transform with respect to η of the transition functions of the processes V_1 and V_2, therefore they uniquely determine the generating operators of the processes.

To compute the function $u_1(h,v)$ we can use Theorem 4.2 of Ch. III with $f=0$, $a=-\infty$, $b=\infty$, $q_1=q$, $q_2=q+h$, $q_l=0$ for $l\neq 1,2$. However the computations can be simplified by using formula (2.14). We should take in (2.14) instead of the function f_+ the Dirac δ-function at point $q+h$, multiplied by η. Actually, this means that in place of $f_+(y)$ we must consider the family $\left\{\frac{\eta}{\varepsilon}\mathbb{I}_{[q+h,q+h+\varepsilon)}(y)\right\}_{\varepsilon>0}$, $y\in\mathbf{R}$, of functions and pass to the limit as $\varepsilon\downarrow 0$ in the problem of computing (2.14). This procedure is analogous to that used in the proof of Theorem 3.1 Ch. III. It is not hard to see that in the domain $y\ge q$ the fundamental solution $\varphi_\delta(y)$ of (2.4) with the condition $\varphi_\delta(q)=1$ corresponding to the δ-function at $q+h$ $(f(y)=\eta\delta_{q+h}(y))$ is the continuous bounded solution of the following problem:

$$\frac{1}{2}\varphi''(y)-\lambda\varphi(y)=0, \qquad y\in(q,\infty)\setminus\{q+h\},$$
$$\varphi'(q+h+0)-\varphi'(q+h-0)=2\eta\varphi(q+h), \qquad \varphi(q)=1.$$

This solution can be represented in the form

$$\varphi_\delta(y)=(1-B)\,e^{(q-y)\sqrt{2\lambda}}+Be^{h\sqrt{2\lambda}}\,e^{-|y-q-h|\sqrt{2\lambda}}.$$

Here we have already taken into account the condition $\varphi_\delta(q)=1$ and some arguments of Example 3.1 Ch. III. The modulus in the exponential guarantees the continuity of the function and the jump of its derivative at $q+h$. The presence of the term $e^{(q-y)\sqrt{2\lambda}}$ guarantees the boundedness at $+\infty$. The condition on the jump of the derivative gives

$$B=-\frac{\eta\,e^{-2h\sqrt{2\lambda}}}{\sqrt{2\lambda}\left(1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{-2h\sqrt{2\lambda}})\right)}. \tag{2.18}$$

Since
$$\sqrt{2\lambda}+\varphi_\delta'(q)=\sqrt{2\lambda}-\sqrt{2\lambda}(1-B)+\sqrt{2\lambda}B=2\sqrt{2\lambda}B,$$

we find by substituting the values obtained above in (2.14), $z\le q$, that

$$u_1(h,v)=\exp\left(-\frac{v\eta e^{-2\sqrt{2\lambda}h}}{1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{2h\sqrt{2\lambda}})}\right). \tag{2.19}$$

Since $u_1(h,v)$ does not depend on q, the process $V_1(h)$, $h\ge 0$, is homogeneous.

Note that for the computation of $u_1(h,v)$ one can use also formula (2.12). In this case, $\psi_\delta(z)=e^{(z-q)\sqrt{2\lambda}}$ for $z\le q$ and $w=\psi_\delta'(q)-\varphi_\delta'(q)=2\sqrt{2\lambda}\,(1-B)$. The answer, according to (2.12) for $z\le q$, is the same: $u_1(h,v)=e^{vB}$.

We proceed similarly to compute $u_2(h,v)$. We use formula (2.16). In place of f_- we take the Dirac δ-function at $q-h$, multiplied by η $(f_-(y)=\eta\delta_{q-h}(y))$. In this case we consider the fundamental solution $\psi_\delta(y)$ of equation (2.4) in the domain $y\le q$ with the condition $\psi_\delta(q)=1$. Then the function $\psi_\delta(y)$ is the continuous bounded solution of the problem

$$\frac{1}{2}\psi(y)-\lambda\psi(y)=0, \qquad y\in(-\infty,q)\setminus\{q-h\},$$
$$\psi'(q-h+0)-\psi'(q-h-0)=2\eta\psi(q-h), \qquad \psi(q)=1.$$

This solution can be represented in the form

$$\psi_\delta(y) = (1-B)e^{(y-q)\sqrt{2\lambda}} + Be^{h\sqrt{2\lambda}}\, e^{-|y-q+h|\sqrt{2\lambda}}.$$

Here, as in the previous case, the condition $\psi_\delta(q)=1$, the condition of boundedness at $-\infty$, and the continuity condition at $q-h$ are taken into account. The condition on the jump of the derivative leads to the expression (2.18) for B. Since

$$\sqrt{2\lambda} - \psi_\delta'(q) = \sqrt{2\lambda} + B\sqrt{2\lambda} - (1-B)\sqrt{2\lambda} = 2\sqrt{2\lambda}B,$$
$$\psi_\delta(0) = e^{-q\sqrt{2\lambda}}\big(1 + B\big(e^{2h\sqrt{2\lambda}} - 1\big)\big),$$

substitution of these values in (2.16), $q \le z$, gives

$$u_2(h,v) = \frac{1}{1 + \frac{\eta}{\sqrt{2\lambda}}\big(1 - e^{-2\sqrt{2\lambda}h}\big)} \exp\bigg(-\frac{v\eta e^{-2\sqrt{2\lambda}h}}{1 + \frac{\eta}{\sqrt{2\lambda}}\big(1 - e^{2h\sqrt{2\lambda}}\big)}\bigg). \tag{2.20}$$

Since this expression is independent of q, the process $V_2(h)$, $h \ge 0$, is homogeneous.

By the definition of the generating operator of a homogeneous Markov process (see § 9 Ch. IV) and the arbitrariness of η, the operators $\mathbb{L}_1$ and $\mathbb{L}_2$, corresponding to the processes V_1 and V_2, can be determined from the respective equations

$$\frac{\partial u_1}{\partial h} = \mathbb{L}_1 u_1, \qquad \frac{\partial u_2}{\partial h} = \mathbb{L}_2 u_2.$$

In view of (2.19),

$$\frac{\partial}{\partial h}u_1(h,v) = \bigg(\frac{2v\eta^2 e^{-4\sqrt{2\lambda}h}}{(1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{-2\sqrt{2\lambda}h}))^2} + \frac{2v\eta\sqrt{2\lambda}e^{-2\sqrt{2\lambda h}}}{1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{-2\sqrt{2\lambda}h})}\bigg)u_1(h,v).$$

On the other hand,

$$\frac{\partial}{\partial v}u_1(h,v) = -\frac{\eta e^{-2\sqrt{2\lambda}h}}{1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{-2\sqrt{2\lambda}h})}u_1(h,v),$$

$$\frac{\partial^2}{\partial v^2}u_1(h,v) = \frac{\eta^2 e^{-4\sqrt{2\lambda}h}}{(1+\frac{\eta}{\sqrt{2\lambda}}(1-e^{-2\sqrt{2\lambda}h}))^2}u_1(h,v).$$

This implies that

$$\mathbb{L}_1 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda}\frac{d}{dv}\Big). \tag{2.21}$$

Similarly, it can be established that

$$\mathbb{L}_2 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda}\frac{d}{dv}\Big) + 2\frac{d}{dv}. \tag{2.22}$$

The theorem is proved. □

The processes V_k, $k = 1, 2, 3$, can be described in a more convenient form. Set

$$p^{(n)}(t, v, g) := t^{-1} g^{n/2} v^{1-n/2} e^{-(v^2+g^2)/2t} I_{n/2-1}(vg/t), \tag{2.23}$$

where $v > 0$, $g > 0$. Here and in what follows, $I_l(x)$ are modified Bessel functions (see Appendix 2). Let $R^{(n)}(t)$, $t \ge 0$, for $n = 0, 1, 2, \dots$ be an n-dimensional Bessel process, that is, a positive continuous homogeneous Markov process with the transition density $p^{(n)}(t, v, g)$. For $n = 1, 2, \dots$ the n-dimensional Bessel process $R^{(n)}(t)$ describes (see Subsection 5 § 16 Ch. IV) the distance from zero of the n-dimensional standard Brownian motion with independent coordinates.

The process $R^{(0)}(t)$, $t \ge 0$, is the 0-dimensional Bessel process, i.e., a continuous Markov process with the transition density $p^{(0)}(t, v, g)$ and with the probability

$$\mathbf{P}\{R^{(0)}(t) = 0 \big| R^{(0)}(0) = v\} = e^{-v^2/2t}$$

of hitting zero. Beginning at a nonnegative starting point, this process a.s. reaches zero and then remains equal to zero.

The process $Q^{(n)}(t) := e^{-\gamma t} R^{(n)}\Big(\dfrac{e^{2\gamma t} - 1}{2\gamma}\Big)$ is (see Subsection 6 § 16 Ch. IV) the radial Ornstein–Uhlenbeck process of order $n/2 - 1$ with the parameters $\gamma > 0$ and $\sigma = 1$. The special interest to us is the square of this process: $Z_n(t) := \big(Q^{(n)}(t)\big)^2$, $t \ge 0$. According to (16.8) Ch. IV, the generator of the quadratic radial Ornstein–Uhlenbeck process has the form

$$\mathbb{L}_2^{(n)} = 2v \frac{d^2}{dv^2} + (n - 2\gamma v) \frac{d}{dv}. \tag{2.24}$$

This formula holds also for $n = 0$.

Note that just the time change $\dfrac{e^{2\gamma t} - 1}{2\gamma}$ and the scale factor $e^{-\gamma t}$ guarantee the transformation of the homogeneous diffusion process (the Bessel process) into the homogeneous one (the radial Ornstein–Uhlenbeck process).

Now, comparing the generating operators (2.21) and (2.22) with the form of the operator (2.24), we get the following assertion.

Proposition 2.1. *The processes $V_k(h)$, $h \ge 0$, $k = 1, 2, 3$, are representable in the form*

$$V_1(h) = e^{-2\sqrt{2\lambda}h} \Big(R^{(0)}\Big(\frac{e^{2\sqrt{2\lambda}h} - 1}{2\sqrt{2\lambda}}\Big)\Big)^2, \qquad V_3(h) = e^{-2\sqrt{2\lambda}h} \Big(\widehat{R}^{(0)}\Big(\frac{e^{2\sqrt{2\lambda}h} - 1}{2\sqrt{2\lambda}}\Big)\Big)^2,$$

$$V_2(h) = e^{-2\sqrt{2\lambda}h} \Big(R^{(2)}\Big(\frac{e^{2\sqrt{2\lambda}h} - 1}{2\sqrt{2\lambda}}\Big)\Big)^2,$$

where $R^{(0)}(t)$, $\widehat{R}^{(0)}(t)$ and $R^{(2)}(t)$, $t \ge 0$, are independent Bessel processes of dimensions 0, 0, and 2 under fixed starting points. The initial values of the processes $V_k(h)$, $h \ge 0$, satisfy the equalities $V_1(0) = V_2(0)$, $V_3(0) = V_2(z)$ and have the density

$$\frac{d}{dv}\mathbf{P}(V_k(0) < v) = \sqrt{2\lambda}\, e^{-v\sqrt{2\lambda}}\, \mathbb{1}_{[0,\infty)}(v).$$

The process $\ell(\tau, y)$, $y \in \mathbf{R}$, given $W(\tau) = z$, can be described in terms of finite-dimensional distributions. Since ℓ is a Markov process, it is sufficient to specify the initial distribution (see (2.1)) and the transition probabilities. The latter can be derived from (4.62)–(4.65) of Ch. III. Since this process is invariant with respect to the replacement x to z, we can assume that $x < z$.

We denote the probability measure corresponding to the starting point $W(0) = x$ and the condition $W(\tau) = z$ by $\mathbf{P}_x^z(\cdot) := \mathbf{P}_x(\cdot | W(\tau) = z)$. Consider the transition probabilities

$$\mathbf{P}_x^z(\ell(\tau, u) \in dg | \ell(\tau, r) = v), \qquad r < u.$$

For brevity set $\Delta := u - r > 0$ and

$$L_l(\Delta, v, g) := \frac{\sqrt{\lambda}}{\sqrt{2}\,\mathrm{sh}(\Delta\sqrt{2\lambda})} \exp\Big(-\frac{\sqrt{\lambda}(ve^{-\Delta\sqrt{2\lambda}} + Ge^{\Delta\sqrt{2\lambda}})}{\sqrt{2}\,\mathrm{sh}(\Delta\sqrt{2\lambda})}\Big) I_l\Big(\frac{\sqrt{2\lambda vg}}{\mathrm{sh}(\Delta\sqrt{2\lambda})}\Big), \quad l = 0, 1.$$

The points x and z divide the real line into three intervals. As a result, for the transition probability we can distinguish six different cases depending on which intervals the points r and u belong to.

1) For $x \le r < u \le z$ and $v > 0$, we have

$$\mathbf{P}_x^z(\ell(\tau, u) \in dg | \ell(\tau, r) = v) = e^{\Delta\sqrt{2\lambda}} L_0(\Delta, v, g)\, dg.$$

2) For $x < z \le r < u$ we have

$$\mathbf{P}_x^z(\ell(\tau, u) = 0 | \ell(\tau, r) = v) = \exp\Big(-\frac{v\sqrt{\lambda}}{e^{\Delta\sqrt{2\lambda}} - 1}\Big),$$

$$\mathbf{P}_x^z(\ell(\tau, u) \in dg | \ell(\tau, r) = v) = \frac{\sqrt{v}}{\sqrt{g}} L_1(\Delta, v, g)\, dg, \qquad v > 0.$$

These expressions for the transition probabilities can be obtained from the representation of the processes V_1 and V_2 in terms of the squared Bessel processes of dimension 0 and 2 respectively, if we take into account the transition density (2.23).

3) For $r < u \le x < z$ we have

$$\mathbf{P}_x^z(\ell(\tau, u) = 0 | \ell(\tau, r) = 0) = \frac{1 - e^{-2(x-u)\sqrt{2\lambda}}}{1 - e^{-2(x-r)\sqrt{2\lambda}}},$$

$$\mathbf{P}_x^z(\ell(\tau, u) \in dg | \ell(\tau, r) = 0) = \frac{\sqrt{2\lambda}\, e^{-2(x-u)\sqrt{2\lambda}}}{1 - e^{-2(x-r)\sqrt{2\lambda}}} \exp\Big(-\frac{g\sqrt{2\lambda}}{1 - e^{-2\Delta\sqrt{2\lambda}}}\Big)\, dg,$$

$$\mathbf{P}_x^z(\ell(\tau, u) \in dg | \ell(\tau, r) = v) = \frac{\sqrt{g}}{\sqrt{v}}\, e^{\Delta\sqrt{2\lambda}} L_1(\Delta, v, g)\, dg, \qquad v > 0. \tag{2.25}$$

Note that these transition densities correspond to the process V_3, considered in the natural time direction (see Remark 2.3). One can verify that the transition density (2.25) corresponds to the process

$$V_4(h) = e^{-2\sqrt{2\lambda}h}\Big(R^{(4)}\Big(\frac{e^{2\sqrt{2\lambda}h} - 1}{2\sqrt{2\lambda}}\Big)\Big)^2.$$

4) For $r < x < u < z$ we have

$$\mathbf{P}_x^z(\ell(\tau,u) \in dg | \ell(\tau,r)=0) = \frac{\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}} \exp\Big(-\frac{g\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big),$$

$$\mathbf{P}_x^z(\ell(\tau,u) \in dg | \ell(\tau,r)=v)\, e^{(2r-x-u)\sqrt{2\lambda}}$$

$$= \frac{\sqrt{g}\,\mathrm{sh}((x-r)\sqrt{2\lambda})}{\sqrt{v}\,\mathrm{sh}((u-r)\sqrt{2\lambda})} L_1(\Delta,v,g)\,dg + \frac{\mathrm{sh}((u-x)\sqrt{2\lambda})}{\mathrm{sh}((u-r)\sqrt{2\lambda})} L_0(\Delta,v,g)\,dg, \qquad v>0.$$

5) For $x < r < z < u$ and $v > 0$ we have

$$\mathbf{P}_x^z(\ell(\tau,u) = 0 | \ell(\tau,r)=v) = \frac{\sqrt{2\lambda}\,\big(1-e^{-2(u-z)\sqrt{2\lambda}}\big)}{\big(1-e^{-2\Delta\sqrt{2\lambda}}\big)} \exp\Big(-\frac{g\sqrt{2\lambda}}{e^{2\Delta\sqrt{2\lambda}}-1}\Big),$$

$$\mathbf{P}_x^z(\ell(\tau,u) \in dg | \ell(\tau,r)=v)\, e^{(rz)\sqrt{2\lambda}}$$

$$= \frac{\sqrt{v}\,\mathrm{sh}((uz)\sqrt{2\lambda})}{\sqrt{g}\,\mathrm{sh}((ur)\sqrt{2\lambda})} L_1(\Delta,v,g)\,dg + \frac{\mathrm{sh}((zr)\sqrt{2\lambda})}{\mathrm{sh}((ur)\sqrt{2\lambda})} L_0(\Delta,v,g)\,dg.$$

6) For $r < x < z < u$ and $v > 0$ we have

$$\mathbf{P}_x^z(\ell(\tau,u) = 0 | \ell(\tau,r)=0) = \frac{1-e^{-2(u-z)\sqrt{2\lambda}}}{1-e^{-2(u-r)\sqrt{2\lambda}}},$$

$$\mathbf{P}_x^z(\ell(\tau,u) = 0 | \ell(\tau,r)=v) = \frac{\big(1-e^{-2(u-x)\sqrt{2\lambda}}\big)\big(1-e^{-2(u-z)\sqrt{2\lambda}}\big)}{\big(1-e^{-2\Delta\sqrt{2\lambda}}\big)^2} \exp\Big(-\frac{g\sqrt{2\lambda}}{e^{2\Delta\sqrt{2\lambda}}-1}\Big),$$

$$\mathbf{P}_x^z(\ell(\tau,u) \in dg | \ell(\tau,r)=0) = \frac{\sqrt{2\lambda}\,\big(e^{-2(u-z)\sqrt{2\lambda}}-e^{-2\Delta\sqrt{2\lambda}}\big)}{\big(1-e^{-2(u-r)\sqrt{2\lambda}}\big)^2} \exp\Big(-\frac{g\sqrt{2\lambda}}{1-e^{-2\Delta\sqrt{2\lambda}}}\Big)dg,$$

$$\mathbf{P}_x^z(\ell(\tau,u) \in dg | \ell(\tau,r)=v)\, e^{(2r-x-z)\sqrt{2\lambda}}$$

$$= \Big(\frac{\sqrt{v}\ \mathrm{sh}((u-x)\sqrt{2\lambda})\,\mathrm{sh}((u-z)\sqrt{2\lambda})}{\sqrt{g}\,\mathrm{sh}^2((u-r)\sqrt{2\lambda})} + \frac{\sqrt{g}\ \mathrm{sh}((x-r)\sqrt{2\lambda})\,\mathrm{sh}((z-r)\sqrt{2\lambda})}{\sqrt{v}\,\mathrm{sh}^2((u-r)\sqrt{2\lambda})}\Big) L_1(\Delta,v,g)\,dg$$

$$+\Big(\frac{\mathrm{sh}((x-r)\sqrt{2\lambda})\,\mathrm{sh}((u-z)\sqrt{2\lambda})}{\mathrm{sh}^2((u-r)\sqrt{2\lambda})} + \frac{\mathrm{sh}((z-r)\sqrt{2\lambda})\,\mathrm{sh}((u-x)\sqrt{2\lambda})}{\mathrm{sh}^2((u-r)\sqrt{2\lambda})}\Big) L_0(\Delta,v,g)\,dg.$$

§ 3. Markov property of Brownian local time stopped at the first exit time

Let $H_{a,b} = \min\{s : W(s) \notin (a,b)\}$ be the first exit time of the Brownian motion W from the interval (a,b). We set $W(0) = x \in (a,b)$. In this section we continue the description of the Brownian local time $\ell(t,y)$ as a process with respect to y. Instead of a fixed time t, here we consider the stopping time $H_{a,b}$.

Theorem 3.1. *Given $W(H_{a,b}) = a$, the process $\{\ell(H_{a,b}, y),\ y \in [a, b]\}$, is a Markov process that can be represented in the form*

$$\ell(H_{a,b}, y) = \begin{cases} V_{a4}(x-y), & \text{for } \ a \le y \le x, \\ V_{a5}(y-x), & \text{for } \ x \le y \le b, \end{cases}$$

where $\{V_{a4}(h), 0 \le h \le x-a\}$ and $\{V_{a5}(h), 0 \le h \le b-x\}$ are nonnegative independent diffusions under fixed starting points. The generating operators of the processes V_{a4} and V_{a5} have the form

$$\overset{\circ}{\mathbb{L}}_{a4} = 2v\frac{d^2}{dv^2} + 2\Big(1 - \frac{v}{x-a-h}\Big)\frac{d}{dv}, \qquad \overset{\circ}{\mathbb{L}}_{a5} = 2v\frac{d^2}{dv^2} - \frac{2v}{b-x-h}\frac{d}{dv}, \tag{3.1}$$

respectively. The initial values coincide ($V_{a4}(0) = V_{a5}(0)$) and their distribution is determined by the formula

$$\mathbf{P}_x\{\ell(H_{a,b}, x) \ge v | W(H_{a,b}) = a\} = \exp\Big(-\frac{(b-a)v}{2(b-x)(x-a)}\Big). \tag{3.2}$$

Remark 3.1. Given $W(H_{a,b}) = b$, the analogous description holds:

$$\ell(H_{a,b}, y) = \begin{cases} V_{b4}(x-y), & \text{for } a \le y \le x, \\ V_{b5}(y-x), & \text{for } x \le y \le b, \end{cases}$$

where $\{V_{b4}(h), 0 \le h \le x-a\}$ and $\{V_{b5}(h), 0 \le h \le b-x\}$ are nonnegative independent diffusions under fixed starting points. They have the generating operators

$$\overset{\circ}{\mathbb{L}}_{b4} = 2v\frac{d^2}{dv^2} - \frac{2v}{x-a-h}\frac{d}{dv}, \qquad \overset{\circ}{\mathbb{L}}_{b5} = 2v\frac{d^2}{dv^2} + 2\Big(1 - \frac{v}{b-x-h}\Big)\frac{d}{dv}, \tag{3.3}$$

respectively, and the same initial values $V_{b4}(0) = V_{b5}(0)$ distributed by (3.2).

Remark 3.2. The processes $\{V_{ak}(h), h \ge 0\}$, $\{V_{bk}(h), h \ge 0\}$, $k = 4, 5$, are nonhomogeneous diffusions and $\overset{\circ}{\mathbb{L}}$ is determined by the formula (9.13) of Ch. IV.

Remark 3.3. From the behavior of the Brownian motion W on the interval (a, b) and the fact that the local time $\ell(H_{a,b}, y)$, $y \in [a, b]$, is strictly positive in any interior states of the Brownian motion except the extreme ones, it follows that the processes $V_{a4}(h)$ and $V_{b5}(h)$ equal zero only at the points $h = x - a$ and $h = b - x$, respectively, and the processes $V_{a5}(h)$ and $V_{b4}(h)$ degenerate to zero when h does not reach the boundary values $b - x$ and $x - a$, respectively.

Remark 3.4. The processes V_{ak}, V_{bk}, $k = 4, 5$, can be expressed in terms of independent Bessel processes $\{R^{(n)}(t), t \ge 0\}$, $n = 0, 2$, whose initial values are assigned the random variables with the appropriate initial distributions:

$$V_{a4}(h) = (x-a-h)^2\Big(R^{(2)}\Big(\frac{1}{x-a-h} - \frac{1}{x-a}\Big)\Big)^2, \qquad \text{for } \ 0 \le h \le x-a,$$

$$V_{a5}(h) = (b-x-h)^2\Big(R^{(0)}\Big(\frac{1}{b-x-h} - \frac{1}{b-x}\Big)\Big)^2, \qquad \text{for } \ 0 \le h \le b-x,$$

$$V_{b4}(h) = (x-a-h)^2\Big(R^{(0)}\Big(\frac{1}{x-a-h} - \frac{1}{x-a}\Big)\Big)^2, \qquad \text{for } \ 0 \le h \le x-a,$$

$$V_{b5}(h) = (b-x-h)^2\Big(R^{(2)}\Big(\frac{1}{b-x-h} - \frac{1}{b-x}\Big)\Big)^2, \qquad \text{for } \ 0 \le h \le b-x.$$

To verify this, it is sufficient to use (9.14) Ch. IV. Since the squared Bessel process $Y_n(t) := (R^{(n)}(t))^2$, by (16.5) Ch. IV, satisfies the stochastic differential equation

$$dY_n(t) = n dt + 2\sqrt{Y_n(t)}\, dW(t),$$

choosing in (9.14) Ch. IV the parameter h instead of s, and $a(h) = (x-a-h)^2$, $b(h) = \frac{1}{x-a-h} - \frac{1}{x-a}$, we see that the processes $V_{a4}(h)$ and $V_{b4}(h)$, represented in such form, have the respective generating operators (3.1) and (3.3). In order to compute the generating operators of $V_{a5}(h)$ and $V_{b5}(h)$, expressed in terms of Bessel processes by formula (9.14) Ch. IV, we should take $a(h) = (b-x-h)^2$ and $b(h) = \frac{1}{b-x-h} - \frac{1}{b-x}$.

Remark 3.5. Without conditions on the end value $W(H_{a,b})$ of a sample paths, the process $\ell(H_{a,b}, y)$, $y \in [a, b]$, is the mixture of two Markov processes, namely,

$$\ell(H_{a,b}, y) = \begin{cases} (y-a)^2 \Big(R^{(\chi)}\Big(\frac{1}{y-a} - \frac{1}{x-a} \Big) \Big)^2, & \text{for } a \le y \le x, \\ (b-y)^2 \Big(R^{(2-\chi)}\Big(\frac{1}{b-y} - \frac{1}{b-x} \Big) \Big)^2, & \text{for } x \le y \le b, \end{cases}$$

where χ is a random variable independent of the Bessel processes $R^{(n)}$, $n = 0, 2$, and having the distribution

$$\mathbf{P}(\chi = 0) = \frac{x-a}{b-a}, \qquad \mathbf{P}(\chi = 2) = \frac{b-x}{b-a}.$$

Remark 3.6. To describe the Brownian local time stopped at the first hitting time H_z of a level z we can choose for $z \le x$ in Theorem 3.1 $a = z$, $b = \infty$, and choose for $x \le z$ in Remark 3.1 $b = z$, $a = -\infty$.

Proof of Theorem 3.1. As usual, we append to the expectation and probability symbols the subscript x to indicate the condition $W(0) = x$. The proof of Theorem 3.1 coincides in the main aspects with that for Theorem 2.1. We use the same notations. To prove the Markov property (see the beginning of the proof of Theorem 2.1 and the equality (2.3)), we verify that for any $q \in (a, b)$ and $v \ge 0$

$$\mathbf{E}_x \Big\{ \exp\Big(-\int_0^{H_{a,b}} f(W(s))\, ds \Big) \Big| \ell(H_{a,b}, q) = v,\, W(H_{a,b}) = a \Big\}$$
$$= \mathbf{E}_x \Big\{ \exp\Big(-\int_0^{H_{a,b}} f_+(W(s))\, ds \Big) \Big| \ell(H_{a,b}, q) = v,\, W(H_{a,b}) = a \Big\}$$
$$\times \mathbf{E}_x \Big\{ \exp\Big(-\int_0^{H_{a,b}} f_-(W(s))\, ds \Big) \Big| \ell(H_{a,b}, q) = v,\, W(H_{a,b}) = a \Big\},$$

which if $v > 0$ is equivalent to

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds\Big);\ell(H_{a,b},q) < v,\, W(H_{a,b}) = a\Big\}$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_+(W(s))\,ds\Big)\Big|\ell(H_{a,b},q) = v,\, W(H_{a,b}) = a\Big\}$$

$$\times \frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_-(W(s))\,ds\Big);\ell(H_{a,b},q) < v,\, W(H_{a,b}) = a\Big\}. \tag{3.4}$$

To compute the expectations included in (3.4) we first compute the function

$$Q(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds - \gamma\ell(H_{a,b},y)\Big); W(H_{a,b}) = a\Big\}, \qquad x \in (a,b),$$

and the analogous functions $Q_-(x)$ and $Q_+(x)$, $x \in (a,b)$, for f_- and f_+, respectively. Then we compute the inverse Laplace transforms with respect to γ.

Let φ and ψ be linearly independent solutions of equation (2.4) with $\lambda = 0$, with φ decreasing and ψ increasing, and let $\omega = \psi'(y)\varphi(y) - \varphi'(y)\psi(y)$ be their Wronskian, which is a constant. Let, in addition, the boundary conditions $\psi(a) = 0$, $\psi(b) = 1$, $\varphi(a) = 1$ and $\varphi(b) = 0$ hold. We can always construct the solutions with such boundary conditions in terms of $\bar\varphi(y)$ and $\bar\psi(y)$, $y \in \mathbf{R}$, the fundamental solutions of equation (2.4) with $\lambda = 0$:

$$\varphi(y) = \frac{\bar\psi(b)\bar\varphi(y) - \bar\psi(y)\bar\varphi(b)}{\bar\psi(b)\bar\varphi(a) - \bar\psi(a)\bar\varphi(b)}, \qquad \psi(y) = \frac{\bar\psi(y)\bar\varphi(a) - \bar\psi(a)\bar\varphi(y)}{\bar\psi(b)\bar\varphi(a) - \bar\psi(a)\bar\varphi(b)}.$$

By Theorem 5.3 Ch. III, the function Q is the unique continuous solution of the problem

$$\frac{1}{2}Q''(x) - f(x)Q(x) = 0, \qquad x \in (a,b)\setminus\{q\}, \tag{3.5}$$

$$Q'(q+0) - Q'(q-0) = 2\gamma Q(q), \tag{3.6}$$

$$Q(a) = 1, \qquad Q(b) = 0. \tag{3.7}$$

A solution of the problem (3.5)–(3.7) can be represented in the form

$$Q(x) = \begin{cases} \varphi(x) + (A - \varphi(q))\dfrac{\psi(x)}{\psi(q)}, & \text{for } \ a \le x \le q, \\ A\dfrac{\varphi(x)}{\varphi(q)}, & \text{for } \ q \le x \le b. \end{cases}$$

In this representation we have taken into account the boundary conditions (3.7) and the continuity condition at q. From (3.6) we compute A and, consequently,

$$Q(x) = \begin{cases} \varphi(x) - \dfrac{\varphi(q)\psi(x)}{\psi(q)} + \dfrac{\omega\varphi(q)\psi(x)}{\psi(q)(\omega + 2\gamma\varphi(q)\psi(q))}, & \text{for } \ a \le x \le q, \\ \dfrac{\omega\varphi(x)}{\omega + 2\gamma\varphi(q)\psi(q)}, & \text{for } \ q \le x \le b. \end{cases}$$

Inverting the Laplace transform with respect to γ, we get

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds\Big); \ell(H_{a,b},q) < v,\, W(H_{a,b}) = a\Big\}$$

$$= \begin{cases} \dfrac{\omega\psi(x)}{2\psi^2(q)}\exp\Big(-\dfrac{v\omega}{2\psi(q)\varphi(q)}\Big), & \text{for} \quad a \le x \le q, \\ \dfrac{\omega\varphi(x)}{2\varphi(q)\psi(q)}\exp\Big(-\dfrac{v\omega}{2\psi(q)\varphi(q)}\Big), & \text{for} \quad q \le x \le b, \end{cases} \qquad v > 0, \tag{3.8}$$

and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(W(s))\,ds\Big); \ell(H_{a,b},q) = 0,\, W(H_{a,b}) = a\Big\}$$

$$= \Big(\varphi(x) - \frac{\varphi(q)\psi(x)}{\psi(q)}\Big)\mathbb{1}_{[a,q]}(x). \tag{3.9}$$

For $f \equiv 0$ we have $\psi(x) = \dfrac{x-a}{b-a}$ and $\varphi(x) = \dfrac{b-x}{b-a}$, and $w = \dfrac{1}{b-a}$. Then from (3.8) and (3.9) it follows that

$$\frac{\partial}{\partial v}\mathbf{P}_x\big(\ell(H_{a,b},q) < v,\, W(H_{a,b}) = a\big)$$

$$= \begin{cases} \dfrac{x-a}{2(q-a)^2}\exp\Big(-\dfrac{(b-a)v}{2(b-q)(q-a)}\Big), & \text{for} \quad a \le x \le q, \\ \dfrac{b-x}{2(b-q)(q-a)}\exp\Big(-\dfrac{(b-a)v}{2(b-q)(q-a)}\Big), & \text{for} \quad q \le x \le b, \end{cases} \qquad v > 0, \tag{3.10}$$

and

$$\mathbf{P}_x\big(\ell(H_{a,b},q) = 0,\, W(H_{a,b}) = a\big) = \frac{q-x}{q-a}\mathbb{1}_{[a,q]}(x). \tag{3.11}$$

For $q = x$ (3.10) implies (3.2).

To obtain the expressions analogous to (3.8) and (3.9) for the function f_-, we proceed as follows. The solution of the problem (3.5)–(3.7) for f_- in place of f can be found in the form

$$Q_-(x) = \begin{cases} \varphi(x) + (A_- - \varphi(q))\dfrac{\psi(x)}{\psi(q)}, & \text{for} \quad a \le x \le q, \\ A_-\dfrac{(b-x)}{(b-q)}, & \text{for} \quad q \le x \le b. \end{cases}$$

By (3.6), we have

$$Q_-(x) = \begin{cases} \varphi(x) - \dfrac{\varphi(q)\psi(x)}{\psi(q)} + \dfrac{\omega\psi(x)}{2\psi^2(q)\Big(\dfrac{\psi'(q)}{2\psi(q)} + \dfrac{1}{2(b-q)} + \gamma\Big)}, & \text{for} \quad a \le x \le q, \\ \dfrac{\omega(b-x)}{2(b-q)\psi(q)\Big(\dfrac{\psi'(q)}{2\psi(q)} + \dfrac{1}{2(b-q)} + \gamma\Big)}, & \text{for} \quad q \le x \le b. \end{cases}$$

Inverting the Laplace transform with respect to γ, we obtain

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_-(W(s))\,ds\Big); \ell(H_{a,b},q)<v,\, W(H_{a,b})=a\Big\}$$

$$=\begin{cases} \dfrac{\omega\psi(x)}{2\psi^2(q)}\exp\Big(-\dfrac{v\psi'(q)}{2\psi(q)}-\dfrac{v}{2(b-q)}\Big), & \text{for}\quad a\le x\le q,\\ \dfrac{\omega(b-x)}{2(b-q)\psi(q)}\exp\Big(-\dfrac{v\psi'(q)}{2\psi(q)}-\dfrac{v}{2(b-q)}\Big), & \text{for}\quad q\le x\le b,\end{cases}\qquad v>0, \tag{3.12}$$

and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_-(W(s))\,ds\Big); \ell(H_{a,b},q)=0,\, W(H_{a,b})=a\Big\}$$

$$=\Big(\varphi(x)-\frac{\varphi(q)\psi(x)}{\psi(q)}\Big)\mathbb{1}_{[a,q]}(x). \tag{3.13}$$

To obtain expressions analogous to (3.8), (3.9) for the function f_+ we represent the solution of the problem (3.5)–(3.7) with the function f_+ in place of f in the form

$$Q_+(x)=\begin{cases} \dfrac{q-x}{q-a}+A_+\dfrac{x-a}{q-a}, & \text{for}\quad a\le x\le q,\\ A_+\dfrac{\varphi(x)}{\varphi(q)}, & \text{for}\quad q\le x\le b.\end{cases}$$

By (3.6), we have

$$Q_+(x)=\begin{cases} \dfrac{q-x}{q-a}+\dfrac{x-a}{2(q-a)^2\Big(\dfrac{1}{2(q-a)}-\dfrac{\varphi'(q)}{2\varphi(q)}+\gamma\Big)}, & \text{for}\quad a\le x\le q,\\ \dfrac{\varphi(x)}{2(q-a)\varphi(q)\Big(\dfrac{1}{2(q-a)}-\dfrac{\varphi'(q)}{2\varphi(q)}+\gamma\Big)}, & \text{for}\quad q\le x\le b.\end{cases}$$

Inverting the Laplace transform with respect to γ, we obtain

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_+(W(s))\,ds\Big); \ell(H_{a,b},q)<v,\, W(H_{a,b})=a\Big\}$$

$$=\begin{cases} \dfrac{x-a}{2(q-a)^2}\exp\Big(-\dfrac{v}{2(q-a)}+\dfrac{v\varphi'(q)}{2\varphi(q)}\Big), & \text{for}\quad a\le x\le q,\\ \dfrac{\varphi(x)}{2(q-a)\varphi(q)}\exp\Big(-\dfrac{v}{2(q-a)}+\dfrac{v\varphi'(q)}{2\varphi(q)}\Big), & \text{for}\quad q\le x\le b,\end{cases}\qquad v>0, \tag{3.14}$$

and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_+(W(s))\,ds\Big); \ell(H_{a,b},q)=0,\, W(H_{a,b})=a\Big\}=\frac{q-x}{q-a}\mathbb{1}_{[a,q]}(x). \tag{3.15}$$

Dividing (3.14) by (3.10), we have

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f_+(W(s))\,ds\Big)\Big|\ell(H_{a,b},q)=v,\,W(H_{a,b})=a\Big\}$$
$$=\begin{cases}\exp\Big(\dfrac{v}{2(b-q)}+\dfrac{v\varphi'(q)}{2\varphi(q)}\Big), & \text{for}\quad a\le x\le q,\\ \dfrac{\varphi(x)(b-q)}{2(b-x)\varphi(q)}\exp\Big(\dfrac{v}{2(b-q)}+\dfrac{v\varphi'(q)}{2\varphi(q)}\Big), & \text{for}\quad q\le x\le b.\end{cases} \tag{3.16}$$

For $a\le x<q$, this formula holds also for $v=0$.

Now, since (3.8) is equal to the product of (3.12) and (3.16), the relation (3.4) holds. The set $\{\ell(H_{a,b},q)=0,\,W(H_{a,b})=a\}$ is empty if $q\le x\le b$. If $a\le x<q$ the equality preceding (3.4) is also easily verified for $v=0$, because of the coincidence of the right-hand sides in (3.9) and (3.13). Thus we prove that $\ell(H_{a,b},y)$, $y\in[a,b]$, is a Markov process given $W(H_{a,b})=a$.

Using the same approach as in the proof of Theorem 2.1, we compute the generating operators of $V_{a4}(h)$ and $V_{a5}(h)$. The difference is that the processes $V_{a4}(h)$ and $V_{a5}(h)$ are not homogeneous. The main part of generating operators for nonhomogeneous diffusions is defined by (9.13) Ch. IV.

For $q,g\in(a,b)$ and $v>0$ we set

$$u(q,v):=\mathbf{E}_x\{e^{-\eta\ell(H_{a,b},g)}|\ell(H_{a,b},q)=v,\,W(H_{a,b})=a\},\qquad \eta>0.$$

Let $q=x+h$, $g=x+h_1$ and $h_1>h$. Then the function u is the Laplace transform with respect to η of the transition function $P(h,v,h_1,dy)$ of the process $V_{a5}(h)=\ell(H_{a,b},x+h)$, $W(0)=x$, i. e.,

$$u(q,v)=\int_0^\infty e^{-\eta y}\,P(h,v,h_1,dy).$$

Thus, the function u uniquely determines the process V_{a5}, in particular, his generator by equation (2.1) of Ch. IV. In order to compute u we use (3.8). The justification of the method of computing of the function u is exactly the same as in the proof of Theorem 2.1. In place of the function f we take the Dirac δ-function at g, multiplied by η ($f(y)=\eta\delta_g(y)$). In this case the corresponding functions $\varphi_\delta(y)$ and $\psi_\delta(y)$ are continuous solutions of the problem

$$\frac12\phi''(y)=0,\qquad y\in(a,b)\setminus\{g\},$$
$$\phi'(g+0)-\phi'(g-0)=2\eta\phi(g),$$

with the boundary conditions $\phi(a)=1$, $\phi(b)=0$ and $\phi(a)=0$, $\phi(b)=1$, respectively.

Such solutions have the form

$$\varphi_\delta(y)=\begin{cases}\dfrac{b-y+2\eta(b-g)(g-y)}{b-a+2\eta(b-g)(g-a)}, & \text{for}\quad a\le y\le g,\\ \dfrac{b-y}{b-a+2\eta(b-g)(g-a)}, & \text{for}\quad g\le y\le b,\end{cases}$$

$$\psi_\delta(y) = \begin{cases} \dfrac{y-a}{b-a+2\eta(b-g)(g-a)}, & \text{for} \quad a \le y \le g, \\ \dfrac{y-a+2\eta(y-g)(g-a)}{b-a+2\eta(b-g)(g-a)}, & \text{for} \quad g \le y \le b. \end{cases}$$

Their Wronskian is $w = (b-a+2\eta(b-g)(g-a))^{-1}$.

Substituting these solutions into (3.8), we find that for $v > 0$

$$\frac{\partial}{\partial v}\mathbf{E}_x\{e^{-\eta\ell(H_{a,b},g)}; \ell(H_{a,b},q) < v, \, W(H_{a,b}) = a\}$$

$$= \begin{cases} \dfrac{x-a}{2(q-a)^2}\exp\Big(-\dfrac{v}{2(q-a)} - \dfrac{v(1+2\eta(b-g))}{2(b-q+2\eta(b-g)(g-q))}\Big), & \text{for} \quad x \le q \le g, \\ \dfrac{b-x}{2(b-q)(q-a+2\eta(q-g)(g-a))} \\ \qquad\qquad \times \exp\Big(-\dfrac{v}{2(b-q)} - \dfrac{v(1+2\eta(g-a))}{2(q-a+2\eta(q-g)(g-a))}\Big), & \text{for} \quad g \le q \le x. \end{cases}$$

Dividing the right-hand side of this expression by (3.10), we obtain

$$u(q,v) = \begin{cases} \exp\Big(-\dfrac{v\eta(b-g)^2}{(b-q)(b-q+2\eta(b-g)(g-q))}\Big), & q \le g, \\ \dfrac{q-a}{q-a+2\eta(q-g)(g-a)}\exp\Big(-\dfrac{v\eta(g-a)^2}{(q-a)(q-a+2\eta(q-g)(g-a))}\Big), & g \le q. \end{cases}$$

By equation (2.1) Ch. IV, the infinitesimal generator $\mathbb{L}^\circ_{a5}$ of the nonhomogeneous process $V_{a5}(h) = \ell(H_{a,b}, x+h)$ can be computed from the equality

$$-\frac{\partial}{\partial q}u(q,v)\Big|_{q=x+h} = \mathbb{L}^\circ_{a5}\, u(x+h,v), \qquad q < g.$$

For $q < g$ we have

$$-\frac{\partial}{\partial q}u = \Big(\frac{v\eta(b-g)^2}{(b-q)^2(b-q+2\eta(b-g)(g-q))} + \frac{v\eta(b-g)^2(1+2\eta(b-g))}{(b-q)(b-q+2\eta(b-g)(g-q))^2}\Big)u$$

$$= \Big(\frac{2v\eta(b-g)^2}{(b-q)^2(b-q+2\eta(b-g)(g-q))} + \frac{2v\eta^2(b-g)^4}{(b-q)^2(b-q+2\eta(b-g)(g-q))^2}\Big)u$$

$$= -\frac{2v}{b-q}\frac{\partial}{\partial v}u + 2v\frac{\partial^2}{\partial v^2}u.$$

This implies that $\mathbb{L}^\circ_{a5} = 2v\dfrac{d^2}{dv^2} - \dfrac{2v}{b-x-h}\dfrac{d}{dv}$.

To write out the infinitesimal generator $\mathbb{L}^\circ_{a4}$ of the process $V_{a4}(h) = \ell(H_{a,b}, x-h)$, which time moves in the opposite direction, we must use the equality

$$\frac{\partial}{\partial q}u(q,v)\Big|_{q=x-h} = \mathbb{L}^\circ_{a4}\, u(x-h,v), \qquad q > g.$$

For $g < q$ we have

$$\frac{\partial}{\partial q}u = \Big(\frac{2}{q-a} - \frac{1+2\eta(g-a)}{q-a+2\eta(q-g)(g-a)} + \frac{v\eta(g-a)^2}{(q-a)^2(q-a+2\eta(q-g)(g-a))}$$

$$+\frac{v\eta(g-a)^2(1+2\eta(g-a))}{(q-a)(q-a+2\eta(q-g)(g-a))^2}\Big)u=\Big(-\frac{2\eta(g-a)^2}{(q-a)(q-a+2\eta(q-g)(g-a))}$$
$$+\frac{2v\eta(g-a)^2}{(q-a)^2(q-a+2\eta(q-g)(g-a))}-\frac{2v\eta^2(g-a)^4}{(q-a)^2(q-a+2\eta(q-g)(g-a))^2}\Big)u$$
$$=2\frac{\partial}{\partial v}u-\frac{2v}{q-a}\frac{\partial}{\partial v}u+2v\frac{\partial^2}{\partial v^2}u.$$

Hence $\mathbb{L}^{\circ}_{a4}=2v\dfrac{d^2}{dv^2}+2\Big(1-\dfrac{v}{x-a-h}\Big)\dfrac{d}{dv}$.

Theorem 3.1 is proved. □

§ 4. Markov property of Brownian local time stopped at the inverse local time

We consider $\varrho(u,z)=\min\{s:\ell(s,z)=u\}$, the moment inverse of the local time at a level z, where $(u,z)\in[0,\infty)\times\mathbf{R}$. This section provides a description of the Brownian local time $\ell(t,y)$ as a process with respect to y at the moment $t=\varrho(u,z)$. It is clear that the path of the Brownian motion W at this moment stops at the level z, i.e., $W(\varrho(u,z))=z$, because $\varrho(u,z)$ is a point of growth of the Brownian local time $\ell(t,z)$. We assume that $W(0)=x$.

Theorem 4.1. *The process $\{\ell(\varrho(u,z),y),\ y\in\mathbf{R}\}$ is a Markov process that can be represented in the form*

$$\ell(\varrho(u,z),y)=\begin{cases}V_6(|x-y|), & \text{for } y\le x\le z \text{ or } z\le x\le y,\\ V_7(|z-y|), & \text{for } x\wedge z\le y\le x\vee z,\\ V_8(|y-z|), & \text{for } y\le z\le x \text{ or } x\le z\le y,\end{cases}$$

where $\{V_k(h),h\ge 0\}$, $k=6,7,8$, are independent homogeneous diffusions under fixed starting points. The initial values of the processes V_k, $k=6,7,8$, satisfy the equalities $V_7(0)=V_8(0)=u$, $V_6(0)=V_7(|z-x|)$, and the generating operators have the form

$$\mathbb{L}_6=2v\frac{d^2}{dv^2},\qquad \mathbb{L}_7=2v\frac{d^2}{dv^2}+2\frac{d}{dv},\qquad \mathbb{L}_8=2v\frac{d^2}{dv^2},$$

respectively.

Remark 4.1. It is useful to consider the formulation of Theorem 4.1 separately for $x<z$ and $x>z$. Thus, for $x<z$ the stopped local time is described by the process $V_8(y-z)$, $y\ge z$, in the direct time, and by the processes $V_7(z-y)$, $x\le y\le z$, and $V_6(x-y)$, $y\le x$, in the reverse time. If $z<x$, then the stopped local time is described by the processes $V_7(y-z)$, $z\le y\le x$, and $V_6(y-x)$, $y\ge x$, in the direct time, and by the process $V_8(z-y)$, $y\le z$, in the reverse time. The point z is chosen as starting value, because, by the definition of the stopping time $\varrho(u,z)$, the local time $\ell(\varrho(u,z),z)$ is equal to u. By the homogeneity and symmetry properties of a Brownian motion it is sufficient to prove the theorem only for one of the cases $x<z$ or $x>z$. From these properties it follows also that the processes V_6 and V_8

with fixed identical starting points coincide, while the process V_7 is reversible in time on any finite interval.

Remark 4.2. The processes $\{V_k(h), h \geq 0\}$, $k = 6, 7, 8$, can be expressed in terms of the independent Bessel processes $\{R^{(l)}(s), s \geq 0\}$, $l = 0, 2$, with the appropriate initial distributions as follows:

$$V_k(h) = (R^{(0)}(h))^2 \quad \text{for } k = 6, 8, \tag{4.1}$$

$$V_7(h) = (R^{(2)}(h))^2. \tag{4.2}$$

This statement is a consequence of the fact that the infinitesimal generators of squared Bessel processes have the appropriate form (see (16.6) Ch. IV).

Proof of Theorem 4.1. The method of the proof of Theorem 4.1 coincides with that for Theorem 2.1. We keep some notations. To establish the Markov property, it is sufficient to prove that for any $q \in \mathbf{R}$ and $v \geq 0$

$$\mathbf{E}_x\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(y)\ell(\varrho(u,z),y)\,dy\Big)\Big|\ell(\varrho(u,z),q) = v\Big\}$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\int_{-\infty}^{\infty} f_+(y)\ell(\varrho(u,z),y)\,dy\Big)\Big|\ell(\varrho(u,z),q) = v\Big\}$$

$$\times \mathbf{E}_z\Big\{\exp\Big(-\int_{-\infty}^{\infty} f_-(y)\ell(\varrho(u,z),y)\,dy\Big)\Big|\ell(\varrho(u,z),q) = v\Big\}.$$

By (1.2), this is equivalent to the following two equalities: for $v > 0$

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds\Big); \ell(\varrho(u,z),q) < v\Big\}$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))\,ds\Big)\Big|\ell(\varrho(u,z),q) = v\Big\}$$

$$\times \frac{\partial}{\partial v}\mathbf{E}_z\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_-(W(s))\,ds\Big); \ell(\varrho(u,z),q) < v\Big\}, \tag{4.3}$$

and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds\Big); \ell(\varrho(u,z),q) = 0\Big\}$$

$$= \mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))\,ds\Big)\Big|\ell(\varrho(u,z),q) = 0\Big\}$$

$$\times \mathbf{E}_z\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_-(W(s))\,ds\Big); \ell(\varrho(u,z),q) = 0\Big\}.$$

We prove this equalities by providing explicit formulas for the expectations. We express them in terms of fundamental solutions of the equation

$$\frac{1}{2}\phi''(y) - f(y)\phi(y) = 0, \qquad y \in \mathbf{R}. \tag{4.4}$$

Let ψ be an increasing and φ be a decreasing nonnegative solutions of (4.4), satisfying the conditions $\psi(q) = \varphi(q) = 1$, and let $w = \psi'(x)\varphi(x) - \psi(x)\varphi'(x)$ be their Wronskian. It is clear that $w = \psi'(q) - \varphi'(q)$.

Let $\gamma \geq 0$. Assume for definiteness that $z < q$. We use Theorem 7.3 Ch. III to compute the function

$$d(u,x) := \mathbf{E}_x \exp\Big(- \int_0^{\varrho(u,z)} f(W(s))ds - \gamma\ell(\varrho(u,z),q)\Big).$$

According to this theorem with $a = -\infty$, $b = \infty$, $q_1 = z$, $q_2 = q$, $\beta_2 = \gamma$, and $\beta_l = 0$, $l \neq 2$, the function $d(u,x)$ has the form

$$d(u,x) = \begin{cases} \dfrac{\widetilde{\psi}(x)}{\widetilde{\psi}(z)} \exp\Big(- \dfrac{wu}{2\widetilde{\varphi}(z)\widetilde{\psi}(z)}\Big), & \text{for} \quad x \leq z, \\ \dfrac{\widetilde{\varphi}(x)}{\widetilde{\varphi}(z)} \exp\Big(- \dfrac{wu}{2\widetilde{\varphi}(z)\widetilde{\psi}(z)}\Big), & \text{for} \quad z \leq x, \end{cases} \tag{4.5}$$

where $\widetilde{\varphi}(x)$, $x \in \mathbf{R}$, is a nonnegative continuous decreasing solution and $\widetilde{\psi}(x)$, $x \in \mathbf{R}$, is a nonnegative continuous increasing solution of the problem

$$\frac{1}{2}\widetilde{\phi}''(x) - f(x)\widetilde{\phi}(x) = 0, \qquad x \neq q, \tag{4.6}$$

$$\widetilde{\phi}'(q+0) - \widetilde{\phi}'(q-0) = 2\gamma\widetilde{\phi}(q). \tag{4.7}$$

Denote for brevity $\Delta := \varphi(z) - \psi(z)$. By the monotonicity properties of ψ and φ and the condition $\psi(q) = \varphi(q) = 1$, we have that $\Delta > 0$ for $z < q$. We let $\widetilde{\psi}(x) = \psi(x)$ for $x < q$, because in this domain (4.7) does not influence the behavior of $\widetilde{\psi}$. The solution $\widetilde{\varphi}$ can be represented in the form

$$\widetilde{\varphi}(x) = \begin{cases} (1+A)\varphi(x) - A\psi(x), & \text{for} \quad x \leq q, \\ \varphi(x), & \text{for} \quad q \leq x. \end{cases}$$

In this representation we have taken into account the continuity of the function $\widetilde{\varphi}$ at the point q. Using (4.7), we compute that $A = 2\gamma/\omega$. Consequently,

$$\widetilde{\varphi}(x) = \varphi(x) + 2\gamma(\varphi(x) - \psi(x))/\omega, \qquad \widetilde{\varphi}(z) = \varphi(z) + 2\gamma\Delta/\omega,$$

$$\widetilde{\omega}(z) := \widetilde{\psi}'(z)\widetilde{\varphi}(z) - \widetilde{\varphi}'(z)\widetilde{\psi}(z) = \omega + 2\gamma,$$

and for $x \leq q$

$$\frac{\widetilde{\varphi}(x)}{\widetilde{\varphi}(z)} = \frac{\varphi(x) + 2\gamma(\varphi(x) - \psi(x))/\omega}{\varphi(z) + 2\gamma\Delta/\omega} = \frac{\varphi(x) - \psi(x)}{\Delta} + \frac{w(\psi(x)\varphi(z) - \varphi(x)\psi(z))}{\Delta(w\varphi(z) + 2\gamma\Delta)}.$$

Now from (4.5) we get

$$\mathbf{E}_x \exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds - \gamma\ell(\varrho(u,z),q)\Big)$$
$$=\begin{cases} \dfrac{\psi(x)}{\psi(z)}\exp\Big(-\dfrac{w(w+2\gamma)\,u}{2\psi(z)(w\varphi(z)+2\gamma\Delta)}\Big), & \text{for} \quad x\le z,\\ \Big(\dfrac{\varphi(x)-\psi(x)}{\Delta}+\dfrac{w(\psi(x)\varphi(z)-\varphi(x)\psi(z))}{\Delta(w\varphi(z)+2\gamma\Delta)}\Big) & \\ \qquad\qquad\times\exp\Big(-\dfrac{w(w+2\gamma)\,u}{2\psi(z)(w\varphi(z)+2\gamma\Delta)}\Big), & \text{for} \quad z\le x\le q,\\ \dfrac{w\varphi(x)}{w\varphi(z)+2\gamma\Delta}\exp\Big(-\dfrac{w(w+2\gamma)\,u}{2\psi(z)(w\varphi(z)+2\gamma\Delta)}\Big), & \text{for} \quad q\le x. \end{cases}$$

Inverting the Laplace transform with respect to γ (see formulas a, 15, 16 of Appendix 3), we have

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds\Big);\ell(\varrho(u,z),q)<v\Big\} \tag{4.8}$$
$$=\begin{cases} \dfrac{w\psi(x)}{2\psi(z)\Delta}\dfrac{\sqrt{u}}{\sqrt{v}}I_1\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{vw\varphi(z)}{2\Delta}\Big), & \text{for} \quad x\le z,\\ \dfrac{w}{2\Delta}\Big(\dfrac{\varphi(x)-\psi(x)}{\Delta}\dfrac{\sqrt{u}}{\sqrt{v}}I_1\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)+\dfrac{\psi(x)\varphi(z)-\varphi(x)\psi(z)}{\Delta}I_0\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\Big) & \\ \qquad\qquad\times\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{vw\varphi(z)}{2\Delta}\Big), & \text{for} \quad z\le x\le q,\\ \dfrac{w\varphi(x)}{2\Delta}I_0\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{vw\varphi(z)}{2\Delta}\Big), & \text{for} \quad q\le x, \end{cases}$$

where I_l are the modified Bessel functions (see Appendix 2). Moreover,

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))ds\Big);\ell(\varrho(u,z),q)=0\Big\}$$
$$=\begin{cases} \dfrac{\psi(x)}{\psi(z)}\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}\Big), & \text{for} \quad x\le z,\\ \dfrac{\varphi(x)-\psi(x)}{\Delta}\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}\Big), & \text{for} \quad z\le x\le q,\\ 0, & \text{for} \quad q\le x. \end{cases} \tag{4.9}$$

Let us compute the analogous expressions for the function f_+. Since formulas (4.8) and (4.9) were obtained for arbitrary nonnegative piecewise continuous functions f in terms of fundamental solutions of (4.4), one can use them for the function f_+. For this we represent ψ_+, φ_+, the fundamental solutions of (4.4) with the function f_+ in place of f, in terms of the solutions ψ, φ. We have

$$\psi_+(y)=\begin{cases} 1, & \text{for} \quad y\le q,\\ A\psi(y)+(1-A)\varphi(y), & \text{for} \quad q\le y, \end{cases}$$

$$\varphi_+(y) = \begin{cases} 1 + B(q-y), & \text{for} \quad y \le q, \\ \varphi(y), & \text{for} \quad q \le y. \end{cases}$$

In this representation we have taken into account the equality $\psi_+(q) = \varphi_+(q) = 1$ and the fact that the functions $\psi_+(y)$ and $\varphi_+(y)$, $y \in \mathbf{R}$, are monotone and continuous. The continuity of the derivative at q enables us to compute the constants A and B. As a result, we have

$$\psi_+(y) = \begin{cases} 1, & \text{for} \quad y \le q, \\ \Big(1 + \dfrac{\varphi'(q)}{w}\Big)\varphi(y) - \dfrac{\varphi'(q)}{w}\psi(y), & \text{for} \quad q \le y, \end{cases}$$

$$\varphi_+(y) = \begin{cases} 1 - \varphi'(q)(q-y), & \text{for} \quad y \le q, \\ \varphi(y), & \text{for} \quad q \le y. \end{cases}$$

The Wronskian of ψ_+, φ_+ is equal to $w_+ = \psi_+'(q) - \varphi_+'(q) = -\varphi'(q)$. Moreover, $\Delta_+ = \varphi_+(z) - \psi_+(z) = -(q-z)\varphi'(q)$.

Substituting ψ_+, φ_+ in place of ψ, φ in (4.8) and (4.9), we obtain

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))ds\Big); \ell(\varrho(u,z),q) < v\Big\} \tag{4.10}$$

$$= \begin{cases} \dfrac{1}{2(q-z)}\dfrac{\sqrt{u}}{\sqrt{v}} I_1\Big(\dfrac{\sqrt{uv}}{q-z}\Big)\exp\Big(-\dfrac{u+v}{2(q-z)} + \dfrac{v\varphi'(q)}{2}\Big), & \text{for} \quad x \le z, \\ \Big(\dfrac{q-x}{2(q-z)^2}\dfrac{\sqrt{u}}{\sqrt{v}} I_1\Big(\dfrac{\sqrt{uv}}{q-z}\Big) + \dfrac{x-z}{2(q-z)^2} I_0\Big(\dfrac{\sqrt{uv}}{q-z}\Big)\Big) & \\ \qquad \times \exp\Big(-\dfrac{u+v}{2(q-z)} + \dfrac{v\varphi'(q)}{2}\Big), & \text{for} \quad z \le x \le q, \\ \dfrac{\varphi(y)}{2(q-z)} I_0\Big(\dfrac{\sqrt{uv}}{q-z}\Big)\exp\Big(-\dfrac{u+v}{2(q-z)} + \dfrac{v\varphi'(q)}{2}\Big), & \text{for} \quad q \le x, \end{cases}$$

for $v > 0$, and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))ds\Big); \ell(\varrho(u,z),q) = 0\Big\}$$

$$= \begin{cases} e^{-u/2(q-z)}, & \text{for} \quad x \le z, \\ \dfrac{q-x}{q-z} e^{-u/2(q-z)}, & \text{for} \quad z \le x \le q, \\ 0, & \text{for} \quad q \le x. \end{cases} \tag{4.11}$$

For $f_+ \equiv 0$ we have $\varphi(y) \equiv 1$ and these formulas imply

$$\frac{\partial}{\partial v}\mathbf{P}_x\Big(\ell(\varrho(u,z),q) < v\Big) = \begin{cases} \dfrac{1}{2(q-z)}\dfrac{\sqrt{u}}{\sqrt{v}} I_1\Big(\dfrac{\sqrt{uv}}{q-z}\Big) e^{-(u+v)/2(q-z)}, & \text{for} \quad x \le z, \\ \Big(\dfrac{q-x}{2(q-z)^2}\dfrac{\sqrt{u}}{\sqrt{v}} I_1\Big(\dfrac{\sqrt{uv}}{q-z}\Big) & \\ \quad + \dfrac{x-z}{2(q-z)^2} I_0\Big(\dfrac{\sqrt{uv}}{q-z}\Big)\Big) e^{-(u+v)/2(q-z)}, & \text{for} \quad z \le x \le q, \\ \dfrac{1}{2(q-z)} I_0\Big(\dfrac{\sqrt{uv}}{q-z}\Big) e^{-(u+v)/2(q-z)}, & \text{for} \quad q \le x, \end{cases} \tag{4.12}$$

for $v>0$, and

$$\mathbf{P}_x\big(\ell(\varrho(u,z),q)=0\big)=\begin{cases} e^{-u/2(q-z)}, & \text{for} \quad x\le z,\\ \dfrac{q-x}{q-z}e^{-u/2(q-z)}, & \text{for} \quad z\le x\le q,\\ 0, & \text{for} \quad q\le x. \end{cases} \tag{4.13}$$

Dividing (4.10) by (4.12), we see that

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))\,ds\Big)\Big|\ell(\varrho(u,z),q)=v\Big\}=\begin{cases} e^{v\varphi'(q)/2}, & \text{for} \quad x\le q,\\ \varphi(x)e^{v\varphi'(q)/2}, & \text{for} \quad q\le x, \end{cases} \tag{4.14}$$

for $v>0$, and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))\,ds\Big)\Big|\ell(\varrho(u,z),q)=0\Big\}=1. \tag{4.15}$$

We now compute the corresponding expressions for the function f_-. For this purpose we express ψ_- and φ_-, the fundamental solutions of (4.4) with the function f_- in place of f, in terms of the fundamental solutions ψ and φ. They have the form

$$\psi_-(y)=\begin{cases} \psi(y), & \text{for} \quad y\le q,\\ 1+\psi'(q)(y-q), & \text{for} \quad q\le y, \end{cases}$$

$$\varphi_-(y)=\begin{cases} \dfrac{\psi'(q)}{w}\varphi(y)-\dfrac{\varphi'(q)}{w}\psi(y), & \text{for} \quad y\le q,\\ 1, & \text{for} \quad q\le y. \end{cases}$$

The Wronskian of ψ_-, φ_- is equal to $w_-=\psi_-'(q)-\varphi_-'(q)=\psi'(q)$. In addition, for $y\le q$

$$\varphi_-(y)-\psi_-(y)=\frac{\psi'(q)}{w}(\varphi(y)-\psi(y))$$

and

$$\psi_-(y)\varphi_-(z)-\varphi_-(y)\psi_-(z)=\frac{\psi'(q)}{w}(\psi(y)\varphi(z)-\varphi(y)\psi(z)),$$

$\Delta_-=\varphi_-(z)-\psi_-(z)=\dfrac{\psi'(q)}{w}\Delta$. Using the functions ψ_- and φ_- in formulas (4.8) and (4.9), we have

$$\frac{\partial}{\partial v}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_-(W(s))ds\Big);\ell(\varrho(u,z),q)<v\Big\} \tag{4.16}$$

$$=\begin{cases} \dfrac{w\psi(x)}{2\psi(z)\Delta}\dfrac{\sqrt{u}}{\sqrt{v}}I_1\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{v(\psi'(q)\varphi(z)-\varphi'(q)\psi(z))}{2\Delta}\Big), & x\le z,\\ \dfrac{w}{2\Delta}\Big(\dfrac{\varphi(x)-\psi(x)}{\Delta}\dfrac{\sqrt{u}}{\sqrt{v}}I_1\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)+\dfrac{\psi(x)\varphi(z)-\varphi(x)\psi(z)}{\Delta}I_0\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\Big) & \\ \qquad\qquad\times\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{v(\psi'(q)\varphi(z)-\varphi'(q)\psi(z))}{2\Delta}\Big), & z\le x\le q,\\ \dfrac{w}{2\Delta}I_0\Big(\dfrac{w}{\Delta}\sqrt{uv}\Big)\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}-\dfrac{v(\psi'(q)\varphi(z)-\varphi'(q)\psi(z))}{2\Delta}\Big), & q\le x, \end{cases}$$

for $v>0$, and

$$\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varrho(u,z)} f_-(W(s))ds\Big);\ell(\varrho(u,z),q)=0\Big\}$$
$$=\begin{cases}\dfrac{\psi(x)}{\psi(z)}\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}\Big), & \text{for}\quad x\le z,\\ \dfrac{\varphi(x)-\psi(x)}{\Delta}\exp\Big(-\dfrac{u\,w}{2\psi(z)\Delta}\Big), & \text{for}\quad z\le x\le q,\\ 0, & q\le x.\end{cases} \tag{4.17}$$

With the help of these expressions it is easy to verify (4.3). Indeed, expression (4.8) is equal to the product of (4.14) and (4.16), because

$$\frac{v\varphi'(q)}{2}-\frac{v(\psi'(q)\varphi(z)-\varphi'(q)\psi(z))}{2\Delta}=-\frac{vw\varphi(z)}{2\Delta}.$$

The equality below (4.3) is also satisfied, because the right-hand sides of (4.9) and (4.17) are equal, and the right-hand side of (4.15) is 1.

We have assumed that $z<q$. The case $q<z$ is verified similarly. But this is not necessary, because the case $q<z$ can be reduced to the case $z<q$ with the help of the spatial homogeneity and symmetry properties of Brownian motion.

The case $z=q$ is degenerate, because $\ell(\varrho(u,z),z)=u$. In this case we have $\varphi(z)=\psi(z)=1$, $\psi'(z)-\varphi'(z)=\omega$ and, consequently, $\omega=\omega_-+\omega_+$. The formula before (4.8) in the limiting case $q-z\downarrow 0$ is transformed to

$$\mathbf{E}_x\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds\Big)=\begin{cases}\dfrac{\psi(x)}{\psi(z)}e^{-\omega u/2}, & \text{for}\quad x\le z,\\ \dfrac{\varphi(x)}{\varphi(z)}e^{-\omega u/2}, & \text{for}\quad z\le x.\end{cases}$$

Since $\psi(x)=\psi_+(x)\psi_-(x)$ for $x\le z$ and $\varphi(x)=\varphi_+(x)\varphi_-(x)$ for $z\le x$,

$$\mathbf{E}_x\exp\Big(-\int_0^{\varrho(u,z)} f(W(s))\,ds\Big)$$
$$=\mathbf{E}_x\exp\Big(-\int_0^{\varrho(u,z)} f_-(W(s))\,ds\Big)\mathbf{E}_x\exp\Big(-\int_0^{\varrho(u,z)} f_+(W(s))\,ds\Big).$$

Therefore, we have established that $\ell(\varrho(u,z),y)$, $y\in\mathbf{R}$, is a Markov process.

We now compute the generating operators of the processes V_6, V_7, V_8. In view of Remark 4.1, it suffices to consider the description of the local time only for the direct time.

We first compute the expression for the function

$$u(h,v):=\mathbf{E}_x\big\{e^{-\eta\ell(\varrho(u,z),q+h)}\big|\ell(\varrho(u,z),q)=v\big\},\quad \eta>0,\quad v>0,$$

when $h > 0$, $x < z < q$ or $z < q < q+h < x$. In these areas the function u is the Laplace transform with respect to η of the transition function of the processes V_8 and V_7, respectively, therefore it uniquely determines the generating operators.

To compute the function $u(h,v)$ we can use Theorem 7.3 of Ch. III with $f \equiv 0$, $a = -\infty$, $b = \infty$, $q_2 = q$, $q_3 = q+h$, $q_l = 0$ for $l \neq 2,3$. However, we can simplify the computations by using the equality (4.14). The justification of the computation method for u is exactly the same as in the proof of Theorem 2.1. In (4.14) in place of $f_+(y)$ we substitute the Dirac δ-function at $q+h$, multiplied by η. It is readily seen that in the domain $y \geq q$ the fundamental solution $\varphi_\delta(y)$ of equation (4.4) with the condition $\varphi_\delta(q) = 1$, corresponding to the Dirac δ-function at $q+h$, is a unique continuous bounded solution of the problem

$$\begin{aligned}&\frac{1}{2}\varphi''(y) = 0, \qquad y \in (q,\infty) \setminus \{q+h\},\\ &\varphi'(q+h+0) - \varphi'(q+h-0) = 2\eta\varphi(q+h), \qquad \varphi(q) = 1.\end{aligned}$$

Solving this problem, we obtain

$$\varphi_\delta(y) = \begin{cases} 1 - \dfrac{2\eta(y-q)}{1+2\eta h}, & \text{for} \quad y \leq q+h,\\ \dfrac{1}{1+2\eta h}, & \text{for} \quad q+h \leq y.\end{cases}$$

Substituting the values for $\varphi'_\delta(q)$ and $\varphi_\delta(x)$, $q+h < x$, in (4.14), we get

$$u(h,v) = \begin{cases} \exp\Big(-\dfrac{\eta v}{1+2\eta h}\Big), & \text{for} \quad x \leq z \leq q,\\ \dfrac{1}{1+2\eta h}\exp\Big(-\dfrac{\eta v}{1+2\eta h}\Big), & \text{for} \quad z \leq q < q+h \leq x.\end{cases}$$

The fact that this expression does not depend on q implies that the processes $V_7(h)$ and $V_8(h)$, $h \geq 0$, are homogeneous.

By the definition of the generating operator of a homogeneous Markov process (see § 9 Ch. IV) and in view of the arbitrariness of η, the operators $\mathbb{L}_7$, $\mathbb{L}_8$, corresponding to the processes V_7, V_8, can be determined from the respective equations

$$\frac{\partial}{\partial h}u = \mathbb{L}_8 u \qquad \text{for} \quad x < z \leq q,$$

and

$$\frac{\partial}{\partial h}u = \mathbb{L}_7 u \qquad \text{for} \quad z \leq q < q+h < x.$$

Hence,

$$\mathbb{L}_8 = 2v\frac{d^2}{dv^2}, \qquad \mathbb{L}_7 = 2v\frac{d^2}{dv^2} + 2\frac{d}{dv}.$$

The theorem is proved. □

§ 5. Distributions of functionals of Brownian local time stopped at an exponential moment

We consider the following question: how to find the distributions of functionals of the Brownian local time? An integral functional of the Brownian local time with respect to the space parameter has the form

$$B(t) := \int_{-\infty}^{\infty} f(\ell(t,y))\,dy, \qquad (5.1)$$

where $f(v)$, $v \in [0,\infty)$, is some nonnegative piecewise-continuous function. For the Laplace transform of this functional we will obtain explicit formulas in terms of solutions of second-order differential equations satisfying certain boundary conditions. Having expressions for the Laplace transforms of nonnegative integral functionals of a process, we can compute the distributions of various supremum type functionals of this process. For example, to compute the supremum of an arbitrary continuous process X we can use (see § 2 Ch. III) the relation

$$\mathbf{P}\Big(\sup_{a\le y\le b} X(y) \le h\Big) = \lim_{\gamma\to\infty} \mathbf{E}\exp\Big(-\gamma\int_a^b \mathbb{1}_{(h,\infty)}(X(y))\,dy\Big). \qquad (5.2)$$

In many cases where $\mathbf{E}\exp\Big(-\gamma\int_a^b \mathbb{1}_{(h,\infty)}(X(y))\,dy\Big)$ is expressed with the help of solutions of certain differential equations, we do not have to compute the expectation explicitly, nor to find the limit, but only to prove that the limit value for this expectation can also be expressed by means of solutions of analogous equations with certain boundary conditions. Such an approach simplifies the computations considerably. We already used this approach in the proof of Theorem 2.1 Ch. III. In this section we obtain results enabling us to compute the joint distribution of functionals $B(t)$ and $\sup_{y\in\mathbf{R}} \ell(t,y)$.

The computation of the distributions of these functionals at a fixed time t reduces to the computation of the distributions of the functionals stopped at the random time τ independent of the Brownian motion W and having the exponential density

$$\frac{d}{dt}\mathbf{P}(\tau < t) = \lambda e^{-\lambda t}\mathbb{1}_{[0,\infty)}(t), \qquad \lambda > 0. \qquad (5.3)$$

The distributions of functionals at a fixed time t can be obtained from formulas for the distributions of the corresponding functionals stopped at the time τ with the help of the inverse Laplace transform with respect to λ.

The next result is due to Borodin (1982).

Theorem 5.1. *Let $f(v)$, $v \in [0,h]$, be a nonnegative piecewise-continuous function, $f(0) = 0$. Then*

$$\mathbf{E}_x\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big); \sup_{y\in\mathbf{R}} \ell(\tau,y) \le h\Big\} = 2\lambda\int_0^h R(v)Q(v)\,dv, \qquad (5.4)$$

where for $v \in [0,h]$ the functions R, Q are the unique bounded continuous solutions of the problem

$$2vR''(v) - (\lambda v + f(v))R(v) = 0, \qquad R(0) = 1, \tag{5.5}$$

$$2vQ''(v) + 2Q'(v) - (\lambda v + f(v))Q(v) = -R(v), \tag{5.6}$$

$$R(h) = 0, \qquad Q(h) = 0. \tag{5.7}$$

Remark 5.1. In the case $h = \infty$ the boundary conditions (5.7) must be replaced by the conditions

$$\limsup_{v\to\infty} e^{v\sqrt{\lambda/2}} R(v) < \infty, \qquad \limsup_{v\to\infty} e^{v\sqrt{\lambda/2}} Q(v) < \infty. \tag{5.8}$$

Remark 5.2. For a piecewise-continuous function f equations (5.5) and (5.6) must be interpreted precisely like equation (1.10) Ch. III (see Remark 1.2 Ch. III).

Proof of Theorem 5.1. Since a Brownian motion is spatially homogeneous, expression (5.4) does not depend on x. So we let $x = 0$. Assume first that $h = \infty$ and f is a bounded twice continuously differentiable function with bounded first and second derivatives. Using (2.1) for $y = z$ and the Markov property of the process $\ell(\tau, y)$, i.e., (2.2) for $q = z$, we find that

$$\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big)\Big|W(\tau)=z\Big\}$$

$$= \sqrt{2\lambda}\int_0^\infty e^{-v\sqrt{2\lambda}}\,\mathbf{E}_0^z\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big)\Big|\ell(\tau,z)=v\Big\}\,dv$$

$$= \sqrt{2\lambda}\int_0^\infty e^{-v\sqrt{2\lambda}}\,\bar r(z,v)\bar q(z,v)\,dv, \tag{5.9}$$

where

$$\bar r(z,v) := \mathbf{E}_0^z\Big\{\exp\Big(-\int_z^\infty f(\ell(\tau,y))\,dy\Big)\Big|\ell(\tau,z)=v\Big\},$$

$$\bar q(z,v) := \mathbf{E}_0^z\Big\{\exp\Big(-\int_{-\infty}^z f(\ell(\tau,y))\,dy\Big)\Big|\ell(\tau,z)=v\Big\}.$$

Let $z > 0$. Then, applying Theorem 2.1, we get

$$\bar r(z,v) = \mathbf{E}\Big\{\exp\Big(-\int_0^\infty f(V_1(h))\,dh\Big)\Big|V_1(0)=v\Big\}.$$

It is clear that $\bar r(z,v)$ is independent of z. Denote $\bar R(v) := \bar r(z,v)$.

Again applying Theorem 2.1, we get

$$\bar q(z,v) = \mathbf{E}\Big\{ \exp\Big(- \int_0^\infty f(V_3(h))\,dh - \int_0^z f(V_2(h))\,dh \Big) \Big| V_2(0) = v \Big\}$$
$$= \int_0^\infty \mathbf{E}_v\Big\{ \exp\Big(- \int_0^\infty f(V_3(h))\,dh - \int_0^z f(V_2(h))\,dh \Big) \Big| V_2(z) = g \Big\} \mathbf{P}_v\big(V_2(z) \in dg\big),$$

where the subscript v indicates that the expectation and the probability are computed with respect to the process V_2 with the initial value $V_2(0) = v$. Using the independence of the processes V_2 and V_3 under fixed starting points and the condition $V_3(0) = V_2(z)$, we have

$$\bar q(z,v) = \int_0^\infty \mathbf{E}\Big\{ \exp\Big(- \int_0^\infty f(V_3(h))\,dh \Big) \Big| V_3(0) = g \Big\}$$
$$\times \mathbf{E}_v\Big\{ \exp\Big(- \int_0^z f(V_2(h))\,dh \Big) \Big| V_2(z) = g \Big\} \mathbf{P}_v\big(V_2(z) \in dg\big).$$

Since for a fixed starting point the infinitesimal characteristics of the processes V_1 and V_3 coincide, we obtain

$$\bar q(z,v) = \int_0^\infty \bar R(g) \mathbf{E}_v\Big\{ \exp\Big(- \int_0^z f(V_2(h))\,dh \Big) \Big| V_2(z) = g \Big\} \mathbf{P}_v\big(V_2(z) \in dg\big)$$
$$= \mathbf{E}\Big\{ \bar R(V_2(z)) \exp\Big(- \int_0^z f(V_2(h))\,dh \Big) \Big| V_2(0) = v \Big\}.$$

We apply Theorem 12.5 Ch. II. Then we have that the function $\bar R(v)$, $v \in (0,\infty)$, is a bounded solution of the homogeneous equation

$$2v\big(\bar R''(v) - \sqrt{2\lambda}\,\bar R'(v)\big) - f(v)\bar R(v) = 0. \tag{5.10}$$

As it was mentioned above, once the 0-dimensional Bessel process hits zero it never leaves zero, consequently, if the process begins at zero it stays at zero. In view of the description of V_1 (see Proposition 2.1), the same is true for it. Since $f(0) = 0$, this implies that $\bar R(0) = 1$.

Further, we apply Theorem 13.1 Ch. II. Then $\bar q(z,v)$, $(z,v) \in [0,\infty) \times [0,\infty)$, is the solution of the problem

$$\frac{\partial}{\partial z}\bar q(z,v) = 2v\Big(\frac{\partial^2}{\partial v^2}\bar q(z,v) - \sqrt{2\lambda}\frac{\partial}{\partial v}\bar q(z,v)\Big) + 2\frac{\partial}{\partial v}\bar q(z,v) - f(v)\bar q(z,v), \tag{5.11}$$

$$\bar q(0,v) = \bar R(v). \tag{5.12}$$

The specifics of the application of Theorems 12.5 and 13.1 of Ch. II are that the processes V_1 and V_2 are nonnegative and their diffusion coefficient $\sigma^2(v) = v$ degenerates at zero.

The substitution $R(v) = e^{-v\sqrt{\lambda/2}}\bar{R}(v)$ leads to the differential problem

$$2vR''(v) - (\lambda v + f(v))R(v) = 0, \qquad R(0) = 1, \tag{5.13}$$

and the substitution $q(z,v) = e^{-v\sqrt{\lambda/2}}\bar{q}(z,v)$ leads to the differential problem

$$\frac{\partial}{\partial z}q(z,v) = 2v\frac{\partial^2}{\partial v^2}q(z,v)+2\frac{\partial}{\partial v}q(z,v)-(\lambda v-\sqrt{2\lambda}+f(v))q(z,v), \qquad q(0,v) = R(v). \tag{5.14}$$

Using the new notations we can rewrite (5.9) in the form

$$\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big)\Big|W(\tau) = z\Big\} = \sqrt{2\lambda}\int_0^\infty R(v)q(z,v)\,dv, \tag{5.15}$$

We set

$$Q(v) := \int_0^\infty e^{-z\sqrt{2\lambda}}q(z,v)\,dz.$$

Since $\bar{q}(z,v)$ is bounded, $Q(v)$, $v \geq 0$, is also bounded. From (5.14) it follows that $Q(v)$, $v \in [0,\infty)$, satisfies (5.6). Using the symmetry property of Brownian motion, the density

$$\frac{d}{dz}\mathbf{P}_0(W(\tau) < z) = \frac{\sqrt{\lambda}}{\sqrt{2}}\,e^{-|z|\sqrt{2\lambda}},$$

and (5.15), we finally get

$$\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big)\Big\}$$

$$=\sqrt{2\lambda}\int_0^\infty e^{-z\sqrt{2\lambda}}\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\tau,y))\,dy\Big)\Big|W(\tau)=z\Big\}dz=2\lambda\int_0^\infty R(v)Q(v)\,dv.$$

This coincides with (5.4) for the case when $h = \infty$ and f is a bounded twice continuously differentiable function with bounded first and second derivatives.

Remark 5.1 is valid, because $|\bar{r}(z,v)| \leq 1$, $|\bar{q}(z,v)| \leq 1$, $(z,v) \in [0,\infty) \times [0,\infty)$.

As in the proof of Theorem 4.1 Ch. IV, the assertion for piecewise-continuous functions f is proved by means of approximating f by continuously differentiable functions.

The uniqueness of a bounded solution of (5.5), (5.6), and (5.8) for $v \in [0,\infty)$ follows from the following facts. Since the solution of equation (5.5) is strictly convex, this equation has only one bounded solution with the initial value equal to 1 (see Proposition 12.1 Ch. II).

The homogeneous equation corresponding to (5.6) can be written in the integral form

$$2v\phi'(v) - \int_0^v (\lambda s + f(s))\phi(s)\,ds = c,$$

where c is a constant. Choosing $c = 0$, we see that this equation has a solution ψ with the following properties: $\psi(+0) = 1$, $\psi'(+0) = 0$, $\psi'(v) \geq \lambda v/4$, and $\psi(v) \geq 1 + \lambda v^2/8$. If this solution is known, another linearly independent solution can be written. For the equation under consideration, the linearly independent solution φ can be represented as

$$\varphi(v) = \psi(v) \int_v^\infty \frac{ds}{s\,\psi^2(s)}, \qquad v > 0.$$

Therefore, $\varphi(v) \sim -\ln v$ as $v \downarrow 0$. Thus the nonhomogeneous equation (5.6) has only one bounded solution on the positive real half-line. Moreover, equation (5.6) for $v \in [0, h]$ has a unique bounded solution for the boundary condition $Q(h) = 0$.

The proof of Theorem 5.1 for $h < \infty$ is based on the obvious generalization of the relation (5.2):

$$E := \mathbf{E}\biggl\{ \exp\biggl(-\int_{-\infty}^{\infty} f(\ell(\tau, y))\,dy \biggr); \sup_{y\in\mathbf{R}} \ell(\tau, y) \leq h \biggr\} = \lim_{\gamma\to\infty} E_\gamma, \tag{5.16}$$

where

$$E_\gamma := \mathbf{E}\biggl\{ \exp\biggl(-\int_{-\infty}^{\infty} (f(\ell(\tau, y)) + \gamma \mathbb{I}_{(h,\infty)}(\ell(\tau, y)))\,dy \biggr) \biggr\}.$$

The equality (5.16) can be justified as follows. If the process $\ell(\tau, y)$ exceeds the level h, then it spends a nonzero amount of time in the interval (h, ∞), i.e., $\int_{-\infty}^{\infty} \mathbb{I}_{(h,\infty)}(\ell(\tau, y))\,dy$ is greater than zero in this case. Consequently, for such sample paths the expression under the expectation sign in E_γ tends to zero as $\gamma \to \infty$. It remains to note that for those sample paths that do not exceed h, the expectation coincides with E.

For E_γ we can use the variant of Theorem 5.1 already proved. According to this statement,

$$E_\gamma = 2\lambda \int_0^\infty R_\gamma(v) Q_\gamma(v)\,dv,$$

where R_γ, Q_γ are bounded solutions of the problem

$$2vR_\gamma''(v) - (\lambda v + f(v) + \gamma \mathbb{I}_{(h,\infty)}(v))R_\gamma(v) = 0, \qquad R_\gamma(0+) = 1, \tag{5.17}$$

$$2vQ_\gamma''(v) + 2Q_\gamma'(v) - (\lambda v + f(v) + \gamma \mathbb{I}_{(h,\infty)}(v))Q_\gamma(v) = -R_\gamma(v). \tag{5.18}$$

In the first part of the proof of Theorem 5.1 we have shown that

$$R_\gamma(v) = e^{-v\sqrt{\lambda/2}}\,\mathbf{E}_0^z\Big\{\exp\Big(-\int_z^\infty (f+\gamma\mathbb{1}_{(h,\infty)})(\ell(\tau,y))\,dy\Big)\Big|\ell(\tau,z)=v\Big\}, \tag{5.19}$$

$$Q_\gamma(v) = e^{-v\sqrt{\lambda/2}}$$
$$\times\int_0^\infty e^{-z\sqrt{2\lambda}}\mathbf{E}_0^z\Big\{\exp\Big(-\int_{-\infty}^z (f+\gamma\mathbb{1}_{(h,\infty)})(\ell(\tau,y))\,dy\Big)\Big|\ell(\tau,z)=v\Big\}dz.$$

Analogously to (5.16) we have

$$R_\gamma(v)\to R(v), \qquad Q_\gamma(v)\to Q(v), \tag{5.20}$$

where

$$R(v) = e^{-v\sqrt{\lambda/2}}\mathbf{E}_0^z\Big\{\exp\Big(-\int_z^\infty f(\ell(\tau,y))\,dy\Big)\mathbb{1}_{[0,h]}\Big(\sup_{y\in(z,\infty)}\ell(\tau,y)\Big)\Big|\ell(\tau,z)=v\Big\},$$

$$Q(v) = e^{-v\sqrt{\lambda/2}}$$
$$\times\int_0^\infty e^{-z\sqrt{2\lambda}}\mathbf{E}_0^z\Big\{\exp\Big(-\int_{-\infty}^z f(\ell(\tau,y))\,dy\Big)\mathbb{1}_{[0,h]}\Big(\sup_{y\in(-\infty,z)}\ell(\tau,y)\Big)\Big|\ell(\tau,z)=v\Big\}dz.$$

The passage to the limit in (5.17), (5.18) is realized just as it was done in the proof of Theorem 2.1 Ch. III. An important role here is played by the relations (5.20), the relations arising from the explicit form of the functions R_γ, Q_γ, and the equalities $\lim_{\gamma\to\infty} R_\gamma(v)=0$, $\lim_{\gamma\to\infty} Q_\gamma(v)=0$ for $v\ge h$. This completes the proof of Theorem 5.1. □

Let us consider one example of application of Theorem 5.1, namely, an explicit formula for the distribution of the supremum of the Brownian local time $\ell(t,y)$ with respect to $y\in\mathbf{R}$.

We use the following standard notations (see Appendix 2): $J_l(x)$ is the Bessel function and $I_l(x)$, $K_l(x)$, $x\in\mathbf{R}$, are the modified Bessel functions of the order l, $0<j_1<j_2<\cdots$ are the positive zeros of the Bessel function $J_0(x)$.

Theorem 5.2. *For $h\ge 0$*

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(\tau,y)>h\Big) = \frac{h\sqrt{2\lambda}\,I_1(h\sqrt{\lambda/2})}{\operatorname{sh}^2(h\sqrt{\lambda/2})I_0(h\sqrt{\lambda/2})} \tag{5.21}$$

and

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)\le h\Big) = 4\sum_{k=1}^\infty \frac{1}{\sin^2 j_k}e^{-2j_k^2t/h^2}$$

$$+4\sum_{k=1}^{\infty}\left[\frac{4t\pi kJ_1(\pi k)}{h^2J_0(\pi k)}-1+\frac{J_1(\pi k)}{\pi kJ_0(\pi k)}-\frac{J_1^2(\pi k)}{J_0^2(\pi k)}\right]e^{-2\pi^2k^2t/h^2}. \tag{5.22}$$

Remark 5.3. In view of the spatial homogeneity of a Brownian motion, the probabilities (5.21) and (5.22) do not depend on the initial value of the Brownian motion W.

Formula (5.21) was derived in Borodin (1982). The inversion formula (5.22) is due to Csáki and Földes (1986).

Proof of Theorems 5.2. We start with an auxiliary result that in certain cases gives a simple method for computing the integrals on the right-hand side of (5.4).

Lemma 5.1. *Let $X(x)$, $Y(x)$, $x>0$, be solutions of the equations*

$$xX''-(\sigma+\theta x)X=F(x), \tag{5.23}$$

$$xY''+Y'-(\delta+\theta x)Y=G(x). \tag{5.24}$$

Then

$$(\theta-(\delta-\sigma)^2)\int XY\,dx=(\sigma+\theta x)XY+(\delta-\sigma)x(X'Y-XY')-xX'Y'$$

$$+(\delta-\sigma)\Big(\int XGdx-\int YFdx\Big)+\int X'Gdx+\int Y'Fdx. \tag{5.25}$$

Formula (5.25) can easily be verified by differentiation.

We apply Theorem 5.1 with $f=0$. The solutions of (5.5)–(5.7) in this case have the form

$$R(v)=\frac{\operatorname{sh}((h-v)\sqrt{\lambda/2})}{\operatorname{sh}(h\sqrt{\lambda/2})},\qquad 0\le v\le h,$$

$$Q(v)=\frac{\operatorname{ch}((h-v)\sqrt{\lambda/2})}{\sqrt{2\lambda}\operatorname{sh}(h\sqrt{\lambda/2})}-\frac{I_0(v\sqrt{\lambda/2})}{\sqrt{2\lambda}\operatorname{sh}(h\sqrt{\lambda/2})\,I_0(h\sqrt{\lambda/2})},\qquad 0\le v\le h.$$

Using (5.4) and Lemma 5.1 with $X=R$, $Y=Q$, $F=0$, $G=-R/2$, $\theta=\lambda/2$, $\sigma=\delta=0$, and taking into account the boundary conditions (5.7), we find that

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(\tau,y)\le h\Big)=2\lambda\int_0^h R(v)Q(v)\,dv=-4hR'(h)Q'(h)-2\int_0^h R'(s)R(s)\,ds$$

$$=1-4hR'(h)Q'(h).$$

Substituting the values for the derivatives of the functions R and Q in the right-hand side of this equality, we get (5.21).

We now prove (5.22) with the help of the well-known formula for inverting the Laplace transform (see formula 1 of Appendix 3), which is based on residue theory. To invert the Laplace transform let $h=1$. For arbitrary h we obtain the distribution by means of the scaling property of the Brownian local time.

Formula (5.21) in this case can be recast as

$$\int_0^\infty e^{-\lambda t}\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)>1\Big)dt=\frac{\sqrt{2}\,I_1(\sqrt{\lambda/2})}{\sqrt{\lambda}\,\mathrm{sh}^2(\sqrt{\lambda/2})\,I_0(\sqrt{\lambda/2})}.$$

Then, using the formula for the inverse Laplace transform, we have

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)>1\Big)=\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta}e^{\lambda t}\frac{\sqrt{2}\,I_1(\sqrt{\lambda/2})}{\sqrt{\lambda}\,\mathrm{sh}^2(\sqrt{\lambda/2})\,I_0(\sqrt{\lambda/2})}\,d\lambda,$$

where γ is some small positive constant. To compute this integral one resorts to residue theory. We first note that

$$\lim_{\lambda\to 0}\frac{\sqrt{2}}{\sqrt{\lambda}}\,I_1(\sqrt{\lambda/2})=\frac{1}{2}.$$

Since $0<j_1<j_2<\cdots$ are the positive zeros of the function $J_0(x)$, the values $-2j_k^2$, $k=1,2,\dots$, are the zeros of the function $I_0(\sqrt{\lambda/2})$. The residues $r_{1,k}(t)$ of the function

$$g(\lambda):=e^{\lambda t}\frac{\sqrt{2}\,I_1(\sqrt{\lambda/2})}{\sqrt{\lambda}\,\mathrm{sh}^2(\sqrt{\lambda/2})\,I_0(\sqrt{\lambda/2})}$$

at the points $-2j_k^2$ are

$$r_{1,k}(t)=e^{-2j_k^2t}\frac{4I_1(ij_k)}{\mathrm{sh}^2(ij_k)\,I_0'(ij_k)}=-\frac{4}{\sin^2 j_k}e^{-2j_k^2t}.$$

Since $\lambda=0$ is a simple root of the function $\mathrm{sh}^2(\sqrt{\lambda/2})$, the residue of $g(\lambda)$ at this point is

$$r_0(t)=\lim_{\lambda\to 0}\frac{\lambda e^{\lambda t}}{2\,\mathrm{sh}^2(\sqrt{\lambda/2})}=1.$$

The points $\lambda_k=-2\pi^2k^2$, $k=1,2,\dots$, are the double roots of the denominator of the function $g(\lambda)$, therefore the residues at these points are

$$r_{2,k}(t)=\lim_{\lambda\to\lambda_k}\frac{d}{d\lambda}\{(\lambda-\lambda_k)^2g(\lambda)\}$$

$$=\lim_{\lambda\to\lambda_k}\frac{d}{d\lambda}\left\{\frac{e^{\lambda t}\,I_1(\sqrt{\lambda/2})}{\sqrt{\lambda/2}\,I_0(\sqrt{\lambda/2})\big[(\mathrm{sh}(\sqrt{\lambda_k/2}))'+(\mathrm{sh}(\sqrt{\lambda_k/2}))''(\lambda-\lambda_k)/2+\cdots\big]^2}\right\}$$

$$=\frac{1}{((\mathrm{sh}(\sqrt{\lambda_k/2}))')^2}\left(\frac{e^{\lambda_k t}\,I_1(\sqrt{\lambda_k/2})}{\sqrt{\lambda_k/2}\,I_0(\sqrt{\lambda_k/2})}\right)'-\frac{(\mathrm{sh}(\sqrt{\lambda_k/2}))''}{((\mathrm{sh}(\sqrt{\lambda_k/2}))')^3}\frac{e^{\lambda_k t}\,I_1(\sqrt{\lambda_k/2})}{\sqrt{\lambda_k/2}\,I_0(\sqrt{\lambda_k/2})}$$

$$=-4\Big\{\frac{4t\pi kJ_1(\pi k)}{J_0(\pi k)}-\Big(1-\frac{J_1(\pi k)}{\pi kJ_0(\pi k)}+\frac{J_1^2(\pi k)}{J_0^2(\pi k)}\Big)\Big\}e^{-2\pi^2k^2t}.$$

Applying the inversion formula based on residues, we have

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)>1\Big)=r_0(t)+\sum_{k=1}^{\infty}(r_{1,k}(t)+r_{2,k}(t)).$$

Considering the probability of the opposite event and substituting the expressions for the residues, we get

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)\le 1\Big)=4\sum_{k=1}^{\infty}\frac{1}{\sin^2 j_k}e^{-2j_k^2 t}$$

$$+4\sum_{k=1}^{\infty}\Big\{\frac{4t\pi k J_1(\pi k)}{J_0(\pi k)}-1+\frac{J_1(\pi k)}{\pi k J_0(\pi k)}-\frac{J_1^2(\pi k)}{J_0^2(\pi k)}\Big\}e^{-2\pi^2k^2t}. \tag{5.26}$$

According to the scaling property of the Brownian local time (see § 1),

$$\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t,y)\le h\Big)=\mathbf{P}\Big(\sup_{y\in\mathbf{R}}\ell(t/h^2,y)<1\Big).$$

Now substituting in (5.26) in place of t the value t/h^2, we get the formula (5.22) for the distribution of $\sup\limits_{y\in\mathbf{R}}\ell(t,y)$. □

From (5.21) it is not hard to get the useful estimate

$$\mathbf{P}\Big(\sup_{x\in\mathbf{R}}\ell(t,x)>h\Big)\le L\frac{h^2}{t}\exp\Big(-\frac{h^2}{2t}\Big), \tag{5.27}$$

where $t\ge 0$, $h\ge\sqrt{t}$, and L is a constant. Indeed, since $\ell(t,x)$ is increasing in t, by (5.21) with $h\sqrt{2\lambda}\ge 1$ we have

$$\mathbf{P}\Big(\sup_{x\in\mathbf{R}}\ell(t,x)>h\Big)\le\lambda e^{\lambda t}\int_t^{\infty}e^{-\lambda s}\mathbf{P}\Big(\sup_{x\in\mathbf{R}}\ell(s,x)>h\Big)\,ds$$

$$\le e^{\lambda t}\mathbf{P}\Big(\sup_{x\in\mathbf{R}}\ell(\tau,x)>h\Big)\le Le^{\lambda t-h\sqrt{2\lambda}}h\sqrt{2\lambda}.$$

For $\lambda=2^{-1}h^2/t^2$ this estimate becomes (5.27).

From (3.17) Ch. III with $x=q=0$ and (10.8) Ch. I it follows that

$$\mathbf{P}_0(\ell(t,0)>h)=\frac{\sqrt{2}}{\sqrt{\pi t}}\int_h^{\infty}e^{-v^2/2t}dv\sim\frac{\sqrt{2t}}{h\sqrt{\pi}}e^{-h^2/2t}, \tag{5.28}$$

where the second relation assumes the condition $h/\sqrt{t}\to\infty$. Thus the exponential function in (5.27) is the best possible. The estimate (5.27) is important for proving the law of the iterated logarithm (see § 9) and for deriving a formula for the exact modulus of continuity of the local time $\ell(t,x)$ with respect to the variable t (see § 10).

§ 6. Distribution of functionals of Brownian local time stopped at the first exit time and at the inverse local time

Using the Markov property of the Brownian local time stopped at the first exit time from an interval or the inverse local time (see § 3 and § 4), one can prove assertions analogous to Theorem 5.1. Below we only formulate them, since they can be deduced by the already familiar approach. In view of the evidence of the treatments, we omit the proofs.

The following assertion concerns the moment $H_{a,b} = \min\{s : W(s) \notin (a,b)\}$.

Theorem 6.1. *Let $f(v)$, $v \in [0,h]$, be a nonnegative piecewise-continuous function. Then for any $\mu > 0$ and $\eta > 0$*

$$\int_0^\infty e^{-\eta b} \int_{-\infty}^0 e^{\mu a} \mathbf{E}_0 \Big\{ \exp\Big(-\int_a^b f(\ell(H_{a,b},y))dy \Big); \sup_{y\in(a,b)} \ell(H_{a,b},y) \le h, W(H_{a,b}) = b \Big\} da db$$

$$= \frac{1}{2} \int_0^h R(v)Q(v)\, dv, \tag{6.1}$$

where the functions $R(v)$ and $Q(v)$, $v \in [0,h]$, are the unique continuous solutions of the problem

$$2vR''(v) - (\mu + f(v))R(v) = 0, \qquad R(0) = \frac{1}{\mu + f(0)}, \tag{6.2}$$

$$2vQ''(v) + 2Q'(v) - (\eta + f(v))Q(v) = 0, \qquad \lim_{v\downarrow 0} Q(v)/\ln v = -1, \tag{6.3}$$

$$R(h) = Q(h) = 0. \tag{6.4}$$

Remark 6.1. In the case $h = \infty$ the boundary conditions (6.4) must be replaced by the conditions

$$\limsup_{v\to\infty} v^{-1/4} e^{\sqrt{2\mu v}} R(v) < \infty, \qquad \limsup_{v\to\infty} v^{1/4} e^{\sqrt{2\eta v}} Q(v) < \infty.$$

Remark 6.2. If we choose the condition $W(H_{a,b}) = a$ on the left-hand side of (6.1), then we must interchange the parameters μ and η in (6.2) and (6.3).

Remark 6.3. Let τ_μ, τ_η be independent of each other and of the Brownian motion W exponentially distributed random variables with the parameters μ and η, respectively. Then, by Fubini's theorem, formula (6.1) can be rewritten in the form

$$\mathbf{E}_0 \Big\{ \exp\Big(-\int_{-\tau_\mu}^{\tau_\eta} f(\ell(H_{-\tau_\mu,\tau_\eta},y))\, dy \Big); \sup_{y\in(-\tau_\mu,\tau_\eta)} \ell(H_{-\tau_\mu,\tau_\eta},y) \le h, W(H_{-\tau_\mu,\tau_\eta}) = \tau_\eta \Big\}$$

$$= \frac{\mu\eta}{2} \int_0^h R(v)Q(v)\, dv, \tag{6.5}$$

For $H_z = \min\{s : W(s) = z\}$, which is the first hitting time of the level z, the following assertion holds.

Theorem 6.2. *Let $f(v)$, $v \in [0,h]$, be a nonnegative piecewise-continuous function, $f(0)=0$. Then*

$$\int_0^\infty e^{-\eta z}\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{z} f(\ell(H_z,y))\,dy\Big);\ \sup_{y\in(-\infty,z)}\ell(H_z,y)\le h\Big\}dz$$

$$= \frac{1}{2}\int_0^h R(v)Q(v)\,dv, \tag{6.6}$$

where the functions $R(v)$ and $Q(v)$, $v\in[0,h]$, are the unique continuous solutions of the problem

$$2vR''(v) - f(v)R(v) = 0, \qquad R(0)=1, \tag{6.7}$$

$$2vQ''(v) + 2Q'(v) - (\eta + f(v))Q(v) = 0, \quad \lim_{v\downarrow 0} Q(v)/\ln v = -1, \tag{6.8}$$

$$R(h) = Q(h) = 0. \tag{6.9}$$

Remark 6.4. Theorem 6.2 is a consequence of Theorem 6.1 as $\mu\downarrow 0$, because in this case $-\tau_\mu \to -\infty$ and $H_{-\tau_\mu,\tau_\eta}\to H_{\tau_\eta}$. (The limit is realized in (6.5)).

Let $\varrho(u,z) = \min\{s : \ell(s,z)=u\}$ be the moment inverse to the Brownian local time at the level z, where $(u,z)\in[0,\infty)\times\mathbf{R}$. For the random moment $\varrho(u,z)$ the following assertion holds.

Theorem 6.3. *Let $f(u)$, $u\in[0,h]$, be a nonnegative piecewise-continuous function, $f(0)=0$. Then*

$$\int_0^\infty e^{-\beta z}\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\varrho(u,z),y))\,dy\Big),\ \sup_{y\in(-\infty,\infty)}\ell(\varrho(u,z),y)\le h\Big\}dz$$

$$= R(u)Q(u)\mathbb{1}_{[0,h]}(u), \tag{6.10}$$

where the functions $R(u)$ and $Q(u)$, $u\in[0,h]$, are the unique bounded continuous solutions of the problem

$$2uR''(u) - f(u)R(u) = 0, \qquad R(0)=1, \tag{6.11}$$

$$2uQ''(u) + 2Q'(u) - (\beta + f(u))Q(u) = -R(u), \tag{6.12}$$

$$R(h-0) = Q(h-0) = 0. \tag{6.13}$$

Remark 6.5. In the case $h=\infty$ the boundary conditions (6.13) must be replaced by the requirement that the functions R and Q be bounded.

If the level at which the inverse local time is considered coincides with the starting point of the Brownian motion, then a more convenient result is available.

Theorem 6.4. *Let $f(v)$, $v \in [0,h]$, be a nonnegative piecewise-continuous function, $f(0)=0$. Then*

$$\mathbf{E}_0\Big\{\exp\Big(-\int_{-\infty}^{\infty} f(\ell(\varrho(u,0),y))\,dy\Big);\ \sup_{y\in(-\infty,\infty)} \ell(\varrho(u,0),y) \le h\Big\}$$

$$= R^2(u)\mathbb{I}_{[0,h]}(u), \tag{6.14}$$

where the function $R(u)$, $u \in [0,h]$, is the unique continuous solution of the problem

$$2uR''(u) - f(u)R(u) = 0, \tag{6.15}$$

$$R(0) = 1, \qquad R(h) = 0. \tag{6.16}$$

Remark 6.6. In the case $h=\infty$ the right-hand side condition in (6.16) must be replaced by the requirement that R be bounded.

A consequence of Theorem 6.1 is the following assertion.

Theorem 6.5. *For $h \ge 0$, $\mu > 0$, $\eta > 0$*

$$\int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\mathbf{P}_0\Big(\sup_{y\in(-a,b)} \ell(H_{-a,b},y) > h, W(H_{-a,b}) = b\Big)\,da\,db$$

$$= \frac{2}{(\eta-\mu)^2}\Big(\frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})} + \frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})} - \frac{1}{\sqrt{2\mu h}I_1(\sqrt{2\mu h})I_0(\sqrt{2\eta h})}\Big). \tag{6.17}$$

Remark 6.7. If we take the event $W(H_{-a,b}) = -a$ on the left-hand side of (6.17), then one has to interchange the parameters μ and η on the right-hand side of (6.17).

Remark 6.8. Without conditions on the exit across a certain boundary the following formula holds:

$$\int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\mathbf{P}_0\Big(\sup_{y\in(-a,b)} \ell(H_{-a,b},y) > h\Big)\,da\,db$$

$$= \frac{2}{(\eta-\mu)^2}\Big(\frac{1}{I_0(\sqrt{2\mu h})} - \frac{1}{I_0(\sqrt{2\eta h})}\Big)\Big(\frac{1}{\sqrt{2\mu h}I_1(\sqrt{2\mu h})} - \frac{1}{\sqrt{2\eta h}I_1(\sqrt{2\eta h})}\Big). \tag{6.18}$$

Proof of Theorem 6.5. We apply Theorem 6.1 with $f \equiv 0$. The solutions of the problem (6.2)–(6.4) in this case are (see Appendix 4, equation 16a with $p=0$, $\nu=-1/2$, or $\nu=0$)

$$R(t) = \frac{\sqrt{2t}}{\sqrt{\mu}}\Big(K_1(\sqrt{2\mu t}) - \frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})}I_1(\sqrt{2\mu t})\Big),$$

$$Q(t) = 2\Big(K_0(\sqrt{2\eta t}) - \frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})}I_0(\sqrt{2\eta t})\Big).$$

To verify the boundary conditions at zero we used the asymptotic behavior at zero point of the Bessel functions K_1, K_0 (see Appendix 2, Section 4). Using Lemma 5.1 with $\theta = 0$, $\delta = \eta/2$, $\sigma = \mu/2$, $F \equiv 0$, $G \equiv 0$, we get

$$\frac{(\eta-\mu)^2}{4}\int_0^h R(t)Q(t)\,dt = hR'(h)Q'(h)$$

$$+\lim_{t\downarrow 0}\Big(\frac{\mu}{2}R(t)Q(t) + \frac{t}{2}(\eta-\mu)\big(R'(t)Q(t) - R(t)Q'(t)\big) - tR'(t)Q'(t)\Big). \tag{6.19}$$

When solving a problem with two boundaries (the problem (6.2)–(6.4) is of this kind), it is useful to apply two-parameter functions associated with the Bessel functions (see Appendix 2, Section 14). Using the definition and the properties of these functions, we have

$$R(t) = \frac{S_{-1}(\sqrt{2\mu h}, \sqrt{2\mu t})}{\mu\sqrt{2\mu h}I_1(\sqrt{2\mu h})}, \qquad Q(t) = \frac{2S_0(\sqrt{2\eta h}, \sqrt{2\eta t})}{I_0(\sqrt{2\eta h})},$$

$$R'(t) = -\frac{C_{-1}(\sqrt{2\mu t}, \sqrt{2\mu h})}{2\mu\sqrt{th}I_1(\sqrt{2\mu h})}, \qquad Q'(t) = -\frac{\sqrt{2\eta}C_0(\sqrt{2\eta t}, \sqrt{2\eta h})}{\sqrt{t}I_0(\sqrt{2\eta h})}.$$

Therefore,

$$R'(h)Q'(h) = \frac{\sqrt{2\mu h}}{2\mu h I_1(\sqrt{2\mu h})}\,\frac{\sqrt{2\eta}\,(\sqrt{2\eta h})^{-1}}{\sqrt{h}I_0(\sqrt{2\eta h})} = \frac{1}{h\sqrt{\mu h}I_1(\sqrt{2\mu h})I_0(\sqrt{2\eta h})}.$$

Using formulas from Appendix 2, Section 4, we conclude that as $t \downarrow 0$

$$R(t) \sim \frac{1}{\mu}, \qquad R'(t) \sim -\frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})} + \ln(\sqrt{\mu t/2}) + \gamma,$$

$$Q(t) \sim -2\Big(\ln(\sqrt{\eta t/2}) + \gamma + \frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})}\Big), \qquad Q'(t) \sim -\eta\frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})} - \frac{1}{t},$$

where γ is the Euler constant. Consequently, for $t \downarrow 0$

$$\frac{\mu}{2}R(t)Q(t) - tR'(t)Q'(t) \sim -\ln(\sqrt{\eta t/2}) - \frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})} - \frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})} + \ln(\sqrt{\mu t/2})$$

$$= -\frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})} - \frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})} + \frac{1}{2}\ln(\mu/\eta).$$

Similarly,

$$t\big(R'(t)Q(t) - R(t)Q'(t)\big) \to \frac{1}{\mu}.$$

Now from (6.1) and (6.19) it follows that

$$\int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\mathbf{P}_0\Big(\sup_{y\in(-a,b)}\ell(H_{-a,b}, y) \le h, W(H_{-a,b}) = b\Big)\,da\,db$$

$$= \frac{1}{\mu(\eta-\mu)} + \frac{2}{(\eta-\mu)^2}\Big(\frac{1}{\sqrt{2\mu h}I_1(\sqrt{2\mu h})I_0(\sqrt{2\eta h})} - \frac{K_0(\sqrt{2\eta h})}{I_0(\sqrt{2\eta h})} - \frac{K_1(\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})} + \frac{\ln(\mu/\eta)}{2}\Big). \tag{6.20}$$

Since

$$\int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\mathbf{P}_0\big(W(H_{-a,b})=b\big)\,da\,db = \int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\frac{a}{b+a}\,da\,db$$

$$= \int_0^\infty ae^{(\eta-\mu)a}\int_a^\infty \frac{e^{-\eta y}}{y}dy\,da = \int_0^\infty \frac{e^{-\eta y}}{y}\int_0^y ae^{(\eta-\mu)a}\,da\,dy$$

$$= \int_0^\infty \frac{e^{-\eta y}}{y}\Big(\frac{ye^{(\eta-\mu)y}}{\eta-\mu} + \frac{1-e^{(\eta-\mu)y}}{(\eta-\mu)^2}\Big)dy = \frac{1}{\mu(\eta-\mu)} + \frac{1}{(\eta-\mu)^2}\int_0^\infty \frac{e^{-\eta y}-e^{-\mu y}}{y}dy$$

$$= \frac{1}{\mu(\eta-\mu)} + \frac{1}{(\eta-\mu)^2}\lim_{\rho\downarrow 0}(\eta^{-\rho}-\mu^{-\rho})\Gamma(\rho) = \frac{1}{\mu(\eta-\mu)} + \frac{\ln(\mu/\eta)}{(\eta-\mu)^2},$$

formula (6.20) is transformed into (6.17). □

Formula (6.17) can be recast as

$$\int_0^\infty\int_0^\infty e^{-\eta b-\mu a}\mathbf{P}_0\Big(\sup_{y\in(-a,b)}\ell(H_{-a,b},y)>h, W(H_{-a,b})=b\Big)\,da\,db$$

$$= \frac{2}{(\eta-\mu)^2}\frac{(C_0(\sqrt{2\mu h},\sqrt{2\eta h})-1/\sqrt{2\mu h})}{I_1(\sqrt{2\mu h})I_0(\sqrt{2\eta h})}. \tag{6.21}$$

Let us derive the expression for (6.21) in the case $\eta=\mu$. For this we compute the limit on the right-hand side of (6.21) as $\eta\to\mu$. We use the following properties of the function $C_0(x,y)$ from Appendix 2, Section 14:

$$C_0(x,x)=\frac{1}{x}, \qquad \frac{\partial}{\partial y}C_0(x,y) = -xyS_1(x,y), \qquad \frac{\partial}{\partial y}S_1(x,y) = -C_1(y,x).$$

Since $S_1(x,x)=0$, we have

$$\frac{\partial}{\partial\eta}C_0(\sqrt{2\mu h},\sqrt{2\eta h}) = -h^{3/2}\sqrt{2\mu}S_1(\sqrt{2\mu h},\sqrt{2\eta h})\Big|_{\eta=\mu} = 0,$$

$$\frac{\partial^2}{\partial\eta^2}C_0(\sqrt{2\mu h},\sqrt{2\eta h}) = h^2\frac{\sqrt{\mu}}{\sqrt{\eta}}C_1(\sqrt{2\eta h},\sqrt{2\mu h})\Big|_{\eta=\mu} = \frac{\sqrt{h}}{2\mu\sqrt{2\mu}}.$$

Now Taylor's formula yields

$$\int_0^\infty\int_0^\infty e^{-\mu(b+a)}\mathbf{P}_0\Big(\sup_{y\in(-a,b)}\ell(H_{-a,b},y)>h, W(H_{-a,b})=b\Big)\,da\,db$$

$$= \frac{\sqrt{h}}{2\mu\sqrt{2\mu} I_1(\sqrt{2\mu h}) I_0(\sqrt{2\mu h})}. \tag{6.22}$$

It is simpler to explain what happens with (6.18) for $\eta = \mu$. Twice using Taylor's formula and the equalities $(I_0(x))' = I_1(x)$, $(xI_1(x))' = xI_0(x)$, we obtain

$$\int_0^\infty \int_0^\infty e^{-\mu(b+a)} \mathbf{P}_0\Big(\sup_{y\in(-a,b)} \ell(H_{-a,b}, y) > h\Big)\, da\, db = \frac{\sqrt{h}}{\mu\sqrt{2\mu} I_1(\sqrt{2\mu h}) I_0(\sqrt{2\mu h})}. \tag{6.23}$$

The right-hand side of this equality is two times larger than the right-hand side of (6.22). According to the symmetry property of Brownian motion, this to be expected.

Using the fact that $\ell(H_{-a,b}, y)$ is an increasing function with respect to a and b, from (6.23) we deduce the estimate

$$\mathbf{P}_0\Big(\sup_y \ell(H_{-\alpha,\beta}, y) > h\Big) \le \mu^2 e^{\mu(\beta+\alpha)} \int_\beta^\infty \int_\alpha^\infty e^{-\mu(b+a)} \mathbf{P}_0\Big(\sup_y \ell(H_{-a,b}, y) > h\Big)\, da\, db$$

$$\le \frac{\sqrt{2\mu h}\, e^{\mu(\beta+\alpha)}}{2I_1(\sqrt{2\mu h}) I_0(\sqrt{2\mu h})} \le L(1+\mu h) e^{\mu(\beta+\alpha) - 2\sqrt{2\mu h}}, \qquad \alpha > 0,\ \beta > 0,$$

where L is a constant. Choosing $\mu = \dfrac{2h}{(\beta+\alpha)^2}$, we finally get

$$\mathbf{P}_0\Big(\sup_y \ell(H_{-\alpha,\beta}, y) > h\Big) \le L\Big(1 + \frac{2h^2}{(\beta+\alpha)^2}\Big) \exp\Big(-\frac{2h}{\alpha+\beta}\Big). \tag{6.24}$$

As to this estimate, we note that $\Big(1 + \dfrac{x^2}{2}\Big) e^{-x}$, $x \in \mathbf{R}$, is a decreasing convex function. It is also interesting to compare the estimate (6.24) with the expression resulting from (5.35) and (5.37) of Ch. III:

$$\mathbf{P}_0\big(\ell(H_{-\alpha,\beta}, y) > h\big) = \begin{cases} \dfrac{\alpha}{(y+\alpha)} \exp\Big(-\dfrac{(\beta+\alpha)h}{2(\beta-y)(y+\alpha)}\Big), & \text{for} \quad 0 \le y \le \beta, \\ \dfrac{\beta}{(\beta-y)} \exp\Big(-\dfrac{(\beta+\alpha)h}{2(\beta-y)(y+\alpha)}\Big), & \text{for} \quad -\alpha \le y \le 0. \end{cases}$$

For $y = \dfrac{\beta-\alpha}{2}$ the arguments of the exponentials in this formula and in (6.24) coincide. Thus the exponential function in (6.24) is unimprovable.

Finally, we give an example of application of Theorem 6.4.

Example 6.1. We consider the problem of distribution of the time spend by a Brownian motion in rarely visited points. We say that y is a *rarely visited point by the Brownian motion* W up to the time t, if $0 < \ell(t, y) < 1$ for its local time.

We consider the functional

$$T(u) = \int_{-\infty}^{\infty} \ell(\varrho(u,0), y) \mathbb{1}_{(0,1)}(\ell(\varrho(u,0), y))\, dy = \int_0^{\varrho(u,0)} \mathbb{1}_{(0,1)}(\ell(\varrho(u,0), W(s)))\, ds,$$

which is equal to the time spent by the Brownian motion in the set of rarely visited points up to the inverse local time moment $\varrho(u,0)$.

It is not hard to compute that for $f(v) = \gamma v \mathbb{1}_{(0,1)}(v)$, $v \ge 0$, the square of the bounded continuous solution of equation (6.15) with initial value equal to one has the form

$$\mathbf{E}_0 \exp(-\gamma T(u)) = \begin{cases} \dfrac{\operatorname{ch}^2((1-u)\sqrt{\gamma/2})}{\operatorname{ch}^2(\sqrt{\gamma/2})}, & \text{for} \quad 0 \le u \le 1, \\ \dfrac{1}{\operatorname{ch}^2(\sqrt{\gamma/2})}, & \text{for} \quad 1 \le u. \end{cases} \tag{6.25}$$

For $u \ge 1$, inverting the Laplace transform with respect to γ (see the corresponding formula of Section 13 of Appendix 2), we obtain

$$\mathbf{P}_0\,(T(u) \in dy) = \mathcal{L}_\gamma^{-1}\Big(\frac{1}{\operatorname{ch}^2(\sqrt{\gamma/2})}\Big) = \frac{d}{dv}\mathcal{L}_\gamma^{-1}\Big(\frac{\operatorname{sh}(v\sqrt{2\gamma})}{\sqrt{2\gamma}\operatorname{ch}(v\sqrt{2\gamma})}\Big)\Big|_{v=1/2} dy$$
$$= \frac{d}{dv}\operatorname{sc}_y(v,v)\big|_{v=1/2} dy = \frac{2\sqrt{2}}{\sqrt{\pi y^3}}\sum_{k=1}^{\infty}(-1)^{k-1}k^2\exp\Big(-\frac{k^2}{2y}\Big)\,dy. \tag{6.26}$$

Exercises.

6.1. Compute the probabilities

$$\lambda\int_0^\infty e^{-\lambda z}\mathbf{P}_0\Big(\sup_{y\in\mathbf{R}}\ell\big(H_z,y\big) < h\Big)dz \qquad \text{and} \qquad \mathbf{P}_0\Big(\sup_{y\in\mathbf{R}}\ell\big(H_z,y\big) < h\Big).$$

6.2. Compute the probabilities

$$\lambda\int_0^\infty e^{-\lambda z}\mathbf{P}_0\Big(\sup_{y\in\mathbf{R}}\ell\big(\varrho(u,z),y\big) < h\Big)dz$$

and

$$\mathbf{P}_0\Big(\sup_{y\in\mathbf{R}}\ell\big(\varrho(u,z),y\big) < h\Big), \qquad z \ge 0.$$

6.3. Using Theorem 6.4, compute the probability

$$\mathbf{P}_0\Big(\sup_{y\in\mathbf{R}}\ell\big(\varrho(u,0),y\big) < h\Big).$$

6.4. Using Theorem 6.4, compute for $\beta \ge 1$ and $b > 0$ the expectation

$$\mathbf{E}_0\exp\Big(-\gamma\int_{-\infty}^{\infty}\ell^{\beta-1}(\varrho(u,0),y)\mathbb{1}_{(0,b)}(\ell(\varrho(u,0),y))\,dy\Big).$$

6.5. Using Theorem 6.4, compute for $\beta \ge 1$ and $b > 0$ the expectation

$$\mathbf{E}_0 \exp\Big(-\gamma \int_{-\infty}^{\infty} \ell^{\beta-1}(\varrho(u,0),y)\mathbb{1}_{(b,\infty)}(\ell(\varrho(u,0),y))\,dy\Big).$$

6.6. Using Theorem 6.4, compute for $\beta \ge 1$ and $0 \le a < b$ the expectation

$$\mathbf{E}_0 \exp\Big(-\gamma \int_{-\infty}^{\infty} \ell^{\beta-1}(\varrho(u,0),y)\mathbb{1}_{(a,b)}(\ell(\varrho(u,0),y))\,dy\Big).$$

§ 7. Distributions of functionals of sojourn time type

Let $W(s)$, $s \ge 0$, be a Brownian motion and let $\ell(s,x)$, $(s,x) \in [0,\infty) \times \mathbf{R}$, be the Brownian local time. This section is devoted to a method enabling us to compute the distributions of functionals $C(t)$, $t \ge 0$, of the form

$$C(t) = \int_0^t f(W(s), \ell(s,r_1),\dots,\ell(s,r_k))\,ds, \qquad r_i \in \mathbf{R},$$

where $f(x,\vec{y})$, $(x,\vec{y}) \in \mathbf{R}^{k+1}$, is a nonnegative measurable function and k is an arbitrary positive integer.

Set

$$\vec{\xi}_k(s) := (x + W(s), y_1 + \ell(s, r_1 - x), \dots, y_k + \ell(s, r_k - x)),$$

where $s \ge 0$, $x \in \mathbf{R}$, $y_l, r_l \in \mathbf{R}$, $l = 1,\dots,k$, and $W(0) = 0$. By the definition of the process $\vec{\xi}_k$, the vector $(x,\vec{y})$ is its initial value $\vec{\xi}_k(0)$. If f is the indicator function of some set in $\mathbf{R}^{k+1}$, then $C(t)$, $t \ge 0$, is the sojourn time of the process $\vec{\xi}_k$ in this set up to the moment t. This is why we call $C(t)$ a functional of sojourn time type.

Before we embark upon the investigation of the process $\vec{\xi}_k$ we consider the process

$$\vec{\eta}_k(s) = \Big(x + W(s), y_1 + \int_0^s g_1(x + W(u))\,du, \dots, y_k + \int_0^s g_k(x+W(u))\,du\Big), \quad s \ge 0.$$

It is a multidimensional diffusion process, whose first component is the Brownian motion and others are integral functionals of the Brownian motion. Obviously, $\vec{\eta}_k(0) = (x,\vec{y})$. For this process we prove an analog of Theorem 13.1 of Ch. II.

Theorem 7.1. *Let $g_l(x)$, $l = 1,\dots,k$, $\Phi(x,\vec{y})$ and $f(x,\vec{y})$, $x \in \mathbf{R}$, $\vec{y} \in \mathbf{R}^k$, be bounded twice continuously differentiable functions with bounded first and second derivatives, $f \ge 0$. Then the function*

$$u(t,x,\vec{y}) := \mathbf{E}\Big\{\Phi(\vec{\eta}_k(t)) \exp\Big(-\int_0^t f(\vec{\eta}_k(s))\,ds\Big)\Big\}$$

satisfies in $(0,\infty)\times\mathbf{R}^{k+1}$ *the differential equation*

$$\frac{\partial}{\partial t}u(t,x,\vec{y})=\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x,\vec{y})+\sum_{l=1}^{k}g_l(x)\frac{\partial}{\partial y_l}u(t,x,\vec{y_l})-f(x,\vec{y})u(t,x,\vec{y}) \tag{7.1}$$

and the initial condition

$$u(0,x,\vec{y})=\Phi(x,\vec{y}). \tag{7.2}$$

Proof. One can verify the initial condition (7.2) by passing to the limit under the expectation sign in the definition of the function $u(t,x,\vec{y})$. By differentiation under the expectation sign we can also easily check that $u(t,x,\vec{y})$ has continuous second derivative with respect to x and first derivative with respect to y_l, $l=1,\dots,k$. Let $0\le s\le t$, $\delta:=t-s$. Set $W_\delta(v):=W(v+\delta)-W(\delta)$, $v\ge 0$, and denote by $\mathcal{F}_\delta$ the σ-algebra of events generated by the process $W(u)$ for $u\le\delta$. Set

$$\vec{\eta}_k(\delta,s):=\Bigl(W_\delta(s),\int_0^s g_1(x+W(\delta)+W_\delta(v))\,dv,\dots,\int_0^s g_k(x+W(\delta)+W_\delta(v))\,dv\Bigr).$$

Then $\vec{\eta}_k(t)=\vec{\eta}_k(\delta)+\vec{\eta}_k(\delta,s)$. Using the independence of the Brownian motion $W_\delta(v)$, $v\ge 0$, and the σ-algebra $\mathcal{F}_\delta$, and applying Lemma 2.1 Ch. I, we get

$$\begin{aligned}u(t,x,\vec{y})&=\mathbf{E}\Bigl\{\exp\Bigl(-\int_0^\delta f(\vec{\eta}_k(v))\,dv\Bigr)\mathbf{E}\Bigl\{\Phi(\vec{\eta}_k(\delta)+\vec{\eta}_k(\delta,s))\\&\qquad\times\exp\Bigl(-\int_0^s f(\vec{\eta}_k(\delta)+\vec{\eta}_k(\delta,v))\,dv\Bigr)\Big|\mathcal{F}_\delta\Bigr\}\Bigr\}\\&=\mathbf{E}\Bigl\{\exp\Bigl(-\int_0^\delta f(\vec{\eta}_k(v))dv\Bigr)u(s,\vec{\eta}_k(\delta))\Bigr\}\\&=\mathbf{E}\Bigl\{u(s,\vec{\eta}_k(\delta))\Bigl(1-\int_0^\delta f(\vec{\eta}_k(v))\,dv+o(\delta)\Bigr)\Bigr\}.\end{aligned} \tag{7.3}$$

Since the function $u(s,x,\vec{y})$ is bounded and continuous with respect to $(s,x,\vec{y})$, while $f(x,\vec{y})$ is also bounded and continuous, we have

$$\mathbf{E}u(s,\vec{\eta}_k(\delta))\int_0^\delta f(\vec{\eta}_k(v))\,dv=\delta f(x,\vec{y})u(s,x,\vec{y})+o(\delta). \tag{7.4}$$

By the Itô formula,

$$u(s,\vec{\eta}_k(\delta))-u(s,x,\vec{y})=\int_0^\delta\frac{\partial}{\partial x}u(s,\vec{\eta}_k(v))\,dW(v)$$

$$+\frac{1}{2}\int_0^\delta \frac{\partial^2}{\partial x^2}u(s,\vec{\eta}_k(v))\,dv+\sum_{l=1}^k\int_0^\delta \frac{\partial}{\partial y_l}u(s,\vec{\eta}_k(v))g_l(x+W(v))\,dv.$$

Taking the expectation of both sides of this equality and using the continuity of the derivatives of $u(t,x,\vec{y})$ in x and y_l, we obtain

$$\mathbf{E}u(s,\vec{\eta}_k(\delta))-u(s,x,\vec{y})=\frac{\delta}{2}\frac{\partial^2}{\partial x^2}u(s,x,\vec{y})+\delta\sum_{l=1}^k g_l(x)\frac{\partial}{\partial y_l}u(s,x,\vec{y})+o(\delta).$$

We now deduce from (7.3) and (7.4) that

$$u(t,x,\vec{y})-u(s,x,\vec{y})=\frac{\delta}{2}\frac{\partial^2}{\partial x^2}u(s,x,\vec{y})$$
$$+\delta\sum_{l=1}^k g_l(x)\frac{\partial}{\partial y_l}u(s,x,\vec{y})-\delta f(x,\vec{y})u(s,x,\vec{y})+o(\delta).$$

From this it follows that for any $s>0$ the right derivative of $u(s,x,\vec{y})$ at s satisfies (7.1). Taking into account the definition of $u(s,x,\vec{y})$ and using double differentiation with respect to x and differentiation with respect to y_l under the expectation sign, we easily get that the functions $\frac{\partial^2}{\partial x^2}u(s,x,\vec{y})$, $\frac{\partial}{\partial y_l}u(s,x,\vec{y})$, $l=1,2,\dots,k$, are continuous in s. This in turn implies the existence of the partial derivative of the function $u(t,x,\vec{y})$ with respect to t, which satisfies (7.1). □

We apply the Laplace transform with respect to t to the problem (7.1), (7.2). As before, let τ be the exponentially distributed with the parameter $\lambda>0$ random time independent of the Brownian motion W.

Proposition 7.1. *The function*

$$U(x,\vec{y}):=\lambda\int_0^\infty e^{-\lambda t}u(t,x,\vec{y})\,dt=\mathbf{E}\Big\{\varPhi(\vec{\eta}_k(\tau))\exp\Big(-\int_0^\tau f(\vec{\eta}_k(s))\,ds\Big)\Big\}$$

for $(x,\vec{y})\in\mathbf{R}^{k+1}$ *satisfies the equation*

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}U(x,\vec{y})+\sum_{l=1}^k g_l(x)\frac{\partial}{\partial y_l}U(x,\vec{y})-(\lambda+f(x,\vec{y}))U(x,\vec{y})=-\lambda\varPhi(x,\vec{y}). \tag{7.5}$$

Indeed, we use the following equality, consequence of the integration by parts formula,

$$\lambda\int_0^\infty e^{-\lambda t}\frac{\partial}{\partial t}u(t,x,\vec{y})\,dt=-\lambda\varPhi(x,\vec{y})+\lambda U(x,\vec{y}).$$

Then we deduce from (7.1) by integration that the function $U(x,\vec{y})$, $(x,\vec{y})\in\mathbf{R}^{k+1}$, satisfies (7.5).

Now we are ready to return to the investigation of the process $\vec{\xi}_k(s)$, $s\ge 0$. The following theorem is the main result of the section. It can be considered as the analog of Proposition 7.1 for the case, when $g_l(x)=\delta_{r_l}(x)$, $l=1,\dots,k$, (Dirac δ-functions) i.e., when the local times at the points r_l are taken in place of the integral functionals.

Theorem 7.2. *Let $f(x,\vec{y})$, $(x,\vec{y})\in\mathbf{R}^{k+1}$, be a nonnegative measurable function that is bounded on any compact subset of $\mathbf{R}^{k+1}$, and let $\varPhi(x,\vec{y})$, $(x,\vec{y})\in\mathbf{R}^{k+1}$, be a bounded measurable function. Then the function*

$$Q(x,\vec{y})=\mathbf{E}\Big\{\varPhi(\vec{\xi}_k(\tau))\exp\Big(-\int_0^\tau f(\vec{\xi}_k(s))\,ds\Big)\Big\},\qquad (x,\vec{y})\in\mathbf{R}^{k+1},\tag{7.6}$$

is continuous in $x\in\mathbf{R}$ and satisfies for any $\vec{y}\in\mathbf{R}^k$ the following equations:

1) *for any $a<b$ such that $(a,b)\subset\mathbf{R}\setminus\{r_1,\dots,r_k\}$,*

$$\frac12\Big(\frac{\partial}{\partial x}Q(b,\vec{y})-\frac{\partial}{\partial x}Q(a,\vec{y})\Big)-\int_a^b(\lambda+f(x,\vec{y}))Q(x,\vec{y})\,dx=-\lambda\int_a^b\varPhi(x,\vec{y})\,dx;\tag{7.7}$$

2) *for any $\alpha<\beta$ and $l=1,\dots,k$,*

$$\frac12\int_\alpha^\beta\Big(\frac{\partial}{\partial x}Q(r_l-0,\vec{y})-\frac{\partial}{\partial x}Q(r_l+0,\vec{y})\Big)\,dy_l=Q(r_l,\vec{y})\Big|_{y_l=\beta}-Q(r_l,\vec{y})\Big|_{y_l=\alpha}.\tag{7.8}$$

Proof. Assume first that $\varPhi(x,\vec{y})$ and $f(x,\vec{y})$ are bounded twice continuously differentiable functions with bounded first and second derivatives. Let g be a twice continuously differentiable function such that $0\le g(x)\le 1$, $g(x)=0$ for $|x|\ge 1$ and $\int_{-\infty}^{\infty} g(x)\,dx=1$, $g(0)=1$. For $\varepsilon>0$ we set

$$g_l^\varepsilon(x):=\frac{1}{\varepsilon}g\Big(\frac{x-r_l}{\varepsilon}\Big),\qquad G_l^\varepsilon(x):=\int_{-\infty}^x g_l^\varepsilon(u)\,du,\qquad l=1,\dots,k,$$

$$\vec{\xi}_k^{\,\varepsilon}(s):=\Big(x+W(s),y_1+\int_0^s g_1^\varepsilon(x+W(u))\,du,\dots,y_k+\int_0^s g_k^\varepsilon(x+W(u))\,du\Big).$$

Obviously,

$$G_l^\varepsilon(x)\to G_l(x)=\mathbb{1}_{[r_l,\infty)}(x),\qquad x\ne r_l\tag{7.9}$$

as $\varepsilon\to 0$. Set

$$u_\varepsilon(t,x,\vec{y}):=\mathbf{E}\Big\{\varPhi(\vec{\xi}_k^{\,\varepsilon}(t))\exp\Big(-\int_0^t f(\vec{\xi}_k^{\,\varepsilon}(s))\,ds\Big)\Big\}.$$

We apply Proposition 7.1. By (7.5), the function

$$U_\varepsilon(x,\vec{y}):=\lambda\int_0^\infty e^{-\lambda t}u_\varepsilon(t,x,\vec{y})\,dt=\mathbf{E}\Big\{\varPhi(\vec{\xi}_k^{\,\varepsilon}(\tau))\exp\Big(-\int_0^\tau f(\vec{\xi}_k^{\,\varepsilon}(s))\,ds\Big)\Big\}$$

satisfies for $(x,\vec{y})\in\mathbf{R}^{k+1}$ the equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}U_\varepsilon(x,\vec{y})+\sum_{l=1}^{k}g_l^\varepsilon(x)\frac{\partial}{\partial y_l}U_\varepsilon(x,\vec{y})-(\lambda+f(x,\vec{y}))U_\varepsilon(x,\vec{y})=-\lambda\Phi(x,\vec{y}). \quad (7.10)$$

We prove that for any $\vec{y}\in\mathbf{R}^k$

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbf{R}}|U_\varepsilon(x,\vec{y})-Q(x,\vec{y})|=0. \quad (7.11)$$

From (1.6) for $k=1$ it follows that

$$\sup_{x\in\mathbf{R}}\mathbf{E}(\ell(t,\Delta+x)-\ell(t,x))^2\le 2^8\Delta\sqrt{t}, \qquad \Delta\ge 0.$$

For any $l=1,\dots,k$

$$\sup_{x\in\mathbf{R}}\mathbf{E}\Big|\int_0^s g_l^\varepsilon(x+W(u))\,du-\ell(s,r_l-x)\Big|=\sup_{x\in\mathbf{R}}\mathbf{E}\Big|\int_{-1}^1 g(v)(\ell(s,\varepsilon v+x)-\ell(s,x))\,dv\Big|$$

$$\le\sup_{x\in\mathbf{R}}\int_{-1}^1\mathbf{E}|\ell(s,\varepsilon v+x)-\ell(s,x)|dv\le Cs^{1/4}\sqrt{\varepsilon}. \quad (7.12)$$

Then, using the restrictions imposed on the functions Φ and f, we get

$$|U_\varepsilon(x,\vec{y})-Q(x,\vec{y})|\le C_k\mathbf{E}\Big(\tau^{1/4}\sqrt{\varepsilon}+\sqrt{\varepsilon}\int_0^\tau s^{1/4}ds\Big)\le 2C_k\Big(\frac{1}{\lambda^{1/4}}+\frac{1}{\lambda^{5/4}}\Big)\sqrt{\varepsilon},$$

and hence (7.11) holds.

Integrating (7.10), we deduce that

$$\frac{1}{2}\Big(\frac{\partial}{\partial x}U_\varepsilon(b,\vec{y})-\frac{\partial}{\partial x}U_\varepsilon(a,\vec{y})\Big)+\sum_{l=1}^{k}\int_a^b\frac{\partial}{\partial y_l}U_\varepsilon(x,\vec{y})\,dG_l^\varepsilon(x)$$

$$-\int_a^b(\lambda+f(x,\vec{y}))U_\varepsilon(x,\vec{y})\,dx=-\lambda\int_a^b\Phi(x,\vec{y})\,dx. \quad (7.13)$$

Since $G_l^\varepsilon(x)$ increases only in the ε-neighborhoods of the points r_l, for each compact set $K\subset\mathbf{R}\setminus\{r_1,\dots,r_k\}$ there is an ε such that

$$K\subset\mathbf{R}\setminus\Big\{\bigcup_{l=1}^{k}(r_l-\varepsilon,r_l+\varepsilon)\Big\}.$$

Suppose that $(a,b) \subset K$ and $(a,c) \subset K$. Then

$$\frac{1}{2}\Big(\frac{\partial}{\partial x}U_\varepsilon(b,\vec{y}) - \frac{\partial}{\partial x}U_\varepsilon(a,\vec{y})\Big) - \int_a^b (\lambda + f(x,\vec{y}))U_\varepsilon(x,\vec{y})\,dx = -\lambda\int_a^b \Phi(x,\vec{y})\,dx, \quad (7.14)$$

$$\frac{1}{2}\Big(U_\varepsilon(c,\vec{y}) - U_\varepsilon(a,\vec{y}) - (c-a)\frac{\partial}{\partial x}U_\varepsilon(a,\vec{y})\Big)$$
$$-\int_a^c\int_a^b (\lambda + f(x,\vec{y}))U_\varepsilon(x,\vec{y})\,dxdb = -\lambda\int_a^c\int_a^b \Phi(x,\vec{y})\,dxdb. \quad (7.15)$$

Since $|U_\varepsilon| \le C$, it follows from (7.14), (7.15) that $\frac{\partial}{\partial x}U_\varepsilon(x,\vec{y})$ as functions of x are equicontinuous on the compact set K and uniformly bounded for $(x,\vec{y}) \in K \times \mathbf{R}^k$. By the Arzelá-Ascoli theorem, the family $\Big\{\frac{\partial}{\partial x}U_\varepsilon(x,\vec{y})\Big\}_{\varepsilon>0}$ as functions of x is a relatively compact in such a set. From this and (7.11) we get that $\frac{\partial}{\partial x}Q(x,\vec{y})$ exists for $x \in K$ and

$$\sup_{x\in K}\Big|\frac{\partial}{\partial x}U_\varepsilon(x,\vec{y}) - \frac{\partial}{\partial x}Q(x,\vec{y})\Big| \to 0 \qquad \text{as } \varepsilon \downarrow 0. \quad (7.16)$$

Passing to the limit in (7.14), we get that $Q(x,\vec{y})$ satisfies (7.7). The continuity of $Q(x,\vec{y})$ in x follows from (7.11).

For any l we choose points a and b in (7.15) such that $r_{l-1} < a < r_l < b < r_{l+1}$. We integrate (7.13) with respect to y_l from α to β. Then for all sufficiently small ε we have

$$\frac{1}{2}\int_\alpha^\beta \Big(\frac{\partial}{\partial x}U_\varepsilon(b,\vec{y}) - \frac{\partial}{\partial x}U_\varepsilon(a,\vec{y})\Big)\,dy_l + \int_a^b \Big(U_\varepsilon(x,\vec{y})\big|_{y_l=\beta} - U_\varepsilon(x,\vec{y})\big|_{y_l=\alpha}\Big)\,dG_l^\varepsilon(x)$$
$$-\int_a^b\int_\alpha^\beta (\lambda + f(x,\vec{y}))U_\varepsilon(x,\vec{y})\,dy_l dx = -\lambda\int_a^b\int_\alpha^\beta \Phi(x,\vec{y})\,dy_l dx.$$

Since $G_l^\varepsilon(x) \to G_l(x) = \mathbb{I}_{[r_l,\infty)}(x)$, we can pass to the limit as $\varepsilon \to 0$ in this relation, taking into account (7.11), (7.16), and the uniform boundedness of the functions $\frac{\partial}{\partial x}U_\varepsilon(b,\vec{y})$, $\frac{\partial}{\partial x}U_\varepsilon(a,\vec{y})$, $\vec{y} \in \mathbf{R}^k$. Therefore,

$$\frac{1}{2}\int_\alpha^\beta \Big(\frac{\partial}{\partial x}Q(b,\vec{y}) - \frac{\partial}{\partial x}Q(a,\vec{y})\Big)dy_l + Q(r_l,\vec{y})\big|_{y_l=\beta} - Q(r_l,\vec{y})\big|_{y_l=\alpha}$$
$$-\int_a^b\int_\alpha^\beta (\lambda + f(x,\vec{y}))Q(x,\vec{y})\,dy_l dx = -\lambda\int_a^b\int_\alpha^\beta \Phi(x,\vec{y})\,dy_l\,dx.$$

Since $Q(x,\vec{y})$ satisfies (7.7), the one-sided derivatives $\frac{\partial}{\partial x}Q(r_l\pm 0,\vec{y})$, $l=1,2,\dots,k$, exist, and we can pass to the limit as $b\downarrow r_l$, $a\uparrow r_l$ in the above relation, which gives us (7.8), and hence proves the theorem for the class of functions Φ, f under consideration.

We extend this class of functions by the limit approximation method. Let $f_n(x,\vec{y})$, $(x,\vec{y})\in\mathbf{R}^{k+1}$, be a sequence of nonnegative measurable functions that are uniformly bounded on an arbitrary compact subset of $\mathbf{R}^{k+1}$, and let $\Phi_n(x,\vec{y})$, $(x,\vec{y})\in\mathbf{R}^{k+1}$ be a sequence of measurable uniformly bounded functions. Suppose that the statement of Theorem 7.2 holds for the functions Φ_n, f_n, and let

$$\Phi(x,\vec{y})=\lim_{n\to\infty}\Phi_n(x,\vec{y}),\qquad f(x,\vec{y})=\lim_{n\to\infty}f_n(x,\vec{y}),\quad \text{for } (x,\vec{y})\in\mathbf{R}^{k+1}.$$

We prove that the statement of Theorem 7.2 then holds also for Φ and f. Let $Q_n(x,\vec{y})$ denote the function (7.6) with Φ_n and f_n in place of Φ and f.

We prove that for any $d>0$ and arbitrary fixed $\vec{y}\in\mathbf{R}^k$,

$$\lim_{n\to\infty}\sup_{x\in[-d,d]}|Q_n(x,\vec{y})-Q(x,\vec{y})|=0. \tag{7.17}$$

For any $\varepsilon>0$ we choose A such that

$$\mathbf{P}\Big(\sup_{0\le s\le\tau}|W(s)|>A-d\Big)<\varepsilon,\qquad \mathbf{P}\Big(\sup_{z\in\mathbf{R}}\ell(\tau,z)>A-|\vec{y}|\Big)<\varepsilon.$$

Let

$$\Delta_A:=\{(x,\vec{v}):-A\le x\le A,-A\le v_l\le A,\ l=1,\dots,k\},$$

$f_n^A(x,\vec{v})=f_n(x,\vec{v})\mathbb{I}_{\Delta_A}(x,\vec{v})$. Denote by $Q_n^A(x,\vec{v})$ the function (7.6) with Φ_n and f_n^A in place of Φ and f. The functions $f^A(x,\vec{v})$ and $Q^A(x,\vec{v})$ are defined similarly. Since ε is arbitrary, instead of (7.17) we need only to prove that for arbitrary fixed $\vec{y}\in\mathbf{R}^k$

$$\lim_{n\to\infty}\sup_{x\in[-d,d]}|Q_n^A(x,\vec{y})-Q^A(x,\vec{y})|=0. \tag{7.18}$$

Set $\Psi_n:=|\Phi_n-\Phi|+|f_n^A-f^A|$. For simplicity we consider the case $k=2$. The general case is not essentially different, because one can use the Markov property of the process $\ell(\tau,y)$, $y\in\mathbf{R}$, given $W(\tau)=z$ (see Theorem 2.1). We have

$$|Q_n^A(x,\vec{y})-Q^A(x,\vec{y})|\le\mathbf{E}|\Phi_n(\vec{\xi}_2(\tau))-\Phi(\vec{\xi}_2(\tau))|+K_1\mathbf{E}\int_0^\tau|f_n^A(\vec{\xi}_2(s))-f^A(\vec{\xi}_2(s))|\,ds$$

$$\le\mathbf{E}|\Phi_n(\vec{\xi}_2(\tau))-\Phi(\vec{\xi}_2(\tau))|+\frac{K_1}{\lambda}\mathbf{E}|f_n^A(\vec{\xi}_2(\tau))-f^A(\vec{\xi}_2(\tau))|$$

$$\le\Big(K+\frac{K}{\lambda}\Big)\mathbf{E}\Psi_n(x+W(\tau),y_1+\ell(\tau,r_1-x),y_2+\ell(\tau,r_2-x))$$

$$=\frac{K(1+\lambda)}{\lambda}\int_{-\infty}^{\infty}\int_{[0,\infty)}\int_{[0,\infty)}\mathbf{P}_x(W(\tau)\in dz,\ell(\tau,r_1)\in dv,\ell(\tau,r_2)\in dg)\Psi_n(z,y_1+v,y_2+g). \tag{7.19}$$

To estimate the supremum with respect to $x \in [-d, d]$ of the right-hand side of this inequality we use the result of Example 4.4 Ch. III. From this example it follows that the joint distribution of $W(\tau)$ and the process $\ell(\tau, y)$, $y \in \mathbf{R}$, has an atom only at the point zero (formulas (4.62)–(4.64) Ch. III), and has a joint density (formula (4.65) Ch. III). We need also the following estimates for the supremum of the coefficients in formulas (4.62)–(4.65) Ch. III: $\Delta = |u - r|$,

$$\sup_{x\in\mathbf{R}} C \le \big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)\big(e^{-|z-r|\sqrt{2\lambda}} + e^{-|z-u|\sqrt{2\lambda}}\big),$$

$$\sup_{x\in\mathbf{R}} D = \big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)\big(e^{-|z-u|\sqrt{2\lambda}} - e^{-(|z-r|+\Delta)\sqrt{2\lambda}}\big),$$

$$\sup_{x\in\mathbf{R}} F = \big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)\big(e^{-|z-r|\sqrt{2\lambda}} - e^{-(|z-u|+\Delta)\sqrt{2\lambda}}\big),$$

$$\sup_{x\in\mathbf{R}} H \le \big(1 - e^{-2\Delta\sqrt{2\lambda}}\big)\big(e^{-\Delta\sqrt{2\lambda}} - e^{-2\Delta\sqrt{2\lambda}}\big)\big(e^{-|z-u|\sqrt{2\lambda}} + e^{-|z-r|\sqrt{2\lambda}}\big).$$

Upon using these estimates with $u = r_2$, $r = r_1$, (7.19) yields (7.18).

Since for any fixed $\vec{y} \in \mathbf{R}^k$ the functions $Q_n(x, \vec{y})$ are continuous with respect to x, it follows from (7.17) that $Q(x, \vec{y})$ is continuous with respect to x. Further, since the functions $Q_n(x, \vec{y})$ are uniformly bounded and satisfy (7.7), it is not hard to see by integrating (7.7) with respect to $b \in [a, c]$ (c is an arbitrary point satisfying $[a, c] \subset \mathbf{R} \setminus \{r_1, \dots, r_k\}$) that the functions $\frac{\partial}{\partial x} Q_n(x, \vec{y})$ are uniformly bounded on $K \times [\alpha, \beta]^k$ for any $\alpha < \beta$ and any compact subset $K \subset \mathbf{R} \setminus \{r_1, \dots, r_k\}$. The functions $\frac{\partial}{\partial x} Q_n(x, \vec{y})$ are equicontinuous on K as functions of x. From this and (7.17) it follows that $\frac{\partial}{\partial x} Q(x, \vec{y})$, $x \in \mathbf{R} \setminus \{r_1, \dots, r_k\}$, exists for any $\vec{y} \in \mathbf{R}^k$, and

$$\lim_{n\to\infty} \sup_{x\in K} \Big| \frac{\partial}{\partial x} Q_n(x, \vec{y}) - \frac{\partial}{\partial x} Q(x, \vec{y}) \Big| = 0.$$

Then the function $Q(x, \vec{y})$ satisfies (7.7) and hence the derivatives $\frac{\partial}{\partial x} Q(r_l \pm 0, \vec{y})$ exist for $l = 1, \dots, k$, and

$$\frac{\partial}{\partial x} Q_n(r_l \pm 0, \vec{y}) \to \frac{\partial}{\partial x} Q(r_l \pm 0, \vec{y}).$$

Since the functions $\frac{\partial}{\partial x} Q_n(x, \vec{y})$ are uniformly bounded in $K \times [\alpha, \beta]^k$, the functions $\frac{\partial}{\partial x} Q_n(r_l \pm 0, \vec{y})$, $\vec{y} \in [\alpha, \beta]^k$, are also uniformly bounded. The functions $Q_n(x, \vec{y})$ satisfy (7.8), therefore passing to the limit in (7.8), we get that (7.8) holds also for the function $Q(x, \vec{y})$.

Thus the theorem is valid for functions Φ, f that are limits of sequences of functions Φ_n, f_n, for which the theorem holds. Moreover, Φ_n are uniformly bounded, while f_n are uniformly bounded on any compact subset in $\mathbf{R}^{k+1}$. Since the theorem is valid for bounded twice continuously differentiable functions, it is valid also for bounded continuous functions Φ and continuous functions f bounded on any compact subset of $\mathbf{R}^{k+1}$. The smallest class of bounded functions closed under taking limits and containing all continuous bounded functions coincides (see Gihman and Skorohod (1969) Ch. 2 § 2 Theorem 4) with the class of all bounded measurable functions. An analogous assertion holds for classes of functions bounded on any compact subset of $\mathbf{R}^{k+1}$. □

The next theorem enables us to find distributions of the functional

$$C(\infty) := \int_0^\infty f(W(s), \ell(s,r_1), \dots, \ell(s,r_k))\, ds, \qquad r_i \in \mathbf{R}.$$

This functional is certainly finite, if the function $f(x,\vec{y})$, $(x,\vec{y}) \in \mathbf{R}^{k+1}$, degenerates to zero when at least one of the variables y_l, $l = 1,\dots,k$, exceeds some value. This is due to the fact that for any r the Brownian local time $\ell(s,r)$, $s \ge 0$, increases infinitely as $s \to \infty$ (see § 9).

Theorem 7.3. *Let $f(x,\vec{y})$, $x \in \mathbf{R}$, $\vec{y} \in \mathbf{R}^k$, be a nonnegative measurable function bounded on any compact subset of $\mathbf{R}^{k+1}$. Assume that $f(x,\vec{y}) = 0$ if $y_l \ge z$ for some l and $z > 0$. Then the function*

$$q(x,\vec{y}) := \mathbf{E}\exp\Big(-\int_0^\infty f(\vec{\xi}_k(s))\, ds\Big), \qquad (x,\vec{y}) \in \mathbf{R}^{k+1},$$

is continuous in x for any fixed $\vec{y} \in \mathbf{R}^k$ and satisfies the following equations:

1) *for any $a < b$ such that $(a,b) \subset \mathbf{R} \setminus \{r_1,\dots,r_k\}$,*

$$\frac{1}{2}\Big(\frac{\partial}{\partial x} q(b,\vec{y}) - \frac{\partial}{\partial x} q(a,\vec{y})\Big) - \int_a^b f(x,\vec{y}) q(x,\vec{y})\, dx = 0; \tag{7.20}$$

2) *for any $\alpha < \beta$ and $l = 1,\dots,k$*

$$\frac{1}{2}\int_\alpha^\beta \Big(\frac{\partial}{\partial x} q(r_l - 0,\vec{y}) - \frac{\partial}{\partial x} q(r_l + 0,\vec{y})\Big)\, dy_l = q(r_l,\vec{y})\big|_{y_l=\beta} - q(r_l,\vec{y})\big|_{y_l=\alpha}. \tag{7.21}$$

Remark 7.1. Theorem 7.3 can be interpreted as the limit case of Theorem 7.2 as $\lambda \downarrow 0$ and $\tau \to \infty$.

Proof of Theorem 7.3. We keep the notations introduced in the proofs of Theorems 7.1 and 7.2. Assume first that f is a bounded twice continuously differentiable function with bounded first and second derivatives such that there exists l and $z > 0$ for which $f(x,\vec{y}) = 0$ for $y_l > z$.

Set

$$q_\varepsilon(x,\vec{y}) := \mathbf{E}\exp\Big(-\int_0^\infty f(\vec{\xi}_k^{\,\varepsilon}(s))\, ds\Big).$$

Let $\mathcal{G}_s^t := \sigma\{\vec{\xi}_k^\varepsilon(v), v \in [s,t]\}$ be the σ-algebra of events generated by the process $\vec{\xi}_k^\varepsilon$ when the time varies from s to t.

Since $\vec{\xi}_k^\varepsilon(s)$ is a homogeneous Markov process, we see that for any $t > 0$

$$\mathbf{E}\Big\{\exp\Big(-\int_t^\infty f(\vec{\xi}_k^\varepsilon(s))\,ds\Big)\Big|\mathcal{G}_0^t\Big\} = \mathbf{E}\Big\{\exp\Big(-\int_t^\infty f(\vec{\xi}_k^\varepsilon(s))\,ds\Big)\Big|\mathcal{G}_t^t\Big\} = q_\varepsilon(\vec{\xi}_k^\varepsilon(t)).$$

By the third property of conditional expectations,

$$q_\varepsilon(x,\vec{y}) = \mathbf{E}\Big\{q_\varepsilon(\vec{\xi}_k^\varepsilon(t))\exp\Big(-\int_0^t f(\vec{\xi}_k^\varepsilon(s))\,ds\Big)\Big\}.$$

In the proof of Theorem 7.1 it was established that q_ε satisfies equation (7.1). In our case q_ε is independent of t, hence for $(x,\vec{y}) \in \mathbf{R}^{k+1}$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}q_\varepsilon(x,\vec{y}) + \sum_{l=1}^k g_l^\varepsilon(x)\frac{\partial}{\partial y_l}q_\varepsilon(x,\vec{y}) - f(x,\vec{y})q_\varepsilon(x,\vec{y}) = 0. \tag{7.22}$$

We now prove that for any $\vec{y} \in \mathbf{R}^k$

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbf{R}}|q_\varepsilon(x,\vec{y}) - q(x,\vec{y})| = 0. \tag{7.23}$$

By (7.12), for any $l = 1,\dots,k$

$$\sup_{x\in\mathbf{R}}\mathbf{P}\Big(\Big|\int_0^s g_l^\varepsilon(x+W(u))\,du - \ell(s,r_l-x)\Big| > 1\Big) \le Cs^{1/4}\sqrt{\varepsilon}. \tag{7.24}$$

By the scaling property of the Brownian local time and by (3.17) Ch. III, for any l

$$\mathbf{P}(y_l + \ell(t,r_l-x) \le z) = \mathbf{P}(\ell(1,(r_l-x)/\sqrt{t}) \le (z-y_l)/\sqrt{t})$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{(z-y_l)^+/\sqrt{t}} e^{-(|r_l-x|/\sqrt{t}+v)^2/2}\,dv \le \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{(z-y_l)^+/\sqrt{t}} e^{-v^2/2}dv. \tag{7.25}$$

Choosing t sufficiently large, we can make this probability as small as desired. From (7.24) we obtain

$$\mathbf{P}\Big(y_l + \int_0^t g_l^\varepsilon(x+W(u))\,du \le z\Big) \le Ct^{1/4}\sqrt{\varepsilon} + \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{(z+1-y_l)^+/\sqrt{t}} e^{-v^2/2}dv. \tag{7.26}$$

In view of the condition $f(x,\vec{y}) = 0$ for $y_l \ge z$, the estimates (7.25), (7.26), and the monotonicity of the local times $\ell(t,x)$ and the function $\int_0^t g_l^\varepsilon(x+W(u))\,du$ with respect to t, it suffices to prove instead of (7.23) that for any $\vec{y} \in \mathbf{R}^k$ and any $t > 0$

$$\lim_{\varepsilon\downarrow 0} \sup_{x\in\mathbf{R}} \left|\mathbf{E}\left\{ \exp\Big(- \int_0^t f(\vec{\xi}_k^\varepsilon(s))\,ds\Big) - \exp\Big(- \int_0^t f(\vec{\xi}_k(s))\,ds\Big)\right\}\right| = 0.$$

This relation is easily established with the help of (7.12) and the boundedness of the derivatives of f. Since (7.22) and (7.10) are identical equations, the arguments establishing Theorem 7.3 for the class of functions f under consideration are analogous to the corresponding arguments in the proof of Theorem 7.2. □

Example 7.1. Let $W(0) = x$. We will prove that the *inverse local time process* $(\ell^{(-1)}(u) := \min\{s : \ell(s,0) > u\},\ u \ge 0)$ is the *stable subordinator with exponent* $1/2$, i.e., it is the right continuous process with positive independent increments, having the α-stable distribution with $\alpha = 1/2$. We also verify that $\ell^{(-1)}(0)$ has a nondegenerate initial distribution for $x \ne 0$. In the notation of the fourth section, $\ell^{(-1)}(u) = \varrho(u+0,0)$.

We apply Theorem 7.3 for the case $k = 1$, $r_1 = 0$,

$$f(x,y) = \sum_{j=0}^m \gamma_j \mathbb{I}_{(u_{j-1},u_j]}(y),$$

where $-\infty = u_{-1} < 0 = u_0 < u_{j-1} < u_j$ and $\gamma_j > 0$, $j = 0,1,\dots,m$, are arbitrary numbers. According to the definition,

$$q(x,y) = \mathbf{E}_x \exp\Big(- \int_0^\infty \sum_{j=0}^m \gamma_j \mathbb{I}_{(u_{j-1},u_j]}(y + \ell(s,0))\,ds\Big), \qquad (x,y) \in \mathbf{R}^2,$$

where the subscript in the expectation indicates that it is computed for the Brownian motion W with the initial value $W(0) = x$.

By the equality $\mathbb{I}_{(u_{j-1},u_j]}(y) = \mathbb{I}_{(-\infty,u_j]}(y) - \mathbb{I}_{(-\infty,u_{j-1}]}(y)$ and the monotonicity of $\ell(s,0)$, $s \ge 0$, we have

$$\mathbf{E}_x \exp\Big(- \gamma_0 \ell^{(-1)}(u_0) - \sum_{j=1}^m \gamma_j \big(\ell^{(-1)}(u_j) - \ell^{(-1)}(u_{j-1})\big)\Big) = q(x,0). \tag{7.27}$$

We solve equation (7.20), $x \ne 0$, taking into account the boundedness of the function $q(x,y)$ and its continuity with respect to x. As a result, we find that

$$q(x,y) = c(y) \exp\Big(- |x| \Big(2 \sum_{j=0}^m \gamma_j \mathbb{I}_{(u_{j-1},u_j]}(y)\Big)^{1/2}\Big),$$

where $c(y)$, $y \in \mathbf{R}$, is some function. It follows from (7.21) that $c(y)$, $y \in \mathbf{R}$, is continuous and for $y \in (u_{k-1}, u_k)$

$$\sqrt{2\gamma_k}\, c(y) = c'(y).$$

Therefore, $c(y) = c_k e^{y\sqrt{2\gamma_k}}$ for $y \in (u_{k-1}, u_k)$, and by the definition of $q(x,y)$, we have that $q(x,y) = 1$ for $y > u_m$, or $c(u_m) = 1$. From the continuity of $c(y)$ we determine the coefficients c_k uniquely. Thus $c_m = e^{-u_m\sqrt{2\gamma_m}}$. By the continuity of $c(y)$ in u_{k-1}, we have

$$c_{k-1}\, e^{u_{k-1}\sqrt{2\gamma_{k-1}}} = c_k\, e^{u_{k-1}\sqrt{2\gamma_k}} \qquad \text{for all} \quad 1 \le k \le m.$$

This implies that

$$c_{k-1} = \exp\Bigl(-\sum_{j=k}^{m} (u_j - u_{j-1})\sqrt{2\gamma_j} - u_{k-1}\sqrt{2\gamma_{k-1}} \Bigr).$$

Since $u_0 = 0$, we finally have

$$q(x,0) = c_0 \exp\bigl(-|x|\sqrt{2\gamma_0}\bigr) = \exp\Bigl(-\sum_{j=1}^{m} (u_j - u_{j-1})\sqrt{2\gamma_j} - |x|\sqrt{2\gamma_0} \Bigr).$$

Thus for the Laplace transform (7.27) of the finite-dimensional distributions of the process inverse of the local time we have the following formula:

$$\mathbf{E}_x \exp\Bigl(-\gamma_0 \ell^{(-1)}(0) - \sum_{j=1}^{m} \gamma_j \bigl(\ell^{(-1)}(u_j) - \ell^{(-1)}(u_{j-1})\bigr)\Bigr)$$
$$= \exp\Bigl(-|x|\sqrt{2\gamma_0} - \sum_{j=1}^{m} (u_j - u_{j-1})\sqrt{2\gamma_j} \Bigr),$$

where $0 = u_0 < u_{j-1} < u_j$, $\gamma_j > 0$, $j = 0, 1, \dots, m$. This proves the required assertion. □

§ 8. Distribution of supremum of Brownian local time increments

In this section we consider the problem of computing the joint distribution of the variables $\ell(\tau, r)$, $\ell(\tau, q)$, and $\sup_{0 \le s \le \tau} (\ell(s,q) - \ell(s,r))$, where r and q are arbitrary points and τ is the exponentially distributed with the parameter $\lambda > 0$ random time independent of the Brownian motion W.

This distribution is uniquely determined by the Laplace transform

$$L(x, \lambda, \mu, \eta) := \mathbf{E}_x \Bigl\{ e^{-\mu\ell(\tau,r) - \eta\ell(\tau,q)}; \sup_{0 \le s \le \tau} (\ell(s,q) - \ell(s,r)) \le h \Bigr\},$$

where $\mu > 0$, $\eta > 0$, and subscript x indicates that $W(0) = x$.

Set $\Delta := q - r > 0$, $\alpha := \dfrac{\sqrt{2\lambda}}{1 - e^{-2\Delta\sqrt{2\lambda}}}$, $\beta := \alpha\, e^{-\Delta\sqrt{2\lambda}}$,

$$\rho := \frac{2\beta}{2\alpha + \mu + \eta + \sqrt{(2\alpha + \mu + \eta)^2 - 4\beta^2}} = \frac{2\alpha + \mu + \eta - \sqrt{(2\alpha + \mu + \eta)^2 - 4\beta^2}}{2\beta}.$$

Theorem 8.1. *For any $x \in \mathbf{R}$ and $h \ge 0$,*

$$L(x,\lambda,\mu,\eta) = 1 - A(x) - B(x)\frac{(\alpha-\beta)(\mu+\alpha+\beta)}{(\alpha+\mu)(\alpha+\eta)-\beta^2} \times \exp\Big(\Big(\mu-\eta-\sqrt{(2\alpha+\mu+\eta)^2-4\beta^2}\Big)\frac{h}{2}\Big), \qquad (8.1)$$

where

$$A(x) = \begin{cases} \dfrac{\mu(\alpha+\eta)+\eta\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2} e^{-(r-x)\sqrt{2\lambda}}, & \text{for} \quad x \le r, \\ \dfrac{(\alpha-\beta)}{\sqrt{2\lambda}((\alpha+\mu)(\alpha+\eta)-\beta^2)} \Big[\mu(\alpha+\beta+\eta)e^{-(x-r)\sqrt{2\lambda}} \\ \qquad +\eta(\alpha+\beta+\mu)e^{-(q-x)\sqrt{2\lambda}}\Big], & \text{for} \quad r \le x \le q, \\ \dfrac{\eta(\alpha+\mu)+\mu\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2} e^{-(x-q)\sqrt{2\lambda}}, & \text{for} \quad q \le x. \end{cases}$$

$$B(x) = \begin{cases} \rho\, e^{-(r-x)\sqrt{2\lambda}}, & \text{for} \quad x \le r, \\ \dfrac{1}{\sqrt{2\lambda}}\big[(\rho\alpha-\beta)e^{-(x-r)\sqrt{2\lambda}} + (\alpha-\rho\beta)e^{-(q-x)\sqrt{2\lambda}}\big], & \text{for} \quad r \le x \le q, \\ e^{-(x-q)\sqrt{2\lambda}}, & \text{for} \quad q \le x. \end{cases}$$

Remark 8.1. For $h = \infty$ we have that

$$L(x,\lambda,\mu,\eta) = \mathbf{E}_x e^{-\mu\ell(\tau,r)-\eta\ell(\tau,q)} = 1 - A(x)$$

and this formula coincides with (3.11) Ch. III.

Corollary 8.1. *In the case $\mu = \eta$ we have $\rho = \dfrac{\beta}{\alpha+\eta+\sqrt{(\alpha+\eta)^2-\beta^2}}$,*

$$\mathbf{E}_x\Big\{e^{-\eta(\ell(\tau,r)+\ell(\tau,q))}; \sup_{0\le s\le\tau}(\ell(s,q)-\ell(s,r)) > h\Big\} = B(x)\frac{\alpha-\beta}{\eta+\alpha-\beta}e^{-h\sqrt{(\alpha+\eta)^2-\beta^2}}. \qquad (8.2)$$

Corollary 8.2. *For $h \ge 0$*

$$\mathbf{P}_x\Big(\sup_{0\le s\le\tau}(\ell(s,q)-\ell(s,r)) > h\Big) = D(x)\exp\bigg(-\frac{h\sqrt{2\lambda}}{\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}}\bigg), \qquad (8.3)$$

where

$$D(x) = \begin{cases} \dfrac{e^{-\Delta\sqrt{2\lambda}}}{1+\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}} e^{-(r-x)\sqrt{2\lambda}}, & \text{for} \quad x \le r, \\ \dfrac{e^{-(q-x)\sqrt{2\lambda}}}{\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}} - \dfrac{e^{-(x-r)\sqrt{2\lambda}}\big(e^{\Delta\sqrt{2\lambda}}-\sqrt{e^{2\Delta\sqrt{2\lambda}}-1}\big)}{\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}}, & \text{for} \quad r \le x \le q, \\ e^{-(x-q)\sqrt{2\lambda}}, & \text{for} \quad q \le x. \end{cases}$$

Remark 8.2. For $q < r$ analogous formulas can easily be derived from the formulas for $q > r$ by using the symmetry property of a Brownian motion.

Using the properties of spatial homogeneity and symmetry of a Brownian motion, it is easy to deduce from (8.3) the following statement for the modulus of the increments of the Brownian local time.

Corollary 8.3. *If* $\Delta\sqrt{2\lambda} \le 1$, *then*

$$\mathbf{P}_x\Big(\sup_{0\le s\le\tau}|\ell(s,q)-\ell(s,r)|>h\Big)\le 4\exp\bigg(-\frac{h\sqrt{2\lambda}}{\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}}-\min(|x-r|,|q-x|)\sqrt{2\lambda}\bigg).$$

Indeed, for $x\in[r,q]$ the function $D(x)$ is increasing, because $D'(x)>0$. Then one has the estimates

$$D(x)\le D(q)=1\le e^{\Delta\sqrt{2\lambda}/2}\exp\big(-\min(|x-r|,|q-x|)\sqrt{2\lambda}\big).$$

Sometimes it is important to have a similar estimate for a fixed time t instead of the random time τ.

Corollary 8.4. *For* $h\Delta/t\le 1$,

$$\mathbf{P}_x\Big(\sup_{0\le s\le t}|\ell(s,q)-\ell(s,r)|>h\Big)$$
$$\le 4\exp\bigg(-\frac{3}{8t^{1/3}}\Big(\frac{h}{\sqrt{\Delta}}\Big)^{4/3}-\min(|x-r|,|q-x|)\frac{1}{2}\Big(\frac{h}{t\sqrt{\Delta}}\Big)^{2/3}\bigg). \qquad (8.4)$$

Indeed, since the probability on the left-hand side of (8.4) is increasing with respect to t, we have

$$\mathbf{P}_x\Big(\sup_{0\le s\le t}|\ell(s,q)-\ell(s,r)|>h\Big)\le \lambda e^{\lambda t}\int_t^\infty e^{-\lambda v}\mathbf{P}_x\Big(\sup_{0\le s\le v}|\ell(s,q)-\ell(s,r)|>h\Big)dv$$
$$\le e^{\lambda t}\mathbf{P}_x\Big(\sup_{0\le s\le\tau}|\ell(s,q)-\ell(s,r)|>h\Big)$$
$$\le 4\exp\bigg(\lambda t-\frac{h\sqrt{2\lambda}}{\sqrt{1-e^{-2\Delta\sqrt{2\lambda}}}}-\min(|x-r|,|q-x|)\sqrt{2\lambda}\bigg),$$

for any $\lambda>0$ and $t>0$. For $\Delta\sqrt{2\lambda}\le 1/2$ the estimate $1-e^{-2\Delta\sqrt{2\lambda}}\le 2\Delta\sqrt{2\lambda}$ holds, therefore,

$$\mathbf{P}_x\Big(\sup_{0\le s\le t}|\ell(s,q)-\ell(s,r)|>h\Big)\le 4\exp\bigg(\lambda t-\frac{h(2\lambda)^{1/4}}{\sqrt{2\Delta}}-\min(|x-r|,|q-x|)\sqrt{2\lambda}\bigg).$$

If we choose the optimal value $\sqrt{2\lambda}=\frac{1}{2}\Big(\frac{h}{t\sqrt{\Delta}}\Big)^{2/3}$, then for $\frac{h\Delta}{t}\le 1$ we get (8.4).

Proof of Theorem 8.1. Let $\theta(y,z):=\mathbb{1}_{(-\infty,0)}(y-z)$ and

$$F(x,y,z):=\mathbf{E}_0\bigg\{\exp(-\mu(y+\ell(\tau,r-x))-\eta(z+\ell(\tau,q-x)))$$
$$\times\exp\bigg(-\gamma\int_0^\tau\theta(y+\ell(s,r-x),z+\ell(s,q-x))\,ds\bigg)\bigg\}.$$

Then

$$\mathbf{E}_0\Big\{e^{-\mu\ell(\tau,r-x)-\eta\ell(\tau,q-x)};\ \sup_{0\le s\le\tau}(\ell(s,q-x)-\ell(s,r-x))\le y-z\Big\}$$
$$=\lim_{\gamma\to\infty}e^{\mu y+\eta z}F(x,y,z). \tag{8.5}$$

This equality holds in view of the following circumstances. If for some value s from $(0,\tau)$ the variable $\ell(s,q-x)-\ell(s,r-x)$ is greater than $y-z$, then the variable $y+\ell(s,r-x)-z-\ell(s,q-x)$ is less than zero a.s. for all s in some random interval. Therefore, the integral of the function $\theta(y+\ell(s,r-x),z+\ell(s,y-x))$ from 0 to τ is positive and the exponential function of $-\gamma$, multiplied by this integral, tends to zero as $\gamma\to\infty$.

To compute the function $F(x,y,z)$ we use Theorem 7.2. The unboundedness of $\Phi(y,z)=e^{-\mu y-\eta z}$, $(y,z)\in\mathbf{R}^2$, is not essential in this case, because the function is multiplicative in each argument.

We first remark that the distribution function $\mathcal{F}(u)$ of the variable $\ell(\tau,b)-\ell(\tau,a)$ has a density at $u\ne 0$, also at all u when $W(0)=0$ and either $b=0$ or $a=0$. This is not hard to establish with the help of the formulas (4.63)–(4.65) of Ch. III. Therefore, for all x the function $F(x,y,z)$ is continuous with respect to (y,z) for $y\ne z$, and the functions $F(q,y,z)$, $F(r,y,z)$ are continuous with respect to $(y,z)\in\mathbf{R}^2$. This is the consequence of the estimate

$$\mathbf{E}_0\Big|\exp\Big(-\int_0^\tau \mathbb{1}_{(-\infty,z_2]}(\ell(s,b)-\ell(s,a))\,ds\Big)-\exp\Big(-\int_0^\tau \mathbb{1}_{(-\infty,z_1]}(\ell(s,b)-\ell(s,a))\,ds\Big)\Big|$$

$$\le\mathbf{E}_0\int_0^\tau \mathbb{1}_{(z_1,z_2]}(\ell(s,b)-\ell(s,a))\,ds=\frac{1}{\lambda}\mathbf{P}_0(\ell(\tau,b)-\ell(\tau,a)\in(z_1,z_2]),\quad z_1\le z_2.$$

It follows from (7.7) that for $c>q$

$$\frac{1}{2}\Big(F(c,y,z)-F(q,y,z)-(c-q)\frac{\partial}{\partial x}F(q+0,y,z)\Big)$$
$$-(\lambda+\gamma\theta(y,z))\int_q^c\int_q^b F(x,y,z)\,dxdb=-\frac{\lambda(c-q)^2}{2}\,e^{-\mu y-\eta z}.$$

In view of the stated above, this relation implies that $\dfrac{\partial}{\partial x}F(q+0,y,z)$ is continuous with respect to (y,z) for $y\ne z$. It can be seen similarly that $\dfrac{\partial}{\partial x}F(q-0,y,z)$ and $\dfrac{\partial}{\partial x}F(r\pm 0,y,z)$ are also continuous with respect to (y,z) for $y\ne z$.

Thus in this case the problem (7.7), (7.8) is transformed to the following one:

1) for $x\ne r,q$,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}F(x,y,z)-(\lambda+\gamma\theta(y,z))F(x,y,z)=-\lambda e^{-\mu y-\eta z}; \tag{8.6}$$

2) for $y \neq z$,

$$\frac{1}{2}\Big(\frac{\partial}{\partial x}F(r-0,y,z) - \frac{\partial}{\partial x}F(r+0,y,z)\Big) = \frac{\partial}{\partial y}F(r,y,z), \tag{8.7}$$

$$\frac{1}{2}\Big(\frac{\partial}{\partial x}F(q-0,y,z) - \frac{\partial}{\partial x}F(q+0,y,z)\Big) = \frac{\partial}{\partial z}F(q,y,z). \tag{8.8}$$

We set

$$\alpha(y,z) := \frac{\sqrt{2(\lambda+\gamma\theta(y,z))}}{1-e^{-2\Delta\sqrt{2(\lambda+\gamma\theta(y,z))}}}, \qquad \beta(y,z) := \alpha(y,z)e^{-\Delta\sqrt{2(\lambda+\gamma\theta(y,z))}}.$$

To keep the notation simple, we sometimes omit the arguments (y,z) in notations of the functions $\theta(y,z)$, $\alpha(y,z)$ and $\beta(y,z)$.

Solving (8.6) with the restriction that F is bounded with respect to x, we get

$$F(x,y,z) = \begin{cases} G(y,z)e^{-(r-x)\sqrt{2(\lambda+\gamma\theta)}} + \dfrac{\lambda e^{-\mu y-\eta z}}{\lambda+\gamma\theta}, & \text{for} \quad x \le r, \\ C(y,z)e^{-(x-r)\sqrt{2(\lambda+\gamma\theta)}} & \\ \qquad +D(y,z)e^{-(q-x)\sqrt{2(\lambda+\gamma\theta)}} + \dfrac{\lambda e^{-\mu y-\eta z}}{\lambda+\gamma\theta}, & \text{for} \quad r \le x \le q, \\ H(y,z)e^{-(x-q)\sqrt{2(\lambda+\gamma\theta)}} + \dfrac{\lambda e^{-\mu y-\eta z}}{\lambda+\gamma\theta}, & \text{for} \quad x \ge q, \end{cases}$$

where G, C, D, and H are certain functions. It follows from the continuity of F with respect to x that

$$C(y,z) = \frac{1}{\sqrt{2(\lambda+\gamma\theta(y,z))}}(G(y,z)\alpha(y,z) - H(y,z)\beta(y,z)), \tag{8.9}$$

$$D(y,z) = \frac{1}{\sqrt{2(\lambda+\gamma\theta(y,z))}}(H(y,z)\alpha(y,z) - G(y,z)\beta(y,z)). \tag{8.10}$$

Then we deduce from (8.7), (8.8) that in the domains $R_1 = \{y \ge z\}$, $R_2 = \{y < z\}$, where $\alpha(y,z)$ and $\beta(y,z)$ take constant values, G and H satisfy the system of equations

$$\frac{\partial}{\partial y}G(y,z) = \alpha G(y,z) - \beta H(y,z) + \frac{\lambda\mu e^{-\mu y-\eta z}}{\lambda+\gamma \mathbb{1}_{(-\infty,0)}(y-z)}, \tag{8.11}$$

$$\frac{\partial}{\partial z}H(y,z) = \alpha H(y,z) - \beta G(y,z) + \frac{\lambda\eta e^{-\mu y-\eta z}}{\lambda+\gamma \mathbb{1}_{(-\infty,0)}(y-z)}. \tag{8.12}$$

By definition, the function $e^{\mu y+\eta z}F(x,y,z)$ is bounded by 1 and, as a function of (y,z), it depends only on the difference $y-z$. Set $Q(y-z) := e^{\mu y+\eta z}G(y,z)$ and $R(y-z) := e^{\mu y+\eta z}H(y,z)$. The functions $Q(h)$ and $R(h)$ are bounded, and in view of (8.11) and (8.12), they satisfy the equations

$$Q'(h) = (\alpha+\mu)Q(h) - \beta R(h) + \frac{\lambda\mu}{\lambda+\gamma \mathbb{1}_{(-\infty,0)}(h)}, \tag{8.13}$$

$$R'(h) = \beta Q(h) - (\alpha+\eta)R(h) - \frac{\lambda\eta}{\lambda + \gamma 1\!\!1_{(-\infty,0)}(h)}. \tag{8.14}$$

It is possible to solve this system with constant coefficients in various ways. One of them is the following. We first express from equations (8.13) and (8.14) the function R. Doing so we see that the function $R(h)$ for $h \neq 0$ is a bounded solution of the equation

$$R''(h) + (\eta-\mu)R'(h) - ((\alpha+\mu)(\alpha+\eta) - \beta^2)R(h) = \frac{\lambda(\eta(\alpha+\mu)+\mu\beta)}{\lambda + \gamma 1\!\!1_{(-\infty,0)}(h)}.$$

The particular solution of this equation is

$$\varkappa_\gamma(h) := -\frac{\lambda(\eta(\alpha+\mu)+\mu\beta)}{(\lambda + \gamma 1\!\!1_{(-\infty,0)}(h))((\alpha+\mu)(\alpha+\eta) - \beta^2)}.$$

The jump of this solution at zero is equal to

$$\delta_\gamma := \varkappa_\gamma(+0) - \varkappa_\gamma(-0) = -\frac{\gamma(\eta(\alpha+\mu)+\mu\beta)}{(\lambda+\gamma)((\alpha+\mu)(\alpha+\eta) - \beta^2)}.$$

Note that the functions $F(q,y,z)$ and $F(r,y,z)$ are continuous in $(y,z) \in \mathbf{R}^2$, therefore the functions

$$R(h) + \frac{\lambda}{\lambda + \gamma 1\!\!1_{(-\infty,0)}(h)} \quad \text{and} \quad Q(h) + \frac{\lambda}{\lambda + \gamma 1\!\!1_{(-\infty,0)}(h)}$$

are continuous in $h \in \mathbf{R}$. Then, in view of continuity of R, the bounded solution R can be represented as

$$R(h) = \varkappa_\gamma(h) - \Big(k_\gamma + (\delta_\gamma + 1) 1\!\!1_{[0,\infty)}(h) + \frac{\lambda}{\lambda+\gamma} 1\!\!1_{(-\infty,0)}(h)\Big)$$
$$\times \exp\Big(\big(\mu - \eta - \sqrt{(2\alpha+\mu+\eta)^2 - 4\beta^2}\,\big)\frac{|h|}{2}\Big). \tag{8.15}$$

where k_γ depend on γ and is independent of h.

According to this formula, $R(0) = \varkappa_\gamma(0) - k_\gamma - \delta_\gamma - 1$. Note also that, if $z = y$ and $x = q$, then from (8.5) and from the representation for the function $F(x,y,z)$, it follows that

$$\lim_{\gamma\to\infty} e^{(\mu+\eta)y} F(q,y,y) = \lim_{\gamma\to\infty} (R(0)+1)$$
$$= \mathbf{E}_0\Big\{\exp(-\mu\ell(\tau, r-q) - \eta\ell(\tau,0));\ \sup_{0\le s\le\tau} (\ell(s,0) - \ell(s,r-q)) = 0\Big\} = 0.$$

Since $W(0) = 0$, the right-hand side equality is valid, because $\ell(s,r-q) = 0$ for s in some neighborhood of zero, while $\ell(s,0) > 0$.

Therefore, we have $k_\gamma \to 0$, because $\varkappa_\gamma(0) - \delta_\gamma \to 0$.

Now from (8.14) we express the function $Q(h)$ for $h \ge 0$. It can be checked that

$$Q(h) = -\frac{\mu(\alpha+\eta)+\eta\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2} - (k_\gamma + \delta_\gamma + 1)\rho \exp\Big(\big(\mu - \eta - \sqrt{(2\alpha+\mu+\eta)^2 - 4\beta^2}\,\big)\frac{h}{2}\Big). \tag{8.16}$$

Here we have used the constant ρ introduced at the beginning of the section.

Thus for $h \geq 0$

$$\lim_{\gamma\to\infty} R(h) = -\frac{\eta(\alpha+\mu)+\mu\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2}$$

$$-\frac{(\alpha-\beta)(\alpha+\beta+\mu)}{(\alpha+\mu)(\alpha+\eta)-\beta^2}\exp\Bigl(\bigl(\mu-\eta-\sqrt{(2\alpha+\mu+\eta)^2-4\beta^2}\bigr)\frac{h}{2}\Bigr), \tag{8.17}$$

$$\lim_{\gamma\to\infty} Q(h) = -\frac{\mu(\alpha+\eta)+\eta\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2}$$

$$-\frac{(\alpha-\beta)(\alpha+\beta+\mu)\rho}{(\alpha+\mu)(\alpha+\eta)-\beta^2}\exp\Bigl(\bigl(\mu-\eta-\sqrt{(2\alpha+\mu+\eta)^2-4\beta^2}\bigr)\frac{h}{2}\Bigr). \tag{8.18}$$

Now it is easy to complete the proof of the theorem. For $x > q$ using the representation for the function $F(x,y,z)$, we get by (8.5) and (8.17) that

$$\mathbf{E}_0\Bigl\{e^{-\mu\ell(\tau,r-x)-\eta\ell(\tau,q-x)};\ \sup_{0\le s\le\tau}(\ell(s,q-x)-\ell(s,r-x))\le h\Bigr\}$$

$$= 1-\frac{\eta(\alpha+\eta)+\mu\beta}{(\alpha+\mu)(\alpha+\eta)-\beta^2}e^{-(x-q)\sqrt{2\lambda}}-\frac{(\alpha-\beta)(\alpha+\beta+\mu)}{(\alpha+\mu)(\alpha+\eta)-\beta^2}$$

$$\times\exp\Bigl(\bigl(\mu-\eta-\sqrt{(2\alpha+\mu+\eta)^2-4\beta^2}\bigr)\frac{h}{2}\Bigr)e^{-(x-q)\sqrt{2\lambda}}, \qquad h\ge 0.$$

Similarly, using the expression for $\lim\limits_{\gamma\to\infty} Q(h)$, we obtain (8.1) for $x \le r$. To prove the formula in the case $r < x < q$ one uses (8.9) and (8.10). □

§ 9. The law of the iterated logarithm for Brownian local time

We consider the asymptotic behavior of the Brownian local time $\ell(t,x)$ for large values of t. The statement below can be formulated non-rigorously as follows: the main component of the rate of growth as $t\to\infty$ of the extremal values of almost all sample paths of the Brownian local time is the nonrandom function $\sqrt{2t\ln\ln t}$, $t > e$. Assertions of this type can be grouped under the general heading the *law of the iterated logarithm*. Assume for definiteness that $W(0) = 0$, although, as it will be seen from the proof, the initial value of the Brownian motion does not matter.

Theorem 9.1. *The following relations hold*:

$$\limsup_{t\to\infty}\frac{\ell(t,0)}{\sqrt{2t\ln\ln t}} = \limsup_{t\to\infty}\frac{\sup_{x\in\mathbf{R}}\ell(t,x)}{\sqrt{2t\ln\ln t}} = 1 \qquad \text{a.s.} \tag{9.1}$$

Proof. Let t_k be an arbitrary increasing sequence of numbers. Since the local time is monotone in t,

$$\limsup_{t\to\infty}\frac{\sup_{x\in\mathbf{R}}\ell(t,x)}{\sqrt{2t\ln\ln t}} \ge \limsup_{t\to\infty}\frac{\ell(t,0)}{\sqrt{2t\ln\ln t}} \ge \limsup_{k\to\infty}\frac{\ell(t_{k+1},0)-\ell(t_k,0)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}}. \tag{9.2}$$

Our first step is to choose for any $0 < \varepsilon < 1$ a sequence of numbers t_k such that the limit on the right-hand side of (9.2) is greater or equal to $1-\varepsilon$ a.s. Let $t_k := \theta^k$, where $\theta > 1$ is such that $\dfrac{(1-\varepsilon)^2\theta}{\theta-1} = 1$. Using the Markov property of the process W and (3.17) Ch. III, we get that

$$\mathbf{P}\Big(\frac{\ell(t_{k+1},0)-\ell(t_k,0)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}} > 1-\varepsilon\Big|W(t), t\le t_k, W(t_k)=x\Big) = \mathbf{P}\Big(\frac{\ell(t_{k+1}-t_k,-x)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}}\ge 1-\varepsilon\Big)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi(t_{k+1}-t_k)}}\int\limits_{(1-\varepsilon)\sqrt{2t_{k+1}\ln\ln t_{k+1}}}^{\infty} \exp\Big(-\frac{(|x|+y)^2}{2(t_{k+1}-t_k)}\Big)\,dy.$$

Since (see (10.8) Ch. I)

$$\frac{\sqrt{2}}{\sqrt{\pi t}}\int\limits_h^\infty e^{-v^2/2t}\,dv \sim \frac{\sqrt{2t}}{h\sqrt{\pi}}\,e^{-h^2/2t} \qquad \text{as } \frac{h}{\sqrt{2t}}\to\infty,$$

we conclude that for $k\to\infty$

$$\mathbf{P}\Big(\frac{\ell(t_k+1,0)-\ell(t_k,0)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}} > 1-\varepsilon\Big|W(t), t\le t_k, W(t_k)=x\Big)$$

$$\sim \frac{\theta^{k/2}\sqrt{2(\theta-1)}}{\sqrt{\pi}\big(|x|+(1-\varepsilon)\theta^{k/2}\sqrt{2\theta\ln\ln\theta^{k+1}}\big)}\exp\Big(-\frac{\big(|x|+(1-\varepsilon)\theta^{k/2}\sqrt{2\theta\ln\ln\theta^{k+1}}\big)^2}{2\theta^k(\theta-1)}\Big)$$

$$\sim \frac{\sqrt{\theta-1}}{\sqrt{\pi\theta\ln(k+1)}(1-\varepsilon)}\exp\Big(-\frac{(1-\varepsilon)^2\theta\ln(k+1)}{\theta-1}\Big) \sim \frac{\sqrt{\theta-1}}{\sqrt{\pi\theta}(1-\varepsilon)(k+1)\sqrt{\ln(k+1)}}.$$

Thus the series from these conditional probabilities diverges for any x. Therefore, taking x to be the value $W(t_k)$, we get

$$\sum_{k=1}^\infty \mathbf{P}\Big(\frac{\ell(t_{k+1},0)-\ell(t_k,0)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}} > 1-\varepsilon\Big|W(t), t\le t_k\Big) = \infty \qquad \text{a.s.}$$

By the Borel–Cantelli–Lévy lemma (see Remark 5.5 Ch. I) this implies

$$\limsup_{k\to\infty}\frac{\ell(t_{k+1},0)-\ell(t_k,0)}{\sqrt{2t_{k+1}\ln\ln t_{k+1}}} \ge 1-\varepsilon \qquad \text{a.s.}$$

Since ε is arbitrary, (9.2) yields that

$$\limsup_{t\to\infty}\frac{\sup_{x\in\mathbf{R}}\ell(t,x)}{\sqrt{2t\ln\ln t}} \ge \limsup_{t\to\infty}\frac{\ell(t,0)}{\sqrt{2t\ln\ln t}} \ge 1.$$

It now remains to prove that for any $\varepsilon > 0$

$$\limsup_{t\to\infty}\frac{\sup_{x\in\mathbf{R}}\ell(t,x)}{\sqrt{2t\ln\ln t}} \le 1+\varepsilon. \tag{9.3}$$

We again choose $t_k := \theta^k$, where $\theta > 1$. Using the monotonicity of $\ell(t,x)$ with respect to t, we get that

$$\limsup_{t\to\infty} \frac{\sup_{x\in\mathbf{R}} \ell(t,x)}{\sqrt{2t\ln\ln t}} \le \limsup_{k\to\infty} \frac{\sup_{x\in\mathbf{R}} \ell(t_k,x)}{\sqrt{2t_{k-1}\ln\ln t_{k-1}}} = \sqrt{\theta}\limsup_{k\to\infty} \frac{\sup_{x\in\mathbf{R}} \ell(t_k,x)}{\sqrt{2t_k\ln\ln t_k}}.$$

By the estimate (5.27),

$$\mathbf{P}\Big(\sqrt{\theta}\,\frac{\sup_{x\in\mathbf{R}} \ell(t_k,x)}{\sqrt{2t_k\ln\ln t_k}} \ge 1+\varepsilon\Big) \le L\frac{2(1+\varepsilon)^2\ln\ln t_k}{\theta}\exp\Big(-\frac{(1+\varepsilon)^2}{\theta}\ln\ln t_k\Big)$$

$$= L\frac{2(1+\varepsilon)^2}{\theta}\ln(k\ln\theta)\exp\Big(-\frac{(1+\varepsilon)^2}{\theta}(\ln k + \ln\ln\theta)\Big) \asymp k^{-(1+\varepsilon)^2/\theta}\ln k.$$

Choosing θ such that $(1+\varepsilon)^2/\theta > 1$, we get that this series of probabilities converges, and hence, by the first part of the Borel–Cantelli lemma (see Remark 1.1 Ch. I)

$$\sqrt{\theta}\limsup_{k\to\infty} \frac{\sup_{x\in\mathbf{R}} \ell(t_k,x)}{2t_k\ln\ln t_k} \le 1+\varepsilon \qquad \text{a.s.}$$

This proves (9.3). Since ε can be chosen arbitrarily close to zero, Theorem 9.1 is proved. □

§ 10. The exact modulus of continuity of the Brownian local time $\ell(t,x)$ with respect to t

The local time is a two-parameter process. For two-parameter processes there are various statements of problems involving moduli of continuity, contrary to the case of one-parameter processes. We are interested in the moduli of continuity separately with respect to each of the parameters. For various applications associated with two-parameter processes it is important to have estimates for moduli of continuity with respect to each of the parameters that are uniform with respect to the other parameter. The next theorem asserts that the function $h(t) := \sqrt{2t\ln(1/t)}$, $0 < t < e^{-1}$, is the *exact modulus of continuity of the Brownian local time* $\ell(t,x)$ with respect to t uniform in x.

Theorem 10.1. *For any* $T > 0$

$$\limsup_{\Delta\downarrow 0} \frac{1}{\sqrt{2\Delta\ln(1/\Delta)}} \sup_{(t,x)\in[0,T]\times\mathbf{R}} (\ell(t+\Delta,x) - \ell(t,x)) = 1 \qquad \text{a.s.} \tag{10.1}$$

Proof. It can be assumed without loss of generality that $T = 1$ and $W(0) = 0$. To establish (10.1) it suffices to prove that for any $0 < \varepsilon < 1$

$$\limsup_{\Delta\downarrow 0} \frac{1}{h(\Delta)} \sup_{(t,x)\in[0,1]\times\mathbf{R}} (\ell(t+\Delta,x) - \ell(t,x)) \ge 1-\varepsilon \qquad \text{a.s.,} \tag{10.2}$$

$$\limsup_{\Delta\downarrow 0} \frac{1}{h(\Delta)} \sup_{(t,x)\in[0,1]\times\mathbf{R}} (\ell(t+\Delta,x) - \ell(t,x)) \le 1+3\varepsilon+2\varepsilon^2 \qquad \text{a.s.} \tag{10.3}$$

We first prove (10.2). By (3.17) Ch. III and (10.7) Ch. I, we have the obvious estimate

$$\mathbf{P}(\ell(2^{-n},0) \ge (1-\varepsilon)h(2^{-n})) = \frac{2}{\sqrt{\pi}} \int\limits_{(1-\varepsilon)\sqrt{\ln 2^n}}^{\infty} e^{-v^2}\,dv$$

$$\ge \frac{C}{(1-\varepsilon)\sqrt{n\ln 2}} \exp(-(1-\varepsilon)^2 n \ln 2) \ge 2^{(\varepsilon-1)n}, \tag{10.4}$$

which is true for some $C > 0$, and all sufficiently large n. Using the Markov property of the Brownian motion W, we get that

$$\mathbf{P}\Big(\max_{1\le k\le 2^n} \sup_{x\in\mathbf{R}} (\ell(k2^{-n},x) - \ell((k-1)2^{-n},x)) \le (1-\varepsilon)h(2^{-n})\Big)$$

$$= \prod_{k=1}^{2^n} \mathbf{P}\Big(\sup_{x\in\mathbf{R}} (\ell(k2^{-n},x) - \ell((k-1)2^{-n},x)) \le (1-\varepsilon)h(2^{-n})\Big)$$

$$= \Big(1 - \mathbf{P}\Big(\sup_{x\in\mathbf{R}} \ell(2^{-n},x) > (1-\varepsilon)h(2^{-n})\Big)\Big)^{2^n}$$

$$\le \big(1 - \mathbf{P}(\ell(2^{-n},0) > (1-\varepsilon)h(2^{-n}))\big)^{2^n} \le (1-2^{(\varepsilon-1)n})^{2^n} \le e^{-2^{\varepsilon n}}.$$

Since the series of variables $e^{-2^{\varepsilon n}}$ converges, by the first part of Borel–Cantelli lemma, we obtain that

$$\limsup_{n\to\infty} \frac{1}{h(2^{-n})} \max_{1\le k\le 2^n} \sup_{x\in\mathbf{R}} (\ell(k2^{-n},x) - \ell((k-1)2^{-n},x)) > 1-\varepsilon \qquad \text{a.s.}$$

This obviously implies (10.2).

We pass to the proof of (10.3). Choose δ so that $0 < \delta < 1$ and $\dfrac{1+\delta}{1-\delta} < (1+\varepsilon)^2$. For brevity set $\|\cdot\| := \sup_{x\in\mathbf{R}} |\cdot|$. By (5.27),

$$\mathbf{P}\Big(\max_{\substack{0\le k\le 2^{n\delta}\\ 0\le i\le 2^n}} \|\ell((i+k)2^{-n},x) - \ell(i2^{-n},x)\| > (1+\varepsilon)h(k2^{-n})\Big)$$

$$\le \sum_{\substack{0\le k\le 2^{n\delta}\\ 0\le i\le 2^n}} \mathbf{P}\big(\|\ell((i+k)2^{-n},x) - \ell(i2^{-n},x)\| > (1+\varepsilon)h(k2^{-n})\big)$$

$$\le L2^n \sum_{k=1}^{2^{n\delta}} \frac{(1+\varepsilon)^2 h^2(k2^{-n})}{k2^{-n}} \exp\Big(-\frac{(1+\varepsilon)^2 h^2(k2^{-n})}{2k2^{-n}}\Big)$$

$$= L2^{n+1} \sum_{k=1}^{2^{n\delta}} (1+\varepsilon)^2 \ln\Big(\frac{2^n}{k}\Big) 2^{-n(1+\varepsilon)^2} k^{(1+\varepsilon)^2}$$

$$\le L_1 n 2^{n(1-(1+\varepsilon)^2+\delta((1+\varepsilon)^2+1))} = L_1 n 2^{-n(1-\delta)((1+\varepsilon)^2-(1+\delta)/(1-\delta))}.$$

We have estimated the probabilities by quantities forming a convergent series. Consequently, by the first part of the Borel–Cantelli lemma, there exists a.s. a number $m = m(\omega)$ such that

$$\|\ell((i+k)2^{-n}, x) - \ell(i2^{-n}, x)\| \leq (1+\varepsilon)h(k2^{-n}). \tag{10.5}$$

for all $n \geq m$, $1 \leq k \leq 2^{n\delta}$, $0 \leq i \leq 2^n$.

Set $\Delta := t - s$. Since $2^{-l(1-\delta)}$ tends monotonically to zero as $l \to \infty$, no matter how small Δ is, it will always be between two successive terms of this sequence. Let $2^{-(n+1)(1-\delta)} \leq \Delta < 2^{-n(1-\delta)}$ and since we are interested only in arbitrarily small values of Δ, we can assume that $n \geq m$. We represent s and t in the form

$$s = i2^{-n} - \sum_{v=1}^{\infty} 2^{-p_v}, \qquad t = (i+k)2^{-n} + \sum_{v=1}^{\infty} 2^{-q_v}, \tag{10.6}$$

where $n < p_1 < p_2 < \cdots$ and $n < q_1 < q_2 < \cdots$. Let $s_0 := i2^{-n}$, $s_l := i2^{-n} - \sum_{v=1}^{l} 2^{-p_v}$ and $t_0 := (i+k)2^{-n}$, $t_l := (i+k)2^{-n} + \sum_{v=1}^{l} 2^{-q_v}$, $l = 1, 2, \ldots$.

The process $\ell(t,x)$ is continuous a.s. in t for all x, therefore,

$$\ell(s,x) = \ell(s_0,x) + \sum_{l=1}^{\infty}(\ell(s_l,x) - \ell(s_{l-1},x)),$$

$$\ell(t,x) = \ell(t_0,x) + \sum_{l=1}^{\infty}(\ell(t_l,x) - \ell(t_{l-1},x)).$$

By the triangle inequality,

$$\|\ell(t,x) - \ell(s,x)\| \leq \|\ell(t_0,x) - \ell(s_0,x)\| + \sum_{l=1}^{\infty} \|(\ell(s_l,x) - \ell(s_{l-1},x)\|$$
$$+ \sum_{l=1}^{\infty} \|\ell(t_l,x) - \ell(t_{l-1},x)\| \leq (1+\varepsilon)h(k2^{-n})$$
$$+ (1+\varepsilon)\sum_{p>n} h(2^{-p}) + (1+\varepsilon)\sum_{q>n} h(2^{-q}).$$

To estimate the differences $\ell(t_0,x) - \ell(s_0,x)$, $\ell(s_l,x) - \ell(s_{l-1},x)$, and $\ell(t_l,x) - \ell(t_{l-1},x)$ for $l \geq 1$, we used (10.5). This can be done, because the points s_l, t_l satisfy conditions under which (10.5) holds. Since $(n+1)(1-\delta) < n$ for sufficiently large n and the function $h(t)$, $0 < t < e^{-1}$, is strictly increasing, we conclude that for some constant $C > 0$

$$\sum_{p>n} h(2^{-p}) \leq Ch(2^{-n}) \leq \varepsilon h(2^{-(n+1)(1-\delta)}) \leq \varepsilon h(\Delta).$$

Finally, we have that

$$\|\ell(t,x) - \ell(s,x)\| \leq (1 + 3\varepsilon + 2\varepsilon^2)h(\Delta).$$

This implies (10.3). The theorem is proved. □

§ 11. The exact modulus of continuity of the Brownian local time $\ell(t,x)$ with respect to x

The formula for the exact modulus of continuity of the Brownian local time $\ell(t,x)$ with respect to the variable t was obtained in § 10. The main result of this section gives a formula for the *exact modulus of continuity of the Brownian local time* $\ell(s,x)$ with respect to x uniform in $s \in [0,t]$. The fundamental difference between the modulus of continuity with respect to x and that with respect to t is that the former is a random variable. However, these moduli are the same with regard to order.

Theorem 11.1. *For any* $t > 0$

$$\limsup_{\Delta\downarrow 0} \sup_{x\in\mathbf{R}} \frac{\sup_{0\le s\le t} |\ell(s,x+\Delta)-\ell(s,x)|}{2\sqrt{(\ell(t,x)+\ell(t,x+\Delta))\Delta\ln 1/\Delta}} = 1 \qquad \text{a.s.} \tag{11.1}$$

Remark 11.1. In (11.1) under the root sign there is the sum $\ell(t,x)+\ell(t,x+\Delta)$ instead of $2\ell(t,x)$, because it is necessary to guarantee that the numerator and the denominator vanish simultaneously. The local time $\ell(t,x)$ may be equal to zero, while $\ell(t,x+\Delta)$ is not equal to zero and conversely.

Proof of Theorem 11.1. Using (8.2), we prove an auxiliary statement. As before, the subscript x of the probability means that $W(0)=x$.

Let τ be the independent of the Brownian motion W (and therefore of the local time $\ell(t,x)$) random moment with the density $\frac{d}{dt}\mathbf{P}(\tau<t) = \lambda e^{-\lambda t}\mathbb{I}_{[0,\infty)}(t)$, $\lambda>0$.

Lemma 11.1. *Let* $\Delta := q-r>0$. *Then for* $u>0$ *and* $2\Delta\sqrt{2\lambda}<1$,

$$\mathbf{P}_x\Big(\sup_{0\le s<\tau} |\ell(s,q)-\ell(s,r)| > u\sqrt{\ell(\tau,r)+\ell(\tau,q)}\Big)$$

$$\le 4\exp\Big(-\frac{u^2}{4\Delta} - \min\{|x-r|,|q-x|\}\sqrt{2\lambda}\Big), \qquad x\in\mathbf{R}. \tag{11.2}$$

Proof. Inverting the Laplace transform in (8.2) with respect to η, we get for the function

$$p(t,h) := \frac{\partial}{\partial t}\mathbf{P}_x\Big(\ell(\tau,r)+\ell(\tau,q)<t,\ \sup_{0\le s\le\tau}(\ell(s,q)-\ell(s,r))>h\Big)$$

the following expression:
for $x\ge q$ (see Appendix 3, formulas f, 14, and 17 with $\mu=0$)

$$p(t,h) = (\alpha-\beta)e^{-(x-q)\sqrt{2\lambda}-\alpha t}\mathbb{I}_{(h,\infty)}(t)\Big(I_0\big(\beta\sqrt{t^2-h^2}\big)$$

$$+\beta\int_0^{t-h} I_0\big(\beta\sqrt{(t-s)^2-h^2}\big)\big(I_0(\beta s)+I_1(\beta s)\big)\,ds\Big), \tag{11.3}$$

and for $x \le r$ (see Appendix 3, formulas f, 14, and 17 with $\mu = 1$)

$$p(t,h) = (\alpha-\beta)e^{-(r-x)\sqrt{2\lambda}-\alpha t}\mathbb{1}_{(h,\infty)}(t)\biggl(\frac{\sqrt{t-h}}{\sqrt{t+h}}I_1\bigl(\beta\sqrt{t^2-h^2}\bigr)$$

$$+\beta\int_0^{t-h}\frac{\sqrt{t-s-h}}{\sqrt{t-s+h}}I_1\bigl(\beta\sqrt{(t-s)^2-h^2}\bigr)(I_0(\beta s)+I_1(\beta s))\,ds\biggr). \tag{11.4}$$

The expression for $p(t,h)$, $r < x < q$, is obtained from (11.3), (11.4) by linear combination according to the definition of the function $B(x)$, $x \in \mathbf{R}$, in Theorem 8.1. Let $x \ge q$. Applying (11.3), we get

$$\mathbf{P}_x\Bigl(\sup_{0\le s<\tau}(\ell(s,q)-\ell(s,r)) > u\sqrt{\ell(\tau,r)+\ell(\tau,q)}\Bigr)$$

$$= \int_0^\infty \mathbf{P}_x\Bigl(\sup_{0\le s<\tau}|\ell(s,q)-\ell(s,r)| > u\sqrt{t},\ \ell(\tau,r)+\ell(\tau,q)\in dt\Bigr)$$

$$= (\alpha-\beta)e^{(q-x)\sqrt{2\lambda}}\biggl[\int_{u^2}^\infty e^{-\alpha t}I_0\bigl(\beta\sqrt{t^2-u^2t}\bigr)\,dt$$

$$+\beta\int_{u^2}^\infty e^{-\alpha t}\int_0^{t-u\sqrt{t}} I_0\bigl(\beta\sqrt{(t-s)^2-u^2t}\bigr)(I_0(\beta s)+I_1(\beta s))\,dsdt\biggr]$$

$$= (\alpha-\beta)e^{-(x-q)\sqrt{2\lambda}}\biggl[\frac{1}{\sqrt{\alpha^2-\beta^2}}e^{-(\alpha+\sqrt{\alpha^2-\beta^2})u^2/2}$$

$$+\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\int_0^\infty \exp\bigl(-\alpha(s+u^2/2)-u\sqrt{\alpha^2-\beta^2}\sqrt{s+u^2/4}\bigr)(I_0(\beta s)+I_1(\beta s)\bigr)\,ds\biggr]$$

$$= \frac{(\alpha-\beta)e^{-(x-q)\sqrt{2\lambda}}}{\sqrt{\alpha^2-\beta^2}}\int_0^\infty \exp\Bigl(-\alpha(s+u^2/2)-u\sqrt{(\alpha^2-\beta^2)(s+u^2/4)}\Bigr)$$

$$\times\bigl(\alpha+\beta+u\sqrt{\alpha^2-\beta^2}/\sqrt{4s+u^2}\bigr)I_0(\beta s)\,ds$$

$$\le \bigl(\sqrt{\alpha^2-\beta^2}+\alpha-\beta\bigr)e^{-\alpha u^2/2-(x-q)\sqrt{2\lambda}}\int_0^\infty e^{-\alpha s}I_0(\beta s)\,ds$$

$$= \frac{\bigl(\sqrt{\alpha^2-\beta^2}+\alpha-\beta\bigr)}{\sqrt{\alpha^2-\beta^2}}e^{-\alpha u^2/2-(x-q)\sqrt{2\lambda}} \le 2e^{-u^2/4\Delta-(x-q)\sqrt{2\lambda}}.$$

This implies (11.2) in the case when the difference itself is considered instead of the modulus of the difference. The estimate for $x \le r$ is derived similarly from (11.4). By the definition of $B(x)$, the estimate for $r < x < q$ is obtained by linear combination of the estimates for $x \ge q$ and $x \le r$. To get an estimate for the modulus of the difference of the local times one uses the symmetry and spatial homogeneity properties of the Brownian motion W. □

As an intermediate result we prove the following lemma.

Lemma 11.2. *For any* $\lambda > 0$

$$\limsup_{\Delta\downarrow 0}\sup_{x\in\mathbf{R}}\frac{\sup_{0\le s\le\tau}|\ell(s,x+\Delta)-\ell(s,x)|}{\sqrt{(\ell(\tau,x)+\ell(\tau,x+\Delta))\Delta\ln(1/\Delta)}}=2 \qquad \text{a.s.} \tag{11.5}$$

Proof. We first establish that this limit is not less than 2 a.s. The following estimate is obvious:

$$\mathbf{P}_0\Big(\limsup_{\Delta\downarrow 0}\sup_{x\in\mathbf{R}}\frac{\sup_{0\le s\le\tau}|\ell(s,x+\Delta)-\ell(s,x)|}{\sqrt{(\ell(\tau,x)+\ell(\tau,x+\Delta))\Delta\ln(1/\Delta)}}\ge 2\Big)$$

$$\ge\sqrt{2\lambda}\int_0^\infty e^{-z\sqrt{2\lambda}}\mathbf{P}_0\Big(\limsup_{\Delta\downarrow 0}\sup_{x\in[0,z]}\frac{|\ell(\tau,x+\Delta)-\ell(\tau,x)|}{(\ell(\tau,x)\Delta\ln(1/\Delta))^{1/2}}\ge 2\sqrt{2},$$

$$\inf_{x\in[0,z]}\ell(\tau,x)>0\Big|W(\tau)=z\Big)\,dz. \tag{11.6}$$

Here we have used the symmetry property of the Brownian motion, which enabled us to reduce the integration over all real z to the integration only over positive z.

According to Proposition 2.1, the process $\ell(\tau,y)$, $y\in[0,z]$, given $W(\tau)=z$ can be represented in the form

$$\ell(\tau,y)=e^{-2\sqrt{2\lambda}y}\Big(R^{(2)}\Big(\frac{e^{2\sqrt{2\lambda}y}-1}{2\sqrt{2\lambda}}\Big)\Big)^2=\big(Q^{(2)}(y)\big)^2, \tag{11.7}$$

where $R^{(2)}(y)$, $y\ge 0$, is a 2-dimensional Bessel process and $Q^{(2)}(y)$, $y\ge 0$, is a radial Ornstein–Uhlenbeck process of order 0 with an exponentially distributed starting point. We know that $Q^{(2)}$ is a continuous strictly positive process, satisfying (see, Subsection 6 § 16 Ch. IV) the equation

$$Q^{(2)}(y_2)-Q^{(2)}(y_1)=\widetilde W(y_2)-\widetilde W(y_1)+\int_{y_1}^{y_2}\Big(\frac{1}{2Q^{(2)}(y)}-\sqrt{2\lambda}Q^{(2)}(y)\Big)dy, \quad 0\le y_1\le y_2,$$

where $\widetilde W$ is a Brownian motion. From this and the formula for the exact modulus of continuity of a Brownian motion (see (10.18) Ch. I) it follows that, for any $L>0$,

$$\limsup_{\Delta\downarrow 0}\sup_{y\in[0,L]}\frac{|Q^{(2)}(y+\Delta)-Q^{(2)}(y)|}{\sqrt{2\Delta\ln(1/\Delta)}}=1 \qquad \text{a.s.}$$

This implies that

$$\limsup_{\Delta\downarrow 0}\sup_{y\in[0,L]}\frac{|\big(Q^{(2)}(y+\Delta)\big)^2-\big(Q^{(2)}(y)\big)^2|}{2Q^{(2)}(y)\sqrt{2\Delta\ln(1/\Delta)}}$$

$$=\limsup_{\Delta\downarrow 0}\sup_{y\in[0,L]}\frac{|Q^{(2)}(y+\Delta)+Q^{(2)}(y)||Q^{(2)}(y+\Delta)-Q^{(2)}(y)|}{2Q^{(2)}(y)\sqrt{2\Delta\ln(1/\Delta)}}=1 \qquad \text{a.s.}$$

In view of (11.7), given the condition $W(\tau)=z>0$

$$\limsup_{\Delta\downarrow 0}\sup_{x\in[0,z]}\frac{|\ell(\tau,x+\Delta)-\ell(\tau,x)|}{2\sqrt{2\ell(\tau,x)\Delta\ln(1/\Delta)}}=1 \qquad \text{a.s.}$$

Consequently, the right-hand side of (11.6) is equal to

$$\sqrt{2\lambda}\int_0^\infty e^{-z\sqrt{2\lambda}}\mathbf{P}_0\Big(\inf_{x\in[0,z]}\ell(\tau,x)>0\Big|W(\tau)=z\Big)\,dz.$$

By the representation (11.7), the process $\ell(\tau,x)$ for $x\in[0,z]$ given the condition $W(\tau)=z$ is strictly positive, therefore the above integral is equal to 1. Thus we have obtained that the limit (11.5) is at least 2 a.s.

We now prove the opposite inequality, that is

$$\limsup_{\Delta\downarrow 0}\sup_{x\in\mathbf{R}}\frac{\sup_{0\le s\le\tau}|\ell(s,x+\Delta)-\ell(s,x)|}{\sqrt{(\ell(\tau,x)+\ell(\tau,x+\Delta))\Delta\ln(1/\Delta)}}\le 2 \qquad \text{a.s.} \tag{11.8}$$

For any $\varepsilon>0$ we choose $\delta>0$ such that $0<(1+\delta)/(1-\delta)<(1+\varepsilon)^2$. Set $\|f(\cdot)\|:=\sup_{0\le s\le\tau}|f(s)|$, $\Delta:=y-x>0$, and $h(t):=2\sqrt{t\ln(1/t)}$, $0<t<e^{-1}$. Using (11.2), we get

$$\mathbf{P}_0\Big(\max_{\substack{1\le k\le 2^{n\delta}\\ -\infty<i<\infty}}\frac{\|\ell(\cdot,(i+k)2^{-n})-\ell(\cdot,i2^{-n})\|}{\sqrt{\ell(\tau,(i+k)2^{-n})+\ell(\tau,i2^{-n})}h(k2^{-n})}>1+\varepsilon\Big)$$

$$\le\sum_{\substack{1\le k\le 2^{n\delta}\\ -\infty<i<\infty}}\mathbf{P}_0\Big(\frac{\|\ell(\cdot,(i+k)2^{-n})-\ell(\cdot,i2^{-n})\|}{\sqrt{\ell(\tau,(i+k)2^{-n})+\ell(\tau,i2^{-n})}}>(1+\varepsilon)h(k2^{-n})\Big)$$

$$\le 4\sum_{1\le k\le 2^{n\delta}}\exp\Big(-\frac{(1+\varepsilon)^2h^2(k2^{-n})}{4k2^{-n}}\Big)\sum_{-\infty<i<\infty}e^{-(|i|\wedge|i+k|)2^{-n}\sqrt{2\lambda}}$$

$$\le C_\lambda 2^{n(1+\delta)}\exp(-(1+\varepsilon)^2\ln 2^{n(1-\delta)})=C_\lambda 2^{-(1-\delta)((1+\varepsilon)^2-(1+\delta)/(1-\delta))n}.$$

By the choice of δ, the series with these terms converges, and by the first part of the Borel–Cantelli lemma (see Remark 1.1 Ch. I), there exist a.s. a number $m=m(\omega)$ such that for all $n>m$ and all $1\le k\le 2^{n\delta}$, $i\in\mathbb{Z}$

$$\|\ell(\cdot,(i+k)2^{-n})-\ell(\cdot,i2^{-n})\|$$
$$\le(1+\varepsilon)h(k2^{-n})\sqrt{\ell(\tau,i2^{-n})+\ell(\tau,(i+k)2^{-n})}. \tag{11.9}$$

Since $2^{-l(1-\delta)}$ tends monotonically to zero as $l\to\infty$, the value Δ will always be between two successive terms of this sequence, no matter how small it is. Let $2^{-(n+1)(1-\delta)}\le\Delta<2^{-n(1-\delta)}$. Since we are interested only in arbitrarily small Δ, we can assume that $n > m$. As in the previous section (see (10.6)) we represent x and y in the form

$$x = i2^{-n} - \sum_{v=1}^{\infty} 2^{-p_v}, \qquad y = (i+k)2^{-n} + \sum_{v=1}^{\infty} 2^{-q_v},$$

where $n < p_1 < p_2 < \cdots$ and $n < q_1 < q_2 < \cdots$. Let $x_0 := i2^{-n}$, $x_l := i2^{-n} - \sum_{v=1}^{l} 2^{-p_v}$, $y_0 := (i+k)2^{-n}$, $y_l := (i+k)2^{-n} + \sum_{v=1}^{l} 2^{-q_v}$. Due to this representation we have the estimate $\Delta \geq k2^{-n}$. We also set $p_0 = q_0 = n$.

Since $x_l \to x$ and the sample paths of $\ell(s,x)$ are a.s. continuous with respect to x, we have

$$\ell(s,x) - \ell(s,x_v) = \sum_{l=v+1}^{\infty} \big(\ell(s,x_l) - \ell(s,x_{l-1})\big).$$

Using (11.9), the inequality $\sqrt{|a|+|b|+|c|} \leq \sqrt{|a|} + \sqrt{|b|} + \sqrt{|c|}$, and the inequalities $h(2^{-p_l}) \leq h(2^{-p_{l-1}})$, we get for $v = 0, 1, \dots$

$$\|\ell(s,x) - \ell(s,x_v)\| \leq (1+\varepsilon) \sum_{l=v+1}^{\infty} h(2^{-p_l}) \sqrt{\ell(\tau,x_l) + \ell(\tau,x_{l-1})}$$

$$\leq (1+\varepsilon) \sum_{l=v+1}^{\infty} h(2^{-p_l}) \Big(\sqrt{2\ell(\tau,x)} + \sqrt{|\ell(\tau,x) - \ell(\tau,x_l)|} + \sqrt{|\ell(\tau,x) - \ell(\tau,x_{l-1})|} \Big)$$

$$\leq (1+\varepsilon) \Big(h_n \sqrt{2\ell(\tau,x)} + 2 \sum_{l=0}^{\infty} h(2^{-p_l}) \sqrt{|\ell(\tau,x) - \ell(\tau,x_l)|} \Big), \tag{11.10}$$

where $h_n := \sum_{l=n}^{\infty} h(2^{-l})$. The estimate (11.10) does not depend on v.

Since $|\ell(\tau,x) - \ell(\tau,x_v)| \leq \|\ell(s,x) - \ell(s,x_v)\|$, we have

$$\sum_{l=0}^{\infty} h(2^{-p_l}) \sqrt{|\ell(\tau,x) - \ell(\tau,x_l)|}$$

$$\leq \sqrt{1+\varepsilon}\, h_n \Big[(2\ell(\tau,x))^{1/4} h_n^{1/2} + \sqrt{2} \Big(\sum_{l=0}^{\infty} h(2^{-p_l}) \sqrt{|\ell(\tau,x) - \ell(\tau,x_l)|} \Big)^{1/2} \Big].$$

Solving this quadratic inequality, we get

$$\sum_{l=0}^{\infty} h(2^{-p_l}) \sqrt{|\ell(\tau,x) - \ell(\tau,x_l)|} \leq 2\sqrt{1+\varepsilon} (2\ell(\tau,x))^{1/4} h_n^{3/2} + 2(1+\varepsilon) h_n^2$$

$$\leq 4(1+\varepsilon) h_n \Big[\Big(h_n \sqrt{\ell(\tau,x) + \ell(\tau,y)} \Big)^{1/2} + h_n \Big]. \tag{11.11}$$

From (11.10) and (11.11) it follows that

$$\sqrt{|\ell(\tau,x)-\ell(\tau,x_v)|}\le 2\sqrt{1+\varepsilon}\sqrt{h_n}\,\big(\sqrt{\ell(\tau,x)+\ell(\tau,y)}\big)^{1/2}$$
$$+4(1+\varepsilon)\Big(h_n^{3/4}\big(\sqrt{\ell(\tau,x)+\ell(\tau,y)}\big)^{1/4}+h_n^{3/2}\Big). \tag{11.12}$$

Note that in (11.11), (11.12) one can replace x by y, x_l by y_l, and the estimates will be the same.

Using the representation

$$\ell(s,y)-\ell(s,x)=\ell(s,y_0)-\ell(s,x_0)+\sum_{l=1}^{\infty}\big(\ell(s,y_l)-\ell(s,y_{l-1})-\ell(s,x_l)+\ell(s,x_{l-1})\big)$$

and (11.9), we get analogously to (11.10) the estimate

$$\|\ell(s,y)-\ell(s,x)\|\le(1+\varepsilon)\Big\{h(k2^{-n})\sqrt{\ell(\tau,x_0)+\ell(\tau,y_0)}$$
$$+\sum_{l=1}^{\infty}\Big(h(2^{-p_l})\sqrt{\ell(\tau,x_l)+\ell(\tau,x_{l-1})}+h(2^{-q_l})\sqrt{\ell(\tau,y_l)+\ell(\tau,y_{l-1})}\Big)\Big\}$$
$$\le(1+\varepsilon)\Big\{h(k2^{-n})\big(\sqrt{\ell(\tau,x)+\ell(\tau,y)}+\sqrt{|\ell(\tau,x)-\ell(\tau,x_0)|}+\sqrt{|\ell(\tau,y)-\ell(\tau,y_0)|}\big)$$
$$+2h_n\sqrt{\ell(\tau,x)+\ell(\tau,y)}+2\sum_{l=0}^{\infty}h(2^{-p_l})\sqrt{|\ell(\tau,x)-\ell(\tau,x_l)|}$$
$$+2\sum_{l=0}^{\infty}h(2^{-q_l})\sqrt{|\ell(\tau,y)-\ell(\tau,y_l)|}\Big\}. \tag{11.13}$$

In the case $\sqrt{\ell(\tau,x)+\ell(\tau,y)}\le h(\Delta)$ the monotonicity of $\ell(s,z)$ with respect to s yields

$$\|\ell(s,y)-\ell(s,x)\|\le\ell(\tau,x)+\ell(\tau,y)\le h(\Delta)\sqrt{\ell(\tau,x)+\ell(\tau,y)},$$

and thus (11.8) holds when the supremum is taken over the set

$$\big\{x:\sqrt{\ell(\tau,x)+\ell(\tau,x+\Delta)}\le h(\Delta)\big\}.$$

Therefore, we need only to consider those x for which

$$\sqrt{\ell(\tau,x)+\ell(\tau,y)}>h(\Delta),\qquad y=x+\Delta.$$

Substituting the estimates (11.11), (11.12) and the analogous estimates for the increments $|\ell(\tau,y)-\ell(\tau,y_l)|$, $l=0,1,\dots$, into (11.13), we obtain

$$\frac{\|\ell(s,y)-\ell(s,x)\|}{\sqrt{\ell(\tau,x)+\ell(\tau,y)}}\le(1+\varepsilon)\Big\{h(k2^{-n})\Big[1+\frac{4\sqrt{1+\varepsilon}\sqrt{h_n}}{\sqrt{h(\Delta)}}+8(1+\varepsilon)\Big(\Big(\frac{h_n}{h(\Delta)}\Big)^{3/4}+\frac{h_n}{h(\Delta)}\Big)\Big]$$

$$+ 2h_n\Big[1 + \frac{4\sqrt{1+\varepsilon}\sqrt{h_n}}{\sqrt{h(\Delta)}} + 8(1+\varepsilon)\Big(\Big(\frac{h_n}{h(\Delta)}\Big)^{3/4} + \frac{h_n}{h(\Delta)}\Big)\Big]\Big\} \le (1+4\varepsilon)h(\Delta).$$

The last inequality is valid for sufficiently large n and small ε. Indeed, since $h(t)$, $0 < t < e^{-1}$, is an increasing function, we conclude that for some constant K

$$h_n \le Kh(2^{-n}) \le \varepsilon h\big(2^{-(n+1)(1-\delta)}\big) \le \varepsilon h(\Delta),$$

and, in addition, $h(k2^{-n}) \le h(\Delta)$, because $\Delta \ge k2^{-n}$.

Since ε can be chosen arbitrarily small, this proves (11.8). Thus (11.5) is proved. □

Now we are ready to complete the proof of the theorem. The relation (11.5) can be written as

$$\lambda \int_0^\infty e^{-\lambda t}\mathbf{P}\bigg(\limsup_{\Delta\downarrow 0}\sup_{x\in\mathbf{R}} \frac{\sup_{0\le s\le t}|\ell(s,x+\Delta)-\ell(s,x)|}{2\sqrt{(\ell(t,x)+\ell(t,x+\Delta))\Delta\ln(1/\Delta)}} = 1\bigg)\,dt = 1.$$

Inverting the Laplace transform with respect to λ, we get that for almost all t the probability under the integral is equal to 1 i.e., (11.1) holds. We prove that it is equal to 1 for all t. Set

$$V_\Delta(t) := \sup_{x\in\mathbf{R}} \frac{\sup_{0\le s\le t}|\ell(s,x+\Delta)-\ell(s,x)|}{2\sqrt{(\ell(t,x)+\ell(t,x+\Delta))\Delta\ln(1/\Delta)}}, \qquad V(t) := \lim_{\delta\downarrow 0}\sup_{0\le\Delta\le\delta} V_\Delta(t).$$

Using the scaling property of the Brownian local time (for any fixed $c > 0$, the distributions of the processes $\sqrt{c}\,\ell(t/c, x/\sqrt{c})$ and $\ell(t,x)$ coincide), we easily deduce that

$$\sup_{0\le\Delta\le\delta} V_\Delta(t) \qquad \text{and} \qquad \sup_{0\le\Delta\le\delta/\sqrt{c}} \Big\{V_\Delta(t/c)\sqrt{\ln\Delta/\ln(\Delta/\sqrt{c})}\Big\}$$

have the same distribution. Therefore, the distributions of the variables $V(t)$ and $V(t/c)$ are the same. This means that (11.1) holds for all t. Theorem 11.1 is proved. □

CHAPTER VI

DIFFUSIONS WITH JUMPS

§ 1. Diffusions with jumps

In this chapter we study a class of diffusions with jumps. The extreme elements of this class are, on the one hand, homogeneous diffusion processes and, on the other hand, Poisson processes with a variable intensity. It will be proved that diffusions with jumps have many good properties inherited both from classical diffusion processes and from Poisson ones. This class is closed with respect to composition with invertible twice continuously differentiable functions. A special random time transformation of a diffusion with jumps gives again a diffusion with jumps (an analog of Lévy's theorem § 8 Ch. II). A statement analogous to Girsanov's transformation of the measure of a classical diffusion (see, § 10 Ch. II) is valid for a class of diffusions with jumps. An important feature of this class also is the availability of effective methods for computing of distributions of some functionals of diffusions with jumps.

The Poisson process was defined in § 7 Ch. I. According to Proposition 7.1 Ch. I, the Poisson process $N(t)$, $t \geq 0$, with the intensity 1 can be represented as follows:

$$N(t) := \max\Big\{ l : \sum_{k=1}^{l} \tau_k \leq t \Big\} \mathbb{I}_{[0,t]}(\tau_1),$$

where τ_k, $k = 1, 2, \dots$, are independent exponentially distributed with parameter 1 random variables, $\frac{d}{dt}\mathbf{P}(\tau_k < t) = e^{-t}\mathbb{I}_{[0,\infty)}(t)$.

Let Y_k, $k = 1, 2, \dots$, be independent identically distributed random variables, which are independent of the process N. The process

$$N_c(t) := \sum_{k=1}^{N(\lambda_1 t)} Y_k$$

is called a *compound Poisson* process with intensity of jumps $\lambda_1 > 0$. Sums with the upper index 0 and the lower index 1 are treated as zero.

This process can be interpreted as a degenerate diffusion with jumps, for which the diffusion component is identically equal to zero.

It is not difficult to verify that the compound Poisson process has independent increments and its characteristic function is given by the formula (7.2) Ch. I

$$\mathbf{E}e^{i\alpha N_c(t)} = \exp\big(\lambda_1 t \big(\mathbf{E}e^{i\alpha Y_1} - 1\big)\big).$$

The most interesting diffusion with jumps is the process

$$J^{(\mu)}(t) := \mu t + \sigma W(t) + \sum_{k=1}^{N(\lambda_1 t)} Y_k, \qquad W(0) = x,$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_6

where $W(t)$, $t \geq 0$, $W(0) = x$, is a Brownian motion independent of the process N and the variables Y_k, $k = 1, 2, \dots$. The process $J^{(\mu)}$ is called a *Brownian motion with linear drift with jumps.* It is a homogeneous right continuous process with independent increments.

The Lévy–Khintchine formula for the characteristic function of this process has the form

$$\mathbf{E} \exp\bigl(i\alpha J^{(\mu)}(t)\bigr) = \exp\Bigl(i\alpha\sigma x + i\alpha\mu t - \frac{1}{2}\alpha^2\sigma^2 t + \lambda_1 t\bigl(\mathbf{E}e^{i\alpha Y_1} - 1\bigr)\Bigr). \tag{1.1}$$

A broad class of diffusions with jumps is considered. One of the main differences of the processes of this class from the process $J^{(\mu)}$ is the following: the values of jumps may depend not only on variables Y_k, $k = 1, 2, \dots$, but also on the position of the diffusion before a jump. The value of a jump is determined by a measurable function $\rho(x, y)$, $(x, y) \in \mathbf{R}^2$, where the first argument is reserved for values of the diffusion before a jump and the second one corresponds to the variables Y_k.

The next generalization is connected with the moments of jumps. Usually these moments are the moments of jumps of the Poisson process N, i.e., the moments follow each other over the intervals τ_k, $k = 1, 2, \dots$. It is possible to consider rather general moments of jumps depending on the behavior of a diffusion between jumps. Such moments are the first hitting times of the levels τ_k by some integral functional of a diffusion.

We consider a homogeneous diffusion X, which is a solution of the following stochastic differential equation: a.s. for every $t \geq 0$

$$X(t) = x + \int_0^t \mu(X(u))\,du + \int_0^t \sigma(X(u))\,dW(u). \tag{1.2}$$

Let $\mu(x)$ and $\sigma(x)$, $x \in \mathbf{R}$, be continuously differentiable functions satisfying the linear growth condition

$$|\mu(x)| + |\sigma(x)| \leq C(1 + |x|) \qquad \text{for all } \ x \in \mathbf{R}.$$

Then, by Theorem 7.3 Ch. II, (1.2) admits a unique strong solution. We also assume that $\inf_{x\in\mathbf{R}} \sigma(x) > 0$ and that the derivative $\Bigl(\dfrac{\mu(x)}{\sigma^2(x)}\Bigr)'$, $x \in \mathbf{R}$, is bounded.

Let $h(x)$, $x \in \mathbf{R}$, be a nonnegative bounded piecewise continuous function. We assume that it is right continuous ($h(x) = h(x+)$, $x \in \mathbf{R}$). The variable

$$\varkappa_1 := \min\Bigl\{s : \int_0^s h(X(v))\,dv = \tau_1\Bigr\}$$

is the moment inverse of the integral functional of the diffusion X. The moment $\varkappa_1$ can be infinite on sets of positive probability, since the integral $\int_0^\infty h(X(v))\,dv$ can be finite. We further assume that $\varkappa_1$ is a.s. finite (see the sufficient conditions (3.2)).

A diffusion with jumps (denoted by J) is defined recurrently as follows. Let $\rho(x,y)$, $(x,y)\in\mathbf{R}^2$, be a measurable function. For $\varkappa_0 := 0 \le t \le \varkappa_1$, set $J(t) := X(t)$, where X is the solution of (1.2). For the time interval $\varkappa_l \le t < \varkappa_{l+1}$, $l = 1, 2, \dots$, the process J is the solution of the following stochastic differential equation:

$$J(t) = \rho\bigl(J(\varkappa_l -), Y_l\bigr) + \int_{\varkappa_l}^{t} \mu(J(u))\,du + \int_{\varkappa_l}^{t} \sigma(J(u))\,dW(u), \tag{1.3}$$

where we set $q- := \lim\limits_{s\uparrow q} s$ and

$$\varkappa_{l+1} := \min\Bigl\{ s \ge \varkappa_l : \int_{\varkappa_l}^{s} h(J(v))\,dv = \tau_{l+1} \Bigr\}. \tag{1.4}$$

The stochastic differential equation (1.3) is unusual, because it is considered on a random interval. A rigorous interpretation of the existence and uniqueness of solution of such equation runs as follows. Consider a family $X_{s,x}(t)$, $t\in[s,\infty)$, of solutions of the stochastic differential equations

$$X_{s,x}(t) = x + \int_{s}^{t} \mu(X_{s,x}(u))\,du + \int_{s}^{t} \sigma(X_{s,x}(u))\,dW(u), \qquad s \ge 0 \qquad x\in\mathbf{R}.$$

The process $X_{s,x}(t)$, $t \ge s$, is adapted to the filtration $\mathcal{G}_0^t = \sigma(W(u), 0 \le u \le t)$, generated by Brownian motion W up to the time t. By Theorem 9.2 Ch. II, the process $X_{s,x}(t)$ is a.s. continuous with respect to $0 \le s \le t < \infty$, $x\in\mathbf{R}$, and for all $s \le v \le t$ the equality $X_{s,x}(t) = X_{v, X_{s,x}(v)}(t)$ holds.

By Remark 9.1 Ch. II, one can consider for $\varkappa_1 \le t$ the equation

$$X_{\varkappa_1,x}(t) = x + \int_{\varkappa_1}^{t} \mu(X_{\varkappa_1,x}(u))\,du + \int_{\varkappa_1}^{t} \sigma(X_{\varkappa_1,x}(u))\,dW(u).$$

Setting $\widetilde{X}_x^{(1)}(s) := X_{\varkappa_1,x}(s+\varkappa_1)$ and making the change of variables $u = v + \varkappa_1$ in the integrals, we get

$$\widetilde{X}_x^{(1)}(s) = x + \int_{0}^{s} \mu(\widetilde{X}_x^{(1)}(v))\,dv + \int_{0}^{s} \sigma(\widetilde{X}_x^{(1)}(v))\,d\widetilde{W}(v), \tag{1.5}$$

where $\widetilde{W}(s) := W(s+\varkappa_1) - W(\varkappa_1)$, $s \ge 0$, is a Brownian motion independent of the σ-algebra $\sigma\bigl(\sigma(Y_1)\bigcup\mathcal{G}_0^{\varkappa_1}\bigr)$ of events generated by the random variable Y_1 and the process W up to the time $\varkappa_1$ (see Remark 7.2 Ch. I). Therefore, instead of a fixed initial value x in (1.5) one can take the random variable $\rho\bigl(X(\varkappa_1), Y_1\bigr)$.

As a result, the process J that satisfies (1.3) can be defined as follows. For $\varkappa_1 \le t < \varkappa_2$ we set

$$J(t) := \widetilde{X}^{(1)}_{\rho(X(\varkappa_1),Y_1)}(t-\varkappa_1) = X_{\varkappa_1,\rho(X(\varkappa_1),Y_1)}(t),$$

where $\varkappa_2$ is determined by (1.4), $l=1$.

For $l=1,2,\dots$, set $\widetilde{X}^{(l)}_x(s) := X_{\varkappa_l,x}(s+\varkappa_l)$, $s \ge 0$,

$$J(t) := \widetilde{X}^{(l)}_{\varkappa_l,\rho(J(\varkappa_l-),Y_l)}(t-\varkappa_l) = X_{\varkappa_l,\rho(J(\varkappa_l-),Y_l)}(t), \qquad \varkappa_l \le t < \varkappa_{l+1}. \quad (1.6)$$

It should be taken into account that $\widetilde{X}^{(l)}_x(s)$ is the solution of the equation

$$\widetilde{X}^{(l)}_x(s) = x + \int_0^s \mu(\widetilde{X}^{(l)}_x(v))\,dv + \int_0^s \sigma(\widetilde{X}^{(l)}_x(v))\,d\widetilde{W}_l(v), \quad (1.7)$$

where $\widetilde{W}_l(s) := W(s+\varkappa_l) - W(\varkappa_l)$, $s \ge 0$. For each $l = 1,2,\dots$ the process $\widetilde{W}_l(s)$ is a Brownian motion independent of the σ-algebra of events generated by the process J up to the time $\varkappa_l$.

Since equations (1.2) and (1.7) coincide for any l, the diffusion $\widetilde{X}^{(l)}_x(s)$ has the same finite-dimensional distributions as the initial diffusion X. It is independent of the σ-algebra of events generated by the process J up to the time $\varkappa_l$. Therefore, by (1.6), the diffusion $J(s+\varkappa_l)$ restarts at any moment $\varkappa_l$ as a usual diffusion X with the starting point $\rho\big(J\big(\varkappa_l-\big),Y_l\big)$, and then continues as the diffusion with jumps.

The probabilistic interpretation of the appearance of the moments of jumps is the following. Since a diffusion with jumps restarts at moments of jumps, it is sufficient to consider it from the initial point 0. We consider a sample path of the diffusion X. By the definition of the moment $\varkappa_1$, for any fixed t the event $\{\varkappa_1 > t\}$ holds iff $\Big\{\int_0^t h(X(v))\,dv < \tau_1\Big\}$ holds. In this case the conditional probability of the event $\{\varkappa_1 > t\}$ relative to the σ-algebra $\sigma(X(\cdot))$ generated by the process X is given by

$$\mathbf{P}(\varkappa_1 > t|\sigma(X(\cdot))) = \mathbf{P}\bigg(\int_0^t h(X(v))\,dv < \tau_1\bigg|\sigma(X(\cdot))\bigg) = \exp\bigg(-\int_0^t h(X(v))\,dv\bigg).$$

We divide the time interval $[0,t]$ into small subintervals of length Δ and then let $\Delta \downarrow 0$. Assume that on each interval $[k\Delta,(k+1)\Delta)$ a jump can occur with probability $h(X(k\Delta))\Delta$ independently of the jumps on other intervals. For the moment of the first jump (denoted by $\varkappa_\Delta$) one has the relations

$$\mathbf{P}(\varkappa_\Delta > t|\sigma(X(\cdot))) = \prod_{k=1}^{[t/\Delta]} \big(1 - h(X(k\Delta))\Delta\big)$$

$$\approx \exp\bigg(-\sum_{k=1}^{[t/\Delta]} h(X(k\Delta))\Delta\bigg) \approx \exp\bigg(-\int_0^t h(X(v))\,dv\bigg).$$

Consequently, for a fixed sample path of the diffusion X the limit of the moments $\varkappa_\Delta$ as $\Delta \downarrow 0$ is distributed as the moment $\varkappa_1$ and this justify the probabilistic interpretation of appearance of the moments of jumps. Therefore, if the position of the diffusion J is determined at the moment t, then in the subsequent infinitesimal time interval $[t, t+dt)$ the jump occurs with probability $h(J(t))\,dt$ independently of the moments of the previous jumps. This fact guarantees the Markov property for diffusions with jumps.

The diffusion with jumps J is characterized by the following parameters: the drift coefficient $\mu(x)$, $x \in \mathbf{R}$, the diffusion coefficient $\sigma^2(x)$, $x \in \mathbf{R}$, the function of jumps $\rho(x,y)$, $(x,y) \in \mathbf{R}^2$, the random variables Y_k, $k = 1, 2, \dots$, which determine the values of jumps, and the function $h(x)$, $x \in \mathbf{R}$, which is responsible for the intensity of the jumps (intensity function).

We note that $\varkappa_l = \sum_{k=1}^{l} \tau_k$ if $h \equiv 1$. Choosing $h(x) \equiv \lambda_1 > 0$, we obtain the diffusion that has jumps over the time intervals $\widetilde{\tau}_k$, $k = 1, 2, \dots$, which are independent exponentially distributed with parameter λ_1 random variables.

Analogously to $N(t)$, $t \ge 0$, the process $C(t) := \max\{l : \varkappa_l \le t\}$, $t \ge 0$, counts the number of jumps performed by the diffusion J up to the time t, and $dC(t)$ equals one if $\varkappa_l$ belongs to the interval $[t, t+dt)$ for some $l = 1, 2, \dots$, and equals zero otherwise.

It is easy to see that the *counting process* $C(t)$, $t \ge 0$, can be represented as

$$C(t) = N(I(t)), \tag{1.8}$$

where $I(t) := \int_0^t h(J(v))\,dv$. Indeed, it follows from the definition of the moments $\varkappa_l$ that $\int_{\varkappa_l}^{\varkappa_{l+1}} h(J(v))\,dv = \tau_{l+1}$ or $\int_0^{\varkappa_l} h(J(v))\,dv = \sum_{k=1}^{l} \tau_k$. Therefore,

$$C(t) = \max\{l : I(\varkappa_l) \le I(t)\} = \max\Big\{l : \sum_{k=1}^{l} \tau_k \le I(t)\Big\} = N(I(t)).$$

The differential form of equation (1.3) is the following:

$$dJ(t) = \mu(J(t))\,dt + \sigma(J(t))\,dW(t) + \big(\rho\big(J(t-), Y_{C(t)}\big) - J(t-)\big)dC(t), \qquad J(0) = x. \tag{1.9}$$

Let $b(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function. The following generalization of Itô's formula holds:

$$\begin{aligned} db(J(t)) &= b'(J(t))(\mu(J(t))\,dt + \sigma(J(t))\,dW(t)) + \frac{1}{2}\,\sigma^2(J(t))b''(J(t))\,dt \\ &\quad + \big(b(\rho(J(t-), Y_{C(t)})) - b(J(t-))\big)\,dC(t). \end{aligned} \tag{1.10}$$

Indeed, (1.10) can be written in the integral form

$$b(J(t)) - b(J(0)) = \int_0^t b'(J(u))\sigma(J(u))\,dW(u) + \int_0^t b'(J(u))\mu(J(u))\,du$$

$$+\frac{1}{2}\int_0^t \sigma^2(J(u))b''(J(u))\,du + \int_{(0,t]} \bigl(b\bigl(\rho\bigl(J(u-), Y_{C(u)}\bigr)\bigr) - b(J(u-))\bigr)\,dC(u). \quad (1.11)$$

Let us show that this formula follows from the definition of the process J and the classical Itô formula. We have

$$b(J(t))-b(J(0)) = b(J(t))-b(J(\varkappa_{C(t)}))+\sum_{l=1}^{C(t)} \bigl(b(J(\varkappa_l-))-b(J(\varkappa_{l-1}))\bigr)$$

$$+\sum_{l=1}^{C(t)} \bigl(b(J(\varkappa_l)) - b(J(\varkappa_l-))\bigr).$$

It is clear that

$$\sum_{l=1}^{C(t)}(b(J(\varkappa_l)) - b(J(\varkappa_l-))) = \sum_{l=1}^{C(t)} \bigl(b(\rho\bigl(J\bigl(\varkappa_l-\bigr), Y_l\bigr)) - b(J(\varkappa_l-))\bigr)$$

$$= \int_{(0,t]} \bigl(b(\rho\bigl(J(u-), Y_{C(u)}\bigr)) - b(J(u-))\bigr)\,dC(u).$$

On the other hand,

$$b(J(\varkappa_l-)) - b(J(\varkappa_{l-1}))$$

$$= b\Bigl(\rho\bigl(J\bigl(\varkappa_{l-1}-\bigr), Y_{l-1}\bigr) + \int_{\varkappa_{l-1}}^{\varkappa_l} \mu(J(u))du + \int_{\varkappa_{l-1}}^{\varkappa_l} \sigma(J(u))dW(u)\Bigr) - b(\rho\bigl(J\bigl(\varkappa_{l-1}-\bigr), Y_{l-1}\bigr)).$$

We can apply the classical Itô formula to the right-hand side of this equality and to the difference $b(J(t)) - b(J(\varkappa_{C(t)}))$. Subsequent summation gives the first three terms in (1.11).

The application of the expectation to the Itô formula is of key importance. It is well known that the expectation of a stochastic integral is equal to zero. Later (see (4.1)), we will derive a formula for the expectation of the last term in (1.10), which correspond to jumps. As a result, we will deduce that

$$d\mathbf{E}_x b(J(t)) = \mathbf{E}_x\{b'(J(t))\mu(J(t))\}\,dt + \frac{1}{2}\mathbf{E}_x\{\sigma^2(J(t))b''(J(t))\}\,dt$$
$$+ \mathbf{E}_x\bigl\{h(J(t))\bigl(b(\rho(J(t), Y)) - b(J(t))\bigr)\bigr\}\,dt, \quad (1.12)$$

where Y is a random variable independent of the process J and distributed as Y_1.

Let $b(x)$, $x \in \mathcal{X} \subseteq \mathbf{R}$, be a twice continuously differentiable function having the inverse function $b^{(-1)}(y)$, $y \in b(\mathcal{X})$, i.e., $b^{(-1)}(b(x)) = x$. Assume that the diffusion with jumps J takes the values in $\mathcal{X}$. We prove that $\widetilde{J}(t) := b(J(t))$ is a diffusion with jumps as well. Therefore, the class of diffusions with jumps is closed with respect to compositions with invertible twice continuously differentiable functions.

To prove this fact, we derive a stochastic differential equation for the process $\widetilde{J}(t)$. Setting in Itô's formula (1.10) $\tilde{\rho}(x,y) := b(\rho(b^{(-1)}(x),y))$,

$$\tilde{\mu}(x) := b'(b^{(-1)}(x))\mu(b^{(-1)}(x)) + \frac{1}{2}b''(b^{(-1)}(x))\sigma^2(b^{(-1)}(x)), \qquad (1.13)$$

$$\tilde{\sigma}(x) := b'(b^{(-1)}(x))\sigma(b^{(-1)}(x)) \qquad (1.14)$$

and, using the equality $J(t) = b^{(-1)}(\widetilde{J}(t))$, we get

$$d\widetilde{J}(t) = \tilde{\mu}(\widetilde{J}(t))\,dt + \tilde{\sigma}(\widetilde{J}(t))\,dW(t)$$
$$+\big(\tilde{\rho}\big(\widetilde{J}(t-),Y_{C(t)}\big) - \widetilde{J}(t-)\big)dC(t), \quad \widetilde{J}(0) = b(x). \qquad (1.15)$$

Consequently, for the process $\widetilde{J}(t) = b(J(t))$ an equation of the form (1.9) is valid and the process $\widetilde{J}(t) = b(J(t))$, $t \geq 0$, is a diffusion with jumps. It is clear that the counting processes for the diffusions $\widetilde{J}$ and J coincide, and $\widetilde{h}(x) = h(b^{(-1)}(x))$.

§ 2. Examples of diffusions with jumps

Suppose that the Brownian motion W, the Poisson process N, and the variables $\{Y_k\}_{k=1}^{\infty}$ are independent. Set $\nu := \mu - \sigma^2/2$ and consider the Brownian motion with linear drift with jumps

$$J^{(\nu)}(t) := (\mu - \sigma^2/2)t + \sigma W(t) + \sum_{k=1}^{N(\lambda_1 t)} Y_k, \qquad W(0) = 0. \qquad (2.1)$$

Let $b(x) = e^x$. Then, according to (1.13)–(1.15) with

$$\rho(x,y) = x + y, \qquad \widetilde{\rho}(x,y) = \exp(\ln x + y) = xe^y,$$

the process $Z(t) := e^{J^{(\nu)}(t)}$ is the solution of the linear equation

$$dZ(t) = \mu Z(t)\,dt + \sigma Z(t)\,dW(t) + Z(t-)\big(e^{Y_{N(\lambda_1 t)}} - 1\big)\,dN(\lambda_1 t), \qquad \widetilde{Z}(0) = 1. \quad (2.2)$$

It is natural to call the process Z a *geometric* (*exponential*) *Brownian motion with jumps* by analogy with a diffusion without jumps (see 3 § 16 Chap. IV), when $Y_k \equiv 0$, $k = 1, 2, \dots$. This process is often used in different models connected with financial mathematics.

The next example arises by analogy with a Bessel process, which is the radial part of a multidimensional Brownian motion with independent coordinates (see Subsection 5 § 16 Ch. IV).

Let $\{W_l(s), s \geq 0\}$, $l = 1, 2, \dots, n$, be a family of independent Brownian motions $n \geq 2$. The process $R^{(n)}$ defined by the formula

$$R^{(n)}(t) := \sqrt{W_1^2(t) + W_2^2(t) + \cdots + W_n^2(t)}, \quad t \geq 0,$$

is called an *n-dimensional Bessel process* or a *Bessel process of order $n/2-1$*.

Let the diffusions with jumps $\{J_l(s), s\ge 0\}$, $l=1,2,\dots,n$, be defined by the equations

$$dJ_l(t)=dW_l(t)+\Big(\sqrt{\beta J_l^2(t-)+Y^{(l)}_{N(t)}}-J_l(t-)\Big)\,dN(t), \tag{2.3}$$

where $\vec{Y}_k := \big(Y_k^{(1)}, Y_k^{(2)},\dots,Y_k^{(n)}\big)$, $k=1,2,\dots$, are independent identically distributed random vectors with nonnegative coordinates and β is an arbitrary nonnegative constant. Suppose that the family $\{W_l(s), s\ge 0\}$, $l=1,2,\dots,n$, the Poisson process N, and the variables $\{\vec{Y}_k\}_{k=1}^{\infty}$ are independent. The choice of the function of jumps $\rho_\beta(x,y)=\sqrt{\beta x^2+y}$, $x\in\mathbf{R}$, $y\ge 0$, is predetermined by our desire to obtain the following statement.

Proposition 2.1. *For $n\ge 2$ the radial part of the multidimensional diffusion with jumps $\vec{J}(t)=(J_1(t),J_2(t),\dots,J_n(t))$ is a diffusion with jumps. It is characterized by the following parameters: the drift coefficient $\mu(x)=\dfrac{n-1}{2x}$, the diffusion coefficient 1, the function of jumps $\rho_\beta(x,y)$, the random variables $S_k^{(n)} := \sum_{l=1}^{n} Y_k^{(l)}$, $k=1,2,\dots$, determining the values of jumps, and the intensity function 1.*

Proof. Set

$$Z_n(t):=\sqrt{J_1^2(t)+J_2^2(t)+\cdots+J_n^2(t)},\quad t\ge 0.$$

We verify that Z_n is the diffusion with jumps determined by the equation

$$dZ_n(t)=\frac{n-1}{2Z_n(t)}\,dt+dW(t)+\Big(\sqrt{\beta Z_n^2(t-)+S^{(n)}_{N(t)}}-Z_n(t-)\Big)\,dN(t). \tag{2.4}$$

To prove this, we apply Itô's formula (1.10) (with $b(x)=x^2$) to each term of the process $Y_n(t):=Z_n^2(t)$ i.e., to the processes $J_l^2(t)$, $l=1,\dots,n$. Summing the differentials, we obtain

$$dY_n(t)=2\sum_{l=1}^{n} J_l(t)\,dW_l(t)+n\,dt+\sum_{l=1}^{n}\big((\beta-1)J_l^2(t-)+Y^{(l)}_{N(t)}\big)dN(t). \tag{2.5}$$

We can now apply the following variant of Proposition 16.2 Ch. IV.

Lemma 2.1. *Let $\mathcal{G}_t^l=\sigma\big(W_l(s), 0\le s\le t, Y_k^{(l)},\tau_k, k\le N(t)\big)$, be the σ-algebra of events generated by the Brownian motion W_l, $l=1,\dots,n$, up to the time t and by the variables $Y_k^{(l)}$, τ_k for $k\le N(t)$. Let $f_l(t)$, $t\ge 0$, be a process progressively measurable with respect to the family of σ-algebras $\mathcal{G}_t^l$, $l=1,2,\dots,n$.*

Then there exists a Brownian motion W such that for every $t>0$ the variable $W(t)$ is $\mathcal{G}_t:=\sigma\Big(\bigcup_{l=1}^{n}\mathcal{G}_t^l\Big)$-measurable and

$$\sum_{l=1}^{n}\int_0^t f_l(s)\,dW_l(s)=\int_0^t\Big(\sum_{l=1}^{n} f_l^2(s)\Big)^{1/2}\,dW(s). \tag{2.6}$$

The process $J_l(t)$, $t \ge 0$, is progressively measurable with respect to $\mathcal{G}_t^l$. Therefore,

$$\sum_{l=1}^{n} \int_0^t J_l(s)\, dW_l(s) = \int_0^t \sqrt{Y_n(s)}\, dW(s).$$

Then equation (2.5) can be rewritten in the form

$$dY_n(t) = 2\sqrt{Y_n(t)}\, dW(t) + n\, dt + \big((\beta - 1)Y_n(t-) + S^{(n)}_{N(t)}\big) dN(t). \tag{2.7}$$

This equation takes the form (1.9) ($\rho(x, y) = \beta x + y$) for the process $Y_n(t)$, $t \ge 0$. Since $Z_n(t) = \sqrt{Y_n(t)}$, we get (2.4), applying Itô's formula (1.15) with $b(x) = \sqrt{x}$.

□

§ 3. Distributions of integral functionals of a diffusion with jumps and of infimum and supremum functionals

Consider a method for computing the joint distribution of the functional

$$A(t) := \int_0^t f(J(s))\, ds, \qquad f \ge 0,$$

and of the infimum and supremum functionals $\inf_{0 \le s \le t} J(s)$, $\sup_{0 \le s \le t} J(s)$.

The general approach for computing the distributions of integral functionals of a Brownian motion was described in § 1 Ch. III. This approach is applicable to a broad class of processes, in particular, to diffusions with jumps. Therefore, we will consider only the main results that enable us to compute the above joint distribution within this general approach.

Let τ be a random moment that is independent of the process $\{J(s),\ s \ge 0\}$ and exponentially distributed with parameter $\lambda > 0$.

We recall that the use of the random time τ in place of t corresponds to the Laplace transform with respect to t of the distribution of a functional of the process J, considered up to the time t. In order to obtain the distribution of the functional for a fixed t one needs to invert the Laplace transform with respect to λ in the expression for the corresponding distribution for the random time τ.

We denote by $\mathbf{P}_x$ and $\mathbf{E}_x$ the probability and the expectation with respect to the process J with the starting point $J(0) = x$. For brevity, in what follows, we use the notation $\mathbf{E}\{\xi; A\} := \mathbf{E}\{\xi 1\!\!1_A\}$.

Let the function $h(x)$, $x \in \mathbf{R}$, be such that $\varkappa_1 < \infty$ a.s. This is equivalent to

$$\int_0^\infty h(X(s))\, ds = \infty \qquad \text{a.s.} \tag{3.1}$$

Sufficient conditions for this, according to Corollary 12.1 Ch. II, are

$$\liminf_{y\to\infty}\frac{1}{y}\int_0^y \frac{h(x)}{\sigma^2(x)}\,dx>0,\qquad \liminf_{y\to\infty}\frac{1}{y}\int_{-y}^0 \frac{h(x)}{\sigma^2(x)}\,dx>0. \tag{3.2}$$

Let $C(t)$, $t\ge 0$, be the counting process of J, which similarly to the Poisson process counts the number of jumps made by the diffusion J up to the time t.

Theorem 3.1. *Let $\varPhi(x)$ and $f(x)$, $x\in[a,b]$, be piecewise-continuous functions. Assume that $f\ge 0$ and $\varPhi$ is bounded when either $a=-\infty$ or $b=\infty$. Let $q(x,y)$, $x\in[a,b]$, $y\in\mathbf{R}$, be a nonnegative measurable function. Then the function*

$$Q(x):=\mathbf{E}_x\biggl\{\varPhi(J(\tau))\exp\biggl(-\int_0^\tau f(J(s))\,ds-\int_{(0,\tau]} q\bigl(J(s-),Y_{C(s)}\bigr)\,dC(s)\biggr);$$

$$a\le\inf_{0\le s\le\tau}J(s),\ \sup_{0\le s\le\tau}J(s)\le b\biggr\},\qquad x\in\mathbf{R},$$

is the unique bounded solution of the equation

$$Q(x)=M(x)+\int_{-\infty}^{\infty}G_z(x)\,\mathbf{E}\{e^{-q(z,Y_1)}Q(\rho(z,Y_1))\}\,dz, \tag{3.3}$$

where $M(x)$, $x\in(a,b)$, is the unique solution of the problem

$$\frac12\sigma^2(x)M''(x)+\mu(x)M'(x)-(\lambda+h(x)+f(x))M(x)=-\lambda\varPhi(x), \tag{3.4}$$

$$M(a)=0,\qquad M(b)=0, \tag{3.5}$$

and $G_z(x)$, $x\in(a,b)$, is the unique continuous solution of the problem

$$\frac12\sigma^2(x)G''(x)+\mu(x)G'(x)-(\lambda+h(x)+f(x))G(x)=0,\qquad x\ne z, \tag{3.6}$$

$$G'(z+0)-G'(z-0)=-2h(z)/\sigma^2(z), \tag{3.7}$$

$$G(a)=0,\qquad G(b)=0. \tag{3.8}$$

Here one sets $M(x)=0$, $G_x(z)=0$ if $x,z\notin(a,b)$.

Remark 3.1. If $a=-\infty$ and $b=\infty$, then the corresponding boundary conditions (3.5) and (3.8) should be replaced by the conditions that the functions M, G_x are bounded as x tends to $-\infty$ or ∞.

This remark is applicable also to other problems with the boundary conditions of this type.

Remark 3.2. The function $Q(x)$, $x\in(a,b)$, is the unique solution of the integro-differential equation

$$\frac12\sigma^2(x)Q''(x)+\mu(x)Q'(x)-(\lambda+h(x)+f(x))Q(x)$$

$$= -\lambda\Phi(x) - h(x)\int_{-\infty}^{\infty} e^{-q(x,y)}Q(\rho(x,y))\mathbb{1}_{[a,b]}(\rho(x,y))\,dF(y) \tag{3.9}$$

with the boundary conditions

$$Q(x) = 0, \qquad x \le a, \qquad x \ge b. \tag{3.10}$$

Here $F(y)$, $y \in \mathbf{R}$, is the distribution function of the variables Y_k, $k = 1, 2, \dots$. If either $a = -\infty$ or $b = \infty$, then the corresponding boundary condition must be replaced by the requirement that Q is a bounded solution at this point.

Indeed, G_x is the Green function, hence for an arbitrary function $R(x)$, $x \in [a, b]$, the function

$$U(x) := \int_a^b G_z(x)\,R(z)\,dz, \qquad x \in (a, b), \tag{3.11}$$

is (see (6.8) Ch. IV) the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)U''(x) + \mu(x)U'(x) - (\lambda + h(x) + f(x))U(x) = -h(x)R(x),$$
$$U(a) = 0, \qquad U(b) = 0.$$

From (3.3) it follows that $Q(x) = 0$ for $x \notin [a, b]$ and

$$Q(x) = M(x) + U(x), \qquad x \in (a, b), \tag{3.12}$$

where

$$R(x) = \mathbf{E}\big\{e^{-q(x,Y_1)}Q(\rho(x,Y_1))\mathbb{1}_{[a,b]}(\rho(x,Y_1))\big\}.$$

The sum of the functions M and U satisfies the problem (3.9), (3.10).

Remark 3.3. If $\rho(z, y) = z$, the diffusion J has no jumps, i.e., it coincides with the initial diffusion X, defined by (1.2). Then, for $q \equiv 0$ Theorem 3.1 becomes Theorem 4.2 Ch. IV.

Indeed, in this case, (3.9) is transformed into the equation

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) - (\lambda + h(x) + f(x))Q(x) = -\lambda\Phi(x) - h(x)Q(x),$$

which coincides with (4.8) Ch. IV.

Remark 3.4. If $\rho(z, y) = z$, the Poisson process N is independent of the diffusion $J \equiv X$. Provided that the sample path $X(\cdot)$ is fixed, the process $C(t)$, $t \ge 0$ is a Poisson process with the variable intensity $h(X(t))$, and the equality

$$\mathbf{E}\Big\{\exp\Big(-\int_{(0,t]} q\big(J(s-), Y_{C(s)}\big)dC(s)\Big)\Big|\sigma(X(\cdot))\Big\} = \exp\Big(-\int_0^t h(X(s))\big(1-r(X(s))\big)ds\Big),$$

holds, where $r(x) := \mathbf{E}e^{-q(x,Y_1)}$ and $\sigma(X(\cdot))$ is the σ-algebra of events generated by the process $X(s)$, $s \ge 0$. In this case (3.9) is transformed into the equation

$$\frac{1}{2}\sigma^2(x)Q''(x) + \mu(x)Q'(x) - (\lambda + h(x)(1 - r(x)) + f(x))Q(x) = -\lambda\Phi(x).$$

This statement also coincides with Theorem 4.2 Ch. IV.

Proof of Theorem 3.1. Set

$$M(x) := \mathbf{E}_x\biggl\{\Phi(X(\tau))\exp\Bigl(-\int_0^\tau (h(X(s)) + f(X(s)))\,ds\Bigr);$$

$$a \le \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) \le b\biggr\}, \qquad x \in (a,b). \tag{3.13}$$

Then M is the solution of the problem (3.4), (3.5) (see Ch. IV Theorem 4.2).

Set

$$G_z(x) := \frac{d}{dz+}\mathbf{E}_x\biggl\{\exp\Bigl(-\int_0^{\varkappa_1} (\lambda + f(X(s)))\,ds\Bigr);$$

$$a \le \inf_{0\le s\le\varkappa_1} X(s), \sup_{0\le s\le\varkappa_1} X(s) \le b, X(\varkappa_1) < z\biggr\}, \tag{3.14}$$

where $\frac{d}{dz+}$ denotes the right derivative. It is clear that $G_z(x) = 0$ if $x \notin (a,b)$ or $z \notin (a,b)$. This function is the solution of the problem (3.6)–(3.8) (see Ch. IV Theorem 6.1).

Since $\sup_{x\in\mathbf{R}} h(x) \le K$ for some K, we have that $\varkappa_1 \ge \tau_1/K$ and

$$\int_{-\infty}^{\infty} G_z(x)\,dz \le \mathbf{E}_x\exp\Bigl(-\int_0^{\varkappa_1}(\lambda + f(X(s)))\,ds\Bigr) \le \mathbf{E}e^{-\lambda\tau_1/K} = \frac{K}{K+\lambda} =: \theta < 1 \tag{3.15}$$

for all x.

Let us prove that equation (3.3) has a unique bounded solution. We apply the method of successive approximations. Set $Q_0(x) := M(x)$ and

$$Q_n(x) := \int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}\{e^{-q(z,Y_1)}Q_{n-1}(\rho(z,Y_1))\}\,dz.$$

Then

$$\sup_{x\in\mathbf{R}}|Q_n(x)| \le \sup_{x\in\mathbf{R}}|Q_{n-1}(x)|\sup_{x\in\mathbf{R}}\int_{-\infty}^{\infty} G_z(x)\,dz \le \theta\sup_{x\in\mathbf{R}}|Q_{n-1}(x)| \le \theta^n\sup_{x\in\mathbf{R}}|M(x)|.$$

Therefore the series $Q(x) = \sum_{n=0}^{\infty} Q_n(x)$ converges uniformly in x and

$$\sup_{x\in\mathbf{R}} |Q(x)| \le \frac{1}{1-\theta} \sup_{x\in\mathbf{R}} |M(x)|. \tag{3.16}$$

It is clear that

$$\sum_{k=0}^{n} Q_k(x) = M(x) + \int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}\Big\{e^{-q(z,Y_1)} \sum_{k=0}^{n-1} Q_k(\rho(z,Y_1))\Big\}\,dz.$$

The passage to the limit in the above equality implies that the function Q is the solution of equation (3.3). If $M \equiv 0$, then

$$\sup_{x\in\mathbf{R}} |Q(x)| \le \sup_{x\in\mathbf{R}} |Q(x)|\,\theta.$$

Since $\theta < 1$, the last relation implies that $Q \equiv 0$ if Q is bounded. Hence, equation (3.3) has a unique bounded solution. This proof also shows that for nonnegative M and G_z, the solution of equation (3.3) is nonnegative.

To simplify the formulas in the below argument, we assume that $a = -\infty$, $b = \infty$. We have

$$Q(x) = \mathbf{E}_x\Big\{\varPhi(X(\tau)) \exp\Big(-\int_0^{\tau} f(X(s))\,ds\Big); \tau < \varkappa_1\Big\}$$

$$+\,\mathbf{E}_x\Big\{\varPhi(J(\tau)) \exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds - \int_{\varkappa_1}^{\tau} f(J(s))\,ds + \int_{(\varkappa_1,\tau]} q\big(J(s-), Y_{C(s)}\big)dC(s)\Big)$$

$$\times e^{-q(X(\varkappa_1),Y_1)}; \varkappa_1 \le \tau\Big\} =: V_1(x) + V_2(x), \qquad x \in \mathbf{R}, \tag{3.17}$$

where $V_1(x)$ and $V_2(x)$ are the first and the second term, respectively. The event $\{\tau < \varkappa_1\}$ is equivalent to the event $\Big\{\int_0^{\tau} h(X(s))\,ds < \tau_1\Big\}$, the moment τ_1 is independent of τ and of the process X. Therefore,

$$\mathbf{P}\big(\tau < \varkappa_1 \big| \sigma(X(\cdot),\tau)\big) = \mathbf{P}\Big(\int_0^{\tau} h(X(s))\,ds < \tau_1 \Big| \sigma(X(\cdot),\tau)\Big)$$

$$= \exp\Big(-\int_0^{\tau} h(X(s))\,ds\Big), \tag{3.18}$$

where $\sigma(X(\cdot),\tau)$ is the σ-algebra of events generated by the process X and the moment τ. Applying Fubini's theorem, first computing the expectation with respect to τ_1, and then the one with respect to the process X and the time τ, we get

$$V_1(x)=\mathbf{E}_x\Big\{\Phi(X(\tau))\exp\Big(-\int_0^\tau(h(X(s))+f(X(s)))\,ds\Big)\Big\}=M(x).$$

Hence, V_1 takes the form (3.13) with $a=-\infty$, $b=\infty$.

In order to transform the second term V_2, we use the fact that the moment τ is independent of the process J, the variables $\{Y_k\}_{k=1}^\infty$, and the moment $\varkappa_1$. By Fubini's theorem,

$$V_2(x)=\lambda\mathbf{E}_x\int_{\varkappa_1}^\infty e^{-\lambda t}\Big\{\exp\Big(-\int_0^{\varkappa_1}f(X(s))\,ds\Big)\Phi(J(t))e^{-q(X(\varkappa_1),Y_1)}$$
$$\times\exp\Big(-\int_{\varkappa_1}^t f(J(s))\,ds\Big)\exp\Big(-\int_{(\varkappa_1,t]}q(J(s-),Y_{C(s)})\,dC(s)\Big)\Big\}dt.$$

By (1.6),

$$J(s+\varkappa_1)=\widetilde J_{\rho(X(\varkappa_1),Y_1)}(s),\qquad s\ge 0,\tag{3.19}$$

where $\widetilde J_x$ is constructed as the original diffusion J from $\widetilde W(s):=W(s+\varkappa_1)-W(\varkappa_1)$, $s\ge0$, $\widetilde\tau_k:=\tau_{k+1}$, and $\widetilde Y_k:=Y_{k+1}$. For $0\le t<\widetilde\varkappa_1$, we set $\widetilde J_x(s):=\widetilde X_x^{(1)}(s)$, where $\widetilde X_x^{(1)}$ is the solution of (1.5), and $\widetilde\varkappa_1:=\min\Big\{s:\int_0^s h(\widetilde X_x^{(1)}(v))\,dv=\widetilde\tau_1\Big\}$. For $l=1,2,\dots$, we set

$$\widetilde J_x(t):=X_{\widetilde\varkappa_l,\rho(\widetilde J_x(\widetilde\varkappa_l-),\widetilde Y_l)}(t),\qquad \widetilde\varkappa_l\le t<\widetilde\varkappa_{l+1}.$$

where

$$\widetilde\varkappa_{l+1}:=\min\Big\{s\ge\widetilde\varkappa_l:\int_{\widetilde\varkappa_l}^s h(\widetilde J_x(v))\,dv=\widetilde\tau_{l+1}\Big\}.$$

The diffusion with jumps $\widetilde J_x(s)$, $s\ge0$, has the same finite-dimensional distribution as the original diffusion J with $J(0)=x$. The process $\widetilde J_x(s)$ is independent of the σ-algebra of events $\widetilde{\mathcal G}=\sigma\Big(\mathcal G_0^{\varkappa_1}\bigcup\sigma\big(Y_1\big)\Big)$, where $\mathcal G_0^{\varkappa_1}$ is the σ-algebra of events generated by the process W up to the moment $\varkappa_1$ (see the definition in § 4 Ch. I), and $\sigma\big(Y_1\big)$ is the σ-algebra of events generated by the random variable Y_1.

Let $\widetilde C$ be the counting process for the diffusion $\widetilde J_{\rho(X(\varkappa_1),Y_1)}(s)$. It is clear that $\widetilde C(s)=C(s+\varkappa_1)-1$.

The above remarks enable us to make the change of the variable $t=u+\varkappa_1$ in the expression for V_2. Then

$$V_2(x)=\lambda\int_0^\infty e^{-\lambda u}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1}(\lambda+f(X(s)))\,ds\Big)\Phi(\widetilde J_{\rho(X(\varkappa_1),Y_1)}(u))e^{-q(X(\varkappa_1),Y_1)}$$

$$\times \exp\Big(-\int_0^u f(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(v))\,dv\Big)\exp\Big(-\int_{(0,u]} q(\widetilde{J}_{\rho(X(\varkappa_1)Y_1)}(v-),Y_{\widetilde{C}(v)+1})\,d\widetilde{C}(v)\Big)\Big\}du.$$

By Fubini's theorem, the integral with respect to u with the weight function $\lambda\, e^{-\lambda u}$ can be replaced by the integrand with $\widetilde{\tau}$ instead of u, where $\widetilde{\tau}$ is the exponentially distributed with the parameter $\lambda > 0$ random variable that is independent of other processes and variables. Thus for V_2 we obtain the following expression:

$$V_2(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1}(\lambda + f(X(s)))\,ds\Big)e^{-q(X(\varkappa_1),Y_1)}\mathbf{E}\Big\{\varPhi(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(\widetilde{\tau}))$$

$$\times \exp\Big(-\int_0^{\widetilde{\tau}} f(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s))ds - \int_{(0,\widetilde{\tau}]} q(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s-),\widetilde{Y}_{\widetilde{C}(s)})d\widetilde{C}(s)\Big)\Big|\widetilde{\mathcal{G}}\Big\}\Big\}.$$

Applying Lemma 2.1 Ch. I, we have

$$V_2(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1}(\lambda + f(X(s)))\,ds\Big)e^{-q(X(\varkappa_1),Y_1)}Q(\rho(X(\varkappa_1),Y_1))\Big\}.$$

Using the independence of the σ-algebra $\mathcal{G}_0^{\varkappa_1}$ and the variable Y_1, we finally get

$$V_2(x) = \int_{-\infty}^{\infty}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1}(\lambda + f(X(s)))\,ds\Big)e^{-q(z,Y_1)}Q(\rho(z,Y_1)); X(\varkappa_1)\in dz\Big\}$$

$$= \int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}\{e^{-q(z,Y_1)}Q(\rho(z,Y_1))\}\,dz.$$

Now (3.17) implies (3.3). Theorem 3.1 is proved. □

§ 4. Expectation of integral with respect to a counting process

We prove the following formula: for any bounded measurable function $q(x,y)$,

$$\mathbf{E}_x\int_{(0,t]} q(J(s-),Y_{C(s)})\,dC(s) = \int_0^t \mathbf{E}_x\big\{h(J(s))m(J(s))\big\}\,ds \tag{4.1}$$

where $m(x) = \mathbf{E}q(x,Y_1)$. From this it follows, in particular, that

$$\mathbf{E}_x\int_{\{t\}} q(J(s-),Y_{C(s)})\,dC(s) = 0.$$

We can assume that $q(x,y)$ is a nonnegative function, otherwise the function can be decomposed into a sum of nonnegative and negative parts.

Set

$$Q_\gamma(x) := \mathbf{E}_x \exp\Big(-\gamma \int\limits_{(0,\tau]} q(J(s-), Y_{C(s)})\, dC(s)\Big).$$

By Theorem 3.1 with $\varPhi(x) \equiv 1$, $f(x) \equiv 0$, $a = -\infty$, and $b = \infty$, the function $Q_\gamma(x)$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$Q_\gamma(x) = M(x) + \int\limits_{-\infty}^{\infty} G_z(x)\, \mathbf{E}\{e^{-\gamma q(z,Y_1)} Q_\gamma(\rho(z,Y_1))\}\, dz, \tag{4.2}$$

where $M(x)$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$\frac{1}{2}\sigma^2(x) M''(x) + \mu(x) M'(x) - (\lambda + h(x)) M(x) = -\lambda, \tag{4.3}$$

and $G_z(x)$, $x \in \mathbf{R}$, is the unique bounded solution of the problem

$$\frac{1}{2}\sigma^2(x) G''(x) + \mu(x) G'(x) - (\lambda + h(x)) G(x) = 0, \qquad x \neq z, \tag{4.4}$$

$$G'(z+0) - G'(z-0) = -2h(z)/\sigma^2(z). \tag{4.5}$$

Since the function h is bounded by a constant K, from (1.8) we have $C(t) \le N(Kt)$. Therefore,

$$\mathbf{E}_x \int\limits_{(0,\tau]} q(J(s-), Y_{C(s)})\, dC(s) = -\frac{d}{d\gamma} Q_\gamma(x)\Big|_{\gamma=0}.$$

Since $Q_0(x) = 1$, differentiating (4.2) with respect to γ and setting $\gamma = 0$ we see that the function $L(x) := \frac{d}{d\gamma} Q_\gamma(x)\Big|_{\gamma=0}$ is the unique bounded solution of the equation

$$L(x) = \widetilde{M}(x) + \int\limits_{-\infty}^{\infty} G_z(x)\, \mathbf{E}\{L(\rho(z,Y_1))\}\, dz, \tag{4.6}$$

where $\widetilde{M}(x) := -\int\limits_{-\infty}^{\infty} G_z(x)\, \mathbf{E} q(z,Y_1)\, dz = -\int\limits_{-\infty}^{\infty} G_z(x)\, m(z)\, dz$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$\frac{1}{2}\sigma^2(x) \widetilde{M}''(x) + \mu(x) \widetilde{M}'(x) - (\lambda + h(x)) \widetilde{M}(x) = h(x) m(x). \tag{4.7}$$

Here we used (3.11) with $f(x) \equiv 0$, $a = -\infty$, $b = \infty$ and with $R(x)$ is replaced by $-m(x)$.

Now we can use Theorem 3.1 once more with $\varPhi(x) = -\frac{1}{\lambda} h(x) m(x)$, $f(x) \equiv 0$, $q(x,y) \equiv 0$, $a = -\infty$, and $b = \infty$. According to this assertion, the function

$$\widetilde{L}(x) := -\frac{1}{\lambda} \mathbf{E}_x\{h(J(\tau)) m(J(\tau))\}$$

is the unique bounded solution of equation (4.6). Thus we proved that $L(x) = \widetilde{L}(x)$, i.e.,

$$\mathbf{E}_x \int\limits_{(0,\tau]} q(J(s-), Y_{C(s)})\, dC(s) = \frac{1}{\lambda}\mathbf{E}_x\{h(J(\tau))m(J(\tau))\}.$$

Inverting the Laplace transform with respect to λ of both sides of this equality, we get the formula (4.1).

§ 5. Distributions of functionals of diffusion with jumps bridges

The bridge $Y_{x,t,z}(s)$, $s \in [0,t]$, from x to z on the interval $[0,t]$ for a stochastic process $Y(s)$, $s \ge 0$, with $Y(0) = x$, was defined in § 11 Ch. I.

Let $J_{x,t,z}(s)$, $s \in [0,t]$, be the bridge of the process J. We consider a method for computing the joint distribution of the integral functional

$$A(t) := \int\limits_0^t f(J_{x,t,z}(s))\, ds, \qquad f \ge 0, \tag{5.1}$$

and of the infimum and supremum functionals $\inf\limits_{0\le s\le t} J_{x,t,z}(s)$, $\sup\limits_{0\le s\le t} J_{x,t,z}(s)$.

The general approach to the problem of computing of distributions of nonnegative functionals of bridges of random processes was described in § 4 Ch. III for the Brownian bridge $W_{x,t,z}$. This approach is valid also for other diffusions.

Under the assumption that the one-dimensional distribution of the process J has a density, the following equality holds

$$\mathbf{E}\wp(J_{x,t,z}(s), 0 \le s \le t) = \frac{\dfrac{d}{dz}\mathbf{E}\{\wp(J(s), 0 \le s \le t); J(t) < z\}}{\dfrac{d}{dz}\mathbf{P}(J(t) < z)} \tag{5.2}$$

for any bounded measurable functional $\wp$ on the space of functions without discontinuities of the second kind.

The main object for computing the distributions of integral functionals of the bridges of J is the function

$$G_z^\gamma(x) := \lambda \int\limits_0^\infty e^{-\lambda t} \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(-\gamma \int\limits_0^t f(J(s))\, ds\Big); J(t) < z\Big\}\, dt$$
$$= \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(-\gamma \int\limits_0^\tau f(J(s))\, ds\Big);\ J(\tau) < z\Big\}.$$

In this formula τ is the random time independent of the process $\{J(s),\ s \ge 0\}$ and exponentially distributed with parameter $\lambda > 0$.

To find the distribution of the integral functional $\int_0^t f(J_{x,t,z}(s))\,ds$ at a fixed time t, we must compute the double inverse Laplace transform of the function $G_z^{\gamma}(x)$ with respect to parameters λ and γ, and then, applying formula (5.2), to divide the result by the density of $J(t)$. Note that the density itself is computed in the same way using the function $G_z^{\gamma}(x)$ with $f(x) \equiv 0$.

The reasoning given above shows that the following statement is of key importance for computing the joint distributions of integral functionals and functionals of infimum and supremum of the bridge of the diffusion process with jumps.

Theorem 5.1. *Let $f(x)$, $x \in [a,b]$, be a nonnegative piecewise-continuous function. Then for $a < y < b$ the derivative*

$$\widetilde{G}_y(x) := \frac{d}{dy}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\tau} f(J(s))\,ds\Big); a \le \inf_{0\le s\le\tau} J(s), \sup_{0\le s\le\tau} J(s) \le b, J(\tau) < y\Big\} \tag{5.3}$$

exists and it is the unique bounded solution of the equation

$$\widetilde{G}_y(x) = G_y^{\lambda}(x) + \int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}\widetilde{G}_y(\rho(z,Y_1))\,dz, \qquad x \in \mathbf{R}, \tag{5.4}$$

where $G_y^{\lambda}(x)$, $x \in [a,b]$, is the unique continuous solution of the problem

$$\frac{\sigma^2(x)}{2}G''(x) + \mu(x)G'(x) - (\lambda + h(x) + f(x))G(x) = 0, \quad x \in (a,b)\setminus\{y\}, \tag{5.5}$$

$$G'(y+0) - G'(y-0) = -2\lambda/\sigma^2(y), \tag{5.6}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{5.7}$$

and $G_z(x)$, $x \in [a,b]$, is the unique continuous solution of the problem (3.6)–(3.8). *Here we set $G_y^{\lambda}(x) = 0$ and $G_y(x) = 0$ if either $x \notin (a,b)$ or $y \notin (a,b)$.*

Remark 5.1. $G_y^{\lambda}(x)$, $x \in (a,b)$, is the Green function of the problem (3.4), (3.5). This function has (see Theorem 6.2 Ch. IV) the following probabilistic representation

$$G_y^{\lambda}(x) = \frac{d}{dy}\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\tau}(h(X(s)) + f(X(s)))\,ds\Big);$$
$$a \le \inf_{0\le s\le\tau} X(s), \sup_{0\le s\le\tau} X(s) \le b, X(\tau) < y\Big\}.$$

The function $G_y(x)$, $x \in (a,b)$, takes the form (3.14). The problems (3.6)–(3.8) and (5.5)–(5.7) differ only in the conditions (3.7) and (5.6) on the jumps of derivatives. Therefore, $G_y^{\lambda}(x) = \frac{\lambda}{h(y)}G_y(x)$.

Remark 5.2. It is clear that $\widetilde{G}_y(x)=0$ for $x\notin(a,b)$ or $y\notin(a,b)$. The function $\widetilde{G}_y(x)$, $x\in(a,b)$, is the unique continuous solution of the problem

$$\frac{1}{2}\sigma^2(x)\widetilde{G}''_y(x)+\mu(x)\widetilde{G}'_y(x)-(\lambda+h(x)+f(x))\widetilde{G}_y(x) = -h(x)\mathbf{E}\widetilde{G}_y(\rho(x,Y_1)),\quad x\neq y, \tag{5.8}$$

$$\widetilde{G}'_y(y+0)-\widetilde{G}'_y(y-0)=-2\lambda/\sigma^2(y), \tag{5.9}$$

$$\widetilde{G}_y(a)=0,\qquad \widetilde{G}_y(b)=0. \tag{5.10}$$

Indeed, $G_y(x)$ is the Green function of the corresponding problem, hence, the function

$$U(x):=\int_a^b G_z(x)\,\mathbf{E}\{\widetilde{G}_y(\rho(z,Y_1))\}\,dz,\qquad x\in[a,b],$$

is (see (6.8) Ch. IV) the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)U''(x)+\mu(x)U'(x)-(\lambda+h(x)+f(x))U(x)=-h(x)\mathbf{E}\widetilde{G}_y(\rho(x,Y_1)),$$
$$U(a)=0,\qquad U(b)=0.$$

By (5.4), we have

$$\widetilde{G}_y(x)=G^\lambda_y(x)+U(x),$$

and the sum of the functions G^λ_y and U satisfies the problem (5.8)–(5.10).

Proof of Theorem 5.1. Our proof is based on Theorem 3.1 with $q(x,y)\equiv 0$. Set

$$Q_\Delta(x):=\mathbf{E}_x\biggl\{\frac{\mathbb{1}_{[y,y+\Delta)}(J(\tau))}{\Delta}\exp\biggl(-\int_0^\tau f(J(s))\,ds\biggr);a\le\inf_{0\le s\le\tau}J(s),\ \sup_{0\le s\le\tau}J(s)\le b\biggr\}.$$

We let $\mathbb{1}_{[y,y+\Delta)}(x):=-\mathbb{1}_{[y+\Delta,y)}(x)$ if $\Delta<0$. It is clear that $Q_\Delta(x)=0$ for $x\notin(a,b)$.

By Theorem 3.1, the function $Q_\Delta(x)$, $x\in(a,b)$, is the unique bounded solution of the equation

$$Q_\Delta(x)=M_\Delta(x)+\int_{-\infty}^{\infty}G_z(x)\,\mathbf{E}Q_\Delta(\rho(z,Y_1))\,dz, \tag{5.11}$$

where according to (3.13)

$$M_\Delta(x):=\mathbf{E}_x\biggl\{\frac{\mathbb{1}_{[y,y+\Delta)}(X(\tau))}{\Delta}\exp\biggl(-\int_0^\tau(h(X(s))+f(X(s)))\,ds\biggr);$$
$$a\le\inf_{0\le s\le\tau}X(s),\ \sup_{0\le s\le\tau}X(s)\le b\biggr\}.$$

From (6.19) of Ch. IV with $a=-\infty$, $b=\infty$ it follows that

$$\sup_{x\in\mathbf{R}}\frac{d}{dz}\mathbf{P}_x\big(X(\tau)<z\big)=\frac{2\lambda}{w(z)\sigma^2(z)}\varphi(z)\psi(z),$$

where $\psi(x)$ is an increasing solution and $\varphi(x)$ is a decreasing solution of equation (6.16) of Ch. IV for $x\in\mathbf{R}$, $f\equiv 0$. The function $w(z)=\psi'(z)\varphi(z)-\psi(z)\varphi'(z)>0$ is the Wronskian of these solutions.

Therefore, if $|x|\le 1$

$$\sup_{x\in\mathbf{R}}|M_\Delta(x)|\le\frac{1}{\Delta}\int_y^{y+\Delta}\sup_{x\in\mathbf{R}}\frac{d}{dz}\mathbf{P}_x\big(X(\tau)<z\big)\,dz\le K_y$$

for $0<|\Delta|<1$ and some constant K_y.

By (3.15) and (3.16), we have

$$|Q_\Delta(x)|\le\frac{1}{1-q}\sup_{x\in\mathbf{R}}|M_\Delta(x)|\le\frac{K+\lambda}{\lambda}K_y. \tag{5.12}$$

Due to the specific structure of the kernel G_z, the integral equation (5.11) is equivalent (see (3.9)) to the integro-differential equation

$$\frac{1}{2}\sigma^2(x)Q''_\Delta(x)+\mu(x)Q'_\Delta(x)-(\lambda+h(x)+f(x))Q_\Delta(x)$$

$$=-\frac{\lambda}{\Delta}\mathbb{I}_{[y,y+\Delta)}(x)-h(x)\mathbf{E}Q_\Delta((\rho(x,Y_1)) \tag{5.13}$$

with the boundary conditions

$$Q_\Delta(a)=0,\qquad Q_\Delta(b)=0. \tag{5.14}$$

We now prove that, the passage to the limit as $\Delta\to 0$ in the problem (5.13), (5.14) yields the problem (5.8)–(5.10). Here one can proceed as in the proof of Theorem 6.1 Ch. IV. There the change of variable $x=y^{(-1)}(y(x))$ was used to transform the corresponding equation to the equation without the first derivative. However, we can take another path and make a change of unknown function that figures in the equation. Considering the new variable $V_\Delta(x)=\exp\Big(\int_0^x\frac{\mu(u)}{\sigma^2(u)}du\Big)Q_\Delta(x)$, we deduce from (5.13) that

$$V''_\Delta(x)=\Big\{\frac{2(\lambda+h(x)+f(x))}{\sigma^2(x)}+\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)^2+\Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'\Big\}V_\Delta(x)$$

$$-\frac{2}{\sigma^2(x)}\exp\Big(\int_0^x\frac{\mu(u)}{\sigma^2(u)}du\Big)\Big(\frac{\lambda}{\Delta}\mathbb{I}_{[y,y+\Delta)}(x)+h(x)\mathbf{E}Q_\Delta(\rho(x,Y_1))\Big).$$

Denote by $R_\Delta(x)$ the right-hand side of this equation. Integrating this equation with respect to x over the interval (a,b), we have

$$V'_\Delta(x_2) - V'_\Delta(x_1) = \int_{x_1}^{x_2} R_\Delta(x)\,dx. \tag{5.15}$$

Integrating equality (5.15) with respect to x_2 over the interval $(x_1, x_3) \subseteq (a,b)$, we get

$$V_\Delta(x_3) - V_\Delta(x_1) - V'_\Delta(x_1)(x_3 - x_1) = \int_{x_1}^{x_3} dx_2 \int_{x_1}^{x_2} R_\Delta(x)\,dx. \tag{5.16}$$

The following arguments partially repeat the arguments in the proof of Theorem 6.3 Ch. III, so some of the details are omitted. The estimate (5.12) and the relation (5.15) imply that the family $\{V'_\Delta(x)\}_{\Delta\neq 0}$ is equicontinuous on any closed subinterval $[\alpha,\beta] \subset (a,b)\setminus\{y\}$. Taking the value x_3 in (5.16) outside $[\alpha,\beta]$, it is easy to see that the family $\{V'_\Delta(x)\}_{\Delta\neq 0}$ is uniformly bounded on $[\alpha,\beta]$. In addition, we can deduce from (5.15) that

$$\sup_{\Delta\neq 0}\ \sup_{a<x<b} |V'_\Delta(x)| < \infty. \tag{5.17}$$

By the Arzelá–Ascoli theorem, the family $\{V'_\Delta(x)\}_{\Delta\neq 0}$, $x \in [\alpha,\beta]$, is relatively compact. Hence, any sequence Δ_n that tends to zero contains a subsequence Δ_{n_k} such that the functions $V'_{\Delta_{n_k}}(x)$, $x \in [\alpha,\beta]$, converge uniformly to some limit $\widetilde{V}(x)$. In addition, relations (5.16) and (5.17) imply that the family $\{V_\Delta(x)\}_{\Delta\neq 0}$, $x \in [a,b]$, is equicontinuous. Consequently, $V_{\Delta_{n_k}}(x)$ converges uniformly in $[a,b]$ to some limit $V(x)$. In general, the limit function may depend on the choice of the subsequence Δ_{n_k}, however, we will see that the function V satisfies the differential problem, which has a unique solution. Thus,

$$\sup_{x\in[a,b]} |V_\Delta(x) - V(x)| \to 0 \quad \text{as} \quad \Delta \to 0, \tag{5.18}$$

and $\widetilde{V}(x) = V'(x)$ for $x \neq y$.

Since the convergence (5.18) is uniform, the function $V(x)$, $x \in [a,b]$, is continuous. By (5.14), this function satisfies the boundary conditions $V(a) = 0$, $V(b) = 0$. We set $V(x) = 0$ for $x \notin (a,b)$.

Denote $\chi(x) := \mathbb{I}_{[y,\infty)}(x)$. Letting $\Delta_{n_k} \to 0$ in (5.15) we obtain

$$V'(x_2) - V'(x_1) - \int_{x_1}^{x_2} \Big\{\frac{2(\lambda + h(x) + f(x))}{\sigma^2(x)} + \Big(\frac{\mu(x)}{\sigma^2(x)}\Big)^2 + \Big(\frac{\mu(x)}{\sigma^2(x)}\Big)'\Big\} V(x)\,dx$$

$$= -\int_{x_1}^{x_2} \frac{2\lambda}{\sigma^2(x)} \exp\Big(\int_0^x \frac{\mu(u)}{\sigma^2(u)}du\Big)\big(d\chi(x) + h(x)\mathbf{E}\widetilde{G}_y(\rho(x, Y_1))dx\big),$$

where $\widetilde{G}_y(x) := \exp\Big(-\int\limits_0^x \frac{\mu(u)}{\sigma^2(u)}du\Big)V(x)$. This equation is equivalent to the problem (5.8), (5.9). Indeed, if $y \notin (x_1, x_2)$, then $\chi(x_2) - \chi(x_1) = 0$, and the function $\widetilde{G}(x)$ satisfies equation (5.8). If $y \in (x_1, x_2)$, then $\chi(x_2) - \chi(x_1) = 1$. Letting $x_1 \uparrow y$ and $x_2 \downarrow y$, we see that

$$V'(y+0) - V'(y-0) = -\frac{2\lambda}{\sigma^2(y)} \exp\bigg(\int\limits_0^y \frac{\mu(u)}{\sigma^2(u)}du\bigg).$$

Therefore the function $\widetilde{G}_y(x)$ satisfies the condition (5.9).

Thus, we proved that the family of functions $Q_\Delta(x)$ has a limit as $\Delta \to 0$ and since $\mathbb{I}_{[y,y+\Delta)}(x) = \mathbb{I}_{(-\infty,y+\Delta)}(x) - \mathbb{I}_{(-\infty,y)}(x)$, this limit takes the form (5.3). At the same time we verified that the limit function satisfies the problem (5.8)–(5.10), which is equivalent to (5.4)–(5.7). □

§ 6. Distributions of integral functionals of a diffusion with jumps stopped at the first exit time

For the diffusion with jumps J, the moment $H_{a,b} := \min\{s : J(s) \notin (a,b)\}$, the *first exit time* from an interval, is very important for various applications. If the initial value $x \notin (a,b)$, we set $H_{a,b} = 0$.

It is important that $\mathbf{P}_x(H_{a,b} < \infty) = 1$, or equivalently

$$\mathbf{P}_x\Big(a \le \inf_{0\le s<\infty} J(s),\ \sup_{0\le s<\infty} J(s) \le b\Big) = 0.$$

To justify this we set in Theorem 3.1 $\Phi \equiv 1$, $f \equiv 0$, $q \equiv 0$. Letting $\lambda \downarrow 0$ we have from (3.4), (3.5) that $M(x) \downarrow 0$ and, in view of (3.3) and (6.7) (below), $Q(x) \downarrow 0$, $x \in (a,b)$. But in this case $\tau \to \infty$ and, consequently,

$$Q(x) \downarrow \mathbf{P}_x\Big(a \le \inf_{0\le s<\infty} J(s),\ \sup_{0\le s<\infty} J(s) \le b\Big).$$

Consider the problem of computing distributions of integral functionals of diffusion with jumps, stopped at the moment $H_{a,b}$. For a process with jumps the first exit from an interval can occur either by crossing the boundary or by a jump over the boundary.

6.1. We begin with the case, in which we do not distinguish the way the first exit from (a,b) occurs, i.e., the exit occurs by a jump over the boundary or by crossing the boundary. The following result concerns actually the Laplace transform of the distribution of nonnegative integral functional of the diffusion with jumps J stopped at the first exit time from the interval (a,b) over the boundary b. The expectation is reduced to those paths of the diffusion that exit the interval just over b.

To find the distribution of $\int\limits_0^{H_{a,b}} f(J(s))\,ds$, one applies the next theorem with $\Phi \equiv 1$ for the product $\gamma f(x)$, $\gamma > 0$, instead of $f(x)$, compute the function $R_b(x)$, and then invert the Laplace transform with respect to the parameter γ.

Theorem 6.1. *Let $\varPhi(x)$, $x \in \mathbf{R}$, and $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \geq 0$ and $\varPhi$ is bounded. Then the function*

$$R_b(x) := \mathbf{E}_x\biggl\{\varPhi(J(H_{a,b}))\exp\biggl(-\int\limits_0^{H_{a,b}} f(J(s))\,ds\biggr); J(H_{a,b}) \geq b\biggr\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the equation

$$R_b(x) = \varPhi(b)M_b(x)\mathbb{1}_{[a,b]}(x) + \varPhi(x)\mathbb{1}_{(b,\infty)}(x) + \int\limits_{-\infty}^{\infty} G_z(x)\mathbf{E}R_b(\rho(z,Y_1))\,dz, \quad (6.1)$$

where $M_b(x)$ is the unique solution of the problem

$$\frac{1}{2}\sigma^2(x)M''(x) + \mu(x)M'(x) - (h(x)+f(x))M(x) = 0, \quad x \in (a,b), \quad (6.2)$$

$$M(a) = 0, \qquad M(b) = 1, \quad (6.3)$$

and $G_z(x)$ is the unique continuous solution of the problem

$$\frac{1}{2}\sigma^2(x)G''(x)+\mu(x)G'(x)-(h(x)+f(x))G(x)=0, \qquad x \in (a,b)\setminus\{z\}, \quad (6.4)$$

$$G'(z+0) - G'(z-0) = -2h(z)/\sigma^2(z), \quad (6.5)$$

$$G(a) = 0, \qquad G(b) = 0, \quad (6.6)$$

and $G_z(x) = 0$ for $x, z \notin (a,b)$.

Proof. Set

$$M_b(x) := \mathbf{E}_x\biggl\{\exp\biggl(-\int\limits_0^{H_{a,b}} (h(X(s)) + f(X(s)))\,ds\biggr); X(H_{a,b}) = b\biggr\}.$$

Then $M_b(x)$ for $x \in (a,b)$ is the solution of the problem (6.2), (6.3) (see Ch. IV Theorem 7.2).

Set

$$G_z(x) := \frac{d}{dz+}\mathbf{E}_x\biggl\{\exp\biggl(-\int\limits_0^{\varkappa_1} f(X(s))\,ds\biggr); \\ a \leq \inf_{0\leq s\leq\varkappa_1} X(s), \sup_{0\leq s\leq\varkappa_1} X(s) \leq b, X(\varkappa_1) < z\biggr\},$$

where $\frac{d}{dz+}$ denotes the right derivative. Then G_z is the solution of (6.4)–(6.6) (see Ch. IV Theorem 6.1, $\lambda = 1$).

It is important that

$$\sup_{x\in\mathbf{R}}\int_{-\infty}^{\infty} G_z(x)\,dz \le \sup_{x\in(a,b)} \mathbf{P}_x\Big(a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\Big)<1. \qquad (6.7)$$

From here it follows that equation (6.1) has (see § 3) a unique bounded solution. This estimate can be derived as follows. Since $\varkappa_1\ge\tau_1/K$,

$$\mathbf{P}_x\Big(a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\Big)$$
$$\le \mathbf{P}_x\Big(a\le \inf_{0\le s\le\tau_K} X(s),\ \sup_{0\le s\le\tau_K} X(s)\le b\Big) =: U_K(x),$$

where τ_K is an independent of the process J exponentially distributed with parameter $K>0$ random moment. The function $U_K(x)$, $x\in(a,b)$, is the solution of (4.11), (4.12) Ch. IV, $\lambda=K$, and it is expressed by the formula

$$U_K(x)=1-\frac{(\psi(b)-\psi(a))\varphi(x)+(\varphi(a)-\varphi(b))\psi(x)}{\psi(b)\varphi(a)-\psi(a)\varphi(b)},\qquad x\in(a,b),$$

where ψ is an increasing solution and φ is a decreasing solution of the equation

$$\frac12\sigma^2(x)\phi''(x)+\mu(x)\phi'(x)-K\phi(x)=0,\qquad x\in\mathbf{R}.$$

In view of the monotonicity of the solutions, the quantity $\sup\limits_{x\in(a,b)} U_K(x)$ is estimated by the value

$$1-\frac{(\psi(b)-\psi(a))\varphi(b)+(\varphi(a)-\varphi(b))\psi(a)}{\psi(b)\varphi(a)-\psi(a)\varphi(b)}=\frac{(\psi(b)-\psi(a))(\varphi(a)-\varphi(b))}{\psi(b)\varphi(a)-\psi(a)\varphi(b)}\in(0,1).$$

Thus the required estimate is proved.

The expectation in the definition of the function $R_b(x)$ can be decomposed into the sum of two expectations: over the set $\{H_{a,b}<\varkappa_1\}$ and over its complement $\{H_{a,b}\ge\varkappa_1\}$, which is equivalent to the event

$$\Big\{a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\Big\}.$$

We have

$$R_b(x)=\mathbf{E}_x\Big\{\varPhi(X(H_{a,b}))\exp\Big(-\int_0^{H_{a,b}} f(X(s))\,ds\Big); X(H_{a,b})=b, H_{a,b}<\varkappa_1\Big\}$$
$$+\mathbf{E}_x\Big\{\varPhi(J(H_{a,b}))\exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds-\int_{\varkappa_1}^{H_{a,b}} f(J(s))\,ds\Big);$$
$$a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b, J(H_{a,b})\ge b\Big\} =: V_1(x)+V_2(x),$$

where V_1 and V_2 are the first and the second term, respectively. The event $\{H_{a,b} < \varkappa_1\}$ is equivalent to the event $\Big\{\int\limits_0^{H_{a,b}} h(X(v))\,dv < \tau_1\Big\}$ and the moment τ_1 is independent of the process X, and has an exponential distribution. Then, using the equation analogous to (3.18) with the moment $H_{a,b}$ instead of τ, we get that $V_1(x) = \varPhi(b)M_b(x)$.

We consider the process $\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s) = J(s+\varkappa_1)$, $s \ge 0$, and the σ-algebra $\widetilde{\mathcal{G}}$, which are defined in the proof of Theorem 3.1.

For the event $\{\varkappa_1 \le H_{a,b}\}$ the following equality holds:

$$H_{a,b} - \varkappa_1 = \widetilde{H}_{a,b} := \min\{s : \widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s) \notin (a,b)\}.$$

Then for V_2 we get

$$V_2(x) = \mathbf{E}_x\biggl\{ \exp\biggl(-\int\limits_0^{\varkappa_1} f(X(s))\,ds\biggr) \mathbb{1}_{\bigl\{a \le \inf\limits_{0\le s\le\varkappa_1} X(s),\ \sup\limits_{0\le s\le\varkappa_1} X(s) \le b\bigr\}}$$

$$\times \mathbf{E}\biggl\{\varPhi(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(\widetilde{H}_{a,b}))\exp\biggl(-\int\limits_0^{\widetilde{H}_{a,b}} f(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s))ds\biggr)\mathbb{1}_{\{\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(\widetilde{H}_{a,b})\ge b\}}\bigg|\widetilde{\mathcal{G}}\biggr\}\biggr\}.$$

The process $\widetilde{J}_x$ is distributed as J with $J(0) = x$, and is independent of the σ-algebra $\mathcal{G}_0^{\varkappa_1}$ generated by the process X up to the moment $\varkappa_1$, and of the variable Y_1, i.e., it is independent of the σ-algebra $\widetilde{\mathcal{G}}$. Applying Lemma 2.1 of Ch. I, we get

$$V_2(x) = \mathbf{E}_x\biggl\{\exp\biggl(-\int\limits_0^{\varkappa_1} f(X(s))\,ds\biggr)\mathbb{1}_{\bigl\{a \le \inf\limits_{0\le s\le\varkappa_1} X(s),\ \sup\limits_{0\le s\le\varkappa_1} X(s) \le b\bigr\}} R_b(\rho(X(\varkappa_1), Y_1))\biggr\}.$$

Now, using the probabilistic representation of the function $G_z(x)$ and the independence of the σ-algebra $\mathcal{G}_0^{\varkappa_1}$ from the variable Y_1, we finally get

$$V_2(x) = \int\limits_{-\infty}^{\infty} G_z(x)\,\mathbf{E}R_b(\rho(z,Y_1))\,dz.$$

Thus, we have the equality

$$R_b(x) = \varPhi(b)M_b(x) + \int\limits_{-\infty}^{\infty} G_z(x)\,\mathbf{E}R_b(\rho(z,Y_1))\,dz, \qquad x \in (a,b).$$

For $x > b$ it is obvious that $R_b(x) = \varPhi(x)$, because $H_{a,b} = 0$. For $x < a$ it is clear that $R_b(x) = 0$. This completes the proof of (6.1). □

For the exit over the boundary a there is an assertion analogous to Theorem 6.1, which can be derived from the latter. Indeed, we reflect the diffusion J symmetrically with respect to zero. Then $-J$ is the diffusion with jumps, whose drift coefficient is equal to $-\mu(-x)$, the diffusion coefficient is equal to $\sigma^2(-x)$, and the function of jumps is given by $-\rho(-x,y)$. This diffusion starts from the point $-x$ and the point $-a$ is the upper exit boundary, the point $-b$ is the lower exit boundary. Applying Theorem 6.1 for the diffusion $-J$ with the upper exit boundary $-a$ and rewriting this assertion in terms the original diffusion J, we obtain the following assertion.

Theorem 6.2. *Let $\Phi(x)$, $x \in \mathbf{R}$, and $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \geq 0$ and Φ is bounded. Then the function*

$$R_a(x) := \mathbf{E}_x\biggl\{\Phi(J(H_{a,b}))\exp\biggl(-\int_0^{H_{a,b}} f(J(s))\,ds\biggr); J(H_{a,b}) \leq a\biggr\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the problem

$$R_a(x) = \Phi(a)M_a(x)\mathbb{1}_{[a,b]}(x) + \Phi(x)\mathbb{1}_{(-\infty,a)}(x) + \int_{-\infty}^{\infty} G_z(x)\mathbf{E}R_a(\rho(z,Y_1))\,dz, \quad (6.8)$$

where $M_a(x)$ is the unique solution of (6.2) *with the boundary conditions*

$$M(a) = 1, \qquad M(b) = 0, \qquad (6.9)$$

and $G_z(x)$ is the unique continuous solution of the problem (6.4)–(6.6), *and $G_z(x) = 0$ for $x, z \notin (a,b)$.*

If there are no restrictions on the form of exit of the diffusion with jumps from the interval (a,b), then the following assertion holds.

Theorem 6.3. *Let $\Phi(x)$, $x \in \mathbf{R}$, and $f(x)$, $x \in [a,b]$, be piecewise-continuous functions. Assume that $f \geq 0$ and Φ is bounded. Then the function*

$$R(x) := \mathbf{E}_x\biggl\{\Phi(J(H_{a,b}))\exp\biggl(-\int_0^{H_{a,b}} f(J(s))\,ds\biggr)\biggr\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the problem

$$R(x) = (\Phi(a)M_a(x) + \Phi(b)M_b(x))\mathbb{1}_{[a,b]}(x) + \Phi(x)\mathbb{1}_{\mathbf{R}\setminus[a,b]}(x) + \int_{-\infty}^{\infty} G_z(x)\mathbf{E}R(\rho(z,Y_1))\,dz, \qquad (6.10)$$

where the functions $M_a(x)$ and $M_b(x)$ are the solutions of (6.2) *with the boundary conditions* (6.3) *and* (6.9), *respectively, $G_z(x)$ is the unique continuous solution of the problem* (6.4)–(6.6), *and $G_z(x) = 0$ for $x, z \notin (a,b)$.*

This assertion follows from Theorems 6.1 and 6.2, because $R(x) = R_a(x) + R_b(x)$.

6.2. Consider the case when the process J leaves the interval by crossing the boundary. In this case, if the exit boundary is the point b, the following result holds.

Theorem 6.4. *Let $f(x)$, $x \in [a,b]$, be a nonnegative piecewise-continuous function. Then the function*

$$R_b^\circ(x) := \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(J(s))\,ds\Big); J(H_{a,b}) = b\Big\}, \qquad x \in \mathbf{R},$$

is the unique bounded solution of the equation

$$R_b^\circ(x) = M_b(x)\mathbb{I}_{[a,b]}(x) + \int_{-\infty}^{\infty} G_z(x)\mathbf{E}R_b^\circ(\rho(z,Y_1))\,dz, \tag{6.11}$$

where $M_b(x)$, $x \in (a,b)$, is the unique solution of the problem (6.2), (6.3). *The function $G_z(x)$, $x \in (a,b)$, is the unique continuous solution of the problem* (6.4)–(6.6), *and $G_z(x) = 0$ for $x, z \notin (a,b)$.*

Remark 6.1. For the exit across the boundary a the analogous statement holds: it is only necessary to replace the boundary condition (6.3) by the boundary condition (6.9).

Proof of Theorem 6.4. The proof of this result is similar to that of Theorem 6.1, therefore we will only indicate the essential details. We represent $R_b^\circ(x)$ as a sum of two terms:

$$R_b^\circ(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(X(s))\,ds\Big); X(H_{a,b}) = b, H_{a,b} < \varkappa_1\Big\}$$

$$+\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds - \int_{\varkappa_1}^{H_{a,b}} f(J(s))\,ds\Big); a \le \inf_{0\le s\le\varkappa_1} X(s),$$

$$\sup_{0\le s\le\varkappa_1} X(s) \le b, J(H_{a,b}) = b\Big\}.$$

By the definition of the moment $H_{a,b}$,

$$R_b^\circ(x) = \mathbf{E}_x\Big\{\exp\Big(-\int_0^{H_{a,b}} f(J(s))\,ds\Big); J(H_{a,b}) = b\Big\}\mathbb{I}_{[a,b]}(x).$$

Using this and the properties of the process $\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s) = J(s+\varkappa_1)$, $s \ge 0$, and of the moment $\widetilde{H}_{a,b}$, we obtain

$$R_b^\circ(x) = M_b(x)\mathbb{I}_{[a,b]}(x) + \mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds\Big)\mathbb{I}_{\{a\le\inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\}}$$

$$\times \mathrm{1\!I}_{[a,b]}(\rho(X(\varkappa_1),Y_1))\mathbf{E}\Big\{\exp\Big(-\int_0^{\widetilde H_{a,b}} f(\widetilde J_{\rho(X(\varkappa_1),Y_1)}(s))ds\Big)\mathrm{1\!I}_{\{\widetilde J_{\rho(X(\varkappa_1),Y_1)}(\widetilde H_{a,b})=b\}}\Big|\widetilde{\mathcal G}\Big\}\Big\}$$

$$= M_b(x)\mathrm{1\!I}_{[a,b]}(x)$$

$$+\mathbf{E}_x\Big\{\exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds\Big)\mathrm{1\!I}_{\{a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\}} R_b^\circ(\rho(X(\varkappa_1),Y_1))\Big\}.$$

Now, using the probabilistic representation of the function $G_z(x)$ and the independence of the σ-algebra $\mathcal{G}_0^{\varkappa_1}$ from the variable Y_1, we finally get (6.11). □

6.3. Consider the case, in which the exit from an interval occurs by a jump over the boundary. In this case, if the exit boundary is the point b, we have the following result.

Theorem 6.5. *Let $\Phi(x)$, $x\in\mathbf{R}$, and $f(x)$, $x\in[a,b]$, be piecewise-continuous functions. Assume that $f\ge 0$ and Φ is bounded. Then the function*

$$R_b^{(1)}(x) := \mathbf{E}_x\Big\{\Phi(J(H_{a,b}))\exp\Big(-\int_0^{H_{a,b}} f(J(s))\,ds\Big); J(H_{a,b})>b\Big\},\qquad x\in\mathbf{R},$$

is the unique bounded solution of the equation

$$R_b^1(x) = \Phi(x)\mathrm{1\!I}_{(b,\infty)}(x) + \int_{-\infty}^{\infty} G_z(x)\mathbf{E}R_b^1(\rho(z,Y_1))\,dz, \qquad (6.12)$$

where $G_z(x)$, $x\in(a,b)$, is the unique continuous solution of the problem (6.4)–(6.6), *and $G_z(x)=0$ for $x,z\notin(a,b)$.*

Remark 6.2. For the exit over the boundary a an analogous result is valid, it is only necessary to replace in (6.12) the indicator function $\mathrm{1\!I}_{(b,\infty)}(x)$ by $\mathrm{1\!I}_{(-\infty,a)}(x)$.

Remark 6.3. Note that Theorem 6.1 follows from Theorems 6.4 and 6.5. Indeed, $R_b(x) = \Phi(b)R_b^\circ(x) + R_b^1(x)$ and the sum of equations (6.12) and (6.11), multiplied by $\Phi(b)$, gives equation (6.1).

Proof of Theorem 6.5. The proof is carried out in the same way as the proofs of Theorems 6.1 and 6.4. Using the fact that the event $\{J(H_{a,b})>b, H_{a,b}<\varkappa_1\}$ is realized only if $H_{a,b}=0$ and $x>b$, we get

$$R_b^1(x) = \Phi(x)\mathrm{1\!I}_{(b,\infty)}(x) + \mathbf{E}_x\Big\{\Phi(J(H_{a,b}))\exp\Big(-\int_0^{\varkappa_1} f(X(s))\,ds - \int_{\varkappa_1}^{H_{a,b}} f(J(s))\,ds\Big);$$

$$a\le \inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b, J(H_{a,b})>b\Big\} = \Phi(x)\mathrm{1\!I}_{(b,\infty)}(x)$$

$$+\mathbf{E}_x\biggl\{\exp\biggl(-\int_0^{\varkappa_1} f(X(s))\,ds\biggr)\mathbb{1}_{\{a\le\inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\}}\mathbf{E}\biggl\{\varPhi(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(\widetilde{H}_{a,b}))$$

$$\times\exp\biggl(-\int_0^{\widetilde{H}_{a,b}} f(\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(s))ds\biggr)\mathbb{1}_{\{\widetilde{J}_{\rho(X(\varkappa_1),Y_1)}(\widetilde{H}_{a,b})>b\}}\biggm|\widetilde{\mathcal{G}}\biggr\}\biggr\}$$

$$=\varPhi(x)\mathbb{1}_{(b,\infty)}(x)$$

$$+\mathbf{E}_x\biggl\{\exp\biggl(-\int_0^{\varkappa_1} f(X(s))\,ds\biggr)\mathbb{1}_{\{a\le\inf_{0\le s\le\varkappa_1} X(s),\ \sup_{0\le s\le\varkappa_1} X(s)\le b\}}R_b^1(\rho(X(\varkappa_1),Y_1))\biggr\}$$

$$=\varPhi(x)\mathbb{1}_{(b,\infty)}(x)+\int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}R_b^1(\rho(z,Y_1))\,dz.$$

The theorem is proved. □

§ 7. Distributions of functionals of a diffusion with jumps stopped at the moment inverse of integral functional

Consider the problem of computing distributions of functionals of a diffusion with jumps stopped at the *moment inverse of integral functional.* This moment is defined as follows

$$\nu(t):=\min\biggl\{s:\int_0^s g(J(v))\,dv=t\biggr\},$$

where g is a nonnegative piecewise continuous function. We assume that g is right continuous ($g(z)=g(z+)$, $z\in\mathbf{R}$).

In addition, we assume that the diffusion coefficient $\sigma^2(x)$, $x\in\mathbf{R}$, and the drift coefficient $\mu(x)$, $x\in\mathbf{R}$, are bounded.

The following result is of key importance for the problem of computing the distributions of functionals for diffusions with jumps stopped at the moment inverse of an integral functional. Set for brevity $\nu:=\nu(\tau)$, where τ is the random moment independent of the process $\{J(s),\ s\ge 0\}$ and exponentially distributed with parameter $\lambda>0$.

Theorem 7.1. *Let $\varPhi(x)$ and $f(x)$, $x\in[a,b]$, be piecewise-continuous functions. Assume that $f\ge 0$ and that $\varPhi$ is bounded when either $a=-\infty$ or $b=\infty$. Let $q(x,y)$ be nonnegative measurable function. Then the function*

$$V(x):=\mathbf{E}_x\biggl\{\varPhi(J(\nu))\exp\biggl(-\int_0^{\nu} f(J(s))\,ds-\int_{(0,\nu]} q\bigl(J(s-),Y_{C(s)}\bigr)\,dC(s)\biggr);$$

$$a \le \inf_{0 \le s \le \nu} J(s), \sup_{0 \le s \le \nu} J(s) \le b, \nu < \infty \Big\}$$

is the unique bounded solution of the equation

$$V(x) = M(x) + \int_{-\infty}^{\infty} G_z(x)\, \mathbf{E}\big\{e^{-q(z,Y_1)} V(\rho(z,Y_1))\big\}\, dz, \tag{7.1}$$

where $M(x)$, $x \in (a,b)$, *is the unique solution of the problem*

$$\frac{1}{2}\sigma^2(x)M''(x) + \mu(x)M'(x) - (\lambda g(x) + h(x) + f(x))M(x) = -\lambda g(x)\varPhi(x), \tag{7.2}$$

$$M(a) = 0, \qquad M(b) = 0, \tag{7.3}$$

and $G_z(x)$, $x \in (a,b)$ *is the unique solution of the problem*

$$\frac{1}{2}\sigma^2(x)G''(x) + \mu(x)G'(x) - (\lambda g(x) + h(x) + f(x))G(x) = 0, \quad x \ne z, \tag{7.4}$$

$$G'(z+0) - G'(z-0) = -2h(z)/\sigma^2(z), \tag{7.5}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{7.6}$$

Here we set $M(x) = 0$ *and* $G_z(x) = 0$ *for* $x, z \notin (a,b)$.

Remark 7.1. For $g \equiv 1$ this result recovers Theorem 3.1.

Remark 7.2. It is clear that $V(x) = 0$ for $x \notin (a,b)$. The function $V(x)$, $x \in (a,b)$, is the unique solution of the equation

$$\frac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) - (\lambda g(x) + h(x) + f(x))V(x)$$
$$= -\lambda g(x)\varPhi(x) - h(x)\mathbf{E}\big\{e^{-q(z,Y_1)}V(\rho(x,Y_1))\big\} \tag{7.7}$$

with the boundary conditions

$$V(a) = 0, \qquad V(b) = 0. \tag{7.8}$$

If either $a = -\infty$ or $b = \infty$, then the corresponding boundary condition must be replaced by the requirement that V is bounded solution at this point.

Proof of Theorem 7.1. To simplify the formulas, we prove the result for the case when $a = -\infty$ and $b = \infty$. The case of arbitrary values of a an b is proved analogously. We first assume that f is a bounded continuous function and $\varPhi$ is a twice continuously differentiable function with bounded derivatives.

For arbitrary $\widetilde{\lambda} > 0$ we set

$$\eta_{\widetilde{\lambda}}(s) := \varPhi(J(s)) \exp\bigg(-\int_0^s \big(\widetilde{\lambda} + f(J(v))\big)\, dv - \int_{(0,s]} q\big(J(v-), Y_{C(v)}\big)\, dC(v)\bigg). \tag{7.9}$$

Using an analog of (1.11), we have

$$\eta_{\widetilde\lambda}(t)-\eta_{\widetilde\lambda}(0)=\int\limits_{[0,t]}\exp\Big(-\int\limits_0^s\big(\widetilde\lambda+f(J(v))\big)\,dv-\int\limits_{(0,s]}q\big(J(v-),Y_{C(v)}\big)\,dC(v)\Big)$$

$$\times\Big[\Phi'(J(s))\sigma(J(s))\,dW(s)+\Phi'(J(s))\mu(J(s))\,ds+\frac12\sigma^2(J(s))\Phi''(J(s))\,ds$$

$$-\big(\widetilde\lambda+f(J(s))\big)\Phi(J(s))\,ds+\big(\Phi\big(\rho\big(J(s-),Y_{C(s)}\big)\big)e^{-q(J(s-),Y_{C(s)})}-\Phi(J(s-))\big)dC(s)\Big].$$

Here we use the following assertion: if $dC(s)=1$, then the function

$$\Phi(J(s))\exp\Big(-\int\limits_{(0,s]}q\big(J(v-),Y_{C(v)}\big)\,dC(v)\Big)$$

has at s a jump of the form

$$\exp\Big(-\int\limits_{(0,s)}q\big(J(v-),Y_{C(v)}\big)\,dC(v)\Big)\big(\Phi\big(\rho\big(J(s-),Y_{C(s)}\big)\big)e^{-q(J(s-),Y_{C(s)})}-\Phi(J(s-))\big).$$

Since g is a nonnegative function, we see that $\int\limits_0^s g(J(v))\,dv$, $s\ge0$, is an increasing function. From here it follows that for any $s\ge0$ and $t\ge0$,

$$\mathbb{1}_{[0,\nu(t))}(s)=\mathbb{1}_{[0,t)}\Big(\int\limits_0^s g(J(v))\,dv\Big).$$

Since the expectation of the stochastic integral is zero, and the expectation of the integral with respect to $dC(s)$ gives the factor $h(J(s))$ (see formula (4.1)), we get

$$\mathbf{E}_x\eta_{\widetilde\lambda}(\nu(t))-\Phi(x)=\mathbf{E}_x\int\limits_0^\infty\mathbb{1}_{[0,t)}\Big(\int\limits_0^s g(J(v))\,dv\Big)\exp\Big(-\int\limits_0^s\big(\widetilde\lambda+f(J(v))\big)dv\Big)$$

$$\times\exp\Big(-\int\limits_{(0,s)}q\big(J(v-),Y_{C(v)}\big)\,dC(v)\Big)\Big[\frac12\sigma^2(J(s))\Phi''(J(s))+\mu(J(s))\Phi'(J(s))$$

$$-\big(\widetilde\lambda+f(J(s))\big)\Phi(J(s))+h(J(s))\big(\mathbf{E}_Y\{\Phi(\rho(J(s),Y))e^{-q(J(s),Y)}\}-\Phi(J(s))\big)\Big]\,ds,$$

where $\mathbf{E}_Y$ denotes the expectation only with respect to the random variable Y distributed as Y_1 and independent of the process J.

The computation of the expectation of the stochastic integral was correct, because all the integrands are bounded.

Taking the Laplace transform with respect to t with the parameter $\lambda > 0$ of both sides of the above equality, we obtain

$$\lambda \int_0^\infty e^{-\lambda t} \mathbf{E}_x \eta_{\widetilde{\lambda}}(\nu(t))\, dt - \Phi(x) = \mathbf{E}_x \int_0^\infty e^{-\widetilde{\lambda} s} \exp\Bigl(- \int_0^s \bigl(\lambda g(J(v)) + f(J(v))\bigr) dv \Bigr)$$

$$\times \exp\Bigl(- \int_{(0,s]} q\bigl(J(v-), Y_{C(v)}\bigr)\, dC(v) \Bigr) \Bigl[\frac{1}{2}\sigma^2(J(s))\Phi''(J(s)) + \mu(J(s))\Phi'(J(s))$$

$$-\bigl(\widetilde{\lambda} + f(J(s))\bigr)\Phi(J(s)) + h(J(s))\bigl(\mathbf{E}_Y \{\Phi(\rho(J(s),Y))e^{-q(J(s),Y)}\} - \Phi(J(s))\bigr)\Bigr]\, ds.$$

Alongside with the moment τ, we consider an exponentially distributed random moment $\widetilde{\tau}$ that is independent of J with the tail distribution $\mathbf{P}(\widetilde{\tau} > s) = e^{-\widetilde{\lambda} s}$. Applying Fubini's theorem, we can rewrite the above equality as

$$\mathbf{E}_x \eta_{\widetilde{\lambda}}(\nu(\tau)) - \Phi(x) = \frac{1}{\widetilde{\lambda}} \mathbf{E}_x \Bigl\{ \Bigl[\frac{1}{2}\sigma^2(J(\widetilde{\tau}))\Phi''(J(\widetilde{\tau})) + \mu(J(\widetilde{\tau}))\Phi'(J(s))$$

$$-\bigl(\widetilde{\lambda} + f(J(\widetilde{\tau}))\bigr)\Phi(J(\widetilde{\tau})) + h(J(\widetilde{\tau}))\bigl(\mathbf{E}_Y \{\Phi(\rho(J(\widetilde{\tau}),Y))e^{-q(J(\widetilde{\tau}),Y)}\} - \Phi(J(\widetilde{\tau}))\bigr)\Bigr]$$

$$\times \exp\Bigl(- \int_{(0,\widetilde{\tau}]} \bigl[\bigl(\lambda g(J(v)) + f(J(v))\bigr)dv + q\bigl(J(v-), Y_{C(v)}\bigr) dC(v)\bigr] \Bigr) \Bigr\}.$$

For the expectation on the right-hand side of this equality, we apply Theorem 3.1 in the variant given by formula (3.9), with τ replaced by $\widetilde{\tau}$ and λ replaced by $\widetilde{\lambda}$. As a result, we see that the function

$$\widetilde{Q}(x) := \mathbf{E}_x \eta_{\widetilde{\lambda}}(\nu(\tau)) - \Phi(x)$$

is the unique bounded solution of the integro-differential equation

$$\frac{1}{2}\sigma^2(x)\widetilde{Q}''(x) + \mu(x)\widetilde{Q}'(x) - (\widetilde{\lambda} + h(x) + \lambda g(x) + f(x))\widetilde{Q}(x) = -\frac{1}{2}\sigma^2(x)\Phi''(x)$$

$$- \mu(x)\Phi'(x) + (\widetilde{\lambda} + f(x))\Phi(x) - h(x)\bigl(\mathbf{E}\{\Phi(\rho(x,Y_1))e^{-q(x,Y_1)}\} - \Phi(x)\bigr)$$

$$- h(x)\mathbf{E}\Bigl\{e^{-q(x,Y_1)}\widetilde{Q}(\rho(x,Y_1))\Bigr\}.$$

Thus the function $V_{\widetilde{\lambda}}(x) := \mathbf{E}_x \eta_{\widetilde{\lambda}}(\nu(\tau)) = \widetilde{Q}(x) + \Phi(x)$ satisfies the equation

$$\frac{1}{2}\sigma^2(x)V''_{\widetilde{\lambda}}(x) + \mu(x)V'_{\widetilde{\lambda}}(x) - (\widetilde{\lambda} + h(x) + \lambda g(x) + f(x))V_{\widetilde{\lambda}}(x)$$

$$= -\lambda g(x)\Phi(x) - h(x)\mathbf{E}\bigl\{e^{-q(x,Y_1)}V_{\widetilde{\lambda}}(\rho(x,Y_1))\bigr\}, \qquad x \in \mathbf{R}. \tag{7.10}$$

From (7.9) it follows that

$$V(x) = \lim_{\widetilde{\lambda}\downarrow 0} \mathbf{E}_x \eta_{\widetilde{\lambda}}(\nu(\tau)) = \lim_{\widetilde{\lambda}\downarrow 0} V_{\widetilde{\lambda}}(x).$$

It should be noted that the expectation in the definition of V is reduced to the event $\{\nu < \infty\}$, because $e^{-\widetilde{\lambda}\nu} = 0$ for $\nu = \infty$ and $e^{-\widetilde{\lambda}\nu} \underset{\widetilde{\lambda}\downarrow 0}{\to} \mathbb{1}_{\{\nu<\infty\}}$.

Passing to the limit in (7.10) as $\widetilde{\lambda} \downarrow 0$, we deduce that V satisfies equation (7.7), or equation (7.1). Thus the theorem is proved for a bounded continuous function f and twice continuously differentiable function $\varPhi$ with bounded derivatives.

Any nonnegative piecewise-continuous function f can be approximated by a sequence of continuous functions $\{f_n\}$ such that $0 \le f_n(x) \le f(x)$, $x \in \mathbf{R}$. Any bounded piecewise-continuous function $\varPhi$ can be approximated by a sequence $\{\varPhi_n\}$ of uniformly bounded twice continuously differentiable functions with bounded derivatives. Applying the limit approximation method described in the proofs of Theorem 1.2 and 3.1 Ch. III, one can prove that V is the unique bounded solution of (7.1) under the assumptions of Theorem 7.1. □

§ 8. Random time change

Let the diffusion with jumps $J(t)$, $t \ge 0$, be determined by the parameters $\bigl(\mu(x),\, \sigma(x),\, \rho(x,y),\, \{Y_k\},\, h(x)\bigr)$, i.e., this diffusion is a solution of the stochastic differential equation

$$dJ(t) = \mu(J(t))\,dt + \sigma(J(t))\,dW(t) + \bigl(\rho\bigl(J(t-), Y_{C(t)}\bigr) - J(t-)\bigr)dC(t), \quad J(0) = x. \tag{8.1}$$

Let $g(x)$, $x \in \mathbf{R}$, be a twice continuously differentiable function with $g'(x) \ne 0$, $x \in \mathbf{R}$, thus, the inverse function $g^{(-1)}(x)$, $x \in g(\mathbf{R})$, is well defined. Consider the integral functional

$$A_t := \int_0^t \bigl(g'(J(s))\sigma(J(s))\bigr)^2\,ds, \qquad t \in [0,\infty),$$

as a function of the upper limit of integration. Assume that $A_\infty = \infty$ a.s., and define the inverse process:

$$a_t := \min\bigl\{s : A_s = t\bigr\}, \quad t \in [0,\infty).$$

Since A_t is a strictly increasing function, $\alpha_{0+} = 0$.

Theorem 8.1. *The process*

$$\widetilde{J}(t) =: g\bigl(J(a_t)\bigr), \quad t \in [0,\infty), \tag{8.2}$$

is a diffusion with jumps determined by the parameters $\bigl(\widetilde{\mu}(x), 1, \widetilde{\rho}(x,y), \{Y_k\}, \widetilde{h}(x)\bigr)$, *where* $\widetilde{\mu}(x) = D(g^{(-1)}(x))$,

$$D(x) = \frac{g''(x)}{2(g'(x))^2} + \frac{\mu(x)}{g'(x)\sigma^2(x)}, \tag{8.3}$$

$$\widetilde{\rho}(x,y) = g(\rho(g^{(-1)}(x),y)), \qquad \widetilde{h}(x) = \frac{h(g^{(-1)}(x))}{(g'(g^{(-1)}(x))\sigma(g^{(-1)}(x)))^2}, \tag{8.4}$$

i.e., $\widetilde{J}(t)$ is the solution of the stochastic differential equation

$$d\widetilde{J}(t) = d\widetilde{W}(t) + \widetilde{\mu}(\widetilde{J}(t))\,dt + \big(\widetilde{\rho}\big(\widetilde{J}(t-), Y_{\widetilde{C}(t)}\big) - \widetilde{J}(t-)\big)d\widetilde{C}(t), \quad \widetilde{J}(0) = g(x), \tag{8.5}$$

where $\widetilde{W}(t)$ is a Brownian motion, and $\widetilde{C}(t) = \max\{l : \widetilde{\varkappa}_l \le t\}$,

$$\widetilde{\varkappa}_l := \min\Big\{s \ge 0 : \int_0^s \widetilde{h}(\widetilde{J}(v))\,dv = \sum_{k=1}^{l} \tau_k\Big\}.$$

Proof. By Itô's formula (1.11),

$$g(J(u)) - g(x) = \int_0^u g'(J(s))\sigma(J(s))\,dW(s) + \int_0^u g'(J(s))\mu(J(s))\,ds$$

$$+ \frac{1}{2}\int_0^u g''(J(s))\sigma^2(J(s))\,ds + \int_{(0,u]} \big(g(\rho(J(s-), Y_{C(s)})) - g(J(s-))\big)dC(s).$$

Replacing u by a_t, we get

$$\widetilde{J}(t) - \widetilde{J}(0) = \int_0^{a_t} g'(J(s))\sigma(J(s))\,dW(s) + \int_0^{a_t} \big(g'(J(s))\sigma(J(s))\big)^2 D(J(s))\,ds$$

$$+ \int_{(0,a_t]} \big(g(\rho(J(s-), Y_{C(s)})) - g(J(s-))\big)dC(s).$$

Since a_t is the inverse function to A_t and $A'_t = \big(g'(J(t)\sigma(J(t))\big)^2$, we have

$$a'_t = \frac{1}{A'_{a_t}} = \frac{1}{\big(g'(J(a_t)\sigma(J(a_t))\big)^2}. \tag{8.6}$$

By Lévy's theorem (see § 8 Ch. II), the process

$$\widetilde{W}(t) := \int_0^{a_t} g'(J(s))\sigma(J(s))\,dW(s), \quad t \in [0,\infty),$$

is a Brownian motion. Consequently,

$$\widetilde{J}(t) - \widetilde{J}(0) = \widetilde{W}(t) + \int_0^t \big(g'(J(a_s))\sigma(J(a_s))\big)^2 D(J(a_s))\,da_s$$

$$+ \int_{(0,t]} \big(g(\rho(J(\alpha_s-), Y_{C(\alpha_s)})) - g(J(\alpha_s-))\big)dC(\alpha_s)$$

$$= \widetilde{W}(t) + \int_0^t \widetilde{\mu}(\widetilde{J}(s))\,ds + \int_{(0,t]} \big(\widetilde{\rho}(\widetilde{J}(s-), Y_{\widetilde{C}(s)})) - \widetilde{J}(s-)\big)d\widetilde{C}(s),$$

where we set $\widetilde{C}(s) := C(\alpha_s)$.

In order to make sure that $\widetilde{h}$ is the function determining the intensity of jumps of the process $\widetilde{J}$, it suffices to verify (see (1.8)) that $\widetilde{C}(t) = N(\widetilde{I}(t))$, where $\widetilde{I}(t) := \int_0^t \widetilde{h}(\widetilde{J}(v))\,dv$. Set $I(t) := \int_0^t h(J(v))\,dv$. Since the equality $C(t) = N(I(t))$ holds, $\widetilde{C}(t) = N(I(\alpha_t))$. Therefore, it is sufficient to verify that $\widetilde{I}(t) = I(\alpha_t)$. This follows from the equalities

$$I(\alpha_t) = \int_0^{a_t} h(J(v))\,dv = \int_0^t h(J(a_s))\,da_s$$

$$= \int_0^t \frac{h(J(a_s))}{\big(g'(J(a_s))\sigma(J(a_s))\big)^2}\,ds = \int_0^t \widetilde{h}(\widetilde{J}(s))\,ds = \widetilde{I}(t). \qquad \square$$

One can prove Theorem 8.1 in a different way with the help of Theorem 7.1 or, more exactly, with the help of Remark 7.2.

By this remark, the following statement holds. Let $\Phi(x)$ and $f(x)$, $x \in \mathbf{R}$, be continuous functions, Φ be bounded, and f be nonnegative. Since $a_\tau < \infty$ a.s., the function

$$V(x) := \mathbf{E}_x\Big\{\Phi(J(a_\tau)) \exp\Big(-\int_0^{a_\tau} f(J(s)\,ds\Big)\Big\} \tag{8.7}$$

is (see (7.7)) the unique bounded solution of the equation

$$\frac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) - \big(\lambda\big(g'(x)\sigma(x)\big)^2 + h(x) + f(x)\big)V(x)$$

$$= -\lambda\big(g'(x)\sigma(x)\big)^2\Phi(x) - h(x)\mathbf{E}V(\rho(x, Y_1)).$$

Changing the variable to

$$Q(x) := V(g^{(-1)}(x)), \qquad x \in \mathbf{R}, \tag{8.8}$$

we see that the function Q satisfies the equation

$$\frac{1}{2}Q''(x) + D(g^{(-1)}(x))Q'(x) - \Big(\lambda + \frac{h(g^{(-1)}(x)) + f(g^{(-1)}(x))}{\big(g'(g^{(-1)}(x))\sigma(g^{(-1)}(x))\big)^2}\Big)Q(x)$$

$$= -\lambda\Phi(g^{(-1)}(x)) - \frac{h(g^{(-1)}(x))}{\big(g'(g^{(-1)}(x))\sigma(g^{(-1)}(x))\big)^2}\mathbf{E}Q(g(\rho(g^{(-1)}(x), Y_1))),$$

or, by (8.3) and (8.4), Q is the solution of the equation

$$\frac{1}{2}Q''(x) + \widetilde{\mu}(x)Q'(x) - \Big(\lambda + \widetilde{h}(x) + \frac{f(g^{(-1)}(x))}{\big(g'(g^{(-1)}(x))\sigma(g^{(-1)}(x))\big)^2}\Big)Q(x)$$

$$= -\lambda \Phi(g^{(-1)}(x)) - \widetilde{h}(x)\mathbf{E}Q(\widetilde{\rho}(x, Y_1)), \qquad x \in \mathbf{R}. \tag{8.9}$$

By Remark 7.2, the solution of (8.9) has the probabilistic representation

$$Q(x) = \mathbf{E}_x\Big\{\Phi(g^{(-1)}(\widetilde{J}(\tau))) \exp\Big(- \int_0^\tau \frac{f(g^{(-1)}(\widetilde{J}(s)))}{\big(g'(g^{(-1)}(\widetilde{J}(s)))\sigma(g^{(-1)}(\widetilde{J}(s)))\big)^2}\, ds\Big)\Big\},$$

where the diffusion $\widetilde{J}$ is determined by the parameters $\big(\widetilde{\mu}(x),\ 1,\ \widetilde{\rho}(x,y),\ \{Y_k\},\ \widetilde{h}(x)\big)$. To understand the connection between the processes $J(t)$ and $\widetilde{J}(t)$, $t \geq 0$, we transform the function $V(x)$ as follows:

$$V(g^{(-1)}(x)) := \mathbf{E}_{g^{(-1)}(x)}\Big\{\Phi(J(a_\tau)) \exp\Big(- \int_0^\tau \frac{f(J(a_s))}{\big(g'(J(a_s))\sigma(J(a_s))\big)^2}\, ds\Big)\Big\}.$$

Let us compare the expressions obtained for the two sides of equality (8.8) in terms of expectations of the corresponding variables. Taking into account that the functions Φ and f are arbitrary, we conclude that the processes $J(a_t)$ and $g^{(-1)}(\widetilde{J}(t))$ are identical in law, i.e., their finite-dimensional distributions coincide, where the diffusion $\widetilde{J}$ is defined by formula (8.5) for some Brownian motion $\widetilde{W}$. We note that in the proof of Theorem 8.1, the application of Lévy's theorem gave us additional information, namely, the equality $J(a_t) = g^{(-1)}(\widetilde{J}(t))$. However, in the study of distributions of functionals such an equality is not so important, because for the investigation of distributions of functionals suffices the coincidence of the finite-dimensional distributions of the processes.

§ 9. Transformation of measure

Consider two homogeneous diffusions,

$$dX_l(t) = \sigma(X_l(t))dW(t) + \mu_l(X_l(t))dt, \qquad X_l(0) = x, \qquad l = 1, 2,$$

and two sequences of independent identically distributed random variables $Y_k^{(l)}$ with absolutely continuous distributions $\frac{dF_2(y)}{dF_1(y)} = p(y)$, $y \in \mathbf{R}$. Let J_l, $l = 1, 2$, be two diffusions with jumps defined by X_l, $Y_k^{(l)}$, $l = 1, 2$, and by the same function $\rho(x, y)$, $(x, y) \in \mathbf{R}^2$, and the functions $h_l(x)$, $x \in \mathbf{R}$, determining the intensity of jumps for each diffusion. Denote by C_l the process which counts the number of jumps performed by the diffusion J_l up to the time t. By (1.8), we have $C_l(t) = N(I_l(t))$, where $I_l(t) := \int_0^t h_l(J_l(v))\, dv$.

The main result of this section is the following.

Let $D([0, t])$ be the Skorohod space of functions without discontinuities of the second type (see § 6 Ch. I). Then for any bounded measurable functional $\wp(Z(s), 0 \leq s \leq t)$ on $D([0, t])$,

$$\mathbf{E}\wp(J_2(s), 0 \leq s \leq t) = \mathbf{E}\big\{\wp(J_1(s), 0 \leq s \leq t)\Theta(t)\big\}, \tag{9.1}$$

where

$$\Theta(t) := \prod_{k=1}^{C_1(t)} p(Y_k^{(1)}) \exp\Big(- \int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big) ds + \int_{(0,t]} \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} dC_1(s) \Big)$$

$$\times \exp\Big(\int_0^t \frac{\mu_2(J_1(s)) - \mu_1(J_1(s))}{\sigma(J_1(s))} \, dW(s) - \int_0^t \frac{(\mu_2(J_1(s)) - \mu_1(J_1(s)))^2}{2\sigma^2(J_1(s))} \, ds \Big). \qquad (9.2)$$

Remark 9.1. If $\rho(x,y) = x$, then the processes J_l, $l = 1,2$, have no jumps, they are homogeneous diffusion processes. In this case, (9.1) turns into the formula (10.12) Ch. II.

Indeed, since in this case the Poisson process N is independent of the diffusion J_1 and the process $C_1(t) = N(I_1(t))$ has the variable intensity $h_1(J_1(t))$,

$$\mathbf{E}_N \exp\Big(\int_{(0,t]} \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} dC_1(s) \Big) = \exp\Big(\int_0^t \Big(\exp\Big(\ln \frac{h_2(J_1(s))}{h_1(J_1(s))}\Big) - 1\Big) h_1(J_1(s)) \, ds \Big)$$

$$= \exp\Big(\int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big) \, ds \Big),$$

where the subscript N in the expectation means that the expectation is taken only with respect to the Poisson process N. In addition, $\mathbf{E} p(Y_k^{(1)}) = 1$. Now, using the independence of the processes J_1, N and the variables $Y_k^{(1)}$, $k = 1,2,\dots$, and applying Fubini's theorem, it is easy to check that (9.1) is transformed into (10.12) Ch. II.

Using the stochastic differentiation formula (1.11), one can rewrite the derivative $\Theta(t)$ without the stochastic integral.

Set

$$\beta(x) := \frac{1}{\sigma^2(x)}(\mu_2(x) - \mu_1(x)) \quad \text{and} \quad b(x) := \int_0^x \beta(y) \, dy.$$

Assume that β is a continuously differentiable function. Then a.s.

$$\Theta(t) = \prod_{k=1}^{C_1(t)} p(Y_k^{(1)}) \exp\Big(b(J_1(t)) - b(x) - \int_0^t \big(h_2(J_1(s)) - h_1(J_1(s))\big) \, ds \Big)$$

$$\times \exp\Big(- \int_0^t \frac{\mu_2^2(J_1(s)) - \mu_1^2(J_1(s))}{2\sigma^2(J_1(s))} \, ds - \frac{1}{2} \int_0^t \sigma^2(J_1(s)) \beta'(J_1(s)) \, ds$$

$$- \int_{(0,t]} \Big(b\big(\rho\big(J_1(s-), Y_{C_1(s)}^{(1)}\big)\big) - b(J_1(s-)) - \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} \Big) dC_1(s) \Big). \qquad (9.3)$$

Since the Laplace transform uniquely determines the function under transformation, the statement given by formula (9.1) is equivalent to the following one: for a random moment τ independent of the processes J_l, $l = 1, 2$, and exponentially distributed with an arbitrary parameter $\lambda > 0$,

$$\mathbf{E}_x \wp(J_2(s), 0 \le s \le \tau) = \mathbf{E}_x \big\{ \wp(J_1(s), 0 \le s \le \tau) \Theta(\tau) \big\}. \tag{9.4}$$

We prove (9.4) for functionals of integral type and for functionals that determine the position of the diffusion at the moment τ. We do this with the help of Theorem 3.1. Let

$$\wp(J_1(s), 0 \le s \le \tau) := \varPsi(J_1(\tau)) \exp\Big(- \int_0^\tau g(J_1(s))\, ds \Big),$$

where $\varPsi$ is bounded and g is nonnegative. By the Markov property of a diffusion with jumps, the functionals that describe the position of a diffusion, uniquely determine the measure associated with the process.

We argue as follows. For the chosen functional $\wp$ we consider the function

$$Q(x) := e^{b(x)} \mathbf{E}_x \big\{ \wp(J_1(s), 0 \le s \le \tau) \Theta(\tau) \big\}, \qquad x \in \mathbf{R}.$$

By Theorem 3.1, we derive an equation for $Q(x)$, $x \in \mathbf{R}$. Then we transform this equation in such a way that the solution, expressed in probabilistic form, is written only in terms of the process J_2. This transformation will lead to (9.4).

The lack of rigor of the further considerations concerns the correctness of the conditions that are needed to apply Theorem 3.1. Recall that this theorem is proved for nonnegative functions f and q. This imposes poorly foreseeable conditions on the parameters of diffusions with jumps, i.e., on $\big(\mu_l(x), \sigma_l(x), h_l(x)\big)$, $l = 1, 2$, and the density $p(y)$, $y \in \mathbf{R}$.

We start with the formula

$$\prod_{k=1}^{C_1(\tau)} p(Y_k^{(1)}) = \exp\Big(\int_{(0,\tau]} \ln p\big(Y_{C_1(s)}^{(1)}\big)\, dC_1(s) \Big).$$

Then $Q(x)$, $x \in \mathbf{R}$, is expressed in the form

$$\begin{aligned} Q(x) = \mathbf{E}_x \Big\{ \varPsi(J_1(\tau)) e^{b(J_1(\tau))} \exp\Big(&- \int_0^\tau g(J_1(s))\, ds \\ &- \int_0^\tau \Big(h_2(J_1(s)) - h_1(J_1(s)) + \frac{\mu_2^2(J_1(s)) - \mu_1^2(J_1(s))}{2\sigma^2(J_1(s))} + \frac{1}{2}\sigma^2(J_1(s))\beta'(J_1(s)) \Big) ds \\ &- \int_{(0,\tau]} \Big(b\big(\rho\big(J_1(s-), Y_{C_1(s)}^{(1)}\big)\big) - b(J_1(s-)) - \ln p\big(Y_{C_1(s)}^{(1)}\big) - \ln \frac{h_2(J_1(s-))}{h_1(J_1(s-))} \Big)\, dC_1(s) \Big) \Big\}. \end{aligned}$$

By Theorem 3.1 with $\Phi(x) = \Psi(x)e^{b(x)}$, $h(x) = h_1(x)$,

$$f(x) = g(x) + h_2(x) - h_1(x) + \frac{\mu_2^2(x) - \mu_1^2(x)}{2\sigma^2(x)} + \frac{1}{2}\sigma^2(x)\beta'(x),$$

$$q(z,y) = b(\rho(z,y)) - b(z) - \ln p(y) - \ln\frac{h_2(z)}{h_1(z)},$$

and $a = -\infty$, $b = \infty$, the function $Q(x)$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$Q(x) = M(x) + \int_{-\infty}^{\infty} G_z(x)e^{b(z)}\,\frac{h_2(z)}{h_1(z)}\,\mathbf{E}\{p(Y_1^{(1)})e^{-b(\rho(z,Y_1^{(1)}))}Q(\rho(z,Y_1^{(1)}))\}\,dz, \quad (9.5)$$

where $M(x)$, $x \in \mathbf{R}$, is the unique bounded solution of the equation

$$\frac{1}{2}\sigma^2(x)M''(x) + \mu_1(x)M'(x) - \Big(\lambda + g(x) + h_2(x) + \frac{\mu_2^2(x) - \mu_1^2(x)}{2\sigma^2(x)} + \frac{1}{2}\sigma^2(x)\beta'(x)\Big)M(x)$$
$$= -\lambda\Psi(x)e^{b(x)}, \quad (9.6)$$

and $G_z(x)$, $x \in \mathbf{R}$, is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2(x)G''(x) + \mu_1(x)G'(x) - \Big(\lambda + g(x) + h_2(x) + \frac{\mu_2^2(x) - \mu_2^2(x)}{2\sigma^2(x)}$$
$$+ \frac{1}{2}\sigma^2(x)\beta'(x)\Big)G(x) = 0, \qquad x \neq z, \quad (9.7)$$

$$G'(z+0) - G'(z-0) = -2h_1(z)/\sigma^2(z). \quad (9.8)$$

Making the change $\widetilde{M}(x) := e^{-b(x)}M(x)$ and $\widetilde{G}_z(x) := e^{-b(x)}G_z(x)e^{b(z)}\frac{h_2(z)}{h_1(z)}$, we see that these functions are the solutions of the following problems:

$$\frac{1}{2}\sigma^2(x)\widetilde{M}''(x) + \mu_2(x)\widetilde{M}'(x) - (\lambda + h_2(x) + g(x))\widetilde{M}(x) = -\lambda\Psi(x), \quad (9.9)$$

$$\frac{1}{2}\sigma^2(x)\widetilde{G}''(x) + \mu_2(x)\widetilde{G}'(x) - (\lambda + h_2(x) + g(x))\widetilde{G}(x) = 0, \quad x \neq z, \quad (9.10)$$

$$\widetilde{G}'(z+0) - \widetilde{G}'(z-0) = -2h_2(z)/\sigma^2(z). \quad (9.11)$$

From (9.5) it follows that $\widetilde{Q}(x) := e^{-b(x)}Q(x)$ is the solution of the equation

$$\widetilde{Q}(x) = \widetilde{M}(x) + \int_{-\infty}^{\infty} \widetilde{G}_z(x)\,\mathbf{E}\{\widetilde{Q}(\rho(z,Y_1^{(2)}))\}\,dz. \quad (9.12)$$

Applying again Theorem 3.1, with $\Phi(x) = \Psi(x)$, $f(x) = g(x)$, $h(x) = h_2(x)$, $q(z,y) = 0$ and $a = -\infty$, $b = \infty$, we have

$$\widetilde{Q}(x) := \mathbf{E}_x\Big\{\Psi(J_2(\tau))\exp\Big(-\int_0^{\tau} g(J_2(s))\,ds\Big)\Big\}. \quad (9.13)$$

Taking into account the equality $\widetilde{Q}(x) = e^{-b(x)}Q(x)$, $x \in \mathbf{R}$, and the definition of the function Q, we finally get

$$\mathbf{E}_x\Big\{\Psi(J_2(\tau))\exp\Big(-\int_0^\tau g(J_2(s))\,ds\Big)\Big\} = \mathbf{E}_x\Big\{\Psi(J_1(\tau))\exp\Big(-\int_0^\tau g(J_1(s))\,ds\Big)\Theta(\tau)\Big\}.$$

This proves (9.4) and, consequently, (9.1).

§ 10. Transition density of Brownian motion with jumps

Consider the *Brownian motion with jumps*

$$J^{(0)}(t) := \sigma W(t) + \sum_{k=1}^{N(\lambda_1 t)} Y_k, \qquad t \geq 0,$$

where W is a Brownian motion, $\sigma > 0$, $\lambda_1 > 0$, $N(t)$, $t \geq 0$, is a Poisson process with intensity 1, and Y_k, $k = 1, 2, \dots$, are independent identically distributed random variables. It is assumed that the Brownian motion, the Poisson process and the variables $\{Y_k\}_{k=1}^{\infty}$ are independent. The process $J^{(0)}$ is the sum of the Brownian motion and the compound Poisson process with intensity $\lambda_1 > 0$ and it has independent increments.

We assume that the density of the variables Y_k, $k = 1, 2, \dots$, takes the form

$$\frac{d}{dy}\mathbf{P}(Y_1 < y) = \frac{1}{2}\eta e^{-\eta|y|}, \qquad \eta > 0.$$

If $J^{(0)}(0) = 0$, then $J^{(0)}(t)$, $t \geq 0$, is a symmetric random process (the process $-J^{(0)}(t)$, $t \geq 0$, has the same finite-dimensional distributions as $J^{(0)}(t)$, $t \geq 0$).

The form of the density of the variables Y_k is dictated by the fact that for the Brownian motion, in view of Theorems 3.1, 5.1, the functions $M(x)$, $G_z(x)$, and $G_y^\lambda(x)$, $x \in \mathbf{R}$, are expressed in terms of exponential functions and this enables us to solve the integral equations (3.3), (5.4).

Let τ be the random time exponentially distributed with parameter $\lambda > 0$ and independent of the process $\{J^{(0)}(s), s \geq 0\}$ and the variables Y_k, $k = 1, 2, \dots$.

This section deals with the computation of the density of the variable $J^{(0)}(\tau)$, i.e.,

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(\tau) < y\big),$$

and the transition density of the process $J^{(0)}$, i.e.,

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t) < y\big), \qquad t > 0.$$

In many examples presented at the end of this chapter the algebraic equation

$$\frac{\eta^2}{(\eta^2 - \rho^2)}\frac{2\lambda_1}{(2\lambda + 2\lambda_1 - \sigma^2\rho^2)} = 1 \tag{10.1}$$

plays an important role. This equation is equivalent to the following one

$$\frac{\sigma^2\rho^2}{2} + \frac{\lambda_1\rho^2}{\eta^2-\rho^2} = \lambda, \tag{10.2}$$

whose positive roots are

$$\rho_1 = \sqrt{\frac{\lambda+\lambda_1}{\sigma^2} + \frac{\eta^2}{2} - \Bigl(\Bigl(\frac{\lambda+\lambda_1}{\sigma^2} + \frac{\eta^2}{2}\Bigr)^2 - \frac{2\lambda\eta^2}{\sigma^2}\Bigr)^{1/2}} \tag{10.3}$$

and

$$\rho_2 = \sqrt{\frac{\lambda+\lambda_1}{\sigma^2} + \frac{\eta^2}{2} + \Bigl(\Bigl(\frac{\lambda+\lambda_1}{\sigma^2} + \frac{\eta^2}{2}\Bigr)^2 - \frac{2\lambda\eta^2}{\sigma^2}\Bigr)^{1/2}}. \tag{10.4}$$

It is not hard to verify that $0<\rho_1<\eta<\rho_2$ and $\rho_1^2\rho_2^2 = 2\lambda\eta^2/\sigma^2$.

Proposition 10.1. *The density of the variable $J^{(0)}(\tau)$ has the form*

$$\frac{d}{dy}\mathbf{P}_x\bigl(J^{(0)}(\tau)<y\bigr) = \frac{\lambda(\eta^2-\rho_1^2)}{\sigma^2(\rho_2^2-\rho_1^2)\rho_1}e^{-|x-y|\rho_1} + \frac{\lambda(\rho_2^2-\eta^2)}{\sigma^2(\rho_2^2-\rho_1^2)\rho_2}e^{-|x-y|\rho_2}. \tag{10.5}$$

Proof. To compute the density (10.5), we apply Theorem 5.1 for the process $J(t) = J^{(0)}(t)$ with $\sigma(x)\equiv\sigma$, $\mu(x)\equiv 0$, $f(x)\equiv 0$, $h(x)=\lambda_1$, $a=-\infty$, $b=\infty$, and $\rho(x,y)=x+y$. In this case the density (10.5) is the function $\widetilde{G}_y(x)$. When $a=-\infty$ and $b=\infty$, we apply Remark 3.1.

The solutions of problems (3.6), (3.7) and (5.5), (5.6) are

$$G_y(x) = \frac{\lambda_1}{\sigma^2\Upsilon}e^{-|x-y|\Upsilon} \quad\text{and}\quad G_y^{\lambda}(x) = \frac{\lambda}{\sigma^2\Upsilon}e^{-|x-y|\Upsilon}, \tag{10.6}$$

respectively, where $\Upsilon := \sqrt{2\lambda+2\lambda_1}/\sigma$.

Since these functions are exponential, we find a solution of equation (5.4) in the form

$$\widetilde{G}_y(x) = Ae^{-|x-y|q_1} + Be^{-|x-y|q_2}, \tag{10.7}$$

where A and B are some constants, and q_l, $l=1,2$, are nonnegative numbers.

We refer to the following equality: for $\mu>0$ and $\rho>0$

$$\int_{-\infty}^{\infty} e^{-|x-z|\mu}e^{-|z-y|\rho}\,dz = \frac{2}{\mu^2-\rho^2}\bigl(\mu e^{-|x-y|\rho} - \rho e^{-|x-y|\mu}\bigr), \tag{10.8}$$

which is easy to verify. Using (10.8), it is not hard to show that

$$\mathbf{E}\widetilde{G}_y(z+Y_1) = \frac{A\eta}{\eta^2-q_1^2}\bigl(\eta e^{-|z-y|q_1} - q_1e^{-|z-y|\eta}\bigr) + \frac{B\eta}{\eta^2-q_2^2}\bigl(\eta e^{-|z-y|q_2} - q_2e^{-|z-y|\eta}\bigr).$$

We apply (10.8) once more to obtain that

$$\int_{-\infty}^{\infty} G_z(x)\,\mathbf{E}\widetilde{G}_y(z+Y_1)\,dz = \frac{A\eta^2}{(\eta^2-q_1^2)}\frac{2\lambda_1}{(2\lambda+2\lambda_1-\sigma^2q_1^2)}\Bigl(e^{-|x-y|q_1} - \frac{q_1}{\Upsilon}e^{-|x-y|\Upsilon}\Bigr)$$
$$+ \frac{B\eta^2}{(\eta^2-q_2^2)}\frac{2\lambda_1}{(2\lambda+2\lambda_1-\sigma^2q_2^2)}\Bigl(e^{-|x-y|q_2} - \frac{q_2}{\Upsilon}e^{-|x-y|\Upsilon}\Bigr)$$
$$- \Bigl(\frac{A\eta q_1}{(\eta^2-q_1^2)} + \frac{B\eta q_2}{(\eta^2-q_2^2)}\Bigr)\frac{2\lambda_1}{(2\lambda+2\lambda_1-\sigma^2\eta^2)}\Bigl(e^{-|x-y|\eta} - \frac{\eta}{\Upsilon}e^{-|x-y|\Upsilon}\Bigr).$$

We substitute this expression, (10.7), and the right-hand side equality in (10.6) into equation (5.4). The coefficients of the same exponential functions in (5.4) must be equal to each other. Therefore, equating the coefficients at $e^{-|x-y|q_1}$, $e^{-|x-y|q_2}$, we get that the values q_1 and q_2 must satisfy equation (10.1), i.e., $q_1 = \rho_1$ and $q_2 = \rho_2$.

The coefficient at $e^{-|x-y|\eta}$ must be equal to zero, hence,

$$\frac{A\rho_1}{(\eta^2-\rho_1^2)} + \frac{B\rho_2}{(\eta^2-\rho_2^2)} = 0. \tag{10.9}$$

In addition, we take into account (10.1) for $\rho = \rho_1$, $\rho = \rho_2$ and equate the coefficients at $e^{-|x-y|\Upsilon}$ to deduce that

$$\lambda = A\rho_1\sigma^2 + B\rho_2\sigma^2. \tag{10.10}$$

Solving the algebraic system (10.9), (10.10), we finally get

$$A = \frac{\lambda(\eta^2-\rho_1^2)}{\sigma^2(\rho_2^2-\rho_1^2)\rho_1}, \qquad B = \frac{\lambda(\rho_2^2-\eta^2)}{\sigma^2(\rho_2^2-\rho_1^2)\rho_2}.$$

Substituting these coefficients to (10.7), we obtain (10.5).

Relation (10.5) can be derived directly. Indeed, we use formula (1.1) with $\mu = 0$. The Laplace transform with respect to t of the characteristic function $\mathbf{E}_x e^{i\alpha J^{(0)}(t)}$ has the expression

$$\mathbf{E}_x e^{i\alpha J^{(0)}(\tau)} = \mathbf{E}\exp\Big(i\alpha x - \frac{1}{2}\alpha^2\sigma^2\tau - \lambda_1\tau\big(1-\mathbf{E}e^{i\alpha Y_1}\big)\Big) = \frac{\lambda e^{i\alpha x}}{\lambda+\alpha^2\sigma^2/2+\lambda_1\big(1-\mathbf{E}e^{i\alpha Y_1}\big)}.$$

In our case,

$$\mathbf{E}e^{i\alpha Y_1} = \int_{-\infty}^{\infty} e^{i\alpha y}\frac{1}{2}\eta e^{-\eta|y|}\,dy = \frac{\eta}{2}\Big(\frac{1}{\eta+i\alpha}+\frac{1}{\eta-i\alpha}\Big) = \frac{\eta^2}{\eta^2+\alpha^2}. \tag{10.11}$$

Therefore,

$$\mathbf{E}_x e^{i\alpha J^{(0)}(\tau)} = \frac{2\lambda e^{i\alpha x}(\eta^2+\alpha^2)}{(2\lambda+\alpha^2\sigma^2)(\eta^2+\alpha^2)+2\lambda_1\alpha^2} = \frac{2\lambda e^{i\alpha x}(\eta^2+\alpha^2)}{\sigma^2(\alpha^2+\rho_1^2)(\alpha^2+\rho_2^2)}$$
$$= \frac{2\lambda(\eta^2-\rho_1^2)}{\sigma^2(\rho_2^2-\rho_1^2)}\frac{e^{i\alpha x}}{(\alpha^2+\rho_1^2)} + \frac{2\lambda(\rho_2^2-\eta^2)}{\sigma^2(\rho_2^2-\rho_1^2)}\frac{e^{i\alpha x}}{(\alpha^2+\rho_2^2)}. \tag{10.12}$$

Inverting in this formula the Fourier transform with respect to α (see (10.11)), we get (10.5). □

Proposition 10.2. *For the transition density of the Brownian motion with jumps we have the expressions*

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t)<y\big) = e^{-\lambda_1 t}\int_0^{\infty} e^{-\eta^2 u}e^{-(x-y)^2/2(2u+\sigma^2 t)}\Big(\frac{\eta^2}{\sqrt{2\pi(2u+\sigma^2 t)}}$$

$$+\frac{2u+\sigma^2t-(x-y)^2}{\sqrt{2\pi}(2u+\sigma^2t)^{5/2}}\Big)I_0\big(2\eta\sqrt{\lambda_1tu}\big)\,du \tag{10.13}$$

and

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t)<y\big)=e^{-\lambda_1t}\biggl(\frac{1}{\sqrt{2\pi t}\,\sigma}e^{-(y-x)^2/2\sigma^2t}$$

$$+\int_0^\infty e^{-\eta^2u}e^{-(y-x)^2/2(2u+\sigma^2t)}\frac{\eta\sqrt{\lambda_1t}}{\sqrt{2\pi u(2u+\sigma^2t)}}I_1\big(2\eta\sqrt{\lambda_1tu}\big)\,du\biggr). \tag{10.14}$$

Remark 10.1. Formula (10.13) coincides with (10.14) which is easily verified by integration by parts if one uses the equality $\big(I_0(x)\big)'=I_1(x)$. The fact that the integral on the right-hand side of (10.14) with respect to y is equal to one, follows from (10.18).

Proof of Proposition 10.2. To illustrate the possible approaches for the computation of the transition density, we prove the formulas (10.13) and (10.14) by different ways.

We first prove (10.13). In (10.5) we invert the Laplace transform with respect to λ. Consider the new parameter $\widetilde\lambda:=\dfrac{\lambda+\lambda_1}{\sigma^2}-\dfrac{\eta^2}{2}$. For brevity we denote $\delta:=\dfrac{\sqrt{2\lambda_1}\eta}{\sigma}$. In terms of these notations we have $\rho_1=\sqrt{\widetilde\lambda+\eta^2-(\widetilde\lambda^2+\delta^2)^{1/2}}$ and $\rho_2=\sqrt{\widetilde\lambda+\eta^2+(\widetilde\lambda^2+\delta^2)^{1/2}}$. Then, using formula *a* of Appendix 3, we obtain

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t)<y\big)=\mathcal{L}_\lambda^{-1}\Big(\frac{1}{\lambda}\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(\tau)<y\big)\Big)\Big|_t$$

$$=\frac12e^{-(\lambda_1-\eta^2\sigma^2/2)t}\mathcal{L}_{\widetilde\lambda}^{-1}\biggl(\Big(\frac{\eta^2}{\rho_1}-\rho_1\Big)\frac{e^{-|x-y|\rho_1}}{\sqrt{\widetilde\lambda^2+\delta^2}}-\Big(\frac{\eta^2}{\rho_2}-\rho_2\Big)\frac{e^{-|x-y|\rho_2}}{\sqrt{\widetilde\lambda^2+\delta^2}}\biggr)\bigg|_{\sigma^2t}, \tag{10.15}$$

where $\mathcal{L}_\lambda^{-1}$ is the operator of the inverse Laplace transform with respect to the parameter λ.

We use (see formulas 3 and 5 of Appendix 3) the equality

$$\Big(\frac{\eta^2}{\sqrt\gamma}-\sqrt\gamma\Big)e^{-|x-y|\sqrt\gamma}=\int_0^\infty e^{-\gamma v}\frac{1}{\sqrt{\pi v}}\Big(\eta^2+\frac{1}{2v}-\frac{(x-y)^2}{4v^2}\Big)e^{-(x-y)^2/4v}\,dv.$$

Using this formula in (10.15) under the sign of the inverse Laplace transform, we get

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t)<y\big)=e^{-(\lambda_1-\eta^2\sigma^2/2)t}\int_0^\infty\frac{1}{2\sqrt{\pi v}}\Big(\eta^2+\frac{1}{2v}-\frac{(x-y)^2}{4v^2}\Big)e^{-(x-y)^2/4v}$$

$$\times e^{-v\eta^2}\mathcal{L}_{\widetilde\lambda}^{-1}\biggl(\frac{e^{-v\left(\widetilde\lambda-\sqrt{\widetilde\lambda^2+\delta^2}\right)}}{\sqrt{\widetilde\lambda^2+\delta^2}}-\frac{e^{-v\left(\widetilde\lambda+\sqrt{\widetilde\lambda^2+\delta^2}\right)}}{\sqrt{\widetilde\lambda^2+\delta^2}}\biggr)\bigg|_{\sigma^2t}dv. \tag{10.16}$$

We apply formula (34) § 5.6 from Bateman and Erdélyi (1954):

$$\mathcal{L}_\lambda^{-1}\left(\frac{e^{v\left(\lambda-\sqrt{\lambda^2+\delta^2}\right)}}{\sqrt{\lambda^2+\delta^2}}\right)\Bigg|_t = J_0\big(\delta\sqrt{t(t+2v)}\big). \qquad (10.17)$$

To invert the Laplace transform of the first term in (10.16) we apply this formula, substituting $-v$ instead of v. To compute the inverse Laplace transform of the second term we first use formula b of Appendix 3. Then we get

$$\mathcal{L}_\lambda^{-1}\left(\frac{e^{-v\left(\lambda+\sqrt{\lambda^2+\delta^2}\right)}}{\sqrt{\lambda^2+\delta^2}}\right)\Bigg|_t = \mathcal{L}_\lambda^{-1}\left(\frac{e^{v\left(\lambda-\sqrt{\lambda^2+\delta^2}\right)}}{\sqrt{\lambda^2+\delta^2}}\right)\Bigg|_{t-2v}\mathbb{1}_{[2v,\infty)}(t).$$

Now, again, we can apply (10.17), and obtain

$$\frac{d}{dy}\mathbf{P}_x\big(J^{(0)}(t)<y\big) = e^{-(\lambda_1-\eta^2\sigma^2/2)t}\int_0^\infty \frac{1}{2\sqrt{\pi v}}\Big(\eta^2+\frac{1}{2v}-\frac{(x-y)^2}{4v^2}\Big)e^{-(x-y)^2/4v}$$

$$\times e^{-v\eta^2}\Big(J_0\big(\delta\sqrt{\sigma^2t(\sigma^2t-2v)}\big)-J_0\big(\delta\sqrt{\sigma^2t(\sigma^2t-2v)}\big)\mathbb{1}_{[2v,\infty)}(\sigma^2t)\Big)\,dv$$

$$= e^{-(\lambda_1-\eta^2\sigma^2/2)t}\int_{\sigma^2t/2}^\infty \frac{1}{2\sqrt{\pi v}}\Big(\eta^2+\frac{2v-(x-y)^2}{4v^2}\Big)e^{-v\eta^2-(x-y)^2/4v}J_0\big(\delta\sigma\sqrt{t(\sigma^2t-2v)}\big)\,dv.$$

Making the change of variable $u = v-\sigma^2t/2$, we get (10.13), because $I_0(x) = J_0(ix)$.

We now prove (10.14). To compute the transition density at a fixed time t, we use (1.1) with $\mu=0$ and (10.11):

$$\mathbf{E}_x e^{i\alpha J^{(0)}(t)} = \exp\Big(i\alpha x-\frac{\alpha^2\sigma^2t}{2}-\lambda_1t\big(1-\mathbf{E}e^{i\alpha Y_1}\big)\Big)$$

$$= \exp\Big(i\alpha x-\lambda_1t-\frac{\alpha^2\sigma^2t}{2}+\frac{\lambda_1t\eta^2}{\eta^2+\alpha^2}\Big).$$

We apply formula 15 of Appendix 3:

$$e^{\rho/\gamma}-1 = \int_0^\infty e^{-\gamma u}\frac{\sqrt{\rho}}{\sqrt{u}}I_1(2\sqrt{\rho u})\,du, \qquad \rho>0. \qquad (10.18)$$

Then for $\gamma=\eta^2+\alpha^2$ the previous equality has the form

$$\mathbf{E}_x e^{i\alpha J^{(0)}(t)} = e^{i\alpha x-\lambda_1t}\bigg(e^{-\alpha^2\sigma^2t/2}+\int_0^\infty e^{-\eta^2u}e^{-\alpha^2(2u+\sigma^2t)/2}\frac{\eta\sqrt{\lambda_1t}}{\sqrt{u}}I_1\big(2\eta\sqrt{\lambda_1tu}\big)\,du\bigg).$$

To invert the Fourier transform with respect to α, one can use the expression for the characteristic function of the Gaussian distribution (see formulas (8.1) and (8.2) Ch. I). This yields (10.14). □

We consider the particular case of formula (10.5) as $\sigma \to 0$. In this case $\rho_1 \sim \dfrac{\eta\sqrt{\lambda}}{\sqrt{\lambda+\lambda_1}}$ and $\rho_2 \sim \sqrt{2\lambda+2\lambda_1}/\sigma$. We deduce from (10.5) with $x=0$ that

$$\frac{d}{dy}\mathbf{P}\biggl(\sum_{k=1}^{N(\lambda_1\tau)} Y_k < y\biggr) = \frac{\eta\lambda_1\sqrt{\lambda}}{2(\lambda+\lambda_1)^{3/2}} \exp\Bigl(-\frac{|y|\eta\sqrt{\lambda}}{\sqrt{\lambda+\lambda_1}}\Bigr).$$

The distribution with such density has a mass point at zero, i.e.,

$$\mathbf{P}\biggl(\sum_{k=1}^{N(\lambda_1\tau)} Y_k = 0\biggr) = \mathbf{P}(\tau < \tau_1) = \frac{\lambda}{\lambda+\lambda_1}.$$

§ 11. Distributions of infimum or supremum of Brownian motion with linear drift with jumps

Consider the process of $J^{(\mu)}(t)$, $t \ge 0$, defined in § 1. This process is the special case of the process J, defined in (1.2), (1.3), for $\sigma(x) \equiv \sigma$, $\mu(x) \equiv \mu$, $h(x) = \lambda_1$ and $\rho(x,y) = x+y$.

Consider as examples the calculation of the probabilities

$$\mathbf{P}_x\Bigl(a \le \inf_{0\le s\le\tau} J^{(\mu)}(s)\Bigr) \qquad \text{and} \qquad \mathbf{P}_x\Bigl(\sup_{0\le s\le\tau} J^{(\mu)}(s) \le b\Bigr),$$

where τ is the exponentially distributed with parameter $\lambda > 0$ random time independent of the process $J^{(\mu)}$. These probabilities play an important role in ruin theory. Let $H_b = \min\{s : J^{(\mu)}(s) > b\}$ be the first exceedance moment of the level b. Then

$$\mathbf{P}_x\Bigl(\sup_{0\le s\le\tau} J^{(\mu)}(s) > b\Bigr) = \mathbf{P}_x(H_b \le \tau) = \mathbf{E}_x e^{-\lambda H_b}.$$

The moment H_b can be interpreted as the ruin moment if the expenses $J^{(\mu)}$ exceed the available capital b.

Consider the problem of computing the distribution of the supremum. The case of the infimum is dealt with similarly. To compute the probability

$$Q(x) = \mathbf{P}_x\Bigl(\sup_{0\le s\le\tau} J^{(\mu)}(s) \le b\Bigr)$$

we apply Theorem 3.1 with $\varPhi(x) \equiv 1$, $f(x) \equiv 0$, $h(x) \equiv \lambda_1$, $q(x,y) \equiv 0$ and $a = -\infty$.

By (3.3), the function $Q(x)$ is the unique bounded solution of the equation

$$Q(x) = M(x) + \int_{-\infty}^{b} G_z(x)\mathbf{E}Q(z+Y_1)\,dz. \tag{11.1}$$

In this case the function M is the unique bounded solution of the problem

$$\frac{1}{2}\sigma^2 M''(x) + \mu M'(x) - (\lambda+\lambda_1)M(x) = -\lambda, \qquad x \in (-\infty, b), \tag{11.2}$$

$$M(b) = 0. \tag{11.3}$$

In addition, $M(x) = 0$ for $x \ge b$. The solution of the problem (11.2), (11.3) has the form

$$M(x) = \Big(\frac{\lambda}{\lambda+\lambda_1} - \frac{\lambda}{\lambda+\lambda_1}\exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\Upsilon}{\sigma^2}\Big)\Big)\mathbb{1}_{(-\infty,b]}(x),$$

where $\Upsilon := \sqrt{(2\lambda+2\lambda_1)\sigma^2 + \mu^2}$.

Under the above assumptions the function $G_z(x)$ is the unique bounded solution of the problem

$$\frac{1}{2}\sigma^2 G''(x) + \mu G'(x) - (\lambda+\lambda_1)G(x) = 0, \qquad x \in (-\infty, b)\setminus\{z\}, \tag{11.4}$$

$$G'(z+0) - G'(z-0) = -2\lambda_1/\sigma^2, \tag{11.5}$$

$$G(b) = 0. \tag{11.6}$$

We also set $G_z(x) = 0$ for $z \ge b$ or $x \ge b$.

It is not hard to compute that

$$G_z(x) = \frac{\lambda_1}{\Upsilon}\Big(\exp\Big(\frac{\mu(z-x)}{\sigma^2} - \frac{|z-x|\Upsilon}{\sigma^2}\Big)$$

$$- \exp\Big(\frac{\mu(z-x)}{\sigma^2} - \frac{(2b-z-x)\Upsilon}{\sigma^2}\Big)\Big)\mathbb{1}_{(-\infty,b]}(x)\mathbb{1}_{(-\infty,b]}(z). \tag{11.7}$$

Since the function M and the kernel G_z of the integral equation (11.1) are expressed in terms of exponential functions, we can find the solution of (11.1), under some additional assumptions on the distribution function of Y_k, in the form of a linear combination of exponential functions.

For this we need the following formula: for $x \le b$

$$\int_{-\infty}^{\infty} G_z(x)e^{\rho z}\,dz = \frac{2\lambda_1}{2\lambda+2\lambda_1-\rho^2\sigma^2-2\rho\mu}\Big(e^{\rho x} - e^{\rho b}\exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\Upsilon}{\sigma^2}\Big)\Big). \tag{11.8}$$

The validity of (11.8) can be justified as follows. Since G_z is the Green function corresponding to the problem (11.2), (11.3), the function (11.8) is the unique bounded solution of the equation (11.2) with $-\lambda_1 e^{\rho x}$ in place of $-\lambda$ on the right-hand side of (11.2), and with the boundary condition (11.3).

Example 11.1. Let the random variables Y_k, $k = 1, 2, \dots$, take only non-positive values. This means that the time of the first exceedance of the level b by the process $J^{(\mu)}$ is transformed into the first hitting time moment. The process $J^{(\mu)}$ is a process with independent increments and with the negative jumps.

The *cumulant* $K(\cdot)$ of the process $J^{(\mu)}$ is determined from the Lévy-Khintchine representation

$$\mathbf{E}_0 e^{i\alpha J^{(\mu)}(t)} = e^{tK(\alpha)}, \qquad \alpha \in \mathbf{R},$$

and in this case, for $\operatorname{Re} v \geq 0$ the cumulant is (see (1.1))

$$K(-iv) = \frac{\sigma^2}{2} v^2 + \mu v + \lambda_1 \big(\mathbf{E} e^{vY_1} - 1\big). \tag{11.9}$$

We find the probability $Q(x) = \mathbf{P}_x\Big(\sup_{0 \leq s \leq \tau} J^{(\mu)}(s) \leq b\Big)$ as the solution of equation (11.1) in the form

$$Q(x) = \big(C - Ae^{-\beta(b-x)}\big) \mathbb{1}_{(-\infty,b]}(x), \qquad\qquad \beta > 0, \tag{11.10}$$

where C, A, and β are some constants. The condition $\beta > 0$ is necessary for the solution (11.10) to be bounded at $-\infty$.

In order to substitute the solution (11.10) in equation (11.1), we compute the function $\mathbf{E}Q(z + Y_1)$. By the negativity of Y_1, we derive that

$$\mathbf{E}Q(z+Y_1) = C\mathbf{E}\mathbb{1}_{(-\infty,b-z]}(Y_1) - Ae^{-\beta(b-z)}\mathbf{E}\big\{e^{\beta Y_1}\mathbb{1}_{(-\infty,b-z]}(Y_1)\big\}$$
$$= C - Ae^{-\beta(b-z)}\mathbf{E}e^{\beta Y_1}, \qquad z \leq b.$$

Using the explicit form of $M(x)$, $x \leq b$, and the formula (11.8) for $\rho = 0$, we have

$$M(x) + C\int_{-\infty}^{\infty} G_x(z)\,dz = \frac{\lambda + C\lambda_1}{\lambda + \lambda_1} - \frac{\lambda + C\lambda_1}{\lambda + \lambda_1} \exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\Upsilon}{\sigma^2}\Big).$$

Substituting this expression and the expression for $\mathbf{E}Q(x+Y_1)$ in (11.1), and taking into account (11.8), we get

$$C - Ae^{-\beta(b-x)} = \frac{\lambda + C\lambda_1}{\lambda + \lambda_1} - \frac{\lambda + C\lambda_1}{\lambda + \lambda_1} \exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\Upsilon}{\sigma^2}\Big)$$
$$- \frac{2\lambda_1 A\,\mathbf{E}e^{\beta Y_1}}{2\lambda + 2\lambda_1 - \beta^2\sigma^2 - 2\beta\mu}\Big(e^{-\beta(b-x)} - \exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\Upsilon}{\sigma^2}\Big)\Big).$$

Equating the coefficients at the constant terms, we have $C = 1$. Proceeding similarly with the coefficients at $e^{-\beta(b-x)}$, we see that β is the solution of the equation

$$\frac{2\lambda_1 \mathbf{E}e^{\beta Y_1}}{2\lambda + 2\lambda_1 - \beta^2\sigma^2 - 2\beta\mu} = 1,$$

or, equivalently, the equation

$$K(-i\beta) = \lambda, \tag{11.11}$$

where the cumulant $K(-i\beta)$ is given by (11.9). Since the function $\mathbf{E}e^{\beta Y_1}$, $\beta \in \mathbf{R}$, is decreasing and takes values in the interval $(0, 1]$, equation (11.11) has only one nonnegative root, thus β is unique.

Choosing in (11.10) the parameter β equal to this root and equating the coefficients at $\exp\Big(\frac{\mu(b-x)}{\sigma^2}-\frac{(b-x)\Upsilon}{\sigma^2}\Big)$, we get $A=1$.

Thus,

$$\mathbf{P}_x\Big(\sup_{0\le s\le\tau} J^{(\mu)}(s)\le b\Big)=(1-e^{-\beta(b-x)})\mathbb{I}_{(-\infty,b]}(x), \tag{11.12}$$

where β is the unique nonnegative root of equation (11.11).

Another approach to the computation of this probability is given in the monograph of Gihman and Skorohod (1975) § 2 Ch. IV.

Example 11.2. Let the distribution function of the random variables Y_k, $k=1,2,\dots$, be arbitrary on the negative real half-line including zero, while on the positive half-line it has the density $\frac{d}{dy}\mathbf{P}(Y_1<y)=\mathbf{P}(Y_1>0)\eta e^{-\eta y}$, $y>0$, for some $\eta>0$.

In this case we find the probability $Q(x)=\mathbf{P}_x\Big(\sup_{0\le s\le\tau} J^{(\mu)}(s)\le b\Big)$ as the solution of equation (11.1) in the following form

$$Q(x)=\big(1-A_1e^{-\beta_1(b-x)}-A_2e^{-\beta_2(b-x)}\big)\mathbb{I}_{(-\infty,b]}(x), \qquad \beta_1>0,\ \beta_2>0.$$

Contrary to the representation (11.10) we choose $C=1$ at the beginning.

For $x\le b$, we have

$$\begin{aligned}\mathbf{E}Q(x+Y_1)&=\mathbf{E}\mathbb{I}_{(-\infty,b-x]}(Y_1)-\sum_{k=1}^2 A_ke^{-\beta_k(b-x)}\mathbf{E}\big\{e^{\beta_kY_1}\mathbb{I}_{(-\infty,b-x]}(Y_1)\big\}\\ &=1-\mathbf{P}(Y_1>0)\Big(1-\sum_{k=1}^2\frac{A_k\eta}{\eta-\beta_k}\Big)e^{-\eta(b-x)}\\ &\quad-\sum_{k=1}^2 A_k\Big(\frac{\mathbf{P}(Y_1>0)\eta}{\eta-\beta_k}+\mathbf{E}\big\{e^{\beta_kY_1}\mathbb{I}_{\{Y_1\le0\}}\big\}\Big)e^{-\beta_k(b-x)}.\end{aligned}$$

Substituting this expression in (11.1) and using (11.8), we obtain first that

$$1-\frac{A_1\eta}{\eta-\beta_1}-\frac{A_2\eta}{\eta-\beta_2}=0, \tag{11.13}$$

and second that

$$\begin{aligned}1-\sum_{k=1}^2 A_ke^{-\beta_k(b-x)}&=1-\exp\Big(\frac{\mu(b-x)}{\sigma^2}-\frac{(b-x)\Upsilon}{\sigma^2}\Big)-\sum_{k=1}^2\frac{2\lambda_1A_k}{2\lambda+2\lambda_1-\beta_k^2\sigma^2-2\beta_k\mu}\\ &\times\Big(\frac{\mathbf{P}(Y_1>0)\eta}{\eta-\beta_k}+\mathbf{E}\big\{e^{\beta_kY_1}\mathbb{I}_{\{Y_1\le0\}}\big\}\Big)\Big(e^{-\beta_k(b-x)}-\exp\Big(\frac{\mu(b-x)}{\sigma^2}-\frac{(b-x)\Upsilon}{\sigma^2}\Big)\Big).\end{aligned}$$

Equating the coefficients at e^{β_kx}, we find that β_k, $k=1,2$, are the solutions of the equation

$$\frac{2\lambda_1}{2\lambda+2\lambda_1-\beta^2\sigma^2-2\beta\mu}\Big(\frac{\mathbf{P}(Y_1>0)\eta}{\eta-\beta}+\mathbf{E}\big\{e^{\beta Y_1}\mathbb{I}_{\{Y_1\le0\}}\big\}\Big)=1, \tag{11.14}$$

or, equivalently, the equation

$$\frac{\sigma^2}{2}\beta^2 + \mu\beta + \lambda_1 \mathbf{E}\{(e^{\beta Y_1} - 1)\mathrm{1\!I}_{\{Y_1 \le 0\}}\} + \frac{\lambda_1 \mathbf{P}(Y_1 > 0)\eta}{\eta - \beta} = \lambda. \tag{11.15}$$

Analyzing graphs of the functions figuring in this equation, we can see that for $\eta > 0$ it has exactly two positive roots β_1, β_2 such that $0 < \beta_1 < \eta < \beta_2$.

Equating the coefficients at $\exp\Big(\frac{\mu(b-x)}{\sigma^2} - \frac{(b-x)\varUpsilon}{\sigma^2}\Big)$, we find that $A_1 + A_2 = 1$. Solving this equation together with (11.13), we get

$$A_1 = \frac{\beta_1(\beta_2 - \eta)}{\eta(\beta_2 - \beta_1)}, \qquad A_2 = \frac{\beta_2(\eta - \beta_1)}{\eta(\beta_2 - \beta_1)}.$$

The final answer is: for $x \le b$

$$\mathbf{P}_x\Big(\sup_{0 \le s \le \tau} J^{(\mu)}(s) \le b\Big) = 1 - \frac{\beta_1(\beta_2 - \eta)}{\eta(\beta_2 - \beta_1)} e^{-\beta_2(b-x)} - \frac{\beta_2(\eta - \beta_1)}{\eta(\beta_2 - \beta_1)} e^{-\beta_1(b-x)}. \tag{11.16}$$

For another approach to computing this probability see Mordecki (2003), Kou and Wang (2003).

Note that for $\eta = 0$ or for the limiting case as $\eta \to \infty$, the variables Y_k, $k = 1, 2, \dots$, take only negative values. In these cases equation (11.15) is transformed to equation (11.11) and (11.16) turns into (11.12).

We consider the special case when $\mu = 0$ and the density of Y_k, $k = 1, 2, \dots$, has the form

$$\frac{d}{dy}\mathbf{P}(Y_1 < y) = \frac{1}{2}\eta e^{-\eta|y|}, \qquad \eta > 0.$$

Equation (11.14) is transformed into (10.1), and $\beta_1 = \rho_1$, $\beta_2 = \rho_2$, where ρ_1 and ρ_2 are defined by (10.3) and (10.4). As a result, (11.16) takes the following form: for $x \le b$

$$\mathbf{P}_x\Big(\sup_{0 \le s \le \tau} J^{(0)}(s) \le b\Big) = 1 - \frac{\rho_1(\rho_2 - \eta)}{\eta(\rho_2 - \rho_1)} e^{-\rho_2(b-x)} - \frac{\rho_2(\eta - \rho_1)}{\eta(\rho_2 - \rho_1)} e^{-\rho_1(b-x)}. \tag{11.17}$$

Using the symmetry property of the process $J^{(0)}$, we conclude that for $x \ge a$

$$\mathbf{P}_x\Big(a \le \inf_{0 \le s \le \tau} J^{(0)}(s)\Big) = 1 - \frac{\rho_1(\rho_2 - \eta)}{\eta(\rho_2 - \rho_1)} e^{-\rho_2(x-a)} - \frac{\rho_2(\eta - \rho_1)}{\eta(\rho_2 - \rho_1)} e^{-\rho_1(x-a)}. \tag{11.18}$$

§12. Examples for the first exit time

Let $H_{a,b} := \min\{s : J^{(0)}(s) \notin (a,b)\}$ be the first exit time from the interval (a,b) by the process $J^{(0)}$ defined in §10. We consider the expression

$$\mathbf{E}_x\{e^{-\gamma(J^{(0)}(H_{a,b}) - b)} e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b}) \ge b\}, \qquad \gamma > 0, \qquad \lambda > 0, \tag{12.1}$$

which determines the joint distribution of the first exit time over the boundary b and the value of the jump over the boundary.

The roots ρ_1 and ρ_2 of the equivalent equations (10.1), (10.2) are also of key importance for an explicit formula for (12.1).

12.1. As it was mentioned, the first exit time from an interval over the boundary b can occur either by crossing the boundary or by jumping over it. To compute (12.1) for the process $J^{(0)}$ without restrictions on the exit procedure over the boundary b we can apply Theorem 6.1 with $\sigma(x) \equiv \sigma$, $\mu(x) \equiv 0$, $h(x) \equiv \lambda_1$, $f(x) \equiv \lambda$, $\rho(z,y) = z + y$ and $\varPhi(x) = e^{-\gamma(x-b)}\mathrm{1\!I}_{[b,\infty)}(x)$.

Proposition 12.1. *Let*

$$D_+ := \frac{\dfrac{\rho_2 \operatorname{th}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2} + \dfrac{1}{\gamma+\eta}}{\dfrac{\rho_1 \operatorname{th}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2} + \dfrac{\rho_2 \operatorname{th}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}, \tag{12.2}$$

$$D_- := \frac{\dfrac{\rho_2 \operatorname{cth}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2} + \dfrac{1}{\gamma+\eta}}{\dfrac{\rho_1 \operatorname{cth}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2} + \dfrac{\rho_2 \operatorname{cth}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}. \tag{12.3}$$

Then

$$\begin{aligned}\mathbf{E}_x\big\{e^{-\gamma(J^{(0)}(H_{a,b})-b)}e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b})\ge b\big\} &= e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x)\\ + \Big\{\frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)} &+ \frac{D_+}{2}\Big(\frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} - \frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}\Big)\\ - \frac{D_-}{2}\Big(\frac{\operatorname{sh}((b+a-2x)\rho_1/2)}{\operatorname{sh}((b-a)\rho_1/2)} &- \frac{\operatorname{sh}((b+a-2x)\rho_2/2)}{\operatorname{sh}((b-a)\rho_2/2)}\Big)\Big\}\mathbb{I}_{[a,b]}(x).\end{aligned} \tag{12.4}$$

We do not prove this formula, because it can be obtained by summation of the expressions from Propositions 12.2 and 12.3, for which detailed proofs will be given.

Using the property of symmetry of the process $J^{(0)}$, we have

$$\begin{aligned}\mathbf{E}_x\big\{e^{-\gamma(a-J^{(0)}(H_{a,b}))}e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b})\le a\big\} &= e^{-\gamma(a-x)}\mathbb{I}_{(-\infty,a)}(x)\\ + \Big\{\frac{\operatorname{sh}((b-x)\rho_2)}{\operatorname{sh}((b-a)\rho_2)} &+ \frac{D_+}{2}\Big(\frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} - \frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}\Big)\\ + \frac{D_-}{2}\Big(\frac{\operatorname{sh}((b+a-2x)\rho_1/2)}{\operatorname{sh}((b-a)\rho_1/2)} &- \frac{\operatorname{sh}((b+a-2x)\rho_2/2)}{\operatorname{sh}((b-a)\rho_2/2)}\Big)\Big\}\mathbb{I}_{[a,b]}(x).\end{aligned} \tag{12.5}$$

We set

$$\Delta(x) := \begin{cases} x-b, & \text{for } x\ge b,\\ a-x, & \text{for } x\le a.\end{cases}$$

Then, summing (12.4) and (12.5), we see that

$$\mathbf{E}_x e^{-\gamma\Delta(J^{(0)}(H_{a,b}))-\lambda H_{a,b}} = D_+\frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} + (1-D_+)\frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}$$

for $x\in(a,b)$. In these formulas the inverse Laplace transform with respect to γ is easily computed. On the other hand, it is not clear how to compute the inverse Laplace transform with respect to λ.

From the previous formula for $\gamma=0$ it follows that

$$\mathbf{E}_x e^{-\lambda H_{a,b}} = \frac{\widetilde D_+\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} + \frac{(1-\widetilde D_+)\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}, \quad x\in[a,b], \tag{12.6}$$

where $\widetilde D_+ := D_+\big|_{\gamma=0}$.

Let τ be the exponentially distributed with parameter $\lambda > 0$ random time independent of the process $J^{(0)}$. Then the equality

$$\mathbf{P}_x\Bigl(a \le \inf_{0\le s\le\tau} J^{(0)}(s), \sup_{0\le s\le\tau} J^{(0)}(s) \le b\Bigr) = \mathbf{P}_x\bigl(\tau \le H_{a,b}\bigr) = 1 - \mathbf{E}_x e^{-\lambda H_{a,b}}$$

holds. Thus, from (12.6) it follows that

$$\mathbf{P}_x\Bigl(a \le \inf_{0\le s\le\tau} J^{(0)}(s), \sup_{0\le s\le\tau} J^{(0)}(s) \le b\Bigr) \tag{12.7}$$

$$= 1 - \frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)} + \widetilde{D}_+\Bigl(\frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)} - \frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)}\Bigr).$$

If in (12.4) a tends to $-\infty$, then the first exit time is transformed into the first exceedance moment of the level b. We denote it by $H_b := \min\{s : J^{(0)}(s) \ge b\}$. Therefore, for $x < b$ we have

$$\mathbf{E}_x e^{-\gamma(J^{(0)}(H_b)-b)-\lambda H_b} = \frac{(\eta-\rho_1)(\gamma+\rho_2)}{(\rho_2-\rho_1)(\gamma+\eta)} e^{-(b-x)\rho_1} + \frac{(\rho_2-\eta)(\gamma+\rho_1)}{(\rho_2-\rho_1)(\gamma+\eta)} e^{-(b-x)\rho_2}.$$

For $\gamma = 0$ this formula is directly connected with formula (11.17) (see the beginning of § 11).

Let us also compute the probabilities

$$\mathbf{P}_x\bigl(J^{(0)}(H_{a,b}) = b\bigr), \qquad \mathbf{P}_x\bigl(J^{(0)}(H_{a,b}) \in dz, J^{(0)}(H_{a,b}) > b\bigr), \qquad z > b.$$

For this we let $\lambda \to 0$ in (12.4). Then $\rho_1 \to 0$, $\rho_2 \to \rho := \sqrt{2\lambda_1/\sigma^2 + \eta^2}$. It is easy to verify that

$$D_+ \to \frac{1}{\Delta_+}\Bigl(1 - \frac{\eta}{\gamma+\eta}\Bigr), \qquad D_- \to \frac{1}{\Delta_-}\Bigl(1 + \frac{2}{\eta(b-a)} - \frac{\eta}{\gamma+\eta}\Bigr),$$

where

$$\Delta_+ := 1 + \frac{\sigma^2\eta}{2\lambda_1}(\rho \operatorname{th}((b-a)\rho/2) + \eta),$$

$$\Delta_- := 1 + \frac{2}{\eta(b-a)} + \frac{\sigma^2\eta}{2\lambda_1}(\rho \operatorname{cth}((b-a)\rho/2) + \eta).$$

Therefore,

$$\mathbf{E}_x\bigl\{e^{-\gamma(J^{(0)}(H_{a,b})-b)}; J^{(0)}(H_{a,b}) \ge b\bigr\} = \frac{x-a}{b-a} + \frac{1}{2\Delta_+}\frac{\gamma}{\gamma+\eta}\Bigl(\frac{\operatorname{ch}((b+a-2x)\rho/2)}{\operatorname{ch}((b-a)\rho/2)} - 1\Bigr)$$

$$+ \frac{1}{2\Delta_-}\Bigl(1 + \frac{2}{\eta(b-a)} - \frac{\eta}{\gamma+\eta}\Bigr)\Bigl(\frac{b+a-2x}{b-a} - \frac{\operatorname{sh}((b+a-2x)\rho/2)}{\operatorname{sh}((b-a)\rho/2)}\Bigr).$$

Inverting the Laplace transform with respect to γ, we have

$$\mathbf{P}_x\bigl(J^{(0)}(H_{a,b}) = b\bigr) = \frac{x-a}{b-a}\Bigl(1 - \frac{1}{\Delta_-}\Bigl(1 + \frac{2}{\eta(b-a)}\Bigr)\Bigr) - \frac{\operatorname{sh}((b-x)\rho/2)\operatorname{sh}((x-a)\rho/2)}{\Delta_+ \operatorname{ch}((b-a)\rho/2)}$$

$$+\Big(1+\frac{2}{\eta(b-a)}\Big)\frac{\operatorname{ch}((b-x)\rho/2)\operatorname{sh}((x-a)\rho/2)}{\Delta_-\operatorname{sh}((b-a)\rho/2)}, \tag{12.8}$$

and

$$\frac{d}{dz}\mathbf{P}_x\big(b<J^{(0)}(H_{a,b})<z\big)=\eta e^{-\eta(z-b)}\Big[\frac{x-a}{(b-a)\Delta_-}$$

$$+\frac{\operatorname{sh}((b-x)\rho/2)\operatorname{sh}((x-a)\rho/2)}{\Delta_+\operatorname{ch}((b-a)\rho/2)}-\frac{\operatorname{ch}((b-x)\rho/2)\operatorname{sh}((x-a)\rho/2)}{\Delta_-\operatorname{sh}((b-a)\rho/2)}\Big],\qquad b<z. \tag{12.9}$$

By the symmetry property of the process $J^{(0)}$, the corresponding formula for the exit over the boundary a can be obtained from (12.8) and (12.9) upon replacing $a\mapsto -b$, $b\mapsto -a$, $x\mapsto -x$.

We point out a simple particular case of (12.9) as $\sigma\to 0$. In this case $\rho\to\infty$, $\Delta_+\to 1$, $\Delta_-\to 1+\dfrac{2}{\eta(b-a)}$ and $H_{a,b}:=\min\{s: N_c(s)\notin(a,b)\}$. Then we have

$$\frac{d}{dz}\mathbf{P}\bigg(x+\sum_{k=1}^{N(\lambda_1 H_{a,b})}Y_k<z\bigg)=\frac{1+\eta(x-a)}{2+\eta(b-a)}\eta e^{-\eta(z-b)},\qquad b<z.$$

It is clear that this formula does not depend on λ_1.

12.2. We apply Theorem 6.4 to compute expression (12.1) under the restriction that $J^{(0)}$ leaves the interval without jump, i.e., it simply crosses the boundary.

For further computations we need the auxiliary formulas:

$$\int_a^b e^{-p|z-v|-q|y-v|}\,dv=\frac{2(pe^{-q|y-z|}-qe^{-p|y-z|})}{p^2-q^2}-\frac{e^{-p(z-a)-q(y-a)}+e^{-p(b-z)-q(b-y)}}{p+q}, \tag{12.10}$$

$$\int_a^b e^{-p|z-v|+q|y-v|}\,dv=\frac{2(pe^{q|y-z|}+qe^{-p|y-z|})}{p^2-q^2}-\frac{e^{-p(z-a)+q(y-a)}+e^{-p(b-z)+q(b-y)}}{p-q}, \tag{12.11}$$

$$\int_a^b e^{-p|z-v|}\frac{\operatorname{sh}((v-a)q)}{\operatorname{sh}((b-a)q)}\,dv=\frac{2p\operatorname{sh}((z-a)q)+qe^{-p(z-a)}}{(p^2-q^2)\operatorname{sh}((b-a)q)}-\frac{e^{-p(b-z)}(q\operatorname{cth}((b-a)q)+p)}{p^2-q^2}, \tag{12.12}$$

$$\int_a^b e^{-p|z-v|}\frac{\operatorname{sh}((b-v)q)}{\operatorname{sh}((b-a)q)}\,dv=\frac{2p\operatorname{sh}((b-z)q)+qe^{-p(b-z)}}{(p^2-q^2)\operatorname{sh}((b-a)q)}-\frac{e^{-p(z-a)}(p+q\operatorname{cth}((b-a)q))}{p^2-q^2}. \tag{12.13}$$

Denote

$$D_x^q(y)=D_y^q(x):=\frac{\operatorname{ch}((b-a-|y-x|)q)-\operatorname{ch}((b+a-y-x)q)}{q\operatorname{sh}((b-a)q)},\qquad x,y\in(a,b). \tag{12.14}$$

If $x\notin(a,b)$ or if $y\notin(a,b)$, we set $D_x^q(y)=D_y^q(x)=0$.

The function $D_y^q(x)$, $x \in [a,b]$, is the unique continuous solution of the problem

$$D''(x) - q^2 D(x) = 0, \qquad x \in (a,b) \setminus \{y\}, \tag{12.15}$$

$$D'(y+0) - D'(y-0) = -2, \tag{12.16}$$

$$D(a) = 0, \qquad D(b) = 0. \tag{12.17}$$

Note that the function $\alpha D_y^q(x)$, $x \in [a,b]$, is the solution of the same problem, but with the jump of the derivative in (12.16) equal to -2α.

From (12.10)–(12.13) we can derive the equality

$$\int_a^b e^{-p|z-v|} D_y^q(v)\,dv = \frac{2}{p^2 - q^2}\Big\{pD_y^q(z) - e^{-p|z-y|}$$

$$+ e^{-p(z-a)} \frac{\operatorname{sh}((b-y)q)}{\operatorname{sh}((b-a)q)} + e^{-p(b-z)} \frac{\operatorname{sh}((y-a)q))}{\operatorname{sh}((b-a)q)}\Big\}, \qquad a \le y \le b. \tag{12.18}$$

From (12.18) with $z = a$ and $z = b$ it follows that

$$\int_a^b e^{pv} D_y^q(v)\,dv = -\frac{2}{p^2 - q^2}\Big(e^{py} - \frac{e^{pa}\operatorname{sh}((b-y)q) + e^{pb}\operatorname{sh}((y-a)q)}{\operatorname{sh}((b-a)q)}\Big). \tag{12.19}$$

Therefore,

$$\int_a^b \operatorname{sh}((v-\delta)p) D_y^q(v)\,dv$$

$$= -\frac{2}{p^2 - q^2}\Big(\operatorname{sh}((y-\delta)p) - \frac{\operatorname{sh}((a-\delta)p)\operatorname{sh}((b-y)q) + \operatorname{sh}((b-\delta)p)\operatorname{sh}((y-a)q)}{\operatorname{sh}((b-a)q)}\Big). \tag{12.20}$$

The validity of formulas (12.18)–(12.20) can also be verified in the following way. Obviously, $D_y^q(x)$, $x \in [a,b]$, is the Green function of the corresponding differential problem, i.e., for any $\varPhi(x)$, $x \in [a,b]$, the function

$$U(x) := \int_a^b D_x^q(v)\,\varPhi(v)\,dv, \qquad x \in [a,b],$$

is the unique solution of the problem

$$U''(x) - q^2 U(x) = -2\varPhi(x), \qquad x \in (a,b),$$

$$U(a) = 0, \qquad U(b) = 0.$$

Solving this problem for the function $\varPhi(x) = e^{px}$, we get (12.19), solving it for $\varPhi(x) = \operatorname{sh}((x-\delta)p)$, we get (12.20). A little more complicated task is to find the solution for $\varPhi(x) = e^{-p|x-z|}$, which implies (12.18).

Proposition 12.2. *Let*

$$D_+^\circ := \frac{\dfrac{\rho_2 \operatorname{th}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}{\dfrac{\rho_1 \operatorname{th}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2}+\dfrac{\rho_2 \operatorname{th}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}, \tag{12.21}$$

$$D_-^\circ := \frac{\dfrac{\rho_2 \operatorname{cth}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}{\dfrac{\rho_1 \operatorname{cth}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2}+\dfrac{\rho_2 \operatorname{cth}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}. \tag{12.22}$$

Then for $x \in (a,b)$

$$\begin{aligned}
&\mathbf{E}_x\big\{e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b})=b\big\} \\
&\quad = \bigg\{\frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)} + \frac{D_+^\circ}{2}\Big(\frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} - \frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}\Big) \\
&\quad - \frac{D_-^\circ}{2}\Big(\frac{\operatorname{sh}((b+a-2x)\rho_1/2)}{\operatorname{sh}((b-a)\rho_1/2)} - \frac{\operatorname{sh}((b+a-2x)\rho_2/2)}{\operatorname{sh}((b-a)\rho_2/2)}\Big)\bigg\}\mathbb{I}_{[a,b]}(x).
\end{aligned} \tag{12.23}$$

Remark 12.1. Relation (12.23) follows from (12.4) as $\gamma \to \infty$ (although (12.4) is not proved).

Indeed, $\{J^{(0)}(H_{a,b}) \geq b\} = \{J^{(0)}(H_{a,b}) = b\} \bigcup \{J^{(0)}(H_{a,b}) > b\}$, and for the outcomes of the event $\{J^{(0)}(H_{a,b}) > b\}$ there is the limit $e^{-\gamma(J^{(0)}(H_{a,b})-b)} \to 0$ as $\gamma \to \infty$.

Proof of Proposition 12.2. Set

$$R_b^\circ(x) := \mathbf{E}_x\big\{e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b})=b\big\}.$$

We apply Theorem 6.4 with $\sigma(x) \equiv \sigma$, $\mu(x) \equiv 0$, $h(x) \equiv \lambda_1$, $f(x) \equiv \lambda$ and $\rho(x,y) = x+y$. Then the function R_b° is the unique bounded solution of (6.11).

The solution of equation (6.11) can be found in the form

$$R_b^\circ(x) = \sum_{k=1}^4 A_k \frac{\operatorname{sh}((x-\delta_k)q_k)}{\operatorname{sh}((b-a)q_k)}\mathbb{I}_{[a,b]}(x), \tag{12.24}$$

where $\delta_1=\delta_3=b$, $\delta_2=\delta_4=a$, and A_k and $q_k>0$ are some constants.

The function M_b is a unique solution of the problem

$$\frac{\sigma^2}{2}M''(x) - (\lambda+\lambda_1)M(x) = 0, \qquad x \in (a,b), \tag{12.25}$$

$$M(a)=0, \qquad\qquad M(b)=1. \tag{12.26}$$

This solution has the form

$$M_b(x) = \frac{\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}, \qquad x \in (a,b),$$

where $\Upsilon := \sqrt{2\lambda + 2\lambda_1}/\sigma$.

In this case $G_z(x)$ is the unique solution of the problem

$$\frac{\sigma^2}{2}G''(x) - (\lambda + \lambda_1)G(x) = 0, \qquad x \in (a,b) \setminus \{z\}, \tag{12.27}$$

$$G'(z+0) - G'(z-0) = -2\lambda_1/\sigma^2, \tag{12.28}$$

$$G(a) = 0, \qquad G(b) = 0. \tag{12.29}$$

It is easy to verify (see (12.14)–(12.17)) that

$$G_z(x) = \frac{\lambda_1\big(\operatorname{ch}((b-a-|z-x|)\Upsilon) - \operatorname{ch}((b+a-z-x)\Upsilon)\big)}{\sigma^2\Upsilon \operatorname{sh}((b-a)\Upsilon)} = \frac{\lambda_1}{\sigma^2}D_z^{\Upsilon}(x)$$

for $x \in (a,b)$, $z \in (a,b)$. For $x \notin (a,b)$ or $z \notin (a,b)$ we set $G_z(x) = 0$.

For $a \le x \le b$, using (12.12) and (12.13), we obtain

$$\mathbf{E}R_b^{\circ}(x+Y_1) = \sum_{k=1}^{4} A_k \mathbf{E}\Big\{\frac{\operatorname{sh}((x+Y_1-\delta_k)q_k)\mathbb{1}_{[a-x,b-x]}(Y_1)}{\operatorname{sh}((b-a)q_k)}\Big\}$$

$$= \sum_{k=1}^{4} \frac{A_k\eta}{2\operatorname{sh}((b-a)q_k)} \int_a^b \operatorname{sh}((v-\delta_k)q_k)e^{-\eta|v-x|}\,dv$$

$$= \sum_{k=1}^{4} \frac{A_k\eta^2}{(\eta^2-q_k^2)}\frac{\operatorname{sh}((x-\delta_k)q_k)}{\operatorname{sh}((b-a)q_k)} - e^{-\eta(x-a)}\sum_{k=1}^{4}\frac{A_k\eta(\eta\operatorname{sh}((a-\delta_k)q_k) - q_k\operatorname{ch}((a-\delta_k)q_k))}{2(\eta^2-q_k^2)\operatorname{sh}((b-a)q_k)}$$

$$- e^{-\eta(b-x)}\sum_{k=1}^{4}\frac{A_k\eta(\eta\operatorname{sh}((b-\delta_k)q_k) + q_k\operatorname{ch}((b-\delta_k)q_k))}{2(\eta^2-q_k^2)\operatorname{sh}((b-a)q_k)}. \tag{12.30}$$

Substituting into equation (6.11) the expressions for $R_b^{\circ}(x)$, $M_b(x)$, $\mathbf{E}R_b^{\circ}(x+Y_1)$ and applying formulas (12.19), (12.20), we get

$$\sum_{k=1}^{4} A_k\frac{\operatorname{sh}((x-\delta_k)q_k)}{\operatorname{sh}((b-a)q_k)} = \frac{\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}$$

$$+\sum_{k=1}^{4}\frac{A_k\eta^2}{(\eta^2-q_k^2)\operatorname{sh}((b-a)q_k)}\frac{2\lambda_1}{(2\lambda+2\lambda_1-q_k^2\sigma^2)}$$

$$\times\Big(\operatorname{sh}((x-\delta_k)q_k) - \frac{\operatorname{sh}((a-\delta_k)q_k)\operatorname{sh}((b-x)\Upsilon) + \operatorname{sh}((b-\delta_k)q_k)\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}\Big)$$

$$-\frac{\eta}{2}\sum_{k=1}^{4}A_k\frac{(\eta\operatorname{sh}((a-\delta_k)q_k) - q_k\operatorname{ch}((a-\delta_k)q_k))}{(\eta^2-q_k^2)\operatorname{sh}((b-a)q_k)}\frac{2\lambda_1e^{\eta a}}{(2\lambda+2\lambda_1-\eta^2\sigma^2)}$$

$$\times\Big(e^{-\eta x} - \frac{e^{-\eta a}\operatorname{sh}((b-x)\Upsilon) + e^{-\eta b}\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}\Big)$$

$$-\frac{\eta}{2}\sum_{k=1}^{4}A_k\frac{(\eta\,\mathrm{sh}((b-\delta_k)q_k)+q_k\,\mathrm{ch}((b-\delta_k)q_k))}{(\eta^2-q_k^2)\,\mathrm{sh}((b-a)q_k)}\frac{2\lambda_1 e^{-\eta b}}{(2\lambda+2\lambda_1-\eta^2\sigma^2)}$$
$$\times\Big(e^{\eta x}-\frac{e^{\eta a}\,\mathrm{sh}((b-x)\Upsilon)+e^{\eta b}\,\mathrm{sh}((x-a)\Upsilon)}{\mathrm{sh}((b-a)\Upsilon)}\Big). \tag{12.31}$$

Equating the coefficients at $\mathrm{sh}((x-\delta_k)q_k)$, we obtain

$$A_k=A_k\frac{\eta^2}{(\eta^2-q_k^2)}\frac{2\lambda_1}{(2\lambda+2\lambda_1-q_k^2\sigma^2)}.$$

This implies that $q_1=q_2=\rho_1$ and $q_3=q_4=\rho_2$, i.e., they are the positive roots of equation (10.1).

Equating the coefficients at $e^{-\eta x}$ and at $e^{\eta x}$, we have

$$\sum_{k=1}^{4}A_k\frac{(\eta\,\mathrm{sh}((a-\delta_k)q_k)-q_k\,\mathrm{ch}((a-\delta_k)q_k))}{(\eta^2-q_k^2)\,\mathrm{sh}((b-a)q_k)}=0, \tag{12.32}$$

$$\sum_{k=1}^{4}A_k\frac{(\eta\,\mathrm{sh}((b-\delta_k)q_k)+q_k\,\mathrm{ch}((b-\delta_k)q_k))}{(\eta^2-q_k^2)\,\mathrm{sh}((b-a)q_k)}=0. \tag{12.33}$$

Equating the coefficients at $\mathrm{sh}((x-a)\Upsilon)$ and at $\mathrm{sh}((b-x)\Upsilon)$ respectively, we have

$$1-\sum_{k=1}^{4}A_k\frac{\mathrm{sh}((b-\delta_k)q_k)}{\mathrm{sh}((b-a)q_k)}=0,\qquad \sum_{k=1}^{4}A_k\frac{\mathrm{sh}((a-\delta_k)q_k)}{\mathrm{sh}((b-a)q_k)}=0. \tag{12.34}$$

Since $\delta_1=\delta_3=b$, $\delta_2=\delta_4=a$, from (12.34) we derive that $1-A_2-A_4=0$, $A_1+A_3=0$. We set $B^\circ:=A_3=-A_1$, $A^\circ:=A_2=1-A_4$. Then for $x\in(a,b)$ from (12.24) it follows that

$$R_b^\circ(x)=\frac{\mathrm{sh}((x-a)\rho_2)}{\mathrm{sh}((b-a)\rho_2)}+B^\circ\Big(\frac{\mathrm{sh}((b-x)\rho_1)}{\mathrm{sh}((b-a)\rho_1)}-\frac{\mathrm{sh}((b-x)\rho_2)}{\mathrm{sh}((b-a)\rho_2)}\Big)$$
$$+A^\circ\Big(\frac{\mathrm{sh}((x-a)\rho_1)}{\mathrm{sh}((b-a)\rho_1)}-\frac{\mathrm{sh}((x-a)\rho_2)}{\mathrm{sh}((b-a)\rho_2)}\Big). \tag{12.35}$$

In this representation the conditions $R_b^\circ(a)=0$ and $R_b^\circ(b)=1$ hold, which corresponds to the definition of the function $R_b^\circ(x)$.

The equalities (12.32) and (12.33) can be transformed into the following ones:

$$\frac{B^\circ(\eta\,\mathrm{sh}((b-a)\rho_1)+\rho_1\,\mathrm{ch}((b-a)\rho_1))}{(\eta^2-\rho_1^2)\,\mathrm{sh}((b-a)\rho_1)}-\frac{A^\circ\rho_1}{(\eta^2-\rho_1^2)\,\mathrm{sh}((b-a)\rho_1)}$$
$$+\frac{B^\circ(\eta\,\mathrm{sh}((b-a)\rho_2)+\rho_2\,\mathrm{ch}((b-a)\rho_1))}{(\rho_2^2-\eta^2)\,\mathrm{sh}((b-a)\rho_2)}+\frac{(1-A^\circ)\rho_2}{(\rho_2^2-\eta^2)\,\mathrm{sh}((b-a)\rho_2)}=0, \tag{12.36}$$

$$-\frac{B^\circ\rho_1}{(\eta^2-\rho_1^2)\,\mathrm{sh}((b-a)\rho_1)}+\frac{A^\circ(\eta\,\mathrm{sh}((b-a)\rho_1)+\rho_1\,\mathrm{ch}((b-a)\rho_1))}{(\eta^2-\rho_1^2)\,\mathrm{sh}((b-a)\rho_1)}$$
$$-\frac{B^\circ\rho_2}{(\rho_2^2-\eta^2)\,\mathrm{sh}((b-a)\rho_2)}-\frac{(1-A^\circ)(\eta\,\mathrm{sh}((b-a)\rho_2)+\rho_2\,\mathrm{ch}((b-a)\rho_1))}{(\rho_2^2-\eta^2)\,\mathrm{sh}((b-a)\rho_2)}=0. \tag{12.37}$$

Summing (12.36) and (12.37), we have $A^\circ + B^\circ = D_+^\circ$, where D_+° is defined by (12.21). Subtracting (12.36) from (12.37), we get $A^\circ - B^\circ = D_-^\circ$, where D_-° is defined by (12.22). By (12.35), the expression for $R_b^\circ(x)$ has the form

$$\begin{aligned}&\mathbf{E}_x\{e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b}) = b\}\\ &= \Big\{\frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)} + \frac{D_+^\circ - D_-^\circ}{2}\Big(\frac{\operatorname{sh}((b-x)\rho_1)}{\operatorname{sh}((b-a)\rho_1)} - \frac{\operatorname{sh}((b-x)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Big)\\ &+ \frac{D_-^\circ + D_+^\circ}{2}\Big(\frac{\operatorname{sh}((x-a)\rho_1)}{\operatorname{sh}((b-a)\rho_1)} - \frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Big)\Big\}\mathbb{I}_{[a,b]}(x). \qquad (12.38)\end{aligned}$$

This expression is easily transformed into (12.23). □

12.3. We apply Theorem 6.5 to compute the expression (12.1) under the condition that $J^{(0)}$ leaves the interval by a jump over the boundary.

Proposition 12.3. *Let*

$$D_+^1 := \frac{1/(\gamma+\eta)}{\dfrac{\rho_1 \operatorname{th}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2} + \dfrac{\rho_2 \operatorname{th}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}, \qquad (12.39)$$

$$D_-^1 := \frac{1/(\gamma+\eta)}{\dfrac{\rho_1 \operatorname{cth}((b-a)\rho_1/2)+\eta}{\eta^2-\rho_1^2} + \dfrac{\rho_2 \operatorname{cth}((b-a)\rho_2/2)+\eta}{\rho_2^2-\eta^2}}. \qquad (12.40)$$

Then

$$\begin{aligned}&\mathbf{E}_x\{e^{-\gamma(J^{(0)}(H_{a,b})-b)}e^{-\lambda H_{a,b}}; J^{(0)}(H_{a,b}) > b\}\\ &= e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x) + \Big\{\frac{D_+^1}{2}\Big(\frac{\operatorname{ch}((b+a-2x)\rho_1/2)}{\operatorname{ch}((b-a)\rho_1/2)} - \frac{\operatorname{ch}((b+a-2x)\rho_2/2)}{\operatorname{ch}((b-a)\rho_2/2)}\Big)\\ &- \frac{D_-^1}{2}\Big(\frac{\operatorname{sh}((b+a-2x)\rho_1/2)}{\operatorname{sh}((b-a)\rho_1/2)} - \frac{\operatorname{sh}((b+a-2x)\rho_2/2)}{\operatorname{sh}((b-a)\rho_2/2)}\Big)\Big\}\mathbb{I}_{[a,b]}(x). \qquad (12.41)\end{aligned}$$

Proof. Denote the left-hand side of (12.41) by $R_b^1(x)$. By Theorem 6.5, the function $R_b^1(x)$ is the unique bounded solution of equation (6.12) with $\Phi(x) = e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x)$. This solution can be found in the form

$$R_b^1(x) = e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x) + \sum_{k=1}^{4} A_k \frac{\operatorname{sh}((x-\delta_k)q_k)}{\operatorname{sh}((b-a)q_k)}\mathbb{I}_{[a,b]}(x), \qquad (12.42)$$

where the additional term $e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x)$ appears, in contrast to the representation (12.24).

As a result, the additional term

$$\mathbf{E}\{e^{-\gamma(x+Y_1-b)}\mathbb{I}_{(b,\infty)}(x+Y_1)\} = \frac{\eta}{2(\gamma+\eta)}e^{-\eta(b-x)}$$

appears in the expression for $\mathbf{E}R_b^1(x+Y_1)$, in contrast to formula (12.30).

Hence, in view of (12.19), in the expression for

$$\int_{-\infty}^{\infty} G_z(x)\mathbf{E}R_b^1(z+Y_1)\,dz,$$

which is the basic component of equation (6.12), the additional term

$$\frac{\eta e^{-\eta b}}{2(\gamma+\eta)}\frac{2\lambda_1}{(2\lambda+2\lambda_1-\eta^2\sigma^2)}\Bigl(e^{\eta x}-\frac{e^{\eta a}\operatorname{sh}((b-x)\Upsilon)+e^{\eta b}\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}\Bigr) \tag{12.43}$$

appeares, in contrast to the same expression for R_b°. Then substituting the solution (12.42) into equation (6.12), we get for $x\in[a,b]$ the analogue of (12.31) with the term (12.43) in place of the first term $\dfrac{\operatorname{sh}((x-a)\Upsilon)}{\operatorname{sh}((b-a)\Upsilon)}$.

Equating the coefficients at $\operatorname{sh}((x-\delta_k)q_k)$, we obtain the same values for the variables q_k, as in the previous case, i.e., they are the solutions of equation (10.1).

Equating the coefficients at $e^{-\eta x}$, we derive equation (12.32), and equating the coefficients at $e^{\eta x}$, we obtain that on the right-hand side of the analogue for (12.33) there is the ratio $\dfrac{1}{\gamma+\eta}$ instead of zero.

Equating the coefficients at $\operatorname{sh}((x-a)\Upsilon)$ and at $\operatorname{sh}((b-x)\Upsilon)$, we see that 1 no longer appeares in the left-hand side of (12.34). As a result, we have $A_3=-A_1=:B^1$, $A_2=A_4=:A^1$. Then from (12.42) it follows that

$$R_b^1(x)=e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x)+\Bigl\{B^1\Bigl(\frac{\operatorname{sh}((b-x)\rho_1)}{\operatorname{sh}((b-a)\rho_1)}-\frac{\operatorname{sh}((b-x)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Bigr)$$
$$+A^1\Bigl(\frac{\operatorname{sh}((x-a)\rho_1)}{\operatorname{sh}((b-a)\rho_1)}-\frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Bigr)\Bigr\}\mathbb{I}_{[a,b]}(x). \tag{12.44}$$

Proceeding as in the previous example, we get $A^1+B^1=D_+^1$ and $A^1-B^1=D_-^1$. Then the expression for $R_b^1(x)$ has the form

$$\mathbf{E}_x\bigl\{e^{-\gamma(J^{(0)}(H_{a,b})-b)}e^{-\lambda H_{a,b}};J^{(0)}(H_{a,b})>b\bigr\}$$
$$=e^{-\gamma(x-b)}\mathbb{I}_{(b,\infty)}(x)+\Bigl\{\frac{D_+^1-D_-^1}{2}\Bigl(\frac{\operatorname{sh}((b-x)\rho_1)}{\operatorname{sh}((b-a)\rho_1)}-\frac{\operatorname{sh}((b-x)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Bigr)$$
$$+\frac{D_-^1+D_+^1}{2}\Bigl(\frac{\operatorname{sh}((x-a)\rho_1)}{\operatorname{sh}((b-a)\rho_1)}-\frac{\operatorname{sh}((x-a)\rho_2)}{\operatorname{sh}((b-a)\rho_2)}\Bigr)\Bigr\}\mathbb{I}_{[a,b]}(x).$$

This expression is easily transformed into (12.41). □

§ 13. Distribution of local time of Brownian motion with jumps

Let $J^{(0)}(s)$, $s\ge 0$ be the *Brownian motion with jumps* defined in § 10. For the process $J^{(0)}$ there exists a *local time*, i.e., a.s. for all $y\in\mathbf{R}$ and $t\ge 0$ there exists the limit

$$\ell^{(0)}(t,y):=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[y,y+\varepsilon)}(J^{(0)}(s))\,ds.$$

This is a consequence of the fact that the same limit exists for the Brownian motion $\sigma W(s)$, $s \geq 0$. Indeed, we set $S_m := \sum_{k=1}^{m} Y_k$, $S_0 = 0$ and $\varkappa_m := \frac{1}{\lambda_1}\sum_{k=1}^{m} \tau_k$, $\varkappa_0 = 0$. Then

$$\ell^{(0)}(t,y) = \sum_{m=1}^{N(\lambda_1 t)} \big(\ell(\varkappa_m, y - S_{m-1}) - \ell(\varkappa_{m-1}, y - S_{m-1})\big)$$
$$+ \ell(t, y - S_{N(\lambda_1 t)}) - \ell(\varkappa_{N(\lambda_1 t)}, y - S_{N(\lambda_1 t)}),$$

where $\ell(t,y)$ is the local time of the Brownian motion σW.

The following statement enables us to compute the joint distributions of an integral functional and the local time of the process $J^{(0)}$ at the exponentially distributed with parameter $\lambda > 0$ random time τ independent of $J^{(0)}$.

Theorem 13.1. *Let $\varPhi(x)$, $f(x)$, $x \in \mathbf{R}$, be piecewise-continuous functions. Assume that $\varPhi$ is bounded and f is nonnegative. Then for $\gamma \geq 0$ the function*

$$Q(x) := \mathbf{E}_x\bigg\{\varPhi(J^{(0)}(\tau)) \exp\bigg(-\int_0^\tau f(J^{(0)}(s))\,ds - \gamma\ell^{(0)}(\tau, r)\bigg)\bigg\}, \qquad x \in \mathbf{R},$$

is the unique bounded continuous solution of the equation

$$Q(x) = M(x) + \int_{-\infty}^{\infty} G_z(x)\mathbf{E}Q(z + Y_1)\,dz, \tag{13.1}$$

where $M(x)$ is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 M''(x) - (\lambda + \lambda_1 + f(x))M(x) = -\lambda\varPhi(x), \qquad x \neq r, \tag{13.2}$$
$$M'(r+0) - M'(r-0) = 2\gamma M(r)/\sigma^2, \tag{13.3}$$

and $G_z(x)$ is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 G''(x) - (\lambda + \lambda_1 + f(x))G(x) = 0, \qquad x \neq z, r, \tag{13.4}$$
$$G'(z+0) - G'(z-0) = -2\lambda_1/\sigma^2, \tag{13.5}$$
$$G'(r+0) - G'(r-0) = 2\gamma G(r)/\sigma^2. \tag{13.6}$$

Remark 13.1. In the case $z = r$ the conditions (13.5), (13.6) must be replaced by the condition

$$G'(r+0) - G'(r-0) = 2\gamma G(r)/\sigma^2 - 2\lambda_1/\sigma^2.$$

Remark 13.2. The function $Q(x)$, $x \in \mathbf{R}$, is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 Q''(x) - (\lambda + \lambda_1 + f(x))Q(x) = -\lambda\varPhi(x) - \lambda_1\mathbf{E}Q(x + Y_1), \qquad x \neq r, \tag{13.7}$$

$$Q'(r+0) - Q'(r-0) = 2\gamma Q(r)/\sigma^2. \tag{13.8}$$

Indeed, for the Brownian motion we have $G_z(x) = G_x(z)$. Since $G_z(x)$ is the Green function of the corresponding problem (see Ch. III, problems (4.23)–(4.27) and (3.1)–(3.3) with $a = -\infty$, $b = \infty$), the function

$$U(x) := \int_{-\infty}^{\infty} G_x(z)\, R(z)\, dz \tag{13.9}$$

is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 U''(x) - (\lambda + \lambda_1 + f(x))U(x) = -\lambda_1 R(x), \qquad x \neq r, \tag{13.10}$$

$$U'(r+0) - U'(r-0) = 2\gamma U(r)/\sigma^2. \tag{13.11}$$

Here R is an arbitrary bounded piecewise-continuous function.

Now it remains to observe that $Q(x) = M(x) + U(x)$ for $R(x) := \mathbf{E}Q(x + Y_1)$, $x \in \mathbf{R}$, and hence $Q(x)$ is the solution of (13.7), (13.8).

Proof of Theorem 13.1. We apply Theorem 3.1 with $\sigma(x) \equiv \sigma$, $\mu \equiv 0$, $q \equiv 0$, $h(x) \equiv \lambda_1$, $\rho(x,y) = x + y$, $a = -\infty$ and $b = \infty$. Then, by (3.9), we see that for $\varepsilon > 0$ the function

$$Q_\varepsilon(x) := \mathbf{E}_x\biggl\{\varPhi(J^{(0)}(\tau)) \exp\biggl(-\int_0^\tau \bigl(f(J^{(0)}(s)) + \frac{\gamma}{\varepsilon}\mathbb{1}_{[r,r+\varepsilon)}(J^{(0)}(s))\bigr)\, ds\biggr)\biggr\}, \quad x \in \mathbf{R},$$

is the unique bounded solution of the integro-differential equation

$$\frac{1}{2}\sigma^2 Q''_\varepsilon(x) - \bigl(\lambda + \lambda_1 + f(x) + \frac{\gamma}{\varepsilon}\mathbb{1}_{[r,r+\varepsilon)}(x)\bigr)Q_\varepsilon(x)$$

$$= -\lambda\varPhi(x) - \lambda_1 \mathbf{E}Q_\varepsilon(x + Y_1). \tag{13.12}$$

Analogously to the proof of Theorem 3.1 of Ch. III, in equation (13.12) we can pass to the limit as $\varepsilon \downarrow 0$, and conclude that $Q(x)$, $x \in \mathbf{R}$, is the unique bounded continuous solution of (13.7), (13.8). □

Example 13.1. Let us compute the distribution of the local time $\ell^{(0)}(\tau, r)$ for the case when the density of Y_k, $k = 1, 2, \dots$, is $\frac{d}{dy}\mathbf{P}(Y_1 < y) = \frac{1}{2}\eta e^{-\eta|y|}$, $\eta > 0$. We first consider the computation of the Laplace transform of the distribution of the local time, i.e., the function $Q(x) = \mathbf{E}_x e^{-\gamma\ell^{(0)}(\tau,r)}$. We apply Theorem 13.1 with $\varPhi \equiv 1$ and $f \equiv 0$.

In this case M is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 M''(x) - (\lambda + \lambda_1)M(x) = -\lambda, \qquad x \neq r, \tag{13.13}$$

$$M'(r+0) - M'(r-0) = 2\gamma M(r)/\sigma^2. \tag{13.14}$$

The solution of (13.13), (13.14) has the form

$$M(x) = \frac{\lambda}{\lambda+\lambda_1} - \frac{\lambda\gamma}{(\lambda+\lambda_1)(\gamma+\sigma\sqrt{2\lambda+2\lambda_1})} e^{-|x-r|\sqrt{2\lambda+2\lambda_1}/\sigma}. \tag{13.15}$$

Here it is not necessary to compute the function $G_z(x)$, since we at once use the formulas (13.9)–(13.11) and compute (see (13.9)) the function $U(x)$ for $R(x) = e^{-|x-r|\rho}$. The function U is the unique bounded continuous solution of the problem

$$\frac{1}{2}\sigma^2 U''(x) - (\lambda+\lambda_1)U(x) = -\lambda_1 e^{-|x-r|\rho}, \qquad x \neq r, \tag{13.16}$$

$$U'(r+0) - U'(r-0) = 2\gamma U(r)/\sigma^2. \tag{13.17}$$

The solution of (13.16), (13.17) has the form

$$U(x) = \int_{-\infty}^{\infty} G_x(z) e^{-|z-r|\rho}\, dz$$

$$= \frac{2\lambda_1}{2\lambda+2\lambda_1-\sigma^2\rho^2}\Big(e^{-|x-r|\rho} - \frac{\gamma+\rho\sigma^2}{\gamma+\sigma\sqrt{2\lambda+2\lambda_1}} e^{-|x-r|\sqrt{2\lambda+2\lambda_1}/\sigma}\Big). \tag{13.18}$$

The solution of equation (13.1), i.e., the function $Q(x) = \mathbf{E}_x e^{-\gamma\ell^{(0)}(\tau,r)}$, can be represented in the form

$$Q(x) = 1 - A_1 e^{-\rho_1|x-r|} - A_2 e^{-\rho_2|x-r|}, \tag{13.19}$$

where ρ_1 and ρ_2 defined by (10.3) and (10.4). As in the previous sections, one could take in (13.19) any positive exponent constants q_1 and q_2 in place of ρ_1 and ρ_2, but the following treatment show that they are equal to the roots of (10.1).

Taking into account the form of the density of the variables Y_1 and the formula (12.10) with $a = -\infty$, $b = \infty$, we have

$$\mathbf{E}e^{-\rho|x+Y_1-r|} = \frac{\eta^2}{\eta^2-\rho^2} e^{-\rho|x-r|} - \frac{\rho\eta}{\eta^2-\rho^2} e^{-\eta|x-r|}$$

and, therefore,

$$\mathbf{E}Q(x+Y_1) = 1 - \frac{A_1\eta^2}{\eta^2-\rho_1^2} e^{-\rho_1|x-r|} + \frac{A_1\rho_1\eta}{\eta^2-\rho_1^2} e^{-\eta|x-r|}$$

$$-\frac{A_2\eta^2}{\eta^2-\rho_2^2} e^{-\rho_2|x-r|} + \frac{A_2\rho_2\eta}{\eta^2-\rho_2^2} e^{-\eta|x-r|}. \tag{13.20}$$

The coefficient at $e^{-\eta|x-r|}$ is set to be equal to zero, i.e.,

$$\frac{A_1\rho_1}{\eta^2-\rho_1^2} - \frac{A_2\rho_2}{\rho_2^2-\eta^2} = 0. \tag{13.21}$$

Substituting (13.15) and (13.20) into (13.1), and using (13.18) with $\rho = 0$, $\rho = \rho_1$, and $\rho = \rho_2$, we get

$$1 - A_1 e^{-\rho_1|x-r|} - A_2 e^{-\rho_2|x-r|} = 1 - \frac{\gamma}{\gamma+\sigma\sqrt{2\lambda+2\lambda_1}} e^{-|x-r|\sqrt{2\lambda+2\lambda_1}/\sigma}$$

$$-\sum_{k=1}^{2}\frac{2\lambda_1\eta^2 A_k}{(\eta^2-\rho_k^2)(2\lambda+2\lambda_1-\sigma^2\rho_k^2)}\Big(e^{-|x-r|\rho_k}-\frac{\gamma+\rho_k\sigma^2}{\gamma+\sigma\sqrt{2\lambda+2\lambda_1}}e^{-|x-r|\sqrt{2\lambda+2\lambda_1}/\sigma}\Big).$$

Since

$$\frac{2\lambda_1\eta^2}{(\eta^2-\rho_k^2)(2\lambda+2\lambda_1-\sigma^2\rho_k^2)}=1,\qquad k=1,2,$$

equating the coefficients at $e^{-|x-r|\sqrt{2\lambda+2\lambda_1}/\sigma}$, we find that

$$A_1(\gamma+\rho_1\sigma^2)+A_2(\gamma+\rho_2\sigma^2)=\gamma. \tag{13.22}$$

The algebraic system of equations (13.21), (13.22) has the solution

$$A_2=\frac{\gamma\rho_1(\rho_2^2-\eta^2)}{(\rho_2-\rho_1)(\gamma(\eta^2+\rho_1\rho_2)+\rho_1\rho_2\sigma^2(\rho_1+\rho_2))},$$

$$A_1=\frac{\gamma\rho_2(\eta^2-\rho_1^2)}{(\rho_2-\rho_1)(\gamma(\eta^2+\rho_1\rho_2)+\rho_1\rho_2\sigma^2(\rho_1+\rho_2))}.$$

By (13.19),

$$\mathbf{E}_x e^{-\gamma\ell^{(0)}(\tau,r)}=1-\frac{\gamma\rho_2(\eta^2-\rho_1^2)}{(\rho_2-\rho_1)(\gamma(\eta^2+\rho_1\rho_2)+\rho_1\rho_2\sigma^2(\rho_1+\rho_2))}e^{-\rho_1|x-r|}$$

$$-\frac{\gamma\rho_1(\rho_2^2-\eta^2)}{(\rho_2-\rho_1)(\gamma(\eta^2+\rho_1\rho_2)+\rho_1\rho_2\sigma^2(\rho_1+\rho_2))}e^{-\rho_2|x-r|}. \tag{13.23}$$

Inverting the Laplace transform with respect to γ (see formula 1 of Appendix 3), we obtain

$$\frac{d}{dy}\mathbf{P}_x\big(\ell^{(0)}(\tau,r)<y\big)$$

$$=\frac{\rho_2(\eta^2-\rho_1^2)e^{-\rho_1|x-r|}+\rho_1(\rho_2^2-\eta^2)e^{-\rho_2|x-r|}}{(\rho_2-\rho_1)(\eta^2+\rho_1\rho_2)^2(\rho_1\rho_2\sigma^2(\rho_1+\rho_2))^{-1}}\exp\Big(-\frac{y\rho_1\rho_2\sigma^2(\rho_1+\rho_2)}{\eta^2+\rho_1\rho_2}\Big), \tag{13.24}$$

$$\mathbf{P}_x\big(\ell^{(0)}(\tau,r)=0\big)=1-\frac{\rho_2(\eta^2-\rho_1^2)e^{-\rho_1|x-r|}+\rho_1(\rho_2^2-\eta^2)e^{-\rho_2|x-r|}}{(\rho_2-\rho_1)(\eta^2+\rho_1\rho_2)}. \tag{13.25}$$

Subtracting from (13.25) the probability (11.17) for $b=r$, $x\le r$, and the probability (11.18) for $a=r$, $x\ge r$, we get

$$\mathbf{P}_x\big(\ell^{(0)}(\tau,r)=0,\ \text{and there are some jumps over the level } r \text{ up to the time } \tau\big)$$

$$=\mathbf{P}_x\big(\text{up to } \tau \text{ there are only jumps over } r \text{ while there are no crossings}\big)$$

$$=\frac{\rho_1\rho_2(\rho_2-\eta)(\eta-\rho_1)}{\eta(\rho_2-\rho_1)(\eta^2+\rho_1\rho_2)}\big(e^{-\rho_1|x-r|}-e^{-\rho_2|x-r|}\big),\qquad x\in\mathbf{R}. \tag{13.26}$$

CHAPTER VII

INVARIANCE PRINCIPLE FOR RANDOM WALKS AND LOCAL TIMES

§ 1. Formulation of problems

The *classical invariance principle* asserts that the distributions of a broad class of continuous functionals of processes constructed from a normalized random walk with finite variance converge to the distributions of these functionals of a Brownian motion. However, nontrivial limit distributions exist also for various discontinuous functionals of random walks, which does not follow directly from the classical invariance principle. Many such functionals of random walks with finite variance have limit distributions expressed in terms of the distributions of functionals of Brownian local time. It turns out that such convergence is a consequence of the weak convergence to Brownian local time of certain processes generated by recurrent random walks with finite variance.

By an *invariance principle* in the wide sense we will mean an assertion that a sequence of random processes constructed from partial sums of independent random variables converges weakly to some random process, whose distributions depend on the original random variables only in terms of finitely many real parameters. Typical parameters are the mean and the variance. Weak convergence of random processes is understood to be weak convergence of the measures generated by these processes in the appropriate function space. The latter is usually a complete separable metric space of functions whose σ-algebra of Borel sets coincide with the minimal σ-algebra containing all the cylindrical sets. For convergence of the processes considered in this chapter we use the following variant of the definition of weak convergence, which is equivalent to the classical one.

The processes $X_n(t)$, $t \in \Sigma \subseteq \mathbf{R}^k$, $n = 1, 2, \dots$, *converge weakly* to the process $X_\infty(t)$, if on some probability space there are processes $X'_n(t)$ and $X'_\infty(t)$, $t \in \Sigma$, such that the finite-dimensional distributions of the processes $X'_n(t)$ and $X_n(t)$ are the same for each $n \in \mathbb{N} \bigcup \{\infty\}$, and for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in\Sigma} |X'_n(t) - X'_\infty(t)| > \varepsilon\Big) = 0. \tag{1.1}$$

In each concrete case there exists a complete separable function space, equipped with the uniform metric, such that weak convergence of the processes, defined by (1.1), implies convergence of the measures generated by the processes in the function space, and conversely.

Let ν_k, $k = 0, 1, 2, \dots$, be a recurrent random walk with unit variance, i.e., $\nu_0 = 0$, $\nu_k = \sum_{l=1}^{k} \xi_l$, where ξ_l, $l = 1, 2, \dots$, are independent identically distributed random variables with $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = 1$. In this chapter, we will prove, in particular, that the process $\widetilde{W}_n(t) := \frac{1}{\sqrt{n}} \nu_{[nt]}$, $t \in [0, T]$, converges weakly to a Brownian

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8_7

motion $W(t)$, $t \in [0,T]$, $W(0)=0$ (the classical invariance principle). Following the definition of weak convergence stated above, the classical (weak) *invariance principle for random walk* ν_k can be formulated as follows.

Given a Brownian motion $W(t)$, $t \ge 0$, we can construct a recurrent random walk ν_k^n, $k = 0, 1, \dots$, such that for each fixed n its finite-dimensional distributions coincide with those of the random walk ν_k, $k = 0, 1, \dots$, and the processes $W_n(t) := \frac{1}{\sqrt{n}} \nu^n_{[nt]}$, $t \in [0,T]$, $n = 1, 2, \dots$, satisfy for any $\varepsilon > 0$ the relation

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,T]} |W_n(t) - W(t)| > \varepsilon \Big) = 0. \tag{1.2}$$

We define the space of functions that contains the sample paths of the processes W_n, W and on which the measures generated by the processes W_n will converge to the measure generated by the Brownian motion W. This is the space $B_r[0,T]$ of right continuous functions on $[0,T]$, having left limits and admitting discontinuities only at rational points of the interval $[0,T]$. Equipped with the metric

$$\rho(x,y) := \sup_{t\in[0,T]} |x(t) - y(t)|,$$

$B_r[0,T]$ becomes a complete separable metric space.

Since the uniform convergence in probability is equivalent (see Proposition 1.1 of Ch. I) to the fact that from every sequence of natural numbers one can extract a subsequence such that there is uniform convergence a.s., (1.2) implies that for any continuous functional $\wp$ on $B_r[0,T]$ the distribution of $\wp(W_n(t), 0 \le t \le T)$ converges weakly to the distribution of $\wp(W(t), 0 \le t \le T)$.

Now we state the invariance principle for local times of random walks. The term *local time of a random walk* is used for the functionals describing the behavior of a random walk near some particular level.

Let $f(y,z)$ be an arbitrary function. In the case of random walks on the integer lattice we assume that f is defined on $\mathbb{Z} \times \mathbb{Z}$ and $\sum_{l=-\infty}^{\infty} |h(l)| < \infty$, where $h(v) := \mathbf{E} f(v, v + \xi_1)$. In the case of continuous random walks (the distribution of steps of the walk has a density), we assume that $f(y,z)$, $(y,z) \in \mathbf{R}^2$, is a measurable function on the plane and $\int_{-\infty}^{\infty} |h(v)|\,dv < \infty$. Let $h := \sum_{l=-\infty}^{\infty} h(l)$ for integer random walks and $h := \int_{-\infty}^{\infty} h(v)\,dv$ for continuous ones. Consider the process

$$q_n(t,x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} f(\nu^n_{k-1} - x_n, \nu^n_k - x_n), \qquad (t,x) \in [0,T] \times \mathbf{R},$$

where $x_n = [x\sqrt{n}]$ for the integer random walks and $x_n = x\sqrt{n}$ for continuous ones. Such functionals of the random walk were considered in Skorohod and Slobodenyuk (1970).

We now give examples of functionals of the random walk ν_k^n that are described by the process $q_n(t,x)$. In the list of examples, we give in parentheses the function f for which $\sqrt{n}\,q_n(t,x)$ is the indicated functional. The examples are

1) the number of times ν_k^n hits the point $[x\sqrt{n}]$ up to the time $[nt]$ $\big(f(y,z)=\mathbb{I}_{\{0\}}(y)\big)$;

2) the number of times ν_k^n hits the interval $(x_n-\alpha, x_n+\beta)$ up to the time $[nt]$ $\big(f(y,z)=\mathbb{I}_{(-\alpha,\beta)}(y),\ \alpha,\ \beta>0\big)$;

3) the number of times ν_k^n crosses the level x_n in $[nt]$ steps $\big(f(y,z)=\mathbb{I}_{(-\infty,0)}(yz)\big)$;

4) the total length of the steps of the random walk ν_k^n that cross the level x_n up to the time $[nt]$ $\big(f(y,z)=|z-y|\,\mathbb{I}_{(-\infty,0)}(yz)\big)$.

The limit process for $q_n(t,x)$ is the process $h\ell(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}$, where ℓ is the Brownian local time (see § 5 Ch. II and § 1 Ch. V). Under certain assumptions on the function f and under the assumption that the random walk just has a second moment, we prove in § 6 and § 7 that for any $T>0$

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}} |q_n(t,x)-h\ell(t,x)|\to 0, \qquad n\to\infty, \tag{1.3}$$

in probability. This is the so-called *weak invariance principle for local times.* Together with the classical invariance principle (1.2) this result enables us to prove convergence of distributions of functionals of random walks when the classical principle cannot be applied. This significantly extends the class of functionals of random walks for which we can prove convergence of distributions.

Under more restrictive assumptions we establish in § 8 a *strong invariance principle*, which gives an estimate of the rate of convergence in (1.3). Various applications of the invariance principle for local times are considered in § 9.

We use the Skorohod embedding scheme (see Skorohod (1965)) described in § 2 to construct the random walks ν_k^n from the Brownian motion W.

The assertions of this chapter can be extended for the processes $q_n(t,x)$ defined as the normalized partial sums of $f(\nu_{k-1}-x_n, \nu_k-x_n,\dots,\nu_{k+l}-x_n)$ with any integer $l\ge 1$.

§ 2. The Skorohod embedding scheme

The *Skorohod embedding scheme* describes how using a Brownian motion one can construct a sequence of stopping times such that the Brownian motion stopped at these moments forms a given random walk with zero mean and finite variance. Without loss of generality, we can assume that the variance of a step of the random walk is equal to 1, because a Brownian motion has the scaling property (see § 1 Ch. V). This scheme will be used to construct from a Brownian motion $W(t)$, $t\in[0,\infty)$, $W(0)=0$, a sequences of random walks ν_k^n, $n=1,2,\dots$, distributed like the original random walk $\nu_k=\sum_{l=1}^{k}\xi_l$, $k=1,2,\dots$, $\nu_0=0$, for every fixed n.

Before describing this scheme in detail, we mention some properties of a Brownian motion W with zero initial value. Let $H:=\min\{s: W(s)\in(-a,b)\}$, $a>0$,

$b > 0$, be the first exit time. We rewrite formula (5.29) of Ch. III as

$$\mathbf{E}e^{-\alpha H} = \frac{\operatorname{ch}((b-a)\sqrt{\alpha/2})}{\operatorname{ch}((a+b)\sqrt{\alpha/2})}. \tag{2.1}$$

Formulas (5.10) of Ch. III are transformed to the equalities

$$\mathbf{P}(W(H) = -a) = \frac{b}{a+b}, \qquad \mathbf{P}(W(H) = b) = \frac{a}{a+b}. \tag{2.2}$$

We need the formulas for the series expansion of the hyperbolic cosine and hyperbolic secant:

$$\operatorname{ch} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \qquad \frac{1}{\operatorname{ch} x} = \operatorname{sech} x = \sum_{k=0}^{\infty} \frac{(-1)^k E_k x^{2k}}{(2k)!},$$

where

$$E_l := \frac{2^{2l+2}(2l)!}{\pi^{2l+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2l+1}}$$

are *Euler's numbers*, $E_0 = E_1 = 1$, $E_2 = 5$, ... Expanding the left-hand side and the right-hand side of (2.1) in powers of α, we get for any integer $q \geq 0$

$$\mathbf{E}H^q = \frac{q!}{2^q} \sum_{l=0}^{q} \frac{(-1)^{q-l} E_l}{(2l)!(2q-2l)!} (a+b)^{2l}(b-a)^{2q-2l}. \tag{2.3}$$

For $q = 1$ (2.3) becomes $\mathbf{E}H = ab$ and for $q = 2$ it becomes $\mathbf{E}H^2 = \frac{ab}{3}(a^2+3ab+b^2)$.

Lemma 2.1. *For $q = 1, 2, \dots$*

$$\mathbf{E}H^q \leq \frac{(q+1)!}{2^{q-4}} ab(a+b)^{2q-2}. \tag{2.4}$$

Proof. For a fixed q we set $C_l := \frac{(-1)^l E_l}{(2l)!(2q-2l)!}$. Formula (2.3) with $a = 0$ implies that $\sum_{l=0}^{q} C_l = 0$, because in this case $H = 0$. Then (2.3) can be rewritten as

$$\mathbf{E}H^q = \frac{q!(-1)^q}{2^q} \sum_{l=0}^{q-1} \Big(\sum_{j=0}^{l} C_j\Big)\big((a+b)^{2l}(b-a)^{2q-2l} - (a+b)^{2l+2}(b-a)^{2q-2l-2}\big).$$

It is obvious that

$$\big|(a+b)^{2l}(b-a)^{2q-2l} - (a+b)^{2l+2}(b-a)^{2q-2l-2}\big| \leq 4ab\,(a+b)^{2q-2},$$

$$\sum_{l=0}^{q} |C_l| \leq \sum_{l=0}^{\infty} \frac{2^{2l+2}}{\pi^{2l+1}} = \frac{4}{\pi(1-4/\pi^2)} \leq 4.$$

Substituting these estimates in the above equality, we get (2.4). □

Using example 5.3 of Ch. III for $x = q = 0$, we find that

$$\mathbf{E}e^{-\beta\ell(H,0)} = \mathbf{E}\{e^{-\beta\ell(H,0)}|W(H) = -a\} = \mathbf{E}\{e^{-\beta\ell(H,0)}|W(H) = b\} = \frac{b+a}{b+a+2\beta ab}$$

for $\beta \geq 0$. Expanding both sides of this equality in powers of β, we get

$$\begin{aligned} \mathbf{E}\ell^q(H,0) &= \mathbf{E}\{\ell^q(H,0)|W(H) = -a\} \\ &= \mathbf{E}\{\ell^q(H,0)|W(H) = b\} = \frac{2^q q! a^q b^q}{(a+b)^q} \end{aligned} \tag{2.5}$$

for any integer $q \geq 0$.

We now proceed to the description of the Skorohod embedding scheme. Let $F(x)$, $x \in \mathbf{R}$, be an arbitrary distribution function with zero mean and unit variance:

$$\int_{-\infty}^{\infty} x\,dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\,dF(x) = 1.$$

We consider two positive random variables a and b with the joint distribution

$$\mathbf{P}\bigl(-a \in [x, x+dx), b \in [y, y+dy)\bigr) = \frac{2(y-x)\,dF(x)\,dF(y)}{p_0 \int_{-\infty}^{\infty} |z|\,dF(z)} 1\!\!1_{(-\infty,0)}(x) 1\!\!1_{(0,\infty)}(y), \tag{2.6}$$

where $p_0 = 1 - F(+0) + F(0)$ is the probability that a random variable with the distribution function F does not equal to zero. It is easy to see that (2.6) determines the joint distribution function by considering the equalities

$$-\int_{(-\infty,0)} x dF(x) = \int_{(0,\infty)} x\,dF(x) = \frac{1}{2}\int_{-\infty}^{\infty} |x|\,dF(x). \tag{2.7}$$

Let $F(x) := \mathbf{P}(\xi_1 < x)$, $x \in \mathbf{R}$, be the distribution function of the step of the random walk ν_k. To simplify notations, we consider two independent random variables μ and η with distributions

$$\mathbf{P}(\mu < x) = \mathbf{P}(-\xi_1 < x), \qquad \mathbf{P}(\eta < x) = \mathbf{P}(\xi_1 < x).$$

Then (2.7) can be written as follows:

$$\mathbf{E}\bigl\{\mu 1\!\!1_{\{\mu>0\}}\bigr\} = \mathbf{E}\bigl\{\eta 1\!\!1_{\{\eta>0\}}\bigr\} = \frac{1}{2}\,\mathbf{E}|\xi_1|. \tag{2.8}$$

Moreover, (2.6) implies that for an arbitrary bounded measurable function $f(x,y)$, $(x,y) \in (0,\infty)\times(0,\infty)$,

$$\mathbf{E}f(a,b) = \frac{2}{p_0\mathbf{E}|\xi_1|}\mathbf{E}\bigl\{(\mu+\eta)f(\mu,\eta)1\!\!1_{\{\mu>0,\eta>0\}}\bigr\}. \tag{2.9}$$

Let the two-dimensional random variable (a,b) be independent of the Brownian motion W. Let H_n be the first time the process W hits the set $\{-a/\sqrt{n}, 0, b/\sqrt{n}\}$ after the first exit time from the interval $(-p_0a/\sqrt{n}, p_0b/\sqrt{n})$, i.e., $H_n := H_{1,n} + H_{2,n}$, where

$$H_{1,n} := \min\Big\{s \ge 0 : W(s) \notin \Big(-\frac{p_0a}{\sqrt{n}}, \frac{p_0b}{\sqrt{n}}\Big)\Big\},$$

$$H_{2,n} := \min\Big\{s \ge 0 : W(s+H_{1,n}) \in \Big\{-\frac{a}{\sqrt{n}}, 0, \frac{b}{\sqrt{n}}\Big\}\Big\}.$$

We remark that for $p_0 = 1$ the time H_n is just the first exit time from the interval $(-a/\sqrt{n}, b/\sqrt{n})$.

Lemma 2.2. *The distribution function of $W(H_1)$ coincides with F and $\mathbf{E}H_1 = 1$.*

Proof. Let $x > 0$. Using the independence of the two-dimensional variable (a,b) of the Brownian motion W, Fubini's theorem, and the strong Markov property of the process W together with (2.2) and (2.9), we get

$$\mathbf{P}\big(W(H_1) \in [x, x+dx)\big) = \mathbf{E}\Big(\mathbb{I}_{[x,x+dx)}(b)\mathbb{I}_{(0,\infty)}(a)\frac{p_0a}{p_0a+p_0b}\frac{p_0b}{b}\Big)$$

$$= \frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\mathbb{I}_{[x,x+dx)}(\eta)\frac{\mu}{\mu+\eta}\mathbb{I}_{(0,\infty)}(\mu)\Big\} = \mathbf{P}\big(\xi_1 \in [x, x+dx)\big).$$

The case $x < 0$ is handled similarly. The case $x = 0$ need not be treated separately, but we dwell on it in order to gain experience in such computations. We have

$$\mathbf{P}\big(W(H_1) = 0\big) = \mathbf{E}\Big\{\frac{p_0b}{p_0a+p_0b}\frac{(1-p_0)a}{a} + \frac{p_0a}{p_0a+p_0b}\frac{(1-p_0)b}{b}\Big\} = 1 - p_0.$$

The proof of the second assertion of the lemma is based on the strong Markov property of the process W and on the formulas (2.3) with $q = 1$ and (2.9). We have

$$\mathbf{E}H_1 = \mathbf{E}H_{1,1} + \mathbf{E}H_{2,1} = \mathbf{E}\big(p_0^2ab\big) + \mathbf{E}\Big(\frac{b}{a+b}p_0(1-p_0)a^2 + \frac{a}{a+b}p_0(1-p_0)b^2\Big)$$

$$= p_0\mathbf{E}(ab) = \frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\mu\eta\mathbb{I}_{\{\mu>0,\eta>0\}}\big\}$$

$$= \mathbf{E}\big\{\mu^2\mathbb{I}_{\{\mu>0\}}\big\} + \mathbf{E}\big\{\eta^2\mathbb{I}_{\{\eta>0\}}\big\} = \mathbf{E}\xi_1^2 = 1. \tag{2.10}$$

□

Remark 2.1. For any integer $n \ge 1$ the distribution function of the variable $\sqrt{n}\,W(H_n)$ coincides with F and $\mathbf{E}H_n = 1/n$.

This follows from Lemma 2.2 and the scaling property of the Brownian motion (see § 10 Ch. I).

Given a Brownian motion W we can now construct random walks ν_k^n distributed like the original random walk ν_k for each $n = 1, 2, \dots$. We consider a sequence $\{(a_k, b_k)\}_{k=1}^\infty$ of independent two-dimensional random variables identically distributed with (a,b). Assume that the sequence of random variables $\{(a_k, b_k)\}_{k=1}^\infty$

is independent of the process W. Let $\nu_k^n := \sqrt{n}\,W\Big(\sum_{l=1}^{k} H_n^{(l)}\Big)$, where for each $l = 1, 2, \dots$ the stopping time $H_n^{(l)}$ is determined recurrently from the Brownian motion

$$W^{(l)}(s) := W\Big(s + \sum_{j=1}^{l-1} H_n^{(j)}\Big) - W\Big(\sum_{j=1}^{l-1} H_n^{(j)}\Big), \qquad s \ge 0,$$

and the random variable (a_l, b_l) as follows: $H_n^{(l)} := H_{1,n}^{(l)} + H_{2,n}^{(l)}$, where

$$H_{1,n}^{(l)} := \min\Big\{s \ge 0 : W^{(l)}(s) \notin \Big(-\frac{p_0 a_l}{\sqrt{n}}, \frac{p_0 b_l}{\sqrt{n}}\Big)\Big\},$$

$$H_{2,n}^{(l)} := \min\Big\{s \ge 0 : W^{(l)}(s + H_{1,n}^{(l)}) \in \Big\{-\frac{a_l}{\sqrt{n}}, 0, \frac{b_l}{\sqrt{n}}\Big\}\Big\}.$$

Using the strong Markov property of the Brownian motion W and Remark 2.1, it is not hard to see that the variables ν_k^n can be represented as the sum $\nu_k^n = \sum_{l=1}^{k} \xi_l^n$ of independent identically distributed random variables

$$\xi_l^n := \sqrt{n}\Big(W\Big(\sum_{j=1}^{l} H_n^{(j)}\Big) - W\Big(\sum_{j=1}^{l-1} H_n^{(j)}\Big)\Big), \qquad l = 1, 2, \dots, \tag{2.11}$$

whose distribution function coincides with F.

We sum up the essence of the Skorohod embedding scheme. This scheme describes the construction of the copies ν_k^n of the original random walk ν_k such that

$$\frac{1}{\sqrt{n}}\nu_k^n = W\Big(\sum_{l=1}^{k} H_n^{(l)}\Big), \qquad \mathbf{E}\sum_{l=1}^{k} H_n^{(l)} = \frac{k}{n}, \qquad k = 1, 2, \dots, \tag{2.12}$$

where $H_n^{(l)}$ are the increments of successive random stopping times of the Brownian motion W.

§ 3. Invariance principle for random walks

The *weak invariance principle for random walks* is characterized by the formula (1.2). In what follows we deals with the random walks ν_k^n constructed according to the Skorohod embedding scheme. For every n these random walks are identically distributed like the initial random walk ν_k. Therefore, considering the random walk ν_k^n under the expectation sign or probability, we omit sometimes the index n to simplify the notations. The same is true for the variables $\xi_k^n = \nu_k^n - \nu_{k-1}^n$. They will be denoted by ξ_k under the expectation sign or probability. Also to simplify the notation we often omit the index 1 on $H_n^{(1)}$.

We first derive a number of auxiliary results that will be often used for proving theorems on convergence to the Brownian motion and to its local time.

We define the truncated two-dimensional random variable $(\bar{a}, \bar{b})$ as

$$\bar{a} := \begin{cases} a, & \text{for } a \le \theta\sqrt{n}, \\ \theta\sqrt{n}, & \text{for } a > \theta\sqrt{n}, \end{cases}$$

$$\bar{b} := \begin{cases} b, & \text{for } b \le \theta\sqrt{n}, \\ \theta\sqrt{n}, & \text{for } b > \theta\sqrt{n}, \end{cases}$$

where $\theta = \theta(n)$ will be chosen in dependence on the assertions to be proved, but such that $\theta \to 0$ and $\theta\sqrt{n} \to \infty$.

We consider the truncated random walk $\bar{\nu}_k^n$, constructed from the Brownian motion W in the same way as the original random walk ν_k^n, but using the truncated two-dimensional random variable $(\bar{a}, \bar{b})$. Let $\{(\bar{a}_k, \bar{b}_k)\}_{k=1}^{\infty}$ be a sequence of independent two-dimensional random variables distributed like $(\bar{a}, \bar{b})$. It is clear that this sequence is independent of W, because this is true for $\{(a_k, b_k)\}_{k=1}^{\infty}$.

Let $\bar{\nu}_k^n := \sqrt{n}\, W\Big(\sum\limits_{l=1}^{k} \overline{H}_n^{(l)}\Big)$, where the stopping time $\overline{H}_n^{(l)}$, $l = 1, 2 \dots$, is defined from the Brownian motion

$$\overline{W}^{(l)}(s) := W\Big(s + \sum_{j=1}^{l-1} \overline{H}_n^{(j)}\Big) - W\Big(\sum_{j=1}^{l-1} \overline{H}_n^{(j)}\Big), \qquad s \ge 0,$$

and the random variables $(\bar{a}_l, \bar{b}_l)$ as follows: $\overline{H}_n^{(l)} := \overline{H}_{1,n}^{(l)} + \overline{H}_{2,n}^{(l)}$,

$$\overline{H}_{1,n}^{(l)} := \min\Big\{s \ge 0 : \overline{W}^{(l)}(s) \notin \Big(-\frac{p_0 \bar{a}_l}{\sqrt{n}}, \frac{p_0 \bar{b}_l}{\sqrt{n}}\Big)\Big\},$$

$$\overline{H}_{2,n}^{(l)} := \min\Big\{s \ge 0 : \overline{W}^{(l)}(s + \overline{H}_{1,n}^{(l)}) \in \Big\{-\frac{\bar{a}_l}{\sqrt{n}}, 0, \frac{\bar{b}_l}{\sqrt{n}}\Big\}\Big\}.$$

We denote $\bar{\xi}_k^n := \bar{\nu}_k^n - \bar{\nu}_k^{n-1}$, $\Omega_n := \{\bar{\nu}_k^n = \nu_k^n, k = 1, 2, \dots, n\}$. Observe that $|\bar{\xi}_k^n| \le \theta\sqrt{n}$. We explain the choice of the set Ω_n. Suppose that $p_0 = 1$ and hence,

$$\overline{H}_n^{(l)} = \min\Big\{s : \overline{W}^{(l)}(s) \notin \Big(-\frac{\bar{a}_l}{\sqrt{n}}, \frac{\bar{b}_l}{\sqrt{n}}\Big)\Big\}.$$

At first glance we can take the set $\Omega_{0,n} := \bigcap\limits_{k=1}^{n} \{a_k \le \theta\sqrt{n}, b_k \le \theta\sqrt{n}\}$ instead of Ω_n. It is clear that $\Omega_{0,n} \subset \Omega_n$. However, if ξ_1 has only the second moment, then the probability of $\Omega_{0,n}$ does not tend to 1 as $n \to \infty$. Indeed, by (2.9),

$$\mathbf{P}(\Omega_{0,n}) = \prod_{k=1}^{n} \mathbf{P}\big(a_k \le \theta\sqrt{n}, b_k \le \theta\sqrt{n}\big) = \Big(\frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu + \eta)\mathbb{1}_{\{0<\mu\le\theta\sqrt{n}, 0<\eta\le\theta\sqrt{n}\}}\big\}\Big)^n$$

$$= \Big(1 - \frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu + \eta)\mathbb{1}_{\{\mu>\theta\sqrt{n}\}\cup\{\eta>\theta\sqrt{n}\}}\mathbb{1}_{\{\mu>0,\eta>0\}}\big\}\Big)^n \sim \exp\Big\{-n\mathbf{P}\big(|\xi_1| > \theta\sqrt{n}\big)$$

$$-\frac{2n}{\mathbf{E}|\xi_1|}\mathbf{E}\{\xi_1; \xi_1 > \theta\sqrt{n}\}\mathbf{P}(-\theta\sqrt{n} \le \xi_1 < 0) + \frac{2n}{\mathbf{E}|\xi_1|}\mathbf{E}\{\xi_1; \xi_1 < -\theta\sqrt{n}\}\mathbf{P}(0 < \xi_1 \le \theta\sqrt{n})\Big\}.$$

Thus $\mathbf{P}(\Omega_{0,n}) \not\to 1$ as $\theta(n) \to 0$.

When considering the set Ω_n, in addition to the behavior of the random variable $(\bar{a}, \bar{b})$ we also take into account the behavior of the Brownian sample paths. For example, for $p_0 = 1$ the set $\{\bar{\nu}_1^n = \nu_1^n\}$ will hold even if $0 < a_1 \le \theta\sqrt{n}$, $b_1 > \theta\sqrt{n}$, whereas the sample paths of W reach the level $-a_1$ before the level $\theta\sqrt{n}$.

Let us estimate the probability of the complement of Ω_n. Applying (2.2) and (2.9), we get

$$\mathbf{P}(\bar{\nu}_1 \ne \nu_1) = \frac{2}{p_0 \mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\Big(\mathbb{I}_{\{\mu>\theta\sqrt{n},\eta>\theta\sqrt{n}\}} + \mathbb{I}_{\{0<\mu\le\theta\sqrt{n},\eta>\theta\sqrt{n}\}}\frac{\mu}{\mu+\theta\sqrt{n}}$$

$$+\mathbb{I}_{\{\mu>\theta\sqrt{n},0<\eta\le\theta\sqrt{n}\}}\frac{\eta}{\eta+\theta\sqrt{n}}\Big)\Big\} \le \frac{C}{p_0\theta^2 n}\mathbf{E}\big\{\xi_1^2; |\xi_1| > \theta\sqrt{n}\big\}.$$

Here and in what follows, C denotes positive constants, not always the same. As a result, we have

$$\mathbf{P}(\Omega_n^{\mathrm{c}}) = \mathbf{P}\Big(\Big(\bigcap_{k=1}^{n}\{\bar{\xi}_k^n = \xi_k^n\}\Big)^{\mathrm{c}}\Big) \le n\,\mathbf{P}(\bar{\nu}_1 \ne \nu_1) \le \frac{C}{p_0\theta^2}\mathbf{E}\big\{\xi_1^2; |\xi_1| > \theta\sqrt{n}\big\}. \quad (3.1)$$

With an appropriate choice of θ (for example, as in Theorem 3.1), the right-hand side of this estimate tends to zero.

Similarly to the truncation operation, defined above, we set

$$\bar{\mu} := \begin{cases} \mu, & \text{for } \mu \le \theta\sqrt{n}, \\ \theta\sqrt{n}, & \text{for } \mu > \theta\sqrt{n}, \end{cases}$$

$$\bar{\eta} := \begin{cases} \eta, & \text{for } \eta \le \theta\sqrt{n}, \\ \theta\sqrt{n}, & \text{for } \eta > \theta\sqrt{n}. \end{cases}$$

Consider the random moments $\overline{H}_n^{(l)}$, $l = 1, \dots, n$, that are independent and identically distributed. For simplicity we sometimes omit the superscript (1) of the quantities $\overline{H}_n^{(1)}$, $\overline{H}_{1,n}^{(1)}$, $\overline{H}_{2,n}^{(1)}$. Using the strong Markov property of the Brownian motion, (2.3) with $q = 1, 2$, (2.9), and (2.10), we get

$$\mathbf{E}\overline{H}_n = \mathbf{E}\overline{H}_{1,n} + \mathbf{E}\overline{H}_{2,n} = \frac{2}{p_0\mathbf{E}|\xi_1|n}\mathbf{E}\Big\{(\mu+\eta)\Big(p_0^2\bar{\mu}\bar{\eta} + \frac{\bar{\eta}}{\bar{\mu}+\bar{\eta}}p_0(1-p_0)\bar{\mu}^2$$

$$+\frac{\bar{\mu}}{\bar{\mu}+\bar{\eta}}p_0(1-p_0)\bar{\eta}^2\Big)\mathbb{I}_{\{\mu>0,\eta>0\}}\Big\} = \frac{2}{\mathbf{E}|\xi_1|n}\mathbf{E}\big\{(\mu+\eta)\bar{\mu}\bar{\eta}\mathbb{I}_{\{\mu>0,\eta>0\}}\big\}$$

$$= \frac{2}{\mathbf{E}|\xi_1|n}\Big(\mathbf{E}\big\{(\mu+\eta)\mu\eta\mathbb{I}_{\{\mu>0,\eta>0\}}\big\} + O\big(\mathbf{E}\big\{\xi_1^2; |\xi_1| > \theta\sqrt{n}\big\}\big)\Big)$$

$$= \frac{1}{n}\Big(1 + \frac{2}{\mathbf{E}|\xi_1|}O\big(\mathbf{E}\{\xi_1^2; |\xi_1| > \theta\sqrt{n}\}\big)\Big), \quad (3.2)$$

$$\mathbf{E}\overline{H}_n^2 \le 2\big(\mathbf{E}\overline{H}_{1,n}^2 + \mathbf{E}\overline{H}_{2,n}^2\big) = \frac{4}{3p_0 n^2\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\big\{p_0^4\bar{\mu}\bar{\eta}\big(\bar{\mu}^2 + 3\bar{\mu}\bar{\eta} + \bar{\eta}^2\big)$$

$$+(p_0-p_0^2)(1+p_0-p_0^2)\bar{\mu}\bar{\eta}(\bar{\mu}^2-\bar{\mu}\bar{\eta}+\bar{\eta}^2)\big\}\mathbb{I}_{\{\mu>0,\eta>0\}}\big\} \le \frac{C}{n^2}\mathbf{E}\big\{|\xi_1||\bar{\xi}_1|^3\big\} \le \frac{C\theta^2}{p_0 n}, \quad (3.3)$$

where

$$|\bar{\xi}_1| = |\xi_1|\mathbb{1}_{\{|\xi_1|\le\theta\sqrt{n}\}} + \theta\sqrt{n}\mathbb{1}_{\{|\xi_1|>\theta\sqrt{n}\}}.$$

We now pass directly to the derivation of the classical invariance principle (1.2). We recall the notation $W_n(t) := \frac{1}{\sqrt{n}}\nu^n_{[nt]}$, $t \in [0,T]$, introduced above. Without loss of generality, we assume that $T = 1$. Set $\tau_n(t) := \sum_{l=1}^{[nt]} H_n^{(l)}$, $t \in [0,1]$. According to the construction of the random walk ν^n_k, $k = 1,2,\dots$, with the help of the Skorohod embedding scheme (see (2.12)), we have $W_n(t) = W(\tau_n(t))$.

Theorem 3.1. *Let* $\mathbf{E}\xi_1^2 < \infty$,

$$\theta := \max\bigl\{\mathbf{E}^{1/3}\{\xi_1^2; |\xi_1| > n^{1/4}\}, n^{-1/4}\bigr\}.$$

Then

$$\mathbf{P}\Bigl(\sup_{0\le t\le 1}|W_n(t) - W(t)| > \theta^{1/5}\Bigr) \le C\bigl(1 + 2/p_0\bigr)\,\theta, \tag{3.4}$$

where C *is a constant.*

Proof. Set $\bar{\tau}_n(t) := \sum_{l=1}^{[nt]} \overline{H}_n^{(l)}$, $t \in [0,1]$. From (3.2) we deduce that

$$\Bigl|\sum_{l=1}^{[nt]}\mathbf{E}\overline{H}_n^{(l)} - \frac{[nt]}{n}\Bigr| \le C\mathbf{E}\{\xi_1^2; |\xi_1| > \theta\sqrt{n}\} \le C\mathbf{E}\{\xi_1^2; |\xi_1| > n^{1/4}\} \le C\theta^3.$$

Then for all sufficiently large n such that $C\theta^3 + 1/n \le \sqrt{\theta}$, we have

$$\mathbf{P}\Bigl(\sup_{0\le t\le 1}|\bar{\tau}_n(t) - t| > 2\sqrt{\theta}\Bigr) \le \mathbf{P}\Bigl(\sup_{1\le k\le n}\Bigl|\sum_{l=1}^{k}\bigl(\overline{H}_n^{(l)} - \mathbf{E}\overline{H}_n^{(l)}\bigr)\Bigr| > \sqrt{\theta}\Bigr).$$

Now, using (5.10) Ch. I and (3.3), we obtain

$$\mathbf{P}\Bigl(\sup_{0\le t\le 1}|\bar{\tau}_n(t) - t| > 2\sqrt{\theta}\Bigr) \le \frac{1}{\theta}\mathbf{E}\Bigl|\sum_{l=1}^{n}\bigl(\overline{H}_n^{(l)} - \mathbf{E}\overline{H}_n^{(l)}\bigr)\Bigr|^2 \le \frac{C\theta}{p_0}. \tag{3.5}$$

By (3.1),

$$\mathbf{P}(\Omega_n^{\rm c}) \le \frac{C}{p_0\theta^2}\mathbf{E}\bigl\{\xi_1^2, |\xi_1| > n^{1/4}\bigr\} \le \frac{C\theta}{p_0}.$$

On the set Ω_n we have $\overline{H}_n^{(l)} = H_n^{(l)}$ for all $l = 1,\dots,n$, and hence, $\bar{\tau}_n(t) = \tau_n(t)$, $t \in [0,1]$. Therefore,

$$\mathbf{P}\Bigl(\sup_{0\le t\le 1}|\tau_n(t) - t| > 2\sqrt{\theta}\Bigr) \le \mathbf{P}(\Omega_n^{\rm c}) + \mathbf{P}\Bigl(\sup_{0\le t\le 1}|\bar{\tau}_n(t) - t| > 2\sqrt{\theta}\Bigr) \le \frac{2C\theta}{p_0}. \tag{3.6}$$

Then, using (3.6) and the representation $W_n(t) = W(\tau_n(t))$, we have

$$\mathbf{P}\Bigl(\sup_{0\le t\le 1}|W_n(t) - W(t)| > \theta^{1/5}\Bigr) \le \frac{2\,C\theta}{p_0} + \mathbf{P}\Bigl(\sup_{|s-t|\le 2\sqrt{\theta}, t\in[0,1]}|W(s) - W(t)| > \theta^{1/5}\Bigr)$$

$$\leq \frac{2\,C\theta}{p_0} + \mathbf{P}\biggl(\bigcup_{k=0}^{[1/\sqrt{\theta}]}\Bigl\{\sup_{s\in[t_k,t_{k+1}]}|W(s)-W(t_k)| > \frac{\theta^{1/5}}{3}\Bigr\}\biggr)$$

$$\leq \frac{2\,C\theta}{p_0} + \frac{1+\sqrt{\theta}}{\sqrt{\theta}}\mathbf{P}\Bigl(\sup_{s\in[0,\sqrt{\theta}]}|W(s)| > \frac{\theta^{1/5}}{3}\Bigr) \leq \frac{2\,C\theta}{p_0} + \frac{3^{30}2}{\theta^6\sqrt{\theta}}\mathbf{E}\sup_{s\in[0,\sqrt{\theta}]} W^{30}(s), \quad (3.7)$$

where $\{t_k\}$, $k = 0, 1, 2, \dots$, is the lattice in $[0, 1]$ with the array spacing $\sqrt{\theta}$. The inequality (4.25) Ch. II with $h \equiv 1$ and $k = 15$ implies the estimate

$$\mathbf{E}\sup_{s\in[0,\sqrt{\theta}]} W^{30}(s) \leq C_1\,\theta^7\sqrt{\theta}.$$

This estimate together with (3.7) implies (3.4). □

Remark 3.1. Since $\theta \downarrow 0$ as $n \to \infty$, (3.4) implies (1.2), i.e., the *weak invariance principle for random walks* holds.

Remark 3.2. Under the fourth finite moment condition $\mathbf{E}\xi_1^4 < \infty$ we can prove the weak invariance principle without using the truncation procedure (see the estimate (3.3)).

Further we consider the *strong invariance principle for random walks*, which gives an a.s. estimate for the rate of convergence of the differences $W_n(t) - W(t)$ to zero.

Theorem 3.2. *Assume that* $\mathbf{E}\,\xi_1^8 < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{n^{1/4}}{\ln n} \sup_{0\leq t\leq 1} |W_n(t) - W(t)| < \infty \qquad \text{a.s.} \quad (3.8)$$

Proof. According to the first part of the Borel–Cantelli lemma (see Remark 1.1 Ch. I), it suffices to prove that for some constant $K > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Bigl(\sup_{0\leq t\leq 1} |W_n(t) - W(t)| > Kn^{-1/4}\ln n\Bigr) < \infty. \quad (3.9)$$

Indeed, from (3.9) if follows that there exists a.s. a number $n_0 = n_0(\omega)$ such that for all $n \geq n_0$

$$\sup_{0\leq t\leq 1} |W_n(t) - W(t)| \leq Kn^{-1/4}\ln n.$$

This implies (3.8).

We prove an auxiliary result, which characterizes closeness of t and $\tau_n(t)$.

Lemma 3.1. *For some constant* $K_1 > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Bigl(\sup_{0\leq t\leq 1} |\tau_n(t) - t| > K_1\frac{\ln n}{\sqrt{n}}\Bigr) < \infty. \quad (3.10)$$

Proof. As mentioned above, if the set Ω_n holds, then $\bar\tau_n(t) = \tau_n(t)$, $t \in [0,1]$. We estimate the probability of the complement of Ω_n with the help of the estimates (3.1)–(3.3) for $\theta = n^{-1/4}$. From (3.1) we have

$$\sum_{n=1}^{\infty} \mathbf{P}(\Omega_n^{\mathrm{c}}) \le \frac{C}{p_0} \sum_{n=1}^{\infty} \mathbf{E}\{\xi_1^4; \xi_1^4 > n\} = \frac{C}{p_0} \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathbf{E}\big\{\xi_1^4 \mathbb{I}_{\{k<\xi_1^4\le k+1\}}\big\}$$

$$= \frac{C}{p_0} \sum_{k=1}^{\infty} k\mathbf{E}\big\{\xi_1^4 \mathbb{I}_{\{k<\xi_1^4\le k+1\}}\big\} \le \frac{C}{p_0}\mathbf{E}\{\xi_1^8; |\xi_1| > 1\}. \tag{3.11}$$

This estimate enables us to replace in (3.10) the moments $\tau_n(t)$ by the moments $\bar\tau_n(t)$. By (3.2),

$$\Big|\mathbf{E}\overline{H}_n - \frac{1}{n}\Big| \le \frac{C}{n^{5/2}}. \tag{3.12}$$

Applying (2.2), (2.4) and (2.9), we get that for any integer $q \ge 2$

$$\mathbf{E}\overline{H}_n^q \le 2^{q-1}\Big(\mathbf{E}\overline{H}_{1,n}^q + \mathbf{E}\overline{H}_{2,n}^q\Big)$$

$$\le \frac{(q+1)!2^3}{p_0\mathbf{E}|\xi_1|n^q}\mathbf{E}\Big\{(\mu+\eta)\Big(p_0^{2q}\bar\mu\bar\eta(\bar\mu+\bar\eta)^{2q-2} + \frac{\bar\mu\bar\eta(1-p_0)p_0}{\bar\mu+\bar\eta}(\bar\mu^{2q-1}+\bar\eta^{2q-1})\Big)\mathbb{I}_{\{\mu>0,\eta>0\}}\Big\}$$

$$\le \frac{(q+1)!2^{2q+2}}{\mathbf{E}|\xi_1|n^q}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta(\bar\mu^{2q-2}+\bar\eta^{2q-2})\mathbb{I}_{\{\mu>0,\eta>0\}}\big\} \le \frac{C(q+1)!\,2^{2q}}{n^{1+q/2}}. \tag{3.13}$$

We prove that for some $\lambda > 0$ and any integer $1 \le m \le n$,

$$\mathbf{E}\exp\Big(\lambda\Big|\sqrt{n}\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big|\Big) \le 2. \tag{3.14}$$

To estimate the corresponding moments we use induction on the power of moments. The induction hypothesis is the following: for all integers $1 \le m \le n$ and $0 \le p \le q-1$

$$\Big|\mathbf{E}\Big(\sqrt{n}\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big)^p\Big| \le L^p p!. \tag{3.15}$$

For $p = 0$ and $p = 1$ this estimate is obvious, providing the induction base. We prove (3.15) for $p = q$.

For integers $m \ge 2$ and $q \ge 1$, we have the equalities

$$\Big(\sum_{k=1}^{m} x_k\Big)^q = \sum_{j=1}^{q}\frac{q!}{j!(q-j)!}x_1^j\Big(\sum_{l=2}^{m} x_l\Big)^{q-j} + \Big(\sum_{l=2}^{m} x_l\Big)^q$$

$$= \cdots = \sum_{j=1}^{q}\frac{q!}{j!(q-j)!}\sum_{k=1}^{m-1} x_k^j\Big(\sum_{l=k+1}^{m} x_l\Big)^{q-j} + x_m^q.$$

The resulting equality can be written in a more compact form, if we set $0^0 = 1$ and set the sums equal to zero if the upper index is less than the lower one. Then

$$\Big(\sum_{k=1}^{m} x_k\Big)^q = \sum_{j=1}^{q} \frac{q!}{j!(q-j)!} \sum_{k=1}^{m} x_k^j \Big(\sum_{l=k+1}^{m} x_l\Big)^{q-j}. \tag{3.16}$$

Set $X_k := \sqrt{n}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)$, $k = 1, \dots, n$. These variables are independent and identically distributed. From (3.12) and (3.13) it follows that

$$\mathbf{E}|X_1|^q \le n^{q/2} 2^{q-1} \big(\mathbf{E}\big(\overline{H}_n^{(1)}\big)^q + \mathbf{E}^q \overline{H}_n^{(1)}\big) \le \frac{C(q+1)!2^{3q}}{n}, \qquad q = 2, 3, \dots.$$

Using this estimate, (3.15) for $p \le q-1$, (3.16), and the fact that the variables X_k are independent and identically distributed, we conclude that for every integer $1 \le m \le n$,

$$\Big|\mathbf{E}\Big(\sum_{k=1}^{m} X_k\Big)^q\Big| = \Big|\sum_{j=1}^{q} \frac{q!}{j!(q-j)!} \sum_{k=1}^{m} \mathbf{E}X_k^j \mathbf{E}\Big(\sum_{l=k+1}^{m} X_l\Big)^{q-j}\Big|$$

$$\le L^q q! \sum_{j=1}^{\infty} L^{-j}(j+1)\frac{C2^{3j}m}{n} = L^q q! \,\frac{Cm}{n}\,\frac{16(1-4/L)}{L(1-8/L)^2}.$$

For $L \ge \dfrac{8\sqrt{C+1}}{\sqrt{C+1}-\sqrt{C}}$ this yields (3.15) for $p = q$. To prove the previous estimate we used the equality

$$\sum_{j=1}^{\infty}(j+1)z^j = \Big(\sum_{l=0}^{\infty} z^l - 1 - z\Big)' = \Big(\frac{1}{1-z}\Big)' - 1 = \frac{z(2-z)}{(1-z)^2}, \qquad 0 < z < 1.$$

We now pass to the derivation of the estimate analogous to (3.15) in which the absolute moments are considered. Using *Stirling's formula*

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\Big(1 + \frac{\alpha_n}{n}\Big), \qquad n = 1, 2, \dots,$$

where $|\alpha_n| \le 1/12$, we get that for some $M > 1$

$$(n!)^{(n-1)/n} \le M^{n-1}(n-1)!. \tag{3.17}$$

Indeed, the logarithm of the ratio $(n!)^{(n-1)/n}/(n-1)!$, $n \ge 2$, has the expression

$$\Big(n - \frac{1}{2}\Big)\ln\Big(\frac{n}{n-1}\Big) - \frac{1}{2n}\ln(2\pi n) + \ln\Big(\Big(1 + \frac{\alpha_n}{n}\Big)\Big/\Big(1 + \frac{\alpha_{n-1}}{n-1}\Big)\Big) - \frac{1}{n}\ln\Big(1 + \frac{\alpha_n}{n}\Big),$$

which is estimated by $(n-1)\ln M$.

Using (3.15) for even p and (3.17), we obtain

$$\mathbf{E}\Big|\sqrt{n}\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big|^{p-1} \le \mathbf{E}^{(p-1)/p}\Big(\sqrt{n}\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big)^p$$

$$\leq L^{p-1}(p!)^{(p-1)/p} \leq (LM)^{p-1}(p-1)!,$$

and, therefore,

$$\mathbf{E}\Big|\sqrt{n}\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big|^p \leq (LM)^p p!$$

for all integers p. This estimate implies (3.14) with $\lambda = (2LM)^{-1}$.

We conclude the proof of the lemma. By (3.12), (3.14), for sufficiently large constants $K_2 < K_1$

$$\mathbf{P}\Big(\sup_{0\leq t\leq 1}|\tau_n(t)-t| > K_1\frac{\ln n}{\sqrt{n}}\Big) \leq \mathbf{P}(\Omega_n^{\rm c})+\mathbf{P}\Big(\sup_{0\leq t\leq 1}\Big|\sum_{k=1}^{[nt]}\big(\overline{H}_n^{(k)}-\mathbf{E}\overline{H}_n^{(k)}\big)\Big| > K_2\frac{\ln n}{\sqrt{n}}\Big)$$

$$\leq \mathbf{P}(\Omega_n^{\rm c}) + \sum_{m=1}^{n}\mathbf{P}\Big(\Big|\sum_{k=1}^{m}\big(\overline{H}_n^{(k)} - \mathbf{E}\overline{H}_n^{(k)}\big)\Big| > K_2\frac{\ln n}{\sqrt{n}}\Big)$$

$$\leq \mathbf{P}(\Omega_n^{\rm c}) + 2n\exp\big(-\lambda K_2\ln n\big) = \mathbf{P}(\Omega_n^{\rm c}) + 2n^{-1-\rho}, \tag{3.18}$$

where $K_2\lambda = 2+\rho$, $\rho > 0$. From (3.18) and (3.11) we get (3.10). □

It is not hard to derive (3.9) with the help of Lemma 3.1. Indeed, proceeding analogously to the proof of (3.7) and using (2.15) Ch. III, we obtain

$$\mathbf{P}\Big(\sup_{0\leq t\leq 1}|W_n(t) - W(t)| > \frac{K\ln n}{n^{1/4}}\Big)$$

$$\leq \mathbf{P}\Big(\sup_{0\leq t\leq 1}|\tau_n(t)-t| > K_1\frac{\ln n}{\sqrt{n}}\Big) + \mathbf{P}\Big(\sup_{|s-t|\leq K_1\ln n/\sqrt{n}}|W(s)-W(t)| > \frac{K\ln n}{n^{1/4}}\Big)$$

$$\leq \mathbf{P}(\Omega_n^{\rm c}) + 2n^{-1-\rho} + \frac{\sqrt{n}}{K_1\ln n}\mathbf{P}\Big(\sup_{s\in[0,K_1\ln n/(2\sqrt{n})]}|W(s)| > \frac{K\ln n}{3n^{1/4}}\Big)$$

$$\leq \mathbf{P}(\Omega_n^{\rm c}) + 2n^{-1-\rho} + \frac{6\sqrt{n}}{\sqrt{\pi}K\sqrt{K_1}\ln^{3/2}n}\exp\Big(-\frac{K^2\ln n}{9K_1}\Big). \tag{3.19}$$

For $K^2 > 27K_1/2$ the right-hand side of (3.19) is summable, which proves (3.9).

□

§ 4. Estimates for distributions of maximum of sums of random variables

In order to prove weak convergence of processes it is necessary, along with convergence of the finite-dimensional distributions, to estimate the deviations of these processes on small time intervals.

We need the following two results giving estimates for the distributions of the maximum of sums of random variables.

Proposition 4.1 (Billingsley). *Suppose that for some $\gamma > 0$, $\beta > 1$, and for all $\lambda > 0$*

$$\mathbf{P}(|S_l - S_j| \geq \lambda) \leq \frac{1}{\lambda^\gamma}\Big(\sum_{t=j+1}^{l} v_t\Big)^\beta, \qquad 0 \leq j < l \leq n,$$

where $\{S_l\}_{l=1}^n$, $S_0 = 0$, is a sequence of random variables and $\{v_t\}_{t=1}^n$ is a collection of nonnegative numbers. Then

$$\mathbf{P}\Big(\max_{1\leq l\leq n} |S_l| \geq \lambda\Big) \leq \frac{K}{\lambda^\gamma}\Big(\sum_{t=1}^{n} v_t\Big)^\beta,$$

where K is a constant depending only on γ and β.

Proof. We prove this assertion by induction on n. For $n = 1$ the estimate is obvious. Assume that the assertion holds for any integer less than n. We prove that it holds also for n. We choose h, $1 \leq h \leq n$, such that

$$\frac{v_1 + \cdots + v_{h-1}}{v} \leq \frac{1}{2} \leq \frac{v_1 + \cdots + v_h}{v},$$

where $v = v_1 + \cdots + v_n > 0$. Set a sum equal to zero if the upper index of the sum is less than the lower one. Put $V_1 := \max_{1\leq j\leq h-1} |S_j|$, $V_2 := \max_{h+1\leq j\leq n} |S_j - S_h|$. For $h = n$ we set $V_2 := 0$. Then

$$\max_{1\leq l\leq n} |S_l| \leq \max\{V_1, V_2 + |S_h|\}.$$

From this it follows that for $0 < \lambda_1 < \lambda$

$$\begin{aligned}\mathbf{P}\Big(\max_{1\leq l\leq n} |S_l| \geq \lambda\Big) &\leq \mathbf{P}(V_1 \geq \lambda) + \mathbf{P}(V_2 + |S_h| \geq \lambda)\\ &\leq \mathbf{P}(V_1 \geq \lambda) + \mathbf{P}(V_2 \geq \lambda_1) + \mathbf{P}(|S_h| \geq \lambda - \lambda_1).\end{aligned}$$

Using the induction hypothesis, we get that

$$\mathbf{P}\Big(\max_{1\leq l\leq n} |S_l| \geq \lambda\Big) \leq \frac{Kv^\beta}{\lambda^\gamma 2^\beta} + \frac{Kv^\beta}{\lambda_1^\gamma 2^\beta} + \frac{v^\beta}{(\lambda - \lambda_1)^\gamma}.$$

It is not hard to establish that

$$\min_{0<\lambda_1<\lambda}\Big(\frac{K}{2^\beta \lambda_1^\gamma} + \frac{1}{(\lambda - \lambda_1)^\gamma}\Big) = \frac{1}{\lambda^\gamma}\Big(\Big(\frac{K}{2^\beta}\Big)^{1/(\gamma+1)} + 1\Big)^{\gamma+1}.$$

We choose λ_1 corresponding to this minimum value. Then

$$\begin{aligned}\mathbf{P}\Big(\max_{1\leq l\leq n} |S_l| \geq \lambda\Big) &\leq \Big(\frac{K}{2^\beta} + \Big(\Big(\frac{K}{2^\beta}\Big)^{1/(\gamma+1)} + 1\Big)^{\gamma+1}\Big)\frac{v^\beta}{\lambda^\gamma}\\ &= \frac{1}{2^\beta}\Big(1 + \Big(1 + \Big(\frac{2^\beta}{K}\Big)^{1/(1+\gamma)}\Big)^{\gamma+1}\Big)\frac{K}{\lambda^\gamma}v^\beta \leq \frac{K}{\lambda^\gamma}v^\beta.\end{aligned}$$

The last inequality is valid for sufficiently large K depending only on β and γ. □

Remark 4.1. Proposition 4.1 is valid for random variables S_l taking values in an arbitrary normed linear space, in which case the norm in the corresponding space is denoted by $|\cdot|$. This is true because the proof used only the triangle inequality, which holds for any norm.

The next proposition is a generalization of Proposition 4.1 to the case of two parameters.

Proposition 4.2. *For an arbitrary sequence* $\{S_{kl}\}_{k=1,l=1}^{m,n}$ *of random variables set*

$$S_{ij}^{kl} := S_{kl} - S_{il} + S_{ij} - S_{kj}, \qquad 1 \le i \le k, \quad 1 \le j \le l.$$

Suppose that for some $\gamma > 0$, $\alpha > 1$, $\beta > 1$, *and for all* $\lambda > 0$

$$\mathbf{P}\big(|S_{ij}^{kl}| \ge \lambda\big) \le \frac{1}{\lambda^\gamma}\Big(\sum_{s=i+1}^{k} u_s\Big)^\alpha \Big(\sum_{t=j+1}^{l} v_t\Big)^\beta,$$

$$0 \le i < k \le m, \qquad 0 \le j < l \le n,$$

where $\{u_s\}_{s=1}^m, \{v_t\}_{t=1}^n$ *are some collections of nonnegative numbers and* $S_{k0} := 0$, $S_{0l} := 0$ *for all* k *and* l. *Then*

$$\mathbf{P}\Big(\max_{\substack{1\le k\le m\\1\le l\le n}} |S_{kl}| \ge \lambda\Big) \le \frac{K}{\lambda^\gamma}\Big(\sum_{s=1}^{m} u_s\Big)^\alpha \Big(\sum_{t=1}^{n} v_t\Big)^\beta,$$

where K *is a constant depending only on* γ, α, *and* β.

Proof. For the proof of this proposition we can use Proposition 4.1 twice. We fix i and k. Obviously $S_{ij}^{kl} = S_{i0}^{kl} - S_{i0}^{kj}$. Then, if we set $S_v := S_{i0}^{kv}$, we have that $S_{ij}^{kl} = S_l - S_j$, and, in view of the condition formulated in Proposition 4.2, we can apply the preceding result. The factor $\Big(\sum\limits_{s=i+1}^{k} u_s\Big)^\alpha$ can be formally included in each of the terms v_t. By Proposition 4.1,

$$\mathbf{P}\Big(\max_{1\le l\le n} |S_{i0}^{kl}| \ge \lambda\Big) \le \frac{K_{\gamma,\beta}}{\lambda^\gamma}\Big(\sum_{s=i+1}^{k} u_s\Big)^\alpha \Big(\sum_{t=1}^{n} v_t\Big)^\beta. \tag{4.1}$$

As mentioned in Remark 4.1, Proposition 4.1 is valid not only for real-valued variables but also for function-valued variables with arbitrary norm. For a fixed k we consider the sequence $S_k(l) := S_{00}^{kl} = S_{kl}$, $1 \le l \le n$, with the norm $\|S_k(\cdot) - S_i(\cdot)\| = \max\limits_{1\le l\le n} |S_k(l) - S_i(l)|$. In the new terms (4.1) takes the form

$$\mathbf{P}\big(\|S_k(\cdot) - S_i(\cdot)\| \ge \lambda\big) \le \frac{K_{\gamma,\beta}}{\lambda^\gamma}\Big(\sum_{t=1}^{n} v_t\Big)^\beta \Big(\sum_{s=i+1}^{k} u_s\Big)^\alpha$$

for any $0 \le i < k \le m$.

We again use Proposition 4.1, formally including the factor $K_{\gamma,\beta}\Big(\sum\limits_{t=1}^{n} v_t\Big)^\beta$ in each of the terms u_s. This yields

$$\mathbf{P}\Big(\max_{1\le k\le m} \|S_k(\cdot)\| \ge \lambda\Big) \le \frac{K_{\gamma,\alpha}K_{\gamma,\beta}}{\lambda^\gamma}\Big(\sum_{t=1}^{n} v_t\Big)^\beta \Big(\sum_{s=1}^{m} u_s\Big)^\alpha,$$

i.e., the assertion of Proposition 4.2. □

§ 5. Auxiliary results

Let $\varphi(v) = \mathbf{E}e^{iv\xi_1}$, $v \in \mathbf{R}$, be the characteristic function of the step of the random walk ν_k, $k = 0, 1, 2\ldots$, with mean zero and variance one. We consider only integer (D) or continuous (C) random walks, assuming that

(D) $|\varphi(v)| = 1$ iff v is a multiple of 2π,

(C) $\displaystyle\int_{-\infty}^{\infty} |\varphi(v)|\, dv < \infty$.

Further, we will consider separately the cases (D) and (C). We assume in condition (D) that the random walk takes values in the integer lattice. Condition (C) means that the distribution function of a step of the random walk has a density.

The significance of the auxiliary results given below is the following: they provide a recursion relations that enable us to estimate the moments of additive functionals of a random walk. An important part of the method for constructing such a recursion relations is the application of the Fourier transform to a function, in which the argument is replaced by the random walk. This transform relocates the random walk from the argument of this function to the argument of the exponential function e^{ivx}, $x \in \mathbf{R}$. Due to this, when computing the expectation of such a function of the random walk ν_k, $k = 0, 1, 2\ldots$, we get the powers of the characteristic function φ. Then for the sum with respect to the steps of the random walk we can apply the formula for the sum of a geometric progression.

1. Integer random walk.

For an arbitrary summable function $\mathfrak{z}(v)$ set

$$\mathfrak{Z}(u) := \sum_{v=-\infty}^{\infty} e^{-ivu}\mathfrak{z}(v), \qquad u \in \mathbf{R}. \tag{5.1}$$

Then, using the inversion formula

$$\mathbf{P}(\nu_k = v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iuv}\varphi^k(u)\, du, \qquad k = 0, 1, \ldots, \tag{5.2}$$

it is easy to get that for any β and any integer $\varkappa$,

$$\mathbf{E}\sum_{k=1}^{n} \beta^{n-k}\mathfrak{z}(\nu_{k-1} - \varkappa) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iu\varkappa}\mathfrak{Z}(u)\frac{\beta^n - \varphi^n(u)}{\beta - \varphi(u)}du. \tag{5.3}$$

By the condition (D) and the fact that $\varphi(u) = 1 - u^2/2 + o(u^2)$ as $u \to 0$,

$$\left|\frac{1 - \varphi^n(u)}{1 - \varphi(u)}\right| \le C(n \wedge u^{-2}), \qquad u \in [-\pi, \pi], \tag{5.4}$$

$$\frac{1 - |\varphi(u)|^n}{1 - |\varphi(u)|} \le C(n \wedge u^{-2}), \qquad u \in [-\pi, \pi], \tag{5.5}$$

where C is a constant depending on the distribution of the step of the random walk. Here and below we use the notation $c \wedge d = \min\{c, d\}$.

For any $\Delta > 0$ and $m \geq 1$

$$\int_{-\pi}^{\pi} (m \wedge v^{-2})\, dv \leq 4\sqrt{m}, \tag{5.6}$$

$$\int_{-\pi}^{\pi} (1 \wedge |v|\Delta)(m \wedge v^{-2})\, dv \leq 2\Big\{ \mathbb{1}_{\{\sqrt{m} \leq \Delta\}} \int_{0}^{\pi} (m \wedge v^{-2})\, dv$$

$$+ \mathbb{1}_{\{\Delta < \sqrt{m}\}} \Big(\int_{0}^{1/\sqrt{m}} v\Delta m dv + \int_{1/\sqrt{m}}^{1/\Delta} \frac{\Delta}{v} dv + \int_{1/\Delta}^{\pi} \frac{1}{v^2} dv \Big) \Big\}$$

$$\leq 2\Big\{ 2\mathbb{1}_{\{\sqrt{m} \leq \Delta\}} \sqrt{m} + \mathbb{1}_{\{\Delta < \sqrt{m}\}} \Big(\frac{3\Delta}{2} + \Delta \ln \frac{\sqrt{m}}{\Delta} \Big) \Big\} \leq 9(\Delta\sqrt{m})^{1/2}, \tag{5.7}$$

$$\int_{-\pi}^{\pi} (1 \wedge |v|\Delta)^2 (m \wedge v^{-2})\, dv \leq 9\Delta. \tag{5.8}$$

To estimate moments of the processes considered in this chapter we will use an approach formulated in general form to fit various applications. We assume that expectations appeared in the subsequent formulas in this section are finite and that the series under consideration converge.

For a fixed n and $v \in \mathbf{R}$ let $\zeta_n(k, v)$, $k = 1, \dots, n$, be independent identically distributed real-valued variables measurable for every k with respect to the σ-algebra of events generated by the Brownian motion

$$W^{(k)}(s) := W\Big(s + \sum_{l=1}^{k-1} H_n^{(l)}\Big) - W\Big(\sum_{l=1}^{k-1} H_n^{(l)}\Big)$$

for $s \in [0, H_n^{(k)}]$. Note that, by construction (see (2.11)), the variables ξ_k^n are also measurable for every k with respect to this σ-algebra. Below we derive recursion relations that enable us to estimate by induction the variables

$$Z_n^{(q)}(m, \varkappa) := \mathbf{E}\Big(\sum_{k=1}^{m} \zeta_n(k, \nu_{k-1}^n - \varkappa) \Big)^q, \tag{5.9}$$

where $1 \leq m \leq n$ and $q \geq 1$ are integers. For integer random walks it is assumed that $\varkappa \in \mathbb{Z}$, where $\mathbb{Z}$ is the lattice of integers. We set

$$g_n(v, y) := e^{iyv} \mathbf{E}\big\{ e^{iy\xi_1^n} \zeta_n(1, v) \big\}, \tag{5.10}$$

$$G_n(u, y) := \sum_{v=-\infty}^{\infty} e^{-iuv} g_n(v, y), \tag{5.11}$$

and for $1 \le k \le m$, $j = 1, 2, \dots$, $r = 0, 1, 2, \dots$, set

$$\mathfrak{z}_n^{j,r}(m, v) := \mathbf{E}\big\{\zeta_n^j(k, v) Z_n^{(r)}(m, -(v + \xi_k^n))\big\}, \tag{5.12}$$

$$\mathfrak{Z}_n^{j,r}(m, u) := \sum_{v=-\infty}^{\infty} e^{-iuv} \mathfrak{z}_n^{j,r}(m, v). \tag{5.13}$$

Note that $\mathfrak{z}_n^{j,r}(m, v) = \mathbf{E}\Big\{\zeta_n^j(1, v) Z_n^{(r)}(m, -(v + \xi_1))\Big\}$. According to the agreement stated before (3.16), we must set $\mathfrak{z}_n^{j,r}(0, v) = 0$ for $r \ne 0$ and $\mathfrak{z}_n^{j,0}(0, v) = \mathbf{E}\big\{\zeta_n^j(1, v)\big\}$. This implies that $\mathfrak{Z}_n^{j,r}(0, u) = 0$ for $r \ne 0$ and $\mathfrak{Z}_n^{j,0}(0, u) = G_n(u, 0)$.

Using (3.16), we can see that

$$Z_n^{(q)}(m, \varkappa) = \sum_{j=1}^{q} \frac{q!}{j!(q-j)!} I_n^{j,q-j}(m, \varkappa), \qquad q \ge 1, \tag{5.14}$$

where

$$I_n^{j,r}(m, \varkappa) := \mathbf{E}\Big\{\sum_{k=1}^{m} \zeta_n^j(k, \nu_{k-1}^n - \varkappa)\Big(\sum_{l=k+1}^{m} \zeta_n(l, \nu_{l-1}^n - \varkappa)\Big)^r\Big\}.$$

Taking into account the independence of the steps of the random walk ν_k^n, $k = 0, 1, 2, \dots$, and applying Lemma 2.1 Ch. I twice, we obtain

$$I_n^{j,r}(m, \varkappa) = \sum_{k=1}^{m} \mathbf{E}\Big\{\zeta_n^j(k, \nu_{k-1}^n - \varkappa)\,\mathbf{E}\Big\{\Big(\sum_{l=k+1}^{m} \zeta_n(l, \nu_{l-1}^n - \varkappa)\Big)^r \Big| \mathcal{F}_k^n\Big\}\Big\}$$

$$= \sum_{k=1}^{m} \mathbf{E}\big\{\zeta_n^j(k, \nu_{k-1}^n - \varkappa)\, Z_n^{(r)}(m - k, -(\nu_{k-1}^n - \varkappa + \xi_k^n))\big\}$$

$$= \sum_{k=1}^{m} \mathbf{E}\big\{\mathbf{E}\big\{\zeta_n^j(k, \nu_{k-1}^n - \varkappa)\, Z_n^{(r)}(m - k, -(\nu_{k-1}^n - \varkappa + \xi_k^n))\big| \mathcal{F}_{k-1}^n\big\}\big\}$$

$$= \sum_{k=1}^{m} \mathbf{E}\mathfrak{z}_n^{j,r}(m - k, \nu_{k-1}^n - \varkappa),$$

where $\mathcal{F}_k^n$ is the σ-algebra of events generated by the Brownian motion W up to the moment $\sum_{l=1}^{k} H_n^{(l)}$. Now, applying (5.2), we get for $I_n^{j,r}$ the formula

$$I_n^{j,r}(m, \varkappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu\varkappa} \sum_{k=1}^{m} \mathfrak{Z}_n^{j,r}(m - k, u)\, \varphi^{k-1}(u)\, du. \tag{5.15}$$

Substituting this into (5.14), we have

$$Z_n^{(q)}(m, \varkappa) = \sum_{j=1}^{q} \frac{q!}{j!(q-j)!2\pi} \int_{-\pi}^{\pi} e^{-iu\varkappa} \sum_{k=1}^{m} \mathfrak{Z}_n^{j,q-j}(m - k, u)\, \varphi^{k-1}(u)\, du. \tag{5.16}$$

In view of (5.12), (5.13), formula (5.16) give a recursion relation expressing $Z_n^{(q)}$ in terms of the variables $Z_n^{(r)}$ with $r < q$. Estimates of $\mathfrak{Z}_n^{1,r}$ will be especially significant in estimating the right-hand side of (5.15). We also write a formula for $\mathfrak{Z}_n^{1,r}$, expressing it in terms of the variables $\mathfrak{Z}_n^{j,l}$ with $l < r$. Note that $\mathfrak{Z}_n^{1,0}(m,u) = G_n(u,0)$, $m = 0, 1, 2, \ldots$. Using (5.16), we have

$$\mathfrak{Z}_n^{1,r}(m,u) = \sum_{v=-\infty}^{\infty} e^{-iuv}\mathbf{E}\big\{\zeta_n(1,v)Z_n^{(r)}(m,-(v+\xi_1^n))\big\}$$

$$= \sum_{v=-\infty}^{\infty} e^{-iuv}\mathbf{E}\Big\{\zeta_n(1,v)\sum_{j=1}^{r}\frac{r!}{j!(r-j)!2\pi}\int_{-\pi}^{\pi} e^{is(v+\xi_1^n)}\sum_{l=1}^{m}\mathfrak{Z}_n^{j,r-j}(m-l,s)\,\varphi^{l-1}(s)\,ds\Big\}$$

$$= \sum_{j=1}^{r}\frac{r!}{j!(r-j)!2\pi}\int_{-\pi}^{\pi} G_n(u,s)\sum_{l=1}^{m}\mathfrak{Z}_n^{j,r-j}(m-l,s)\varphi^{l-1}(s)\,ds. \tag{5.17}$$

We consider separately the estimate of the variable (5.9) for $q = 2$. Set

$$b_n(v) := \mathbf{E}\zeta_n^2(1,v), \qquad B_n(u) := \sum_{v=-\infty}^{\infty} e^{-iuv} b_n(v).$$

Applying (5.14) and (5.15) with $j = 1$, $r = 1$, and then using (5.3) with $\beta = 1$, we get

$$Z_n^{(2)}(m,\varkappa) = \sum_{k=1}^{m}\mathbf{E}\zeta_n^2(k,\nu_{k-1}^n - \varkappa) + 2I_n^{1,1}(m,\varkappa)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iu\varkappa}B_n(u)\frac{1-\varphi^m(u)}{1-\varphi(u)}du + \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-iu\varkappa}\sum_{k=1}^{m}\mathfrak{Z}_n^{1,1}(m-k,u)\varphi^{k-1}(u)\,du.$$

Now, using (5.17) with $r = 1$, i.e., the formula

$$\mathfrak{Z}_n^{1,1}(m-k,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_n(u,s)\,G_n(s,0)\frac{1-\varphi^{m-k}(s)}{1-\varphi(s)}\,ds,$$

we finally have

$$Z_n^{(2)}(m,\varkappa) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iu\varkappa}B_n(u)\frac{1-\varphi^m(u)}{1-\varphi(u)}du + \frac{1}{2\pi^2}\int_{-\pi}^{\pi} ds\,\frac{G_n(s,0)}{1-\varphi(s)}$$

$$\times\int_{-\pi}^{\pi} du\, e^{-iu\varkappa}G_n(u,s)\Big\{\frac{1-\varphi^m(u)}{1-\varphi(u)} - \frac{\varphi^m(s)-\varphi^m(u)}{\varphi(s)-\varphi(u)}\Big\}.$$

Using (5.4) and (5.5), we obtain

$$\Big|\frac{1}{1-\varphi(s)}\Big(\frac{1-\varphi^m(u)}{1-\varphi(u)}-\frac{\varphi^m(s)-\varphi^m(u)}{\varphi(s)-\varphi(u)}\Big)\Big|$$

$$=\Big|\sum_{k=1}^{m}\varphi^{k-1}(u)\frac{1-\varphi^{m-k}(s)}{1-\varphi(s)}\Big|\le C(m\wedge u^{-2})(m\wedge s^{-2}),\quad s,u\in[-\pi,\pi].$$

As a result, we have the estimate

$$Z_n^{(2)}(m,\varkappa)\le CB_n(0)\sqrt{m}+C\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|G_n(s,0)||G_n(u,s)|(m\wedge u^{-2})(m\wedge s^{-2})\,ds\,du. \tag{5.18}$$

2. Continuous random walk.

The following formulas concern random walks satisfying the condition (C). For an arbitrary integrable function $\mathfrak{z}(v)$, $v\in\mathbf{R}$, we set

$$\mathfrak{Z}(u):=\int_{-\infty}^{\infty}e^{-iuv}\mathfrak{z}(v)\,dv,\qquad u\in\mathbf{R}.$$

Since each of the random variables ν_k, $k=1,2,\dots$, has a bounded density, representing in the form

$$p_k(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iux}\varphi^k(u)\,du,$$

we have

$$\mathbf{E}\mathfrak{z}(\nu_k-\varkappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\mathfrak{Z}(u)\varphi^k(u)\,du.$$

For $k=0$ this formula is not true in the general case. If we assume that $\mathfrak{Z}(u)$, $u\in\mathbf{R}$, is an integrable function, then there exists the inverse Fourier transform

$$\mathfrak{z}(v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iuv}\mathfrak{Z}(u)\,du,$$

and for any β and $\varkappa$

$$\mathbf{E}\sum_{k=1}^{n}\beta^{n-k}\mathfrak{z}(\nu_{k-1}-\varkappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\mathfrak{Z}(u)\frac{\beta^n-\varphi^n(u)}{\beta-\varphi(u)}\,du.$$

This relation is identical to (5.3) and in the continuous case it enables us to keep the structure of all the formulas of the discrete case. However, if the inverse Fourier transform of the function $\mathfrak{z}$ does not exist, we must estimate some terms in these formulas separately. Therefore, we start with the following formulas:

$$\mathbf{E}\sum_{k=2}^{n}\beta^{n-k}\mathfrak{z}(\nu_{k-1}-\varkappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\mathfrak{Z}(u)\varphi(u)\frac{\beta^{n-1}-\varphi^{n-1}(u)}{\beta-\varphi(u)}du. \tag{5.19}$$

$$\mathbf{E}\sum_{k=1}^{n}\beta^{n-k}\mathfrak{z}(\nu_{k-1}-\varkappa)=\beta^{n-1}\mathfrak{z}(-\varkappa)+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\mathfrak{Z}(u)\varphi(u)\frac{\beta^{n-1}-\varphi^{n-1}(u)}{\beta-\varphi(u)}du. \tag{5.20}$$

The function $|\varphi(u)|$, $u\in\mathbf{R}$, is integrable and by the Riemann–Lebesgue theorem $\lim_{u\to\pm\infty}|\varphi(u)|\to 0$. Since $\varphi(u)=1-u^2/2+o(u^2)$ as $u\to 0$, we have

$$\left|\frac{1-\varphi^n(u)}{1-\varphi(u)}\right|\le C(n\wedge(1+u^{-2}),\qquad u\in\mathbf{R}, \tag{5.21}$$

$$\frac{1-|\varphi(u)|^n}{1-|\varphi(u)|}\le C(n\wedge(1+u^{-2}),\qquad u\in\mathbf{R}. \tag{5.22}$$

For any $\Delta>0$ and $m\ge 1$ the analogs of (5.6)–(5.8) are the following estimates:

$$\int_{-\infty}^{\infty}\big(m\wedge(1+v^{-2})\big)|\varphi(v)|\,dv\le C\sqrt{m}, \tag{5.23}$$

$$\int_{-\infty}^{\infty}(1\wedge|v|\Delta)\big(m\wedge(1+v^{-2})\big)|\varphi(v)|\,dv\le C\big(1+(\Delta\sqrt{m})^{1/2}\big), \tag{5.24}$$

$$\int_{-\infty}^{\infty}(1\wedge|v|\Delta)^2\big(m\wedge(1+v^{-2})\big)|\varphi(v)|\,dv\le C(1+\Delta). \tag{5.25}$$

The approach described by means of (5.9)–(5.17) in the discrete case requires the following changes when estimating the moments of the processes in the continuous case. Let the variables $Z_n^{(q)}$, g_n and $\mathfrak{z}_n^{j,r}$ be defined by (5.9), (5.10) and (5.12), respectively, and $\varkappa\in\mathbf{R}$. We set

$$G_n(u,y):=\int_{-\infty}^{\infty}e^{-iuv}g_n(v,y)\,dv, \tag{5.26}$$

$$\mathfrak{Z}_n^{j,r}(m,u):=\int_{-\infty}^{\infty}e^{-iuv}\mathfrak{z}_n^{j,r}(m,v)\,dv. \tag{5.27}$$

Using (3.16), we get the following analog of (5.14) and (5.15) for the case (C):

$$Z_n^{(q)}(m,\varkappa)=\sum_{j=1}^{q}\frac{q!}{j!(q-j)!}\Big\{\mathfrak{z}_n^{j,q-j}(m-1,-\varkappa)+I_n^{j,q-j}(m,\varkappa)\Big\}, \tag{5.28}$$

where

$$I_n^{j,r}(m,\varkappa) := \mathbf{E}\Big\{\sum_{k=2}^{m}\zeta_n^j(k,\nu_{k-1}^n-\varkappa)\Big(\sum_{l=k+1}^{m}\zeta_n(l,\nu_{l-1}^n-\varkappa)\Big)^r\Big\}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\sum_{k=2}^{m}\mathfrak{Z}_n^{j,r}(m-k,u)\,\varphi^{k-1}(u)\,du. \tag{5.29}$$

The first term in (5.28) is considered separately. The relation corresponding to (5.17) in this connection has the form:

$$\mathfrak{Z}_n^{1,r}(m,u)=\int_{-\infty}^{\infty}e^{-iuv}\mathbf{E}\big\{\zeta_n(1,v)Z_n^{(r)}(m,-(v+\xi_1^n))\big\}\,dv$$

$$=\int_{-\infty}^{\infty}e^{-iuv}\mathbf{E}\Big\{\zeta_n(1,v)\sum_{j=1}^{r}\frac{r!}{j!(r-j)!}\Big\{\mathfrak{z}_n^{j,r-j}(m-1,\xi_1^n+v)+I_n^{j,r-j}(m,-(\xi_1^n+v))\Big\}\Big\}dv$$

$$=\sum_{j=1}^{r}\frac{r!}{j!(r-j)!}\Big\{\int_{-\infty}^{\infty}e^{-iuv}\mathbf{E}\big\{\zeta_n(1,v)\mathfrak{z}_n^{j,r-j}(m-1,\xi_1^n+v)\big\}dv$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}G_n(u,v)\sum_{l=2}^{m}\mathfrak{Z}_n^{j,r-j}(m-l,v)\varphi^{l-1}(v)\,dv\Big\}. \tag{5.30}$$

We consider separately the estimate of the variable (5.9) for $q=2$. We slightly modify this variable, removing the first term. This simplifies a little the expression. We set

$$B_n(u) := \int_{-\infty}^{\infty}e^{-iuv}b_n(v)\,dv.$$

Then, applying (5.19) with $\beta=1$ and (5.29), we get

$$\mathbf{E}\Big(\sum_{k=2}^{m}\zeta_n(k,\nu_{k-1}^n-\varkappa)\Big)^2=\sum_{k=2}^{m}\mathbf{E}b_n(\nu_{k-1}^n-\varkappa)+2I_n^{1,1}(m,\varkappa)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}B_n(u)\varphi(u)\frac{1-\varphi^{m-1}(u)}{1-\varphi(u)}du+\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-iu\varkappa}\sum_{k=2}^{m-1}\mathfrak{Z}_n^{1,1}(m-k,u)\varphi^{k-1}(u)\,du.$$

Now, using (5.30) with $r=1$, i.e., the formula

$$\mathfrak{Z}_n^{1,1}(m-k,u)=\int_{-\infty}^{\infty}e^{-iuv}\mathbf{E}\big\{\zeta_n(1,v)\,g_n(\xi_1^n+v,0)\big\}dv$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} G_n(u,v)\sum_{l=2}^{m-k} G_n(v,0)\varphi^{l-1}(v)\,dv$$

$$= \mathfrak{Z}_n^{1,1}(1,u) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} G_n(u,v)\,G_n(v,0)\,\varphi(v)\frac{1-\varphi^{m-k-1}(v)}{1-\varphi(v)}\,dv,$$

we finally get

$$\mathbf{E}\Big(\sum_{k=2}^{m}\zeta_n(k,\nu_{k-1}-\varkappa)\Big)^2 = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-iu\varkappa}\big(B_n(u)+2\,\mathfrak{Z}_n^{1,1}(1,u)\big)\varphi(u)\,\frac{1-\varphi^{m-1}(u)}{1-\varphi(u)}\,du$$

$$+\frac{1}{2\pi^2}\int\limits_{-\infty}^{\infty}\frac{\varphi(v)G_n(v,0)}{1-\varphi(v)}\int\limits_{-\infty}^{\infty} e^{-iu\varkappa}G_n(u,v)\sum_{k=2}^{m-1}\varphi^{k-1}(u)(1-\varphi^{m-k-1}(v))\,du\,dv. \tag{5.31}$$

It is easy to see that

$$\mathfrak{Z}_n^{1,1}(1,u)\le\int\limits_{-\infty}^{\infty}\mathbf{E}\big|\zeta_n(1,v)\,g_n(\xi_1^n+v,0)\big|dv\le 2\int\limits_{-\infty}^{\infty}\big(\mathbf{E}\zeta_n^2(1,v)+|g_n(v,0)|^2\big)dv\le 4B_n(0).$$

This together with (5.19)–(5.23) imply

$$\mathbf{E}\Big(\sum_{k=2}^{m}\zeta_n(k,\nu_{k-1}-\varkappa)\Big)^2 \le CB_n(0)\sqrt{m}$$

$$+C\int\limits_{-\infty}^{\infty}|\varphi(v)||G_n(v,0)|\int\limits_{-\infty}^{\infty}|G_n(u,v)||\varphi(u)|(m\wedge(1+v^{-2}))(m\wedge(1+u^{-2}))\,du\,dv. \tag{5.32}$$

§ 6. Weak invariance principle for local times (integer random walk)

The weak invariance principle for local times is described by the formula (1.3). In this section we assume that the condition (D) holds and that the random walk has just a second moment. We start with the presentation of the special and the most natural case when the local time of the integer random walk is treated as the normalized number of times the random walk hits the selected point. We set

$$\ell_n(t,x) := \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\mathrm{I\!I}_{\{0\}}\big(\nu_{k-1}^n-[x\sqrt{n}]\big), \qquad (t,x)\in[0,\infty)\times\mathbf{R}. \tag{6.1}$$

The variable $\ell_n(t,x)$ is the normalized by $\sqrt{n}$ number of times the random walk ν_k^n hits the point $[x\sqrt{n}]$ up to the time $[nt]$.

The special meaning of the process $\ell_n(t,x)$ in the study of additive functionals of random walks is determined by the following relation, which is the analogue of (1.2) of Ch. V for the Brownian local time.

For an arbitrary function f and $t_n := [nt]/n$

$$\int_0^{t_n} f(W_n(s))\,ds = \frac{1}{n}\sum_{k=1}^{[nt]} f\Big(\frac{1}{\sqrt{n}}\nu_{k-1}^n\Big) = \frac{1}{n}\sum_{k=1}^{[nt]}\sum_{l=-\infty}^{\infty} f\big(\tfrac{l}{\sqrt{n}}\big)\mathrm{I\!I}_{\{0\}}\big(\nu_{k-1}^n - l\big)$$

$$= \int_{-\infty}^{\infty} f\Big(\frac{[x\sqrt{n}]}{\sqrt{n}}\Big)\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\mathrm{I\!I}_{\{0\}}\big(\nu_{k-1}^n - [x\sqrt{n}]\big)\,dx = \int_{-\infty}^{\infty} f\Big(\frac{[x\sqrt{n}]}{\sqrt{n}}\Big)\ell_n(t,x)\,dx. \qquad (6.2)$$

The convergence $\ell_n(t,x) \to \ell(t,x)$ in probability for a fixed $t \in [0,1]$ and $x \in \mathbf{R}$ can be established as follows.

We use the obvious equality

$$\mathrm{I\!I}_0(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ium}\,du, \qquad m \in \mathbb{Z}.$$

Let $x_n := [x\sqrt{n}]/\sqrt{n}$ be the nearest to the left of x point from the lattice with the array spacing $1/\sqrt{n}$. Then

$$\ell_n(t,x) = \frac{1}{2\pi\sqrt{n}}\sum_{k=1}^{[nt]}\int_{-\pi}^{\pi} e^{iu(\nu_{k-1}^n - [x\sqrt{n}])}\,du = \frac{1}{2\pi}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\int_0^{t_n} e^{i\lambda(\nu_{[ns]}^n/\sqrt{n} - x_n)}\,ds\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\lambda x_n}\int_0^{t_n} e^{i\lambda W_n(s)}\,ds\,d\lambda. \qquad (6.3)$$

For $A > 0$ we set

$$\ell_n^{(A)}(t,x) := \frac{1}{2\pi}\int_{-A}^{A} e^{-i\lambda x_n}\int_0^{t_n} e^{i\lambda W_n(s)}\,ds\,d\lambda,$$

$$\ell^{(A)}(t,x) := \frac{1}{2\pi}\int_{-A}^{A} e^{-i\lambda x}\int_0^{t} e^{i\lambda W(s)}\,ds\,d\lambda.$$

By the invariance principle for random walks (see (3.4)), we have that for an arbitrarily large A, any fixed t, and x

$$\ell_n^{(A)}(t,x) \to \ell^{(A)}(t,x)$$

in probability. By Lemma 1.1 of Ch. V, the process $\ell^{(A)}(t,x)$ converges in the mean square as $A\to\infty$ to the Brownian local time $\ell(t,x)$. Therefore, to prove the convergence $\ell_n(t,x)\to\ell(t,x)$ in probability it is sufficient to prove that $\mathbf{E}\big(\ell_n(t,x)-\ell_n^{(A)}(t,x)\big)^2\to 0$ as $n\to\infty$ and $A\to\infty$. For this, in particular, we must be able to estimate the value

$$\mathbf{E}\Big(\int_A^B e^{-i\lambda x_n}\int_0^{t_n} e^{i\lambda W_n(s)}\,ds\,d\lambda\Big)^2, \qquad B>A.$$

For the Brownian motion W the corresponding estimate was given in (1.4) Ch. V. The approach for obtaining an analogous estimate for the process W_n has been developed in § 5 (see (5.18)), and is based on the Fourier transform. It has been mentioned that the use of the Fourier transform enables us to relocate (this is clearly illustrated by the equality (6.3)) the random walk from the argument of the indicator or some other function to the argument of the exponential function.

Since we need stronger results than convergence in probability, we will also estimate the mean square distance $\mathbf{E}\big(\ell_n(t,x)-\ell(\tau_n(t),x_n)\big)^2$, where $\tau_n(t)$ is the random moment close to t (see (3.6)). The convergence of $\ell_n(t,x)\to\ell(t,x)$ in probability is proved in Lemma 6.1.

The main assertion of this section is the following particular case of the *weak invariance principle for local times* of random walks.

Theorem 6.1. *For any* $\varepsilon>0$

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}}|\ell_n(t,x)-\ell(t,x)|>\varepsilon\Big)=0. \tag{6.4}$$

Proof. We first prove a preliminary statement.

Lemma 6.1. *For any* $(t,x)\in[0,1]\times\mathbf{R}$

$$\ell_n(t,x)\to\ell(t,x), \qquad n\to\infty, \tag{6.5}$$

in probability.

Proof. To simplify formulas, we first assume that $\mathbf{P}(\xi_1=0)=0$. In this case in the formulas of § 2 we have $p_0=1$. The necessary changes needed for the investigation of the general case will be specified at the end of the proof. Let $H_n^{(l)}$ be the sequence of random moments and $\{(\bar a_k,\bar b_k)\}_{k=1}^\infty$ be the sequence of independent two-dimensional truncated random variables defined in § 3. Set

$$\widetilde H_n^{(k)} := \min\big\{s\ge 0: W^{(k)}(s)\notin\big(-\bar a_k/\sqrt n,\bar b_k/\sqrt n\big)\big\},$$

where $W^{(k)}(s)=W\big(s+\sum_{l=1}^{k-1}H_n^{(l)}\big)-W\big(\sum_{l=1}^{k-1}H_n^{(l)}\big)$, $W^{(1)}(s)=W(s)$. Note that $\widetilde H_n^{(k)}\le H_n^{(k)}$ and hence (see § 4 Ch. I, property 9),

$$\sigma(W^{(k)}(s),0\le s\le\widetilde H_n^{(k)})\subseteq\sigma(W^{(k)}(s),0\le s\le H_n^{(k)}).$$

The moments $\widetilde{H}_n^{(k)}$ differs from the moments $\overline{H}_n^{(k)}$ defined in § 3, because $\widetilde{H}_n^{(k)}$ are constructed from the processes $W^{(k)}$ instead of the processes $\overline{W}^{(k)}$.

Let $\ell^{(k)}(\widetilde{H}_n^{(k)}, x)$ be the local time of the Brownian motion $W^{(k)}$ at x before time $\widetilde{H}_n^{(k)}$. We set

$$\zeta_n(k,v) := \frac{1}{\sqrt{n}}\mathbb{I}_{\{0\}}(v) - \ell^{(k)}(\widetilde{H}_n^{(k)}, -v/\sqrt{n}),$$

$$V_n(t,x) := \sum_{k=1}^{[nt]} \zeta_n(k, \nu_{k-1}^n - [x\sqrt{n}]).$$

Note that the kth term of this sum involves the local time at the moment $\widetilde{H}_n^{(k)}$ instead of the moment $H_n^{(k)}$, which would be more natural. This comes from the necessity to estimate the second moment of $\ell^{(k)}(\widetilde{H}_n^{(k)}, -v/\sqrt{n})$ under the assumption that the step of the random walk ν_k has only the second moment. If higher finite moments exist, one can use $H_n^{(k)}$.

For a fixed n and $v \in \mathbf{R}$ the variables $\zeta_n(k,v)$, $k = 1,\dots,n$, are independent and identically distributed, and for every k they are measurable with respect to the σ-algebra of events $\sigma(W^{(k)}(s), 0 \le s \le H_n^{(k)})$. By construction (the Skorohod embedding scheme (see (2.11))), the variables $\xi_k^n = \nu_k^n - \nu_{k-1}^n$ are also measurable with respect to this σ-algebra.

It is clear that the equalities $\widetilde{H}_n^{(k)} = H_n^{(k)}$, $k = 1,\dots,n$, hold on the set $\Omega_n = \{\bar{\nu}_k^n = \nu_k^n, k = 1,2,\dots,n\}$ (see § 3) and on this set

$$V_n(t,x) = \ell_n(t,x) - \ell(\tau_n(t), [x\sqrt{n}]/\sqrt{n}),$$

where $\tau_n(t) = \sum_{k=1}^{[nt]} H_n^{(k)}$.

We choose θ as in Theorem 3.1. Then, by (3.6) and by the properties of a.s. uniform continuity of $\ell(t,x)$ (see (10.1) and (11.1) of Ch. V), we get that for any $\varepsilon_1 > 0$

$$\lim_{n\to\infty} \mathbf{P}\Bigl(\sup_{(t,x)\in[0,1]\times\mathbf{R}} \Bigl|\ell(\tau_n(t), [x\sqrt{n}]/\sqrt{n}) - \ell(t,x)\Bigr| > \varepsilon_1\Bigr) = 0. \tag{6.6}$$

This enables us to reduce the investigation of the asymptotic behavior of the difference $\ell_n(t,x) - \ell(t,x)$ to that of the limit behavior of the process $V_n(t,x)$. Thus for the validity of (6.5) it is sufficient to verify that

$$\mathbf{E}V_n^2(t,x) \to 0, \qquad n \to \infty. \tag{6.7}$$

We now use the notations and formulas of Subsection 1 of § 5. Then, by (5.9), $\mathbf{E}V_n^2(t,x) = Z_n^{(2)}([nt],[x\sqrt{n}])$ and we can apply formula (5.18). For this we need some estimates.

Proposition 6.1. *For all* $n \in \mathbb{N}$,

$$B_n(0) \le \frac{C}{\sqrt{n}}\Bigl(\frac{1}{\sqrt{n}} + \theta\Bigr). \tag{6.8}$$

Proof. Obviously,

$$B_n(0) \le \frac{1}{n} + \sum_{v=-\infty}^{\infty} \mathbf{E}\ell^2\big(\widetilde{H}_n^{(1)} - \frac{v}{\sqrt{n}}\big).$$

In this sum it is convenient to replace the summation index v to $-v$. Then the Brownian local time should be computed at the level $v/\sqrt{n}$. By the scaling property of the Brownian local time (see § 1 Ch. V), (2.2), (2.5), (2.9), and by the strong Markov property of the Brownian motion, we have

$$n\mathbf{E}\ell^2\big(\widetilde{H}_n^{(1)}, \frac{v}{\sqrt{n}}\big) = \mathbf{E}\ell^2\big(\widetilde{H}_1^{(1)}, v\big)$$

$$= \frac{16}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\frac{(\bar{\mu}+v)^2(\bar{\eta}-v)^2}{(\bar{\mu}+\bar{\eta})^2}\Big(\frac{\bar{\mu}}{\bar{\mu}+v}\mathbb{1}_{\{0\le v\le\bar{\eta},\bar{\mu}>0\}} + \frac{\bar{\eta}}{\bar{\eta}-v}\mathbb{1}_{\{-\bar{\mu}\le v<0,\bar{\eta}>0\}}\Big)\Big\}$$

$$\le \frac{16}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\big(\bar{\eta}\bar{\mu}\mathbb{1}_{\{0\le v\le\bar{\eta},\bar{\mu}>0\}} + \bar{\mu}\bar{\eta}\mathbb{1}_{-\bar{\mu}\le v<0,\bar{\eta}>0\}}\big)\big\}. \qquad (6.9)$$

Here we have taken into account that the exit boundaries for the moment $\widetilde{H}_1^{(1)}$ are determined by the independent random variables $-\bar{\mu}$ and $\bar{\eta}$ with $\bar{\mu} > 0$ and $\bar{\eta} > 0$. Moreover, these variables are also independent of the Brownian motion $W^{(1)} \equiv W$. If $v \ne 0$, then the Brownian motion first hits the point v and only after that it hits one of the boundaries, otherwise the local time equals zero. The exit probabilities are given by (2.2). In our case these probabilities are $\frac{\bar{\mu}}{\bar{\mu}+v}$ for $v \ge 0$ and $\frac{\bar{\eta}}{\bar{\eta}-v}$ for $v < 0$. Then, restarting from the point v (the strong Markov property), the Brownian motion accumulate the local time at this point up to the first exit time from the interval $(-\bar{\mu}, \bar{\eta})$. In conclusion, to compute the second moment of the Brownian local time we use the formula (2.5), $q = 2$.

From (6.9) it follows that

$$n\sum_{v=-\infty}^{\infty} \mathbf{E}\ell^2\big(\widetilde{H}_n^{(1)}\frac{v}{\sqrt{n}}\big) \le C\mathbf{E}\{(|\mu|+|\eta|)(\bar{\eta}^2|\bar{\mu}|+\bar{\mu}^2|\bar{\eta}|)\} \le C\theta\sqrt{n}.$$

This proves (6.8). □

Proposition 6.2. *For any $y, u \in \mathbf{R}$,*

$$|G_n(u,y)| \le \frac{C}{\sqrt{n}}\big((1\wedge|y|\theta\sqrt{n}) + (1\wedge|u|\theta\sqrt{n}) + \theta\big). \qquad (6.10)$$

Proof. Since $\xi_1^n = \sqrt{n}W(H_n^{(1)})$, using (5.10), the scaling property of the Brownian motion and its local time, we deduce from (2.2), (2.5), $q = 1$, and (2.9) that

$$\sqrt{n}g_n(-v,y) = e^{-ivy}\varphi(y)\mathbb{1}_{\{0\}}(v) - \sqrt{n}e^{-ivy}\mathbf{E}\big\{e^{iy\sqrt{n}W(H_n^{(1)})}\ell\big(\widetilde{H}_n^{(1)}, \frac{v}{\sqrt{n}}\big)\big\}$$

$$= e^{-ivy}\varphi(y)\mathbb{1}_{\{0\}}(v) - e^{-ivy}\mathbf{E}\big\{e^{iyW(H_1^{(1)})}\ell\big(\widetilde{H}_1^{(1)}, v\big)\big\}. \qquad (6.11)$$

Of special importance here is the function $\sqrt{n}\, g_n(-v,0)$, therefore we compute it separately. Arguing as in the derivation of (6.9), we have

$$\sqrt{n} g_n(-v,0) = \mathbb{I}_{\{0\}}(v) - \mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big) = \mathbb{I}_{\{0\}}(v)$$

$$-\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{(\mu+\eta)\bar\mu(\bar\eta-v)}{(\bar\mu+\bar\eta)}\mathbb{I}_{\{0\le v\le\bar\eta,\bar\mu>0\}} + \frac{(\mu+\eta)\bar\eta(\bar\mu+v)}{(\bar\mu+\bar\eta)}\mathbb{I}_{\{-\bar\mu\le v<0,\bar\eta>0\}}\Big)\Big\}. \qquad (6.12)$$

Then, taking into account (2.10), we obtain

$$\sqrt{n} G_n(0,0) = 1 - \frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big(\bar\mu\sum_{v=0}^{\bar\eta}(\bar\eta-v) + \bar\eta\sum_{v=-\mu}^{-1}(\bar\mu+v)\Big)\mathbb{I}_{\{\mu>0,\eta>0\}}\Big\} = 1$$

$$-\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big(\bar\mu\,\frac{\bar\eta+1}{2}\bar\eta + \bar\eta\,\frac{\bar\mu-1}{2}\bar\mu\Big)\mathbb{I}_{\{\mu>0,\eta>0\}}\Big\} = 1 - \frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta\mathbb{I}_{\{\mu>0,\eta>0\}}\big\}$$

$$= 1 - \frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\mu\eta\mathbb{I}_{\{\mu>0,\eta>0\}}\big\} + O\big(\mathbf{E}\big\{\xi_1^2\mathbb{I}_{\{|\xi_1|>\theta\sqrt{n}\}}\big\}\big)$$

$$= O\big(\mathbf{E}\big\{\xi_1^2\mathbb{I}_{\{|\xi_1|>\theta\sqrt{n}\}}\big\}\big) = O\big(\mathbf{E}\big\{\xi_1^2; |\xi_1| > n^{1/4}\big\}\big) = O(\theta^3). \qquad (6.13)$$

The computation of the function $\sqrt{n}\, g_n(-v,y)$ is similar, but slightly more complicated. For it we derive the following expression:

$$\sqrt{n} g_n(-v,y) = e^{-ivy}\varphi(y)\mathbb{I}_{\{0\}}(v) - \frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\frac{(\bar\mu+v)(\bar\eta-v)}{\bar\mu+\bar\eta}$$

$$\times\Big(\frac{\bar\mu}{\bar\mu+v}\mathbb{I}_{\{\bar\eta\ge v\ge 0,\bar\mu>0\}} + \frac{\bar\eta}{\bar\eta-v}\mathbb{I}_{\{-\bar\mu\le v<0,\bar\eta>0\}}\Big)\Big(\frac{\eta-v}{\mu+\eta}e^{-iy(\mu+v)} + \frac{\mu+v}{\mu+\eta}e^{iy(\eta-v)}\Big)\Big\}.$$

In order to prove (6.10), it now suffices to estimate $G_n(u,y) - G_n(0,0)$. Using the inequality $\big|e^{ix}-1\big| \le 2(1\wedge|x|)$, we get

$$\sqrt{n}|G_n(u,y) - G_n(0,0)| = \sqrt{n}\Big|\sum_{v=-\infty}^{\infty}\big(e^{iuv}g_n(-v,y) - g_n(-v,0)\big)\Big| \le 2(1\wedge|y|\mathbf{E}|\xi_1|)$$

$$+\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{1}{\bar\mu+\bar\eta}\mathbb{I}_{\{\eta>0,\mu>0\}}\sum_{v=-\bar\mu}^{\bar\eta}\Delta_v\big(\bar\mu(\bar\eta-v)\mathbb{I}_{\{v\ge0\}} + \bar\eta(\bar\mu+v)\mathbb{I}_{\{v<0\}}\big)\Big\}, \qquad (6.14)$$

where

$$\Delta_v := (\eta-v)\big(e^{ivu-iy(\mu+v)}-1\big) + (\mu+v)\big(e^{ivu+iy(\eta-v)}-1\big).$$

For $-\bar\mu \le v \le \bar\eta$

$$|\Delta(v)| \le 2(\eta-v)\big((1\wedge(|vu|+|y|(\mu+v)))\mathbb{I}_{\{0<\mu\le\theta\sqrt{n}\}} + \mathbb{I}_{\{\theta\sqrt{n}<\mu\}}\big)$$

$$+2(\mu+v)\big((1\wedge(|vu|+|y|(\eta-v)))\mathbb{I}_{\{0<\eta\le\theta\sqrt{n}\}} + \mathbb{I}_{\{\theta\sqrt{n}<\eta\}}\big)$$

$$\le 2(\mu+\eta)\big((1\wedge(|u|+2|y|)\theta\sqrt{n}) + \mathbb{I}_{\{\theta\sqrt{n}<\mu\}} + \mathbb{I}_{\{\theta\sqrt{n}<\eta\}}\big).$$

Therefore,

$$\sqrt{n}|G_n(u,y)-G_n(0,0)|\le 2(1\wedge|y|\mathbf{E}|\xi_1|)$$

$$+\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\frac{\mathrm{1\!I}_{\{\eta>0,\mu>0\}}}{\bar\mu+\bar\eta}\big((1\wedge(|u|+2|y|)\theta\sqrt{n}))+\mathrm{1\!I}_{\{\theta\sqrt{n}<\mu\}}+\mathrm{1\!I}_{\{\theta\sqrt{n}<\eta\}}\big)$$

$$\times\sum_{v=-\bar\mu}^{\bar\eta}\big(\bar\mu(\bar\eta-v)\mathrm{1\!I}_{\{v\ge0\}}+\bar\eta(\bar\mu+v)\mathrm{1\!I}_{\{v<0\}}\big)\Big\}=2(1\wedge|y|\mathbf{E}|\xi_1|)$$

$$+\frac{2}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta\mathrm{1\!I}_{\{\eta>0,\mu>0\}}\big((1\wedge(|u|+2|y|)\theta\sqrt{n})+\mathrm{1\!I}_{\{\theta\sqrt{n}<\mu\}}+\mathrm{1\!I}_{\{\theta\sqrt{n}<\eta\}}\big)\big\}$$

$$\le C\big\{(1\wedge|y|\theta\sqrt{n})+(1\wedge|u|\theta\sqrt{n})+\theta\big\}. \tag{6.15}$$

Here we used the estimate $\mathbf{E}|\xi_1|\le 1\le\theta\sqrt{n}$. Proposition 6.2 is proved. □

Now, using (5.18) and (5.6)–(5.8), we find that

$$\mathbf{E}V_n^2(t,x)=Z_n^{(2)}([nt],[x\sqrt{n}])\le\frac{C}{\sqrt{n}}\Big(\theta+\frac{1}{\sqrt{n}}\Big)\int_{-\pi}^{\pi}(n\wedge u^{-2})\,du+\frac{C}{n}\int_{-\pi}^{\pi}du\int_{-\pi}^{\pi}dv$$

$$\times\big((1\wedge|u|\theta\sqrt{n})+\theta\big)\big((1\wedge|v|\theta\sqrt{n})+(1\wedge|u|\theta\sqrt{n})+\theta\big)(n\wedge u^{-2})(n\wedge v^{-2})\le C\theta.$$

Thus (6.7) holds and therefore (6.5) is proved. □

The relation (6.4) will be proved as follows. We deduce from the uniform boundedness in probability of the process $W(t)$, $t\in[0,1]$, and from (3.4) that for any $\rho>0$ there is a constant $A=A(\rho)$ such that for all n

$$\mathbf{P}\Big(\sup_{t\in[0,1]}|W(t)|>A\Big)<\rho,\qquad \mathbf{P}\Big(\sup_{t\in[0,1]}|W_n(t)|>A\Big)<\rho. \tag{6.16}$$

Then on the set

$$\Omega_n(\rho):=\Big\{\sup_{t\in[0,1]}|W(t)|\le A\Big\}\bigcap\Big\{\sup_{t\in[0,1]}|W_n(t)|\le A\Big\}$$

the Brownian local time $\ell(t,x)$ and the process $\ell_n(t,x)$ are equal to zero for $x\notin[-A,A]$. This enables us to replace in (6.4) the supremum over $x\in\mathbf{R}$ by the supremum over $x\in[-A,A]$, because the probability of the complementary event obeys the estimate $\mathbf{P}\big(\Omega_n^c(\rho)\big)\le 2\rho$. Now, in view of (6.5), the following assertion plays a key role in the proof of Theorem 6.1.

Lemma 6.2. *For any $\varepsilon>0$ and $\rho>0$ there exist $\varrho=\varrho(\varepsilon,\rho)$ and $n_0=n_0(\varepsilon,\rho,\varrho)$, such that for all $n>n_0$*

$$\mathbf{P}\Big(\sup_{Q(\varrho)}|\ell_n(t,x)-\ell_n(s,y)|>\varepsilon\Big)<\rho, \tag{6.17}$$

where $Q(\varrho) = \{(s,t),(x,y) : |t-s| \le \varrho, |x-y| \le \varrho,\ s,t \in [0,1],\ x,y \in [-A,A]\}$.

From this and (6.5) it is not hard to deduce Theorem 6.1. Indeed, since the process $\ell(t,x)$ is uniformly continuous,

$$\mathbf{P}\Big(\sup_{Q(\varrho)} |\ell(t,x) - \ell(s,y)| > \varepsilon\Big) < \rho \tag{6.18}$$

for all sufficiently small ϱ. On the rectangle $[0,1]\times[-A,A]$ we consider the lattice $\Sigma = \{t_j, x_j\}_{j=1}^{n}$ with the array spacing ϱ and, using (6.5), we choose $n_1 = n_1(\varepsilon,\rho,\varrho)$ such that for all $n > n_1$

$$\mathbf{P}\Big(\sup_{(t_j,x_j)\in\Sigma} |\ell_n(t_j,x_j) - \ell(t_j,x_j)| > \varepsilon\Big) < \rho. \tag{6.19}$$

It follows from the definition of the processes $\ell(t,x)$ and $\ell_n(t,x)$ that

$$\{(t,x) : \ell(t,x) > 0, 0 \le t \le 1\} \subset \{0 \le t \le 1, \inf_{0\le s\le t} W(s) \le x \le \sup_{0\le s\le t} W(s)\},$$

$$\{(t,x) : \ell_n(t,x) > 0, 0 \le t \le 1\} \subset \{0 \le t \le 1, \inf_{0\le s\le t} W_n(s) \le x \le \sup_{0\le s\le t} W_n(s)\}.$$

Therefore, we get from (6.16)–(6.19) that for $n > n_0 \vee n_1$

$$\mathbf{P}\Big(\sup_{[0,1]\times\mathbf{R}} |\ell_n(t,x) - \ell(t,x)| > 3\varepsilon\Big) \le 5\rho,$$

and this is the assertion of the theorem.

Proof of Lemma 6.2. In view of the estimate

$$\mathbf{P}\Big(\sup_{Q(\varrho)} |\ell_n(t,x) - \ell_n(s,y)| > \varepsilon\Big)$$

$$\le 2\sum_{k\le 1/\varrho}\sum_{|l|\le A/\varrho} \mathbf{P}\Big(\sup_{\substack{k\varrho\le t\le (k+1)\varrho \\ l\varrho\le x\le (l+1)\varrho}} |\ell_n(t,x) - \ell_n(k\varrho, l\varrho)| > \frac{\varepsilon}{3}\Big),$$

it suffices to verify that for any k and l

$$\mathbf{P}\Big(\sup_{\substack{k\varrho\le t\le (k+1)\varrho \\ l\varrho\le x\le (l+1)\varrho}} |\ell_n(t,x) - \ell_n(k\varrho, l\varrho)| > \frac{\varepsilon}{3}\Big) \le \frac{\widetilde{K}}{\varepsilon^6}\varrho^3, \qquad 1/\sqrt{n} < \varrho, \tag{6.20}$$

where $\widetilde{K}$ is a constant. Set

$$\square_n(s,t,x,y) := \ell_n(t,y) - \ell_n(t,x) - \ell_n(s,y) + \ell_n(s,x).$$

Since

$$|\ell_n(t,y) - \ell_n(s,x)| \le |\square_n(s,t,x,y)| + |\ell_n(s,y) - \ell_n(s,x)| + |\ell_n(t,x) - \ell_n(s,x)|,$$

to establish (6.20) it suffices to prove that for any fixed $(s,x)\in[0,1]\times[-A,A]$ and all $\lambda>0$, $1/n\le\varrho_1\le 1$, $1/\sqrt{n}\le\varrho_2\le 1$

$$\mathbf{P}\Big(\sup_{\substack{s\le t\le s+\varrho_1\\ x\le y\le x+\varrho_2}}|\Box_n(s,t,x,y)|>\lambda\Big)\le\frac{K_1}{\lambda^6}\,\varrho_1^{3/2}\,\varrho_2^3, \tag{6.21}$$

$$\mathbf{P}\Big(\sup_{x\le y\le x+\varrho_2}|\ell_n(s,y)-\ell_n(s,x)|>\lambda\Big)\le\frac{K_1}{\lambda^6}\,\varrho_2^3, \tag{6.22}$$

$$\mathbf{P}\Big(\sup_{s\le t\le s+\varrho_1}|\ell_n(t,x)-\ell_n(s,x)|>\lambda\Big)\le\frac{K_2}{\lambda^6}\,\varrho_1^3. \tag{6.23}$$

Indeed, from these estimates with $\lambda=\varepsilon/9$ it follows that

$$\mathbf{P}\Big(\sup_{\substack{k\varrho\le t\le(k+1)\varrho\\ l\varrho\le x\le(l+1)\varrho}}|\ell_n(t,x)-\ell_n(k\varrho,l\varrho)|>\frac{\varepsilon}{3}\Big)\le\frac{K_19^6}{\varepsilon^6}\varrho^{4+1/2}+\frac{K_19^6}{\varepsilon^6}\varrho^3+\frac{K_29^6}{\varepsilon^6}\varrho^3.$$

Thus for $\widetilde{K}=(2K_1+K_2)9^6$

$$\mathbf{P}\Big(\sup_{Q(\varrho)}|\ell_n(t,x)-\ell_n(s,y)|>\varepsilon\Big)\le 4A(\rho)\widetilde{K}\frac{\varrho}{\varepsilon^6}.$$

Choosing $\varrho=\varrho(\varepsilon,\rho)$ such that $4A(\rho)\widetilde{K}\frac{\varrho}{\varepsilon^6}\le\rho$, we get (6.17).

As to the estimates (6.21)–(6.23) it should be noticed that the process $\ell_n(t,y)$ is constant on the rectangles $(t,y)\in\Big[\frac{k}{n},\frac{k+1}{n}\Big)\times\Big[\frac{l}{\sqrt{n}},\frac{l+1}{\sqrt{n}}\Big)$, $l\in\mathbb{Z}$, $k=0,1,2,\dots$, i.e., it is determined by the values on the lattice vertices $\Big\{\frac{k}{n},\frac{l}{\sqrt{n}}\Big\}$.

By Proposition 4.2, the estimate (6.21) is valid if

$$\mathbf{P}(|\Box_n(s,t,x,y)|>\lambda)\le C_p\lambda^{-p}\big(|y-x|\sqrt{t-s}\big)^{p/2} \tag{6.24}$$

holds for some $p\ge 6$ and for any (s,x), (t,y) such that $1/n\le t-s\le 1$, $1/\sqrt{n}\le|y-x|\le 1$. We derive (6.24) by Chebyshev's inequality from the following statement.

Proposition 6.3. *For any* $1/\sqrt{n}\le|y-x|\le 1$, $1/n\le t-s\le 1$

$$\mathbf{E}|\Box_n(s,t,x,y)|^p\le L^pp!\big(|y-x|\sqrt{t-s}\big)^{p/2},\qquad p=1,2,\dots, \tag{6.25}$$

where L is a constant.

Since $\ell_n(0,x)=0$, the estimate (6.22) follows from (6.21) with $s=0$, $t=s$, $\varrho_1=1$. By the monotonicity of the processes $\ell_n(t,x)$ with respect to t, the estimate (6.23) is a consequence of Chebyshev's inequality and the following result. Actually, the monotonicity is not needed in view of the Proposition 4.1.

Proposition 6.4. *For any $s \in [0,1]$ and $1/n \le \varrho$,*

$$\mathbf{E}\big(\ell_n(s+\varrho,x)-\ell_n(s,x)\big)^p \le 2^{3p/2}p!\varrho^{p/2}, \qquad p=1,2,\dots, \tag{6.26}$$

Proof of Proposition 6.3. Suppose for definiteness that $y > x$. Let $\Delta := [y\sqrt{n}] - [x\sqrt{n}]$, $d(v) := 1\!\!I_{\{0\}}(v-\Delta) - 1\!\!I_{\{0\}}(v)$, $v \in \mathbb{Z}$. Then

$$\square_n(s,t,x,y) = \frac{1}{\sqrt{n}}\sum_{k=[ns]+1}^{[nt]} d(\nu_{k-1}^n - [x\sqrt{n}]) = \frac{1}{\sqrt{n}}\sum_{l=1}^{[nt]-[ns]} d\big(\nu_{l-1}^{n+} - \big([x\sqrt{n}] - \nu_{[ns]}^n\big)\big),$$

where $\nu_l^{n+} := \nu_{[ns]+l}^n - \nu_{[ns]}^n$, $l=1,\dots,n$, is a random walk independent of $\nu_{[ns]}^n$. We can use Lemma 2.1 Ch. I. Therefore, it suffices to prove that for any $1/\sqrt{n} \le |y-x| \le 1$, $1/n \le t \le 1$, $p=1,2,\dots$, and some constant L

$$\Big|\mathbf{E}\Big(\frac{1}{\sqrt{n}}\sum_{l=1}^{[nt]} d\big(\nu_{l-1} - [x\sqrt{n}]\big)\Big)^p\Big| \le L^p p!\big(|y-x|\sqrt{t}\big)^{p/2}. \tag{6.27}$$

We prove (6.27) by induction on p. For this we use the relations (5.9)–(5.17). We take $\zeta_n(k,v)$, $k=1,\dots,n$, to be the nonrandom functions $n^{-1/2}d(v)$ not depending on k, and let $m := [nt]$, $\varkappa := [x\sqrt{n}]$.

With the introduced notations,

$$Z_n^{(p)}(m,\varkappa) = \mathbf{E}\Big(\frac{1}{\sqrt{n}}\sum_{l=1}^{[nt]} d\big(\nu_{l-1} - [x\sqrt{n}]\big)\Big)^p.$$

We have the estimate

$$|G_n(z,y)| = \frac{1}{\sqrt{n}}\Big|\sum_{v=-\infty}^{\infty} e^{i(y-z)v}\varphi(y)\big(1\!\!I_{\{0\}}(v-\Delta) - 1\!\!I_{\{0\}}(v)\big)\Big|$$

$$= \frac{1}{\sqrt{n}}|\varphi(y)(e^{i(y-z)\Delta}-1)| \le \frac{2}{\sqrt{n}}((1\wedge|z|\Delta) + (1\wedge|y|\Delta)). \tag{6.28}$$

Let $q \ge 2$. Suppose that for all $1 \le k \le m$

$$\big|\mathfrak{Z}_n^{1,q-2}(k,z)\big| \le 2L^{q-2}(q-2)!\big(|x-y|\sqrt{t}\big)^{(q-2)/2}\frac{1}{\sqrt{n}}\Big((1\wedge|z|\Delta) + \frac{\sqrt{|x-y|}}{t^{1/4}}\Big), \tag{6.29}$$

$$\big|Z_n^{(p)}(k,\varkappa)\big| \le L^p p!\big(|y-x|\sqrt{t}\big)^{p/2}, \quad 1 \le k \le m, \quad \varkappa \in \mathbb{Z}, \quad p \le q-1. \tag{6.30}$$

Using this induction hypothesis, we prove (6.29) for $q+1$ instead of q and prove (6.30) for $p=q$.

Consider the induction base for $q=2$. We note that $\mathfrak{Z}_n^{1,0}(k,z) = G_n(z,0)$. Moreover, by (5.3), β =1, (5.4), (5.7) and (6.28), $y=0$,

$$\big|Z_n^{(1)}(k,\varkappa)\big| = \Big|\mathbf{E}\Big(\frac{1}{\sqrt{n}}\sum_{l=1}^{k} d(\nu_{l-1}-\varkappa)\Big)\Big| = \frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} e^{-iz\varkappa}G_n(z,0)\frac{1-\varphi^k(z)}{1-\varphi(z)}dz\Big|$$

$$\leq \frac{1}{\pi\sqrt{n}} \int_{-\pi}^{\pi} (1 \wedge |z|\Delta)(m \wedge z^{-2})\, dz \leq \frac{9}{\pi\sqrt{n}}(\Delta\sqrt{m})^{1/2} \leq 4\big(|y-x|\sqrt{t}\big)^{1/2}.$$

Thus, if $q = 2$ the induction hypothesis holds for $L \geq 4$.

For $j \geq 2$, $r \leq q-1$, $1 \leq k \leq m$, using the notation (5.13) and the estimate (6.30), we get that

$$\big|\mathfrak{Z}_n^{j,r}(k,z)\big| \leq \frac{1}{n^{j/2}} \sum_{v=-\infty}^{\infty} \mathbf{E}|d^j(v)| L^r r! \big(|y-x|\sqrt{t}\big)^{r/2}$$

$$\leq \frac{2L^r r!}{n}\big(|y-x|\sqrt{t}\big)^{(r+j-2)/2} \leq 2L^r r! \big(|y-x|\sqrt{t}\big)^{(r+j-1)/2} \frac{1}{\sqrt{n}}\Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}.$$

We now estimate $\big|\mathfrak{Z}_n^{1,q-1}(k,z)\big|$. Applying (5.17), using (5.5), (6.28) and the preceding estimates for the variables $\mathfrak{Z}_n^{j,q-1-j}$, $j = 1, \dots, q-1$, we get

$$\big|\mathfrak{Z}_n^{1,q-1}(k,z)\big| \leq \frac{1}{2\pi} \sum_{j=1}^{q-1} \frac{(q-1)!}{j!(q-1-j)!} \int_{-\pi}^{\pi} |G_n(z,u)| \sum_{l=1}^{k} \big|\mathfrak{Z}_n^{j,q-1-j}(k-l,u)\big| |\varphi(u)|^{l-1}\, du$$

$$\leq \frac{2}{\pi} \sum_{j=1}^{q-1} \frac{(q-1)!}{j!(q-1-j)!} \int_{-\pi}^{\pi} \frac{1}{\sqrt{n}}((1 \wedge |z|\Delta) + (1 \wedge |u|\Delta)) L^{q-1-j}(q-1-j)!$$

$$\times \big(|y-x|\sqrt{t}\big)^{(q-2)/2} \Big(\frac{(1 \wedge |u|\Delta)}{\sqrt{n}} + \frac{1}{\sqrt{n}}\Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}\Big) \sum_{l=1}^{k} |\varphi(u)|^{l-1}\, du$$

$$\leq L^{q-1}(q-1)! \big(|y-x|\sqrt{t}\big)^{(q-2)/2} \frac{2}{\pi n} \int_{-\pi}^{\pi} \big((1 \wedge |z|\Delta) + (1 \wedge |u|\Delta)\big) \Big((1 \wedge |u|\Delta)$$

$$+ \Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}\Big) \big(k \wedge \frac{1}{u^2}\big)\, du \sum_{j=1}^{q-1} \frac{1}{L^j}.$$

Using the estimates (5.6)–(5.8) and the inequalities $\Delta \leq 2|x-y|\sqrt{n}$, $|y-x|\sqrt{t} \geq 1/n$, we get that for $1 \leq k \leq m$

$$\big|\mathfrak{Z}_n^{1,q-1}(k,z)\big| \leq L^{q-1}(q-1)! \big(|y-x|\sqrt{t}\big)^{(q-2)/2} \frac{2}{\pi n} \Big\{ \Big((1 \wedge |z|\Delta)$$

$$+ \Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}\Big) 9(\Delta\sqrt{m})^{1/2} + 4(1 \wedge |z|\Delta)\Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}\sqrt{m} + 9\Delta \Big\} \frac{1}{L-1}$$

$$\leq L^{q-1}(q-1)! \big(|y-x|\sqrt{t}\big)^{(q-1)/2} \frac{1}{(L-1)\sqrt{n}} \Big\{ \Big((1 \wedge |z|\Delta) + \Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2}\Big) 9\sqrt{2}$$

$$+ (1 \wedge |z|\Delta)4 + \Big(\frac{|y-x|}{\sqrt{t}}\Big)^{1/2} \frac{9\Delta}{|y-x|\sqrt{n}} \Big\}$$

$$\le L^{q-1}(q-1)!\bigl(|y-x|\sqrt{t}\bigr)^{(q-1)/2}\frac{32}{L-1}\frac{1}{\sqrt{n}}\Bigl((1\wedge|z|\Delta)+\Bigl(\frac{|y-x|}{\sqrt{t}}\Bigr)^{1/2}\Bigr).$$

For $L=17$ this estimate takes the necessary form (6.29).

Substituting the estimates for $\mathfrak{Z}_n^{j,q-j}$, $j=1,\dots,q$, in (5.16) and using (5.6), (5.7), we have

$$\bigl|Z_n^{(q)}(m,\varkappa)\bigr|\le\sum_{j=1}^{q}\frac{q!L^{q-j}}{j!\pi\sqrt{n}}\bigl(|x-y|\sqrt{t}\bigr)^{(q-1)/2}\int_{-\pi}^{\pi}\Bigl((1\wedge|z|\Delta)+\Bigl(\frac{|y-x|}{\sqrt{t}}\Bigr)^{1/2}\Bigr)(m\wedge z^{-2})\,dz$$

$$\le\frac{L^q q!}{\pi(L-1)}\bigl(|x-y|\sqrt{t}\bigr)^{q/2}\Bigl(\frac{9(\Delta\sqrt{m})^{1/2}}{(|x-y|\sqrt{t})^{1/2}\sqrt{n}}+\frac{4\sqrt{m}}{\sqrt{nt}}\Bigr)\le\frac{(9\sqrt{2}+4)}{\pi(L-1)}L^q q!\bigl(|x-y|\sqrt{t}\bigr)^{q/2}.$$

Since $L\ge 17$, this yields (6.30) for $p=q$. This completes the induction proof, because, with our notations, the estimate (6.30) coincide with (6.27). □

Proof of Proposition 6.4. In the relations (5.9)–(5.17) we set $\zeta_n(k,v):=\frac{1}{\sqrt{n}}\mathbb{I}_{\{0\}}(v)$. To verify (6.26) it suffices to prove that for any integers $1\le m\le n$ and $\varkappa$

$$Z_n^{(p)}(m,\varkappa)=\mathbf{E}\Bigl(\frac{1}{\sqrt{n}}\sum_{k=1}^{m}\mathbb{I}_{\{0\}}(\nu_{k-1}-\varkappa)\Bigr)^p\le 2^p p!\Bigl(\frac{m}{n}\Bigr)^{p/2},\quad p=1,2,\dots \tag{6.31}$$

Indeed, by Lemma 2.1 of Ch. I, we get

$$\mathbf{E}\bigl(\ell_n(s+\varrho,x)-\ell_n(s,x)\bigr)^p=\mathbf{E}\Bigl\{\mathbf{E}\Bigl(\frac{1}{\sqrt{n}}\sum_{k=1}^{[n(s+\varrho)]-[ns]}\mathbb{I}_{\{0\}}\bigl(\nu_{k-1}^{n+}-\bigl([x\sqrt{n}]-\nu_{[ns]}^{n}\bigr)\bigr)\Bigr)^p\Bigm|\mathcal{F}_s\Bigr\}$$

$$=\mathbf{E}Z_n^{(p)}\bigl([n(s+\varrho)]-[ns],[x\sqrt{n}]-\nu_{[ns]}^n\bigr)\le 2^p p!(2\varrho)^{p/2}.$$

We assume that (6.31) holds for all $p\le q-1$ and prove it for $p=q$. By the induction hypothesis and (5.13) for $1\le k\le m$, we have

$$\mathfrak{Z}_n^{j,q-j}(m-k,z)=\frac{1}{n^{j/2}}\mathbf{E}Z_n^{(q-j)}(m-k,-\xi_1)\le\frac{2^{q-j}(q-j)!}{n^{j/2}}\Bigl(\frac{m}{n}\Bigr)^{(q-j)/2},\quad 1\le j\le q-1,$$

and therefore, by (5.16) and (5.5), (5.6),

$$Z_n^{(q)}(m,\varkappa)\le\sum_{j=1}^{q}\frac{q!2^{q-j}}{j!2\pi}\frac{m^{(q-j)/2}}{n^{q/2}}\int_{-\pi}^{\pi}\frac{1-|\varphi(z)|^m}{1-|\varphi(z)|}\,dz\le 2^q q!\Bigl(\frac{m}{n}\Bigr)^{q/2}\sum_{j=1}^{q}2^{-j}.$$

Thus we have (6.31) for $p=q$. This completes the proof by induction. Thus the estimate (6.26) is valid and Proposition 6.4 is proved. □

The estimate (6.17), as already explained, follows from (6.25) and (6.26). □

Theorem 6.1 is proved for the case $\mathbf{P}(\xi_1=0)=0$.

It is time to mention that Remark 3.2 is valid for the proof of the weak invariance principle for the local times. Indeed, for the proof of (6.6) we used the estimate (3.5), which can be established under the condition $\mathbf{E}\xi_1^4 < \infty$ without the truncation procedure of the steps of the random walk. Propositions 6.1 and 6.2 are proved without the truncation procedure if $\mathbf{E}|\xi_1|^3 < \infty$.

Let us consider the case $p_0 = 1 - \mathbf{P}(\xi_1 = 0) < 1$. This case leads only to complication in the formulas used for proving Propositions 6.1 and 6.2. The formulations of the lemmas and the structure of the proofs remain the same.

Taking into account the method of constructing the random walk ν_k^n and the strong Markov property of the Brownian motion, we get in this case that for $v \geq 0$

$$n\mathbf{E}\ell^2\big(\widetilde{H}_n^{(1)}, \tfrac{v}{\sqrt{n}}\big) = \mathbf{E}\ell^2\big(\widetilde{H}_1^{(1)}, v\big) = \frac{16}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big\{\Big(\frac{\bar\mu(p_0\bar\mu+v)(p_0\bar\eta-v)(\bar\eta-v)}{p_0(\bar\mu+\bar\eta)}$$

$$+\frac{\bar\mu(1-p_0)}{\bar\eta p_0}(\bar\eta-v)v^2\Big)\mathbb{I}_{\{0\leq v<p_0\bar\eta,\mu>0\}} + \frac{\bar\mu v}{\bar\eta}(\bar\eta-v)^2\mathbb{I}_{\{0<p_0\bar\eta\leq v\leq\bar\eta,\mu>0\}}\Big\}\Big\}. \qquad (6.32)$$

For $v < 0$, using the symmetry property of the Brownian motion, it is not hard to get by the substitution $\mu \mapsto \eta$, $\eta \mapsto \mu$, $v \mapsto -v$ in (6.32) that

$$n\mathbf{E}\ell^2\big(\widetilde{H}_n^{(1)}, \tfrac{v}{\sqrt{n}}\big) = \frac{16}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big\{\Big(\frac{\bar\eta(p_0\bar\eta-v)(p_0\bar\mu+v)(\bar\mu+v)}{p_0(\bar\mu+\bar\eta)}$$

$$+\frac{\bar\eta(1-p_0)}{\bar\mu p_0}(\bar\mu+v)v^2\Big)\mathbb{I}_{\{-p_0\bar\mu<v<0,\eta>0\}} + \frac{\bar\eta|v|}{\bar\mu}(\bar\mu+v)^2\mathbb{I}_{\{-\bar\mu\leq v\leq -p_0\bar\mu<0,\eta>0\}}\Big\}\Big\}.$$

With the help of these expressions we can derive the estimate (6.9), which enables us to prove Proposition 6.1 for $p_0 < 1$.

As for the proof of Proposition 6.2, we present a detailed computation for the expectation $\mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big)$. Applying (2.9), the strong Markov property of the Brownian motion, and the formula for the probability of the first exit to the corresponding boundary, we get that for $v \geq 0$

$$\mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big) = \frac{2}{p_0\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\Big(\mathbb{I}_{\{0\leq v<p_0\bar\eta,\mu>0\}}\frac{p_0\bar\mu}{(p_0\bar\mu+v)}\Big\{\frac{(p_0\bar\eta-v)}{p_0(\bar\mu+\bar\eta)}\frac{2(p_0\bar\eta-v)(v+p_0\bar\mu)}{p_0(\bar\mu+\bar\eta)}$$

$$+\frac{(v+p_0\bar\mu)}{p_0(\bar\mu+\bar\eta)}\Big[\frac{2(p_0\bar\eta-v)(v+p_0\bar\mu)}{p_0(\bar\mu+\bar\eta)} + \frac{\bar\eta(1-p_0)}{(\bar\eta-v)}\frac{2(\bar\eta-v)v}{\bar\eta}\Big]\Big\} \qquad (6.33)$$

$$+\mathbb{I}_{\{p_0\bar\eta\leq v\leq\bar\eta,\bar\mu>0\}}\frac{p_0\bar\mu}{p_0(\bar\eta+\bar\eta)}\frac{p_0\bar\eta}{v}\frac{2(\bar\eta-v)v}{\bar\eta}\Big)\Big\} = \frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{(\mu+\eta)\bar\mu(\bar\eta-v)}{(\bar\mu+\bar\eta)}\mathbb{I}_{\{0\leq v\leq\bar\eta,\mu>0\}}\Big\}.$$

By the symmetry property of the Brownian motion,

$$\mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big) = \frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{(\mu+\eta)\bar\eta(\bar\mu+v)}{\bar\mu+\bar\eta}\mathbb{I}_{\{-\bar\mu\leq v<0,\eta>0\}}\Big\}, \qquad v \leq 0. \qquad (6.34)$$

From here it follows that the expression for $\sqrt{n}g_n(-v,0) = \mathbb{I}_{\{0\}}(v) - \mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big)$ is the same as for p_0 =1. Then the equality (6.13) holds, i.e.,

$$\sqrt{n}G_n(0,0) = 1 - \frac{2}{D\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta\mathbb{I}_{\{\mu>0,\eta>0\}}\big\} = O\big(\mathbf{E}\{\xi_1^2; |\xi_1| > n^{1/4}\}\big) = O(\theta^3).$$

This formula plays a key role in the proof of Proposition 6.2.

To compute $\sqrt{n}g_n(-v,y)$ explicitly is very difficult, but in fact is not necessary. We need only the estimate (6.15). Applying (6.11), we have

$$\sqrt{n}|G_n(u,y)-G_n(0,0)| = \Big|\sqrt{n}\sum_{v=-\infty}^{\infty}\big(e^{iuv}g_n(-v,y)-g_n(-v,0)\big)\Big| = \Big|\mathbf{E}\big(e^{iy\xi_1}-1\big)$$

$$-\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{\big(e^{iv(u-y)+iyW(H_1^{(1)})}-1\big)\ell\big(\widetilde{H}_1^{(1)},v\big)\big\}\Big| \le 2(1\wedge|y|\mathbf{E}|\xi_1|)$$

$$+\sum_{v=-\infty}^{\infty}2\Big((1\wedge(|uv|+|yv|+|y|\theta\sqrt{n}))\mathbf{E}\ell\big(\widetilde{H}_1^{(1)},v\big)+\mathbf{E}\big\{\mathbb{1}_{\{\theta\sqrt{n}\le|W(H_1^{(1)})|\}}\ell\big(H_1^{(1)},v\big)\big\}\Big).$$

It is important that in the last term of this estimate the stopping moment $\widetilde{H}_1^{(1)}$ of the local time is replaced by the larger moment $H_1^{(1)}$. Although the local time increases, it enables us to simplify the computations, because this stopping time coincides with the time in the argument of the Brownian motion located in the indicator function in front of the local time. The term $2(1\wedge|y|\mathbf{E}|\xi_1|)$ was already considered in (6.15). Besides that, there are, in fact, two different sums of terms. The first sum, in view of (6.33), (6.34) and the inequalities $0<\bar{\mu}\le\theta\sqrt{n}$, $0<\bar{\eta}\le\theta\sqrt{n}$, is bounded by $2(1\wedge(|u|\theta\sqrt{n}+|y|\theta\sqrt{n}))$, which corresponds to (6.15). To estimate the second sum one must compute $\mathbf{E}\big\{\mathbb{1}_{\{\theta\sqrt{n}\le|W(H_1^{(1)})|\}}\ell\big(H_1^{(1)},v\big)\big\}$. Here the presence of the indicator function excludes from the consideration those sample paths which at the time $H_1^{(1)}$ hit zero. Proceeding as in the derivation of the formula (6.33), we get that for $0\le v$

$$\mathbf{E}\big\{\mathbb{1}_{\{\theta\sqrt{n}\le|W(H_1^{(1)})|\}}\ell\big(H_1^{(1)},v\big)\big\}$$

$$=\frac{2}{p_0\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\Big(\mathbb{1}_{\{0\le v<p_0\eta,\mu>0\}}\frac{p_0\mu}{(p_0\mu+v)}\Big\{\frac{(p_0\eta-v)}{p_0(\mu+\eta)}\frac{2(p_0\eta-v)(v+p_0\mu)p_0}{p_0(\mu+\eta)}\mathbb{1}_{\{\theta\sqrt{n}\le\mu\}}$$

$$+\frac{(v+p_0\mu)}{p_0(\mu+\eta)}\Big\{\frac{2(p_0\eta-v)(v+p_0\mu)}{p_0(\mu+\eta)}\frac{p_0\eta-v}{\eta-v}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}+\frac{\eta(1-p_0)}{(\eta-v)}\Big[\frac{2(p_0\eta-v)(v+p_0\mu)}{p_0(\mu+\eta)}$$

$$+\frac{2(\eta-v)v}{\eta}\Big]\frac{v}{\eta}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}\Big\}\Big\}+\mathbb{1}_{\{p_0\eta\le v\le\eta,\mu>0\}}\frac{p_0\mu}{p_0(\eta+\eta)}\frac{p_0\eta}{v}\frac{2(\eta-v)v}{\eta}\frac{v}{\eta}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}\Big)\Big\}$$

$$=\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\mathbb{1}_{\{0\le v<p_0\eta,\mu>0\}}\Big(\frac{\mu(p_0\eta-v)(v+p_0\mu)}{p_0(\mu+\eta)}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}+\frac{\mu(p_0\eta-v)^2}{p_0(\mu+\eta)}\mathbb{1}_{\{\theta\sqrt{n}\le\mu\}}\Big)$$

$$+\mathbb{1}_{\{p_0\eta\le v\le\eta,\mu>0\}}\frac{\mu(\eta-v)v}{\eta}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}\Big\}$$

$$\le\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\mu(\eta-v)\mathbb{1}_{\{0\le v<\eta\}}\Big(\mathbb{1}_{\{\theta\sqrt{n}\le\eta,\mu>0\}}+\mathbb{1}_{\{\theta\sqrt{n}\le\mu\}}\Big)\Big\}.$$

For $v<0$, using the symmetry property of the Brownian motion, it is easy to get (in the above relation one should replace $\mu\mapsto\eta$, $\eta\mapsto\mu$, $v\mapsto-v$) that

$$\mathbf{E}\big\{\mathbb{1}_{\{\theta\sqrt{n}\le|W(H_1^{(1)})|\}}\ell\big(H_1^{(1)},v\big)\big\}=\mathbf{E}\Big\{\frac{4\mathbb{1}_{\{-p_0\mu\le v<0,\eta>0\}}}{\mathbf{E}|\xi_1|}\Big(\frac{\eta(p_0\eta-v)(v+p_0\mu)}{p_0(\mu+\eta)}\mathbb{1}_{\{\theta\sqrt{n}\le\mu\}}$$

$$+\frac{\eta(v+p_0\mu)^2}{p_0(\mu+\eta)}\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}\Big)+\mathbb{1}_{\{-\mu p_0<v<0,\eta>0\}}\frac{\eta(v+\mu)|v|}{\mu}\mathbb{1}_{\{\theta\sqrt{n}\le\mu\}}\Big\}$$
$$\le\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\eta(v+\mu)\mathbb{1}_{\{-\mu\le v<0\}}\Big(\mathbb{1}_{\{\theta\sqrt{n}\le\mu,\eta>0\}}+\mathbb{1}_{\{\theta\sqrt{n}\le\eta\}}\Big)\Big\}.$$

These relations enable us to estimate the second sum by $C\theta$. This completes the proof of (6.15) for $p_0\ne 1$, and hence proves Proposition 6.2 for this case. Thus Theorem 6.1 is completely proved. □

We now prove the *weak invariance principle for local times* of random walks in a more general case. We recall our notations

$$h(v):=\mathbf{E}f(v,v+\xi_1),\qquad h:=\sum_{v=-\infty}^{\infty}h(v),$$

$$q_n(t,x):=\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}f\big(\nu_{k-1}^n-[x\sqrt{n}],\nu_k^n-[x\sqrt{n}]\big),\qquad (t,x)\in[0,1]\times\mathbf{R}.$$

Theorem 6.2. *Suppose that*

$$\sum_{l=-\infty}^{\infty}\mathbf{E}|f(l,l+\xi_1)|<\infty, \tag{6.35}$$

$$\sum_{l=-\infty}^{\infty}\mathbf{E}f^2(l,l+\xi_1)<\infty. \tag{6.36}$$

Then for any $\varepsilon>0$

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{t\in[0,1]}|W_n(t)-W(t)|>\varepsilon\Big)=0, \tag{6.37}$$

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}}|q_n(t,x)-h\ell(t,x)|>\varepsilon\Big)=0. \tag{6.38}$$

Remark 6.1. The invariance principle for random walks is included in Theorem 6.2 especially. The fact that (6.37) is realized jointly with (6.38) significantly strengthens the result. Here it is important that the processes $W_n(t)$, $q_n(t,x)$, $(t,x)\in[0,1]\times\mathbf{R}$, are determined by the same sequences of random walks constructed from the Brownian motion $W(t)$, $t\in[0,1]$ with the help of the Skorohod embedding scheme.

Theorem 6.2 can be formulated differently. This is connected with the scaling property of the Brownian motion: for any fixed $c>0$ the process $c^{-1/2}W(ct)$ is a Brownian motion, and the process $c^{-1/2}\ell(tc,x\sqrt{c})$ is its local time. We consider instead of the sequences of random walks ν_k^n, $k=0,1,\dots,n$, the first representative, i.e., the random walk ν_k^1, $k=0,1,\dots,n$. We set

$$\widetilde{q}_n(t,x):=\sum_{k=1}^{[nt]}f\big(\nu_{k-1}^1-[x\sqrt{n}],\nu_k^1-[x\sqrt{n}]\big),\qquad (t,x)\in[0,1]\times\mathbf{R}.$$

By the scaling property of the Brownian motion and by the method of construction of the random walk ν_k^n (see § 2), the finite-dimensional distributions of the processes $n^{-1/2}\big(\nu_{[ns]}^1 - W(ns), \widetilde{q}_n(t,x) - h\ell(nt, x\sqrt{n})\big)$ coincide with those of the processes $\big(W_n(s) - W(s), q_n(t,x) - h\ell(t,x)\big)$, $(s,t,x) \in [0,1]^2 \times \mathbf{R}$. Therefore, Theorem 6.2 can be reformulated as follows.

Theorem 6.3. *Suppose that the conditions* (6.35), (6.36) *hold. Then for any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} |\nu_{[nt]}^1 - W(nt)| > \varepsilon\sqrt{n}\Big) = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\widetilde{q}_n(t,x) - h\ell(nt, x\sqrt{n})| > \varepsilon\sqrt{n}\Big) = 0.$$

Proof of Theorem 6.2. We set

$$H_n(t,x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} h(\nu_{k-1}^n - [x\sqrt{n}]), \qquad (t,x) \in [0,1] \times \mathbf{R},$$

$$r_n(y,z) := \frac{1}{\sqrt{n}}(f(y,z) - h(y)), \qquad y \in \mathbb{Z}, \quad z \in \mathbb{Z},$$

$$R_n(t,x) := \sum_{k=1}^{[nt]} r_n(\nu_{k-1}^n - [x\sqrt{n}], \nu_k^n - [x\sqrt{n}]), \qquad (t,x) \in [0,1] \times \mathbf{R}.$$

Proposition 6.5. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |H_n(t,x) - h\ell(t,x)| > \varepsilon\Big) = 0.$$

Proof. Using the identity

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} h\big(\nu_{k-1}^n - [x\sqrt{n}]\big) = \sum_{j=-\infty}^{\infty} h(j) \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \mathbb{I}_{\{0\}}\big(\nu_{k-1}^n - \big[j + x\sqrt{n}\big]\big)$$

$$= \sum_{j=-\infty}^{\infty} h(j)\ell_n\big(t, \frac{j}{\sqrt{n}} + x\big),$$

we get

$$\sup_{(t,x)\in[0,1]\times\mathbf{R}} |H_n(t,x) - h\ell(t,x)| \le \sum_{j=-\infty}^{\infty} |h(j)| \sup_{(t,x)\in[0,1]\times\mathbf{R}} \Big|\ell_n\big(t, \frac{j}{\sqrt{n}} + x\big) - \ell(t,x)\Big|$$

$$\le \sum_{j=-\infty}^{\infty} |h(j)| \Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\ell_n(t,x) - \ell(t,x)| + \sup_{(t,x)\in[0,1]\times\mathbf{R}} \Big|\ell\big(t, \frac{j}{\sqrt{n}} + x\big) - \ell(t,x)\Big|\Big).$$

The right-hand side of this estimate tends to zero in probability in view of (6.4), the uniform continuity of $\ell(t,x)$, and condition (6.35). □

The next lemma together with Proposition 6.5 proves Theorem 6.2.

Lemma 6.3. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |R_n(t,x)| > \varepsilon \Big) = 0. \tag{6.39}$$

Proof. We divide the proof of this lemma into several parts. We consider first the supremum of the random function $R_n(t,x)$ over the set $\{0 \le t \le 1\} \times \{|x| > 2A\}$, where for an arbitrary $\rho > 0$ the value $A = A(\rho)$ is defined in (6.16).

Proposition 6.6. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{|x|>2A} \sup_{0\le t\le 1} |R_n(t,x)| > \varepsilon \Big) = 0. \tag{6.40}$$

Proof. Set $\chi_k := \mathbb{1}_{\{|\nu_k^n| \le A\sqrt{n}\}}$,

$$s_n(j,l) := \sum_{k=1}^{j} \chi_{k-1} r_n(\nu_{k-1}^n - l, \nu_k^n - l), \qquad h^{(2)}(l) := \mathbf{E} f^2(l, l+\xi_1).$$

Since $\mathbf{E} r_n(y, y+\xi_k) = 0$, for fixed l and n the variables $s_n(j,l)$, $j = 1,\dots,n$, form a martingale with respect to the family of σ-algebras $\mathcal{F}_j$ generated by the random walk ν_k^n, $k = 1,2,\dots$, up to the time j. Using (6.16), Doob's inequality for martingales (see (5.8) of Ch. I), Lemma 2.1 of Ch. I, and (5.3) with $\beta = 1$, $\varkappa = 0$, we get

$$\mathbf{P}\Big(\sup_{|x|>2A} \sup_{0\le t\le 1} \Big| \sum_{k=1}^{[nt]} r_n(\nu_{k-1}^n - [x\sqrt{n}], \nu_k^n - [x\sqrt{n}]) \Big| > \varepsilon \Big)$$

$$\le \rho + \sum_{|l|>2A\sqrt{n}} \mathbf{P}\Big(\sup_{0\le j\le n} |s_n(j,l)| > \varepsilon \Big) \le \rho + \sum_{|l|>2A\sqrt{n}} \frac{1}{\varepsilon^2} \mathbf{E} s_n^2(n,l)$$

$$= \rho + \sum_{|l|>2A\sqrt{n}} \frac{1}{\varepsilon^2 n} \mathbf{E} \sum_{k=1}^{n} \chi_{k-1} (h^{(2)}(\nu_{k-1}^n - l) - h^2(\nu_{k-1}^n - l))$$

$$= \rho + \frac{1}{\varepsilon^2} \sum_{|l|>2A\sqrt{n}} \frac{1}{2\pi n} \int_{-\pi}^{\pi} \mathfrak{Z}_l(u) \frac{1-\varphi^n(u)}{1-\varphi(u)}\, du, \tag{6.41}$$

where

$$\mathfrak{Z}_l(u) := \sum_{|v|\le A\sqrt{n}} e^{-iuv} (h^{(2)}(v-l) - h^2(v-l)).$$

Substituting the estimate

$$\sum_{|l|>2A\sqrt{n}} |\mathfrak{Z}_l(u)| \le 2A\sqrt{n} \sum_{|l|\ge A\sqrt{n}} \mathbf{E} f^2(l, l+\xi_1)$$

in (6.41) and using (5.4), (5.6), we get

$$\mathbf{P}\Big(\sup_{|x|>2A}\sup_{0\le t\le 1}|R_n(t,x)|>\varepsilon\Big)\le \rho+\frac{4A}{\pi\varepsilon^2}\sum_{|l|\ge A\sqrt{n}}\mathbf{E}f^2(l,l+\xi_1).$$

Since ρ is arbitrary, this together with (6.36) implies (6.40). □

We now estimate the supremum of the random function $R_n(t,x)$ over the set $\{0\le t\le 1\}\times\{|x|\le 2A\}$. We start with the explanation of the forthcoming conclusions. If the condition

$$\sum_{l=-\infty}^{\infty}\mathbf{E}f^4(l,l+\xi_1)<\infty,$$

would be satisfied, then one could prove the estimate

$$\mathbf{E}R_n^4(1,\varkappa/\sqrt{n})\le\frac{C}{n}.$$

Since $\mathbf{E}r_n(y,y+\xi_k)=0$, for a fixed $\varkappa$ and n the variables $R_n(j/n,\varkappa/\sqrt{n})$, $j=1,\dots,n$, form a martingale with respect to the family of the σ-algebras $\mathcal{F}_j$ generated by the random walk ν_k up to the time j. By Doob's inequality for martingales (see (5.8), $p=4$, Ch. I),

$$\mathbf{P}\Big(\sup_{|x|\le 2A}\sup_{0\le t\le 1}\big|R_n(t,x)\big|>\varepsilon\Big)\le\frac{1}{\varepsilon^4}\sum_{|\varkappa|\le 2A\sqrt{n}}\mathbf{E}R_n^4(1,\varkappa/\sqrt{n})\le\frac{4AC}{\varepsilon^4\sqrt{n}}. \tag{6.42}$$

This together with Proposition 6.6 is sufficient for the proof of Lemma 6.3.

Since the above condition is not required to be satisfied, we must use the method of truncation of the function $f(y,z)$, representing it as the sum of two functions:

$$\hat f_n(y,z):=f(y,z)\mathbb{1}_{\{|f(y,z)|\le n^{1/4}\}}\qquad\text{and}\qquad \check f_n(y,z):=f(y,z)\mathbb{1}_{\{|f(y,z)|>n^{1/4}\}}.$$

Set $\hat h_n^{(2)}(y):=\mathbf{E}\hat f_n^2(y,y+\xi_1)$, $\hat h_n(y):=\mathbf{E}\hat f_n(y,y+\xi_1)$, $\check h_n^{(2)}(y):=\mathbf{E}\check f_n^2(y,y+\xi_1)$, $\check h_n(y):=\mathbf{E}\check f_n(y,y+\xi_1)$. We decompose the function $r_n(y,z)$ into a sum of two functions:

$$\hat r_n(y,z):=\frac{1}{\sqrt{n}}\big(\hat f_n(y,z)-\hat h_n(y)\big)\qquad\text{and}\qquad \check r_n(y,z):=\frac{1}{\sqrt{n}}\big(\check f_n(y,z)-\check h_n(y)\big).$$

In this decomposition the equalities $\mathbf{E}\hat r_n(y,y+\xi_1)=0$, $\mathbf{E}\check r_n(y,y+\xi_1)=0$ and the level of truncation $n^{1/4}$ are important. We denote by $\hat R_n(t,x)$ and $\check R_n(t,x)$ the processes, corresponding to the functions $\hat r_n$ and $\check r_n$. Since $R_n(t,x)=\hat R_n(t,x)+\check R_n(t,x)$, it is sufficient to prove the analogue of (6.39) for each of these summands.

Proposition 6.7. *For any $\varepsilon>0$,*

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{|x|\le 2A}\sup_{0\le t\le 1}\big|\check R_n(t,x)\big|>\varepsilon\Big)=0. \tag{6.43}$$

Proof. We set

$$\check{\mathfrak{Z}}_n(u) := \sum_{v=-\infty}^{\infty} e^{-iuv}\mathbf{E}\check{r}_n^2(v, v+\xi_1) = \sum_{v=-\infty}^{\infty} e^{-iuv}\big(\check{h}_n^{(2)}(v) - \check{h}_n^2(v)\big).$$

Using Doob's inequality for martingales, (5.3) with $\beta = 1$, and (5.4), we get

$$\mathbf{P}\Big(\sup_{|x|\le 2A}\sup_{0\le t\le 1}|\check{R}_n(t,x)| > \varepsilon\Big) \le \tfrac{1}{\varepsilon^2}\sum_{|l|\le 2A\sqrt{n}} \mathbf{E}\Big(\sum_{k=1}^{n}\check{r}_n(\nu_{k-1}-l, \nu_k - l)\Big)^2$$

$$= \tfrac{1}{\varepsilon^2}\sum_{|l|\le 2A\sqrt{n}} \frac{1}{2\pi n}\int_{-\pi}^{\pi} e^{-iul}\check{\mathfrak{Z}}_n(u)\frac{1-\varphi^n(u)}{1-\varphi(u)}du \le \frac{A}{\pi\varepsilon^2\sqrt{n}}\int_{-\pi}^{\pi}\big|\check{\mathfrak{Z}}_n(u)\big|(n\wedge u^{-2})\,du.$$

Obviously,

$$\big|\check{\mathfrak{Z}}_n(u)\big| \le \sum_{v=-\infty}^{\infty}\mathbf{E}\big\{f^2(v, v+\xi_1)\mathbb{1}_{\{|f(v,v+\xi_1)|>n^{1/4}\}}\big\}.$$

Therefore, in view of (5.6), we have

$$\mathbf{P}\Big(\sup_{|x|\le 2A}\sup_{0\le t\le 1}\big|\check{R}_n(t,x)\big| > \varepsilon\Big) \le \tfrac{4A}{\pi\varepsilon^2}\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{f^2(v, v+\xi_1)\mathbb{1}_{\{|f(v,v+\xi_1)|>n^{1/4}\}}\big\}.$$

By (6.36), this implies (6.43). □

The analogous assertion for the process $\hat{R}_n(t,x)$ is the following.

Proposition 6.8. *For any* $\varepsilon > 0$,

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{|x|\le 2A}\sup_{0\le t\le 1}\big|\hat{R}_n(t,x)\big| > \varepsilon\Big) = 0. \tag{6.44}$$

Proof. By Doob's inequality for martingales (see (5.8), $p = 4$, Ch. I),

$$\mathbf{P}\Big\{\sup_{|x|\le 2A}\sup_{0\le t\le 1}\big|\hat{R}_n(t,x)\big| > \varepsilon\Big\} \le \varepsilon^{-4}\sum_{|\varkappa|\le 2A\sqrt{n}}\mathbf{E}\hat{R}_n^4(1, \varkappa/\sqrt{n}). \tag{6.45}$$

To estimate the fourth moment of $\hat{R}_n(1,\varkappa/\sqrt{n})$ we use the relations (5.9)–(5.17) and the notations introduced there. We set $\zeta_n(k,v) := \hat{r}_n(v, v+\xi_k)$. According to (5.9), $\mathbf{E}\hat{R}_n^q(m/n, \varkappa/\sqrt{n}) = Z_n^{(q)}(m,\varkappa)$. From (5.16) it follows that

$$Z_n^{(4)}(m,\varkappa) \le \sum_{j=1}^{4}\frac{4!}{j!(4-j)!}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\big|\mathfrak{Z}_n^{j,4-j}(m-k,u)\big||\varphi(u)|^{k-1}\,du. \tag{6.46}$$

The goal is to estimate the variables $\big|\mathfrak{Z}_n^{j,4-j}(m-k,u)\big|$, $j = 1,2,3,4$. We remark that, in view of the equality $\hat{v}_n(y) := \mathbf{E}\hat{r}_n(y, y+\xi_1) = 0$ and Lemma 2.1 Ch. I,

$$Z_n^{(1)}(m,\varkappa) := \mathbf{E}\sum_{k=1}^{m}\zeta_n(k, \nu_{k-1}-\varkappa) = \sum_{k=1}^{m}\mathbf{E}\hat{v}_n(\nu_{k-1}-\varkappa) = 0,$$

and therefore $\mathfrak{Z}_n^{j,1} \equiv 0$, $j = 1, 2, 3, 4$.

If $j \ge 2$, then using (6.36) we get

$$\big|\mathfrak{Z}_n^{j,0}(m-k,u)\big| \le \frac{16}{n^{j/2}} \sum_{v=-\infty}^{\infty} \mathbf{E}\big\{|f(v,v+\xi_1)|^j \mathrm{1\!I}_{\{|f(v,v+\xi_1)|\le n^{1/4}\}} \le \frac{C}{n^{j/4}\sqrt{n}}. \quad (6.47)$$

Since for $m \le n$

$$Z_n^{(2)}(m,\varkappa) = \frac{1}{n}\sum_{k=1}^{m} \mathbf{E}(\hat{h}_n^{(2)}(\nu_{k-1}-\varkappa) - \hat{h}_n^2(\nu_{k-1}-\varkappa)),$$

using (5.3), (5.4), and (5.6), we have

$$Z_n^{(2)}(m,\varkappa) \le \frac{1}{2\pi n}\sum_{v=-\infty}^{\infty} \mathbf{E}f^2(v,v+\xi_1)\int_{-\pi}^{\pi}\Big|\frac{1-\varphi^m(u)}{1-\varphi(u)}\Big|du \le \frac{C}{\sqrt{n}}.$$

Therefore,

$$\big|\mathfrak{Z}_n^{2,2}(m-k,u)\big| \le \frac{C}{n^{3/2}}\sum_{v=-\infty}^{\infty}\mathbf{E}f^2(v,v+\xi_1) \le \frac{C}{n^{3/2}}. \quad (6.48)$$

Further, since $\mathbf{E}\hat{r}_n(y,y+\xi_1) = 0$, we have

$$G_n(u,y) = \sum_{v=-\infty}^{\infty} e^{i(y-u)v}\mathbf{E}\big\{(e^{iy\xi_1}-1)\hat{r}_n(v,v+\xi_1)\big\}.$$

We choose θ as in Theorem 3.1. Then

$$|G_n(u,y)| \le \frac{2}{\sqrt{n}}\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{\big((1\wedge|y||\xi_1|)\mathrm{1\!I}_{\{|\xi_1|\le\theta\sqrt{n}\}} + \mathrm{1\!I}_{\{\theta\sqrt{n}<|\xi_1|\}}\big)|f(v,v+\xi_1)|\big\}$$

$$\le \frac{C}{\sqrt{n}}((1\wedge|y|\theta\sqrt{n}) + \alpha_n(\theta)), \quad (6.49)$$

where

$$\alpha_n(\theta) := \sum_{v=-\infty}^{\infty}\mathbf{E}\big\{|f(v,v+\xi_1)|\mathrm{1\!I}_{\{\theta\sqrt{n}<|\xi_1|\}}\big\} \to 0.$$

This limit holds in view of (6.35) and the inequality $\theta \ge n^{-1/4}$.

Using the equality $\mathfrak{Z}_n^{1,1} \equiv 0$, formula (5.17) with $r = 2$, the estimates (6.47) with $j = 2$, (6.49), and (5.5)–(5.7), we get

$$\big|\mathfrak{Z}_n^{1,2}(m-k,u)\big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|G_n(u,s)|\sum_{l=1}^{m-k}\big|\mathfrak{Z}_n^{2,0}(m-k-l,s)\big||\varphi(s)|^{l-1}ds$$

$$\le \frac{C}{n^{3/2}}\int_{-\pi}^{\pi}((1\wedge|s|\theta\sqrt{n}) + \alpha_n(\theta))\frac{1-|\varphi^n(s)|}{1-|\varphi(s)|}ds \le \frac{C}{n}\Big(\sqrt{\theta} + \alpha_n(\theta)\Big).$$

Substituting this estimate in (5.17) with $r = 3$, we obtain

$$\left|\mathfrak{Z}_n^{1,3}(m-k,u)\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_n(u,s)| \sum_{l=1}^{m-k} \bigl(3\left|\mathfrak{Z}_n^{1,2}(m-k-l,s)\right| + \left|\mathfrak{Z}_n^{3,0}(m-k-l,s)\right|\bigr)|\varphi^{l-1}(s)|\,ds$$

$$\le \frac{C}{n\sqrt{n}}\int_{-\pi}^{\pi} \bigl((1\wedge|s|\theta\sqrt{n})+\alpha_n(\theta)\bigr)\bigl(\alpha_n(\theta)+\sqrt{\theta}\bigr)\bigl(n\wedge\tfrac{1}{s^2}\bigr)ds \le \frac{C}{n}\bigl(\alpha_n(\theta)+\sqrt{\theta}\bigr)^2.$$

We now have all the estimates we need to use in the inequality (6.46) and thus to estimate $\mathbf{E}\hat{R}_n^4(1,\varkappa/\sqrt{n}) = Z_n^{(4)}(n,\varkappa)$. We get

$$Z_n^{(4)}(n,\varkappa) \le \frac{C}{n}\bigl(\alpha_n(\theta)+\sqrt{\theta}\bigr)\int_{-\pi}^{\pi}(n\wedge u^{-2})\,du \le \frac{C}{\sqrt{n}}\bigl(\alpha_n(\theta)+\sqrt{\theta}\bigr).$$

Substituting this estimate in the right-hand side of (6.45), we get

$$\mathbf{P}\Bigl(\sup_{|x|\le 2A}\sup_{0\le t\le 1}\left|\hat{R}_n(t,x)\right| > \varepsilon\Bigr) \le \frac{4AC}{\varepsilon^4}\bigl(\alpha_n(\theta)+\sqrt{\theta}\bigr).$$

Proposition 6.8 and therefore Lemma 6.3 are proved. □

This, in turn, completes the proof of Theorem 6.2. □

§ 7. Weak invariance principle for local times (continuous random walk)

In this section we assume that the condition (C) holds and the random walk ν_k has a second moment. The essential difference from the condition (D) is that the local time of the random walk depends on the parameter x belonging to $\mathbf{R}$ instead of the discrete lattice. The proof of the convergence of local times of random walks is essentially based on the results of § 4, having the lattice structure of the parameter of the process. In this regard we consider first a discrete lattice as a domain for the parameter x. Let $h(v)$, $v\in\mathbf{R}$, be a bounded integrable function.

Set

$$H_n(t,x) := \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} h\Bigl(\nu_{k-1}^n - \Bigl[\frac{x\sqrt{n}}{\delta}\Bigr]\delta\Bigr), \qquad t\in[0,1], x\in\mathbf{R},$$

where $0<\delta<1$ is an arbitrary number. The process $H_n(t,x)$ is determined by the parameter x taking values in the discrete lattice $\mathbb{Z}_n^\delta = \{j\delta/\sqrt{n}\}_{j\in\mathbb{Z}}$.

Theorem 7.1. *Suppose that*

$$\int_{-\infty}^{\infty} \sqrt{|v|}|h(v)|\,dv < \infty. \qquad (7.1)$$

Then for any $\varepsilon > 0$ *and* $0 < \delta < 1$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |H_n(t,x) - h\ell(t,x)| > \varepsilon \Big) = 0, \qquad (7.2)$$

where $h = \int_{-\infty}^{\infty} h(v)\,dv$.

Proof. The proof of (7.2) is quite analogous to the proof of (6.4), so we keep the structure of the proof of Theorem 6.1. It suffices to point out the essential aspects.

Lemma 7.1. *For any* $(t,x) \in [0,1] \times \mathbf{R}$

$$H_n(t,x) \to h\ell(t,x), \qquad n \to \infty, \qquad (7.3)$$

in probability.

Proof. We use the notations and formulas of § 5 for continuous random walks. We note first that for the continuous random walk $\mathbf{P}(\xi_1 = 0) = 0$, i.e., p_0 =1. Let

$$\theta := \max\Big\{ \int_{|v|>n^{1/4}} |h(v)|\,dv,\ \mathbf{E}^{1/3}\{\xi_1^2; |\xi_1| > n^{1/4}\},\ n^{-1/4} \Big\}.$$

Set

$$\zeta_n(k,v) := \frac{1}{\sqrt{n}} h(v) - h\,\ell^{(k)}(\widetilde{H}_n^{(k)}, -v/\sqrt{n}),$$

where the random time $\widetilde{H}_n^{(k)}$ and the local time $\ell^{(k)}(\widetilde{H}_n^{(k)}, x)$ are defined in the beginning of the proof of Lemma 6.1. Set, in addition,

$$V_n(t,x) := \sum_{k=2}^{[nt]} \zeta_n\Big(k, \nu_{k-1}^n - \Big[\frac{x\sqrt{n}}{\delta}\Big]\delta\Big), \qquad x \in \mathbf{R}.$$

This process, like in the case of the integer random walk, plays an important role in the proof of (7.3). We don't include the term corresponding to $k = 1$ in the definition of the function V_n for the reason discussed in § 5 before the formula (5.19). For the proof of (7.3) this term is negligible, because, by the boundedness of the function h and the estimate (6.9),

$$\sup_{v\in\mathbf{R}} \frac{1}{\sqrt{n}}|h(v)| \le \frac{K}{\sqrt{n}}, \qquad \sup_{v\in\mathbf{R}} \mathbf{E}\ell^2(\widetilde{H}_n^{(1)}, v) \le \frac{C}{n}.$$

As it was explained in the proof of Lemma 6.1 relation (7.3) is a consequence of (6.7). To derive (6.7) we apply (5.32). Propositions 6.1 and 6.2 are valid. Indeed,

$$B_n(0) \leq \frac{2}{n}\int\limits_{-\infty}^{\infty} h^2(v)\,dv + 2h^2\int\limits_{-\infty}^{\infty} \mathbf{E}\ell^2\big(\widetilde{H}_n^1, -\tfrac{v}{\sqrt{n}}\big)\,dv.$$

Now to prove (6.8) one should use the estimate (6.9).

Let us prove (6.10). The analog of (6.12) is the formula

$$\sqrt{n}g_n(-v,0) = h(v) - h\mathbf{E}\ell\big(\widetilde{H}_1^{(1)}, v\big) = h(v)$$

$$-\frac{4h}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{(\mu+\eta)\bar\mu(\bar\eta - v)}{(\bar\mu+\bar\eta)}\mathbb{1}_{\{0\le v\le\bar\eta,\bar\mu>0\}} + \frac{(\mu+\eta)\bar\eta(\bar\mu+v)}{(\bar\mu+\bar\eta)}\mathbb{1}_{\{-\bar\mu\le v<0,\bar\eta>0\}}\Big)\Big\}. \quad (7.4)$$

Therefore, the analog of (6.13) has the form

$$\sqrt{n}G_n(0,0) = h - \frac{4h}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big(\bar\mu\int\limits_0^{\bar\eta}(\bar\eta - v) + \bar\eta\int\limits_{-\mu}^{0}(\bar\mu+v)\Big)\mathbb{1}_{\{\mu>0,\eta>0\}}\Big\}$$

$$= h - \frac{2h}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta\mathbb{1}_{\{\mu>0,\eta>0\}}\big\} = h\,O\big(\mathbf{E}\big\{\xi_1^2\mathbb{1}_{\{|\xi_1|>\theta\sqrt{n}\}}\big\}\big) = h\,O(\theta^3). \quad (7.5)$$

Next we prove (6.15). We have

$$\sqrt{n}|G_n(u,y) - G_n(0,0)| = \sqrt{n}\Big|\int\limits_{-\infty}^{\infty}\mathbf{E}\big\{\big(e^{i(u-y)v+iy\xi_1} - 1\big)\zeta_n(1,v)\big\}dv\Big|$$

$$\le \int\limits_{-\infty}^{\infty}\mathbf{E}\big\{\big|e^{i(u-y)v+iy\xi_1} - 1\big||h(v)|\,dv + |h|\Big|\int\limits_{-\infty}^{\infty}\mathbf{E}\big\{\big(e^{i(u-y)v+iy\xi_1} - 1\big)\ell(\widetilde{H}_1^{(1)}, v)\big\}\,dv\Big|$$

$$\le C(1\wedge(|u|+|y|)\theta\sqrt{n}) + 2\int\limits_{|v|>n^{1/4}}|h(v)|\,dv$$

$$+\frac{4|h|}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{1}{\bar\mu+\bar\eta}\mathbb{1}_{\{\eta>0,\mu>0\}}\int\limits_{-\bar\mu}^{\bar\eta}\Delta_v\big(\bar\mu(\bar\eta - v)\mathbb{1}_{\{v\ge0\}} + \bar\eta(\bar\mu+v)\mathbb{1}_{\{v<0\}}\big)\Big\}dv,$$

where the variables Δ_v are defined in § 6 after the formula (6.14). In order to get the first term on the right-hand side of this relation the estimates $\big|e^{ix}-1\big| \le 2(1\wedge|x|)$ and $n^{1/4}\le\theta\sqrt{n}$ were used. The second term is estimated by 2θ and the third one is evaluated like the similar term in the estimate (6.15). As a result, the estimate (6.15) holds for the continuous random walk and therefore (6.10) is valid. This completes the proof of Propositions 6.1 and 6.2.

Applying (5.32), (6.8), (6.10) and (5.21)–(5.25), we obtain

$$\mathbf{E}V_n^2(t,x) \le C\Big(\theta + \tfrac{1}{\sqrt{n}}\Big) + \tfrac{C}{n}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\big((1\wedge|u|\theta\sqrt{n}) + \theta\big)\big((1\wedge|v|\theta\sqrt{n})$$

$$+ (1\wedge|u|\theta\sqrt{n}) + \theta\big)|\varphi(u)||\varphi(v)|(n\wedge(1+u^{-2}))(n\wedge(1+v^{-2})) \le C\theta.$$

Thus (6.7) holds and therefore (7.3) is proved. □

Another key result in the proof of Theorem 6.1 is Lemma 6.2. Here a significant peculiarity arises: because the support of the function $h(v)$, $v\in\mathbf{R}$, is unbounded, it is necessary to use the truncation of the function $h(v)$. Thus we represent the function $h(v)$, $v\in\mathbf{R}$, as the sum of two functions:

$$\hat{h}_n(v) := h(v)\mathbb{I}_{\{|v|\le 3A\sqrt{n}\}} \qquad \text{and} \qquad \check{h}_n(v) := h(v)\mathbb{I}_{\{|v|>3A\sqrt{n}\}},$$

where the constant $A = A(\rho)$ is chosen such that (6.16) holds. The processes, corresponding to the functions $\hat{h}_n$ and $\check{h}_n$, are denoted by $\hat{H}_n(t,x)$ and $\check{H}_n(t,x)$ respectively. Clearly, $H_n(t,x) = \hat{H}_n(t,x) + \check{H}_n(t,x)$. On the set $\Omega_n(\rho)$, defined just after (6.16), $\check{H}_n(t,x) = 0$ for $|x| > 4A$, because on this set $\sup_{1\le k\le n}|\nu_k^n| \le A\sqrt{n}$. Therefore, it is sufficient to consider the process $\hat{H}_n$ only for $|x|\le 4A$. For this process we establish the following analogue of Lemma 6.2.

Lemma 7.2. *For any $\varepsilon > 0$, $\rho > 0$, and $\delta > 0$, there exist $\varrho = \varrho(\varepsilon,\rho,\delta)$ and $n_0 = n_0(\varepsilon,\rho,\varrho)$, such that for all $n > n_0$*

$$\mathbf{P}\Big(\sup_{Q(\varrho)}|\hat{H}_n(t,x) - \hat{H}_n(s,y)| > \varepsilon\Big) < \rho, \tag{7.6}$$

where

$$Q(\varrho) = \big\{(s,t),(x,y) : |t-s|\le\varrho,\ |x-y|\le\varrho,\ s,t\in[0,1],\ x,y\in[-4A,4A]\big\}.$$

Proof. We use the same arguments as in the discrete case. The distinctive feature in this case is that the process $\hat{H}_n(t,x)$, $(t,x)\in[0,1]\times\mathbf{R}$, is defined on the lattice $\Big\{\dfrac{k}{n}, \dfrac{l\delta}{\sqrt{n}}\Big\}$, whose array spacing with respect to the second coordinate is proportional to δ. The variable $\square_n(s,t,x,y)$ is defined by the process $\hat{H}_n$ similarly to that it was defined in the proof of Lemma 6.2 by the local time ℓ_n.

In this case the analog of Proposition 6.3 is the following assertion.

Proposition 7.1. *For any $\dfrac{\delta}{\sqrt{n}} \le |y-x| \le 1$, $\dfrac{1}{n}\le t-s\le 1$*

$$\mathbf{E}|\square_n(s,t,x,y)|^p \le C_p\Big(\frac{|y-x|\sqrt{t-s}}{\delta}\Big)^{p/2}, \qquad p = 1,2,\dots. \tag{7.7}$$

where C_p depends only on the parameter p.

Proof. For $y>x$ set $\Delta:=(y-x)\sqrt{n}$, $d(v):=\hat h_n(v-\Delta)-\hat h_n(v)$, $v\in\mathbf{R}$. For the proof of (7.7) it is sufficient to check (see (6.27)) that for any $\delta/\sqrt{n}\le y-x\le 1$, $1/n\le t\le 1$, $p=1,2,\dots$, and some constant L

$$\Bigl|\mathbf{E}\Bigl(\frac{1}{\sqrt{n}}\sum_{l=1}^{[nt]} d\Bigl(\nu_{l-1}-\Bigl[\frac{x\sqrt{n}}{\delta}\Bigr]\delta\Bigr)\Bigr)^p\Bigr|\le L^p p!\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{p/2}. \tag{7.8}$$

As in § 6, to prove (7.8) we use induction on p. For this we apply the relations (5.19)–(5.30). We set $\zeta_n(k,v):=n^{-1/2}d(v)$, $k=1,\dots,n$, and $m:=[nt]$, $\varkappa:=[x\sqrt{n}/\delta]\delta$. In view of our notations, $\mathfrak{z}_n^{j,0}(m,v)=n^{-j/2}d^j(v)$,

$$Z_n^{(p)}(m,\varkappa)=\mathbf{E}\Bigl(\frac{1}{\sqrt{n}}\sum_{l=1}^{[nt]} d\Bigl(\nu_{l-1}-\Bigl[\frac{x\sqrt{n}}{\delta}\Bigr]\delta\Bigr)\Bigr)^p.$$

Since the function $h(v)$, $v\in\mathbf{R}$, is integrable, we have the estimate

$$|G_n(z,y)|=\frac{1}{\sqrt{n}}\Bigl|\int_{-\infty}^{\infty} e^{i(y-z)v}\varphi(y)\,d(v)\,dv\Bigr|$$
$$=\frac{1}{\sqrt{n}}\Bigl|\int_{-\infty}^{\infty}\varphi(y)\bigl(e^{i(y-z)(u+\Delta)}-e^{i(y-z)u}\bigr)\,\hat h_n(u)\,du\Bigr|\le\frac{2K}{\sqrt{n}}((1\wedge|z|\Delta)+(1\wedge|y|\Delta)), \tag{7.9}$$

where K is the constant bounding the function h.

Let $q\ge 2$. The induction hypothesis is: for all $1\le k\le m$,

$$\bigl|\mathfrak{Z}_n^{1,q-2}(k,z)\bigr|\le 2L^{q-2}(q-2)!\Bigl(\frac{|x-y|\sqrt{t}}{\delta}\Bigr)^{(q-2)/2}\Bigl(\frac{(1\wedge|z|\Delta)}{\sqrt{n}}+\frac{\sqrt{|x-y|}}{t^{1/4}\sqrt{\delta}\sqrt{n}}\Bigr), \tag{7.10}$$

$$\bigl|Z_n^{(p)}(k,\varkappa)\bigr|\le L^p p!\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{p/2},\quad 1\le k\le m,\quad \varkappa\in\mathbb{Z},\quad p\le q-1. \tag{7.11}$$

We must prove (7.10) for $q+1$ instead of q and prove (7.11) for $p=q$.

Consider the induction base with $q=2$. Note that $\mathfrak{Z}_n^{1,0}(k,z)=G_n(z,0)$. Moreover, by (5.28), $q=1$, (5.29), (5.21), (5.24), and (7.9), $y=0$,

$$\bigl|Z_n^{(1)}(k,\varkappa)\bigr|=\Bigl|\mathfrak{z}_n^{1,0}(k-1,-\varkappa)+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iz\varkappa}G_n(z,0)\varphi(z)\frac{1-\varphi^{k-1}(z)}{1-\varphi(z)}dz\Bigr|\le\frac{2K}{\sqrt{n}}$$
$$+\frac{K}{\pi\sqrt{n}}\int_{-\infty}^{\infty}(1\wedge|z|\Delta)\bigl(m\wedge\bigl(1+\frac{1}{z^2}\bigr)\bigr)dz\le\frac{KC}{\sqrt{n}}(3+(\Delta\sqrt{m})^{1/2})\le 4KC\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{1/2}.$$

Here we used the estimate $\frac{1}{\sqrt{n}}\le\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{1/2}$. Thus for $q=2$ the induction hypothesis holds for $L\ge 4KC$.

For $j \ge 2$, $r \le q-1$, $1 \le k \le m$, using (5.27) and (7.11), we get that

$$\bigl|\mathfrak{Z}_n^{j,r}(k,z)\bigr| \le \frac{1}{n^{j/2}} \int_{-\infty}^{\infty} |d^j(v)|\,dv L^r r!\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{r/2}$$

$$\le \frac{(2K)^j L^r r!}{n}\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{(r+j-2)/2} \le (2K)^j L^r r!\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{(r+j-1)/2}\frac{1}{\sqrt{n}}\Bigl(\frac{|y-x|}{\delta\sqrt{t}}\Bigr)^{1/2},$$

where, to simplify the notation, we assume that the integral of the function $|h(v)|$, $v \in \mathbf{R}$, is estimated by the same constant K as the function itself. Here in the last inequality we used the estimate $\dfrac{1}{\sqrt{n}} \le \dfrac{|y-x|}{\delta}$.

The estimation of $\bigl|\mathfrak{Z}_n^{1,q-1}(k,z)\bigr|$ is done in much the same way as in the proof of Proposition 6.3. However, according to formula (5.30), it is necessary to estimate the additional term, which we consider now. Applying (7.11), we obtain

$$\Bigl|\sum_{j=1}^{q-1}\frac{(q-1)!}{j!(q-1-j)!}\int_{-\infty}^{\infty} e^{-iuv}\mathbf{E}\bigl\{\zeta_n(1,v)\mathfrak{z}_n^{j,q-1-j}(m-1,\xi_1+v)\bigr\}dv\Bigr|$$

$$\le \sum_{j=1}^{q-1}\frac{(q-1)!}{j!}\frac{(2K)^j}{n^{(j+1)/2}}\int_{-\infty}^{\infty}|d(v)|\,dvL^{q-1-j}\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{(q-1-j)/2}$$

$$\le L^q(q-1)!\sum_{j=1}^{q-1}\frac{(2K)^{j+1}}{L^j}\frac{1}{n}\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{(q-2)/2}$$

$$\le L^q(q-1)!\Bigl(\frac{|y-x|\sqrt{t}}{\delta}\Bigr)^{(q-1)/2}\frac{1}{\sqrt{n}}\Bigl(\frac{|y-x|}{\delta\sqrt{t}}\Bigr)^{1/2}\frac{(2K)^2}{L-2K}.$$

It is clear that the constant L is chosen according to the inequality $\dfrac{(2K)^2}{L-2K} < 1$ and the inequalities that arise when evaluating $\bigl|\mathfrak{Z}_n^{1,q-1}(k,z)\bigr|$.

The inductive proof of (7.11) is completed in the same way as in the proof of Proposition 6.3 with the help of (5.28) and (5.29). □

The following analog of Proposition 6.4 is also true.

Proposition 7.2. *For any $s \in [0,1]$ and $1/n \le \varrho$*

$$\mathbf{E}\bigl(\hat{H}_n(s+\varrho,x)-\hat{H}_n(s,x)\bigr)^p \le (2K\sqrt{2})^p p!\varrho^{p/2}, \qquad p=1,2,\dots. \tag{7.12}$$

Therefore, in the continuous case the analog of (6.20) turns into the following inequality: for any k and l

$$\mathbf{P}\Bigl(\sup_{\substack{l\varrho\le x\le(l+1)\varrho\\ k\varrho\le t\le(k+1)\varrho}}|\hat{H}_n(t,x)-\hat{H}_n(k\varrho,l\varrho)|>\frac{\varepsilon}{3}\Bigr) \le \frac{\widetilde{K}}{\varepsilon^6}\Bigl(\frac{\varrho}{\delta}\Bigr)^3, \qquad \frac{\delta}{\sqrt{n}}<\varrho,$$

where $\widetilde{K}$ is a constant. As a result, we have

$$\mathbf{P}\Big(\sup_{Q(\varrho)} |\hat{H}_n(t,x) - \hat{H}_n(s,y)| > \varepsilon\Big) \le 16A(\rho)\widetilde{K}\tfrac{\varrho}{\varepsilon^6\delta^3}.$$

Choosing $\varrho = \varrho(\varepsilon,\rho,\delta)$ such that $16A(\rho)\widetilde{K}\frac{\varrho}{\varepsilon^6\delta^3} \le \rho$, we get (7.6). Lemma 7.2 is proved. □

We now consider the process $\check{H}_n(t,x)$, $t \in [0,1]$, $x \in \mathbf{R}$, and verify that it is uniformly small for large n. It is sufficient to consider this process for $x \in \mathbb{Z}_n^\delta$. We set

$$\check{H}_n(z) := \frac{1}{\sqrt{n}} \sum_{k=2}^n |\check{h}_n(\nu_{k-1}^n - z\sqrt{n})|, \qquad z \in \mathbb{Z}_n^\delta.$$

Then

$$\sup_{t\in[0,1]} |\check{H}_n(t,z)| \le \frac{K}{\sqrt{n}} + \check{H}_n(z). \tag{7.13}$$

Proposition 7.3. *For any* $\varepsilon > 0$, $\rho > 0$, *and for all* $0 < \delta < 1$,

$$\limsup_{n\to\infty} \mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n^\delta} \check{H}_n(z) \ge \varepsilon\Big) \le \rho. \tag{7.14}$$

Proof. Let $\chi_k := \mathbb{1}_{\{|\nu_k^n| \le A\sqrt{n}\}}$ and

$$\zeta_n(k,v) := \frac{1}{\sqrt{n}} \mathbb{1}_{\{|v|\le A\sqrt{n}\}} |\check{h}_n(v - z\sqrt{n})|.$$

We apply the results of § 5. The estimates

$$|B_n(u)| = \bigg|\int_{-\infty}^{\infty} e^{-iuv}\frac{1}{n}\mathbb{1}_{\{|v|\le A\sqrt{n}\}}\check{h}_n^2(v - z\sqrt{n})\,dv\bigg| \le \frac{1}{n}\int_{-A\sqrt{n}}^{A\sqrt{n}} \check{h}_n^2(v - z\sqrt{n})\,dv,$$

$$|G_n(s,u)| = \bigg|\int_{-A\sqrt{n}}^{A\sqrt{n}} e^{i(u-s)v}\frac{\varphi(u)}{\sqrt{n}}|\check{h}_n(v - z\sqrt{n})|\,dv\bigg| \le \frac{1}{\sqrt{n}}\int_{-A\sqrt{n}}^{A\sqrt{n}} |\check{h}_n(v - z\sqrt{n})|\,dv$$

hold. We use (6.16) and (5.32). Then, applying the Chebyshev inequality, we get

$$\mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n^\delta} \check{H}_n(z) \ge \varepsilon\Big) \le \rho + \frac{1}{\varepsilon^2}\sum_{z\in\mathbb{Z}_n^\delta} \mathbf{E}\Big(\frac{1}{\sqrt{n}}\sum_{k=2}^n \chi_{k-1}|\check{h}_n(\nu_{k-1} - z\sqrt{n})|\Big)^2 \le \rho$$

$$+\frac{C}{\varepsilon^2}\sum_{z\in\mathbb{Z}_n^\delta}\bigg[\frac{1}{\sqrt{n}}\int_{-A\sqrt{n}}^{A\sqrt{n}} \big(\check{h}_n^2(v - z\sqrt{n}) + |\check{h}_n(v - z\sqrt{n})|\big)\,dv + \bigg(\int_{-A\sqrt{n}}^{A\sqrt{n}} |\check{h}_n(v - z\sqrt{n})|\,dv\bigg)^2\bigg]$$

$$\le \rho + \frac{C}{\varepsilon^2\sqrt{n}} \sum_{\substack{|z|>2A \\ z\in\mathbb{Z}_n^\delta}} \Bigg[\int\limits_{-(A+z)\sqrt{n}}^{(A-z)\sqrt{n}} \big(h^2(u)+|h(u)|\big)\,\mathbb{I}_{\{|u|>3A\sqrt{n}\}}\,du$$

$$+\sqrt{n}\bigg(\int\limits_{-(A+z)\sqrt{n}}^{(A-z)\sqrt{n}} |h(u)|\,\mathbb{I}_{\{|u|>3A\sqrt{n}\}}\,du\bigg)^2\Bigg].$$

Here the important feature is that the integrals in the right-hand side of this inequality are equal to zero for $-2A\le z\le 2A$, so they are not present in the sum.

Since in the interval $(kA,(k+1)A)$, $k\in\mathbb{Z}$, there are $[A\sqrt{n}/\delta]$ points of the lattice $\mathbb{Z}_n^\delta$, we have the estimate

$$\sum_{\substack{z\le -2A \\ z\in\mathbb{Z}_n^\delta}} \int\limits_{-(A+z)\sqrt{n}}^{(A-z)\sqrt{n}} h^2(u)\,du \le \frac{2A\sqrt{n}}{\delta}\sum_{k=1}^{\infty} \int\limits_{Ak\sqrt{n}}^{A(k+1)\sqrt{n}} h^2(v)\,dv = \frac{2A\sqrt{n}}{\delta}\int\limits_{A\sqrt{n}}^{\infty} h^2(v)\,dv.$$

Then from this and a similar estimate for the sum over $z>2A$, $z\in\mathbb{Z}_n^\delta$ we get

$$\mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n^\delta} \check{H}_n(z)\ge\varepsilon\Big) \le \rho + \frac{CA}{\varepsilon^2\delta}\bigg\{ \int\limits_{|v|\ge A\sqrt{n}} \big(h^2(v)+|h(v)|\big)\,dv$$

$$+\frac{1}{A}\bigg(\int\limits_{|v|\ge A\sqrt{n}} \sqrt{|v|}\,|h(v)|\,dv\bigg)^2\bigg\} \le \rho + \frac{A}{\varepsilon^2\delta}o(1).$$

Proposition 7.3 is proved. □

We complete the proof of Theorem 7.1. As already mentioned, on the set $\Omega_n(\rho)$ the process $\hat{H}_n(t,x)$, $(t,x)\in[0,1]\times\mathbf{R}$, differs from zero only if $x\in[-4A,4A]$. Since by (6.16) $\mathbf{P}\big(\Omega_n^{\rm c}(\rho)\big)\le 2\rho$, we have

$$\lim_{n\to\infty}\mathbf{P}\Big(\sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |H_n(t,z)-h\ell(t,z)|>4\varepsilon\Big) \le 2\rho + \lim_{n\to\infty}\mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n^\delta}|\check{H}_n(z)|>\varepsilon\Big)$$

$$+\mathbf{P}\Big(\sup_{Q(\varrho)}|\ell(t,x)-\ell(s,y)|>\varepsilon\Big) + \lim_{n\to\infty}\mathbf{P}\Big(\sup_{Q(\varrho)}|\hat{H}_n(t,x)-\hat{H}_n(s,y)|>\varepsilon\Big)$$

$$+\lim_{n\to\infty}\mathbf{P}\Big(\sup_{(t_j,x_j)\in\Sigma}|\hat{H}_n(t_j,x_j)-h\ell(t_j,x_j)|>\varepsilon\Big)\le 6\rho, \tag{7.15}$$

where $\Sigma=\{t_j,x_j\}_{j=1}^N$ is a lattice on $[0,1]\times\{[-4A,4A]$ with array spacing not greater than ϱ. To derive (7.15) we used (7.13), (7.14), (6.18) and (7.6). In addition, we used the fact that (7.13), (7.14), and (7.3) imply

$$\hat{H}_n(t,z)\to h\ell(t,z)$$

in probability, and we can choose $n_1 = n_1(\varepsilon, \rho, \varrho)$ such that for all $n > n_1$

$$\mathbf{P}\Big(\sup_{(t_j,x_j)\in\Sigma} |\hat{H}_n(t_j,x_j) - h\ell(t_j,x_j)| > \varepsilon\Big) < \rho.$$

Since ε and ρ are arbitrary, from (7.15) it follows (7.2). Theorem 7.1 is proved. □

From Theorem 7.1 it is easy to deduce the analogue of Theorem 6.1 for the continuous random walk.

Let

$$\ell_n^{(\sigma)}(t,x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \mathrm{1\!I}_{[0,\sigma)}(\nu_{k-1}^n - x\sqrt{n}), \qquad \sigma \in \mathbf{R},\, x \in \mathbf{R},$$

where for $\sigma < 0$ we set $\mathrm{1\!I}_{[0,\sigma)}(v) := \mathrm{1\!I}_{(\sigma,0]}(v)$, $v \in \mathbf{R}$.

Theorem 7.2. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\ell_n^{(\sigma)}(t,x) - |\sigma|\ell(t,x)| > \varepsilon\Big) = 0. \tag{7.16}$$

Proof. Let $0 < \delta < 1$ be an arbitrary number. We consider $\mathbb{Z}_n^\delta = \{j\delta/\sqrt{n}\}_{j\in\mathbb{Z}}$.

From Theorem 7.1 it follows that for any $\varepsilon > 0$ and σ

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |\ell_n^{(\sigma)}(t,z) - |\sigma|\ell(t,z)| > \varepsilon\Big) = 0. \tag{7.17}$$

Let us prove that (7.17) implies (7.16). Indeed, for $z \in \mathbb{Z}_n^\delta$

$$\sup_{z\le x<z+\frac{\delta}{\sqrt{n}}} |\ell_n^{(\sigma)}(t,x) - \ell_n^{(\sigma)}(t,z)| \le \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \Big(\mathrm{1\!I}_{[\sigma,\sigma+\delta)}(\nu_{k-1}^n - z\sqrt{n}) + \mathrm{1\!I}_{[0,\delta)}(\nu_{k-1}^n - z\sqrt{n})\Big)$$

$$\le \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \mathrm{1\!I}_{[0,2\delta)}(\nu_{k-1}^n - (z+z_n)\sqrt{n}) + \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \mathrm{1\!I}_{[0,\delta)}(\nu_{k-1}^n - z\sqrt{n}),$$

where $z_n \in \mathbb{Z}_n^\delta$ is the nearest point to the left of $\sigma/\sqrt{n}$, i.e., the point such that $z_n\sqrt{n} \le \sigma < z_n\sqrt{n} + \delta$. Therefore, for $\delta < \sigma/2$,

$$\sup_{z\in\mathbb{Z}_n^\delta} \sup_{z\le x<z+\frac{\delta}{\sqrt{n}}} |\ell_n^{(\sigma)}(t,x) - \ell_n^{(\sigma)}(t,z)| \le 2 \sup_{z\in\mathbb{Z}_n^\delta} \ell_n^{(2\delta)}(t,z)$$

$$\le 4\delta \sup_{z\in\mathbb{Z}_n^\delta} \ell(t,z) + 2 \sup_{z\in\mathbb{Z}_n^\delta} |\ell_n^{(2\delta)}(t,z) - 2\delta\ell(t,z)|.$$

Consequently,

$$\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\ell_n^{(\sigma)}(t,x) - |\sigma|\ell(t,x)| \le \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |\ell_n^{(\sigma)}(t,z) - |\sigma|\ell(t,z)|$$

$$+4\,\delta \sup_{z\in\mathbb{Z}_n^\delta} \ell(1,z) + 2 \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |\ell_n^{(2\delta)}(t,z) - 2\delta\ell(t,z)|$$

$$+ |\sigma| \sup_{t\in[0,1]} \sup_{|x-z|\le\frac{\delta}{\sqrt{n}}} |\ell(t,x) - \ell(t,z)|. \tag{7.18}$$

According to (7.17), the first and the third terms on the right-hand side of (7.18) tend to zero in probability. By (11.1) Ch. V, the fourth term tends to zero a.s. According to the estimate (5.27) Ch. V, for any $\varepsilon > 0$

$$\mathbf{P}\Big(4\delta \sup_{z\in\mathbf{R}} \ell(1,z) > \frac{\varepsilon}{2}\Big) \le \frac{L\varepsilon^2}{64\,\delta^2} \exp\Big(-\frac{\varepsilon^2}{128\,\delta^2}\Big).$$

The right-hand side of this inequality can be made arbitrarily small by suitable choosing δ. Thus (7.16) follows from (7.17). □

We now derive the analogue of Theorem 6.2 for the continuous random walk. We set

$$q_n(t,x) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} f(\nu_{k-1}^n - x\sqrt{n}, \nu_k^n - x\sqrt{n}), \qquad (t,x)\in[0,T]\times\mathbf{R}.$$

In this case we impose an additional restrictions on the function $f(y,z)$. Suppose that there exists nonnegative functions $C(y,z)$, $D_j(y,z)$, $(y,z)\in\mathbf{R}^2$, and numbers α_j, β_j, $j=1,2,\dots,r$, such that

$$\sup_{-\delta<v<\delta} |f(y+v,z+v) - f(y,z)| \le C(y,z)\delta$$

$$+ \sum_{j=1}^{r} D_j(y,z)\big(\mathbb{1}_{(\alpha_j-\delta,\alpha_j+\delta)}(y) + \mathbb{1}_{(\beta_j-\delta,\beta_j+\delta)}(z)\big) \tag{7.19}$$

for all $(y,z)\in\mathbf{R}^2$ and all sufficiently small $\delta > 0$.

Condition (7.19) imposes a restriction that the function f has discontinuities only along lines parallel to the coordinate axes. All the examples given in §1 satisfy this condition.

Theorem 7.3. *Let* (7.19) *holds. Assume that*

$$\int_{-\infty}^{\infty} \big(1+\sqrt{|v|}\big)\,\mathbf{E}|f(v,v+\xi_1)|\,dv < \infty, \tag{7.20}$$

$$\int_{-\infty}^{\infty} \mathbf{E} f^2(v,v+\xi_1)\,dv < \infty, \tag{7.21}$$

$$\mathbf{P}\Big(\sup_{v\in\mathbf{R}} |f(v,v+\xi_1)| > L\Big) \to 0 \qquad \textit{as } L\to\infty. \tag{7.22}$$

Moreover, suppose that for all $(y,z)\in\mathbf{R}^2$,

$$|\mathbf{E}f(y,y+\xi_1)|\le Q,\qquad C(y,z)\le Q,\qquad D_j(y,z)\le Q,\quad j=1,\dots,r,\tag{7.23}$$

and

$$\int_{-\infty}^{\infty}\bigl(1+\sqrt{|v|}\bigr)\,\mathbf{E}C(v,v+\xi_1)\,dv<\infty,\tag{7.24}$$

$$\int_{-\infty}^{\infty}\mathbf{E}C^2(v,v+\xi_1)\,dv<\infty,\tag{7.25}$$

where Q *is a constant.*

Then

$$\lim_{n\to\infty}\mathbf{P}\Bigl(\sup_{t\in[0,1]}|W_n(t)-W(t)|>\varepsilon\Bigr)=0,\tag{7.26}$$

$$\lim_{n\to\infty}\mathbf{P}\Bigl(\sup_{(t,x)\in[0,1]\times\mathbf{R}}|q_n(t,x)-h\ell(t,x)|>\varepsilon\Bigr)=0\tag{7.27}$$

for any $\varepsilon>0$.

Remark 7.1. Condition (7.22) is necessary for (7.27) to hold. Indeed, $q_n(1/n,x)=\frac{1}{\sqrt{n}}f(-x\sqrt{n},\xi_1^n-x\sqrt{n})$ and

$$\mathbf{P}\Bigl(\sup_{v\in\mathbf{R}}|f(v,v+\xi_1)|>2\varepsilon\sqrt{n}\Bigr)\le\mathbf{P}\Bigl(\sup_{x\in\mathbf{R}}|\ell(1/n,x)|>\varepsilon/h\Bigr)$$

$$+\mathbf{P}\Bigl(\sup_{x\in\mathbf{R}}|q_n(1/n,x)-h\ell(1/n,x)|>\varepsilon\Bigr).$$

In view of (7.27) and the continuity of the process $\ell(t,x)$, $t\ge0$, $x\in\mathbf{R}$, the right-hand side of this inequality tends to zero as $n\to\infty$.

Proof of Theorem 7.3. The relation (7.26) was proved in § 3 for an arbitrary random walk for which the second moment exists. We prove (7.27). We can assume without loss of generality that $r=1$, $\alpha_1=\beta_1=0$ in condition (7.19). The condition (7.19) enables us to reduce the proof of (7.27) to the case when the supremum with respect to x in (7.27) is taken only over the lattice $\mathbb{Z}_n^\delta=\{j\delta/\sqrt{n}\}_{j\in\mathbb{Z}}$ with the array spacing $\delta/\sqrt{n}$. To prove (7.27) it suffices to establish that for any $\varepsilon>0$, $\rho>0$, and for all sufficiently small $\delta=\delta(\varepsilon,\rho)$

$$\limsup_{n\to\infty}\mathbf{P}\Bigl(\sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta}|q_n(t,z)-h\ell(t,z)|>\varepsilon\Bigr)\le\rho.\tag{7.28}$$

Let us prove this. Set

$$C_n(m,z):=\delta\sum_{k=1}^{m}C(\nu_{k-1}^n-z\sqrt{n},\nu_k^n-z\sqrt{n}),\qquad q_n^{(C)}(t,z):=\frac{1}{\sqrt{n}}C_n([nt],z).$$

We assume for convenience that the integral in (7.24) is estimated by the same constant Q as in (7.23). By (7.23) and (7.24),

$$h_C := \delta \int_{-\infty}^{\infty} \mathbf{E}C(v, v+\xi_1)\, dv \le \delta Q,$$

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{m} D_1(\nu_{k-1}^n - z\sqrt{n}, \nu_k^n - z\sqrt{n})\big(\mathbb{I}_{(-\delta,\delta)}(\nu_{k-1}^n - z\sqrt{n}) + \mathbb{I}_{(-\delta,\delta)}(\nu_k^n - z\sqrt{n})\big)$$

$$\le 2Q\Big(\ell_n^{(\delta)}\big(\frac{m}{n}, z\big) + \ell_n^{(-\delta)}\big(\frac{m}{n}, z\big) + \frac{1}{\sqrt{n}}\Big).$$

From (7.19) it follows that for any $t \in [0,1]$

$$\sup_{z\in\mathbb{Z}_n^\delta} \sup_{z\le x<z+\frac{\delta}{\sqrt{n}}} |q_n(t,x) - q_n(t,z)|$$

$$\le \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \sup_{z\in\mathbb{Z}_n^\delta} \sup_{0\le v\le\delta} \big|f(\nu_{k-1}^n - (z\sqrt{n}+v), \nu_k^n - (z\sqrt{n}+v)) - f(\nu_{k-1}^n - z\sqrt{n}, \nu_k^n - z\sqrt{n})\big|$$

$$\le \sup_{z\in\mathbb{Z}_n^\delta} q_n^{(C)}(t,z) + 2Q \sup_{z\in\mathbb{Z}_n^\delta} \ell_n^{(\delta)}(t,z) + 2Q \sup_{z\in\mathbb{Z}_n^\delta} \ell_n^{(-\delta)}(t,z) + \frac{2Q}{\sqrt{n}}. \tag{7.29}$$

We set $\bar\delta := h_C + 4Q\delta$. Then

$$\sup_{(t,x)\in[0,1]\times\mathbf{R}} |q_n(t,x) - h\ell(t,x)| \le \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |q_n(t,z) - h\ell(t,z)| + \bar\delta \sup_{z\in\mathbb{Z}_n^\delta} \ell(1,z)$$

$$+ \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |q_n^{(C)}(t,z) - h_C\ell(t,z)| + 2Q \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |\ell_n^{(\delta)}(t,z) - \delta\ell(t,z)|$$

$$+ \frac{2Q}{\sqrt{n}} + 2Q \sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |\ell_n^{(-\delta)}(t,z) - \delta\ell(t,z)|$$

$$+ h \sup_{t\in[0,1]} \sup_{|x-z|\le\frac{\delta}{\sqrt{n}}} |\ell(t,x) - \ell(t,z)|. \tag{7.30}$$

The function $C(y,z)$ satisfies (7.20)–(7.22). Therefore, if we have (7.28), the first and the third terms on the right-hand side of (7.30) tend to zero in probability. By (7.16), the fourth and the sixth terms converge to zero. In view of (11.1) of Ch. V, the 7th term tends to zero a.s. According to (5.27) of Ch. V, for any $\varepsilon > 0$ and $\bar\delta > 0$

$$\mathbf{P}\Big(\bar\delta \sup_{z\in\mathbf{R}} \ell(1,z) > \frac{\varepsilon}{2}\Big) \le \frac{L\varepsilon^2}{4\,\bar\delta^2} \exp\Big(-\frac{\varepsilon^2}{8\,\bar\delta^2}\Big). \tag{7.31}$$

Since $\bar\delta \le 5Q\delta$, the right-hand side of this inequality can be made as small as necessary by choosing δ. Thus for the validity of (7.27) it suffices to prove (7.28).

Let us pass to the proof of (7.28). Set

$$r_n(y,z) := \frac{1}{\sqrt{n}}(f(y,z)-h(y)), \qquad R_n(t,z) := \sum_{k=2}^{[nt]} r_n(\nu_{k-1}^n - z\sqrt{n}, \nu_k^n - z\sqrt{n}),$$

$$H_n(t,z) := \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} h(\nu_{k-1}^n - z\sqrt{n}).$$

In the definition of the process R_n we excluded the first term, since by the conditions (7.22) and the boundedness of $h(y)$, $y \in \mathbf{R}$, (see (7.23)) it does not play a significant role. The necessity of such an exclusion was discussed in the Subsection 2 of § 5.

For the process $H_n(t,z)$, $(t,z) \in [0,1] \times \mathbb{Z}_n^\delta$, we can apply Theorem 7.1, i.e., the relation (7.2) holds. Since

$$q_n(t,z) = H_n(t,z) + R_n(t,z) + \frac{1}{\sqrt{n}} f(-z\sqrt{n}, \xi_1^n - z\sqrt{n}) - \frac{1}{\sqrt{n}} h(-z\sqrt{n}),$$

to establish (7.28) it suffices to prove the following statement.

Lemma 7.3. *For any $\varepsilon > 0$ and $0 < \delta < 1$*

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{(t,z)\in[0,1]\times\mathbb{Z}_n^\delta} |R_n(t,z)| > \varepsilon \Big) = 0. \tag{7.32}$$

Proof. This lemma is proved analogously to Lemma 6.3, so we point out only significant differences.

We first consider the supremum of the random process $R_n(t,z)$ over the set $\{0 \le t \le 1\} \times \{\{|z| > 2A\} \bigcap \mathbb{Z}_n^\delta\}$, where for an arbitrary $\rho > 0$ the value $A = A(\rho)$ is defined by (6.16).

Proposition 7.4. *For any $\varepsilon > 0$ and $0 < \delta < 1$*

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{\{|z|>2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} |R_n(t,x)| > \varepsilon \Big) = 0. \tag{7.33}$$

Proof. Applying (5.19) for $\beta = 1$ and $\varkappa = 0$, we get the following analog of (6.41):

$$\mathbf{P}\Big(\sup_{\{|z|>2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} |R_n(t,x)| > \varepsilon \Big)$$
$$\le \rho + \frac{1}{\varepsilon^2} \sum_{|l|>2A\sqrt{n}/\delta} \frac{1}{2\pi n} \int_{-\infty}^{\infty} \mathfrak{Z}_l(u)\varphi(u)\frac{1-\varphi^{n-1}(u)}{1-\varphi(u)}\, du,$$

where

$$\mathfrak{Z}_l(u) := \int_{-A\sqrt{n}}^{A\sqrt{n}} e^{-iuv}(h^{(2)}(v-l\delta) - h^2(v-l\delta))\, dv.$$

Substituting the estimate

$$\sum_{|l|>2A\sqrt{n}/\delta} |\mathfrak{Z}_l(u)| \le \frac{2A\sqrt{n}}{\delta} \int\limits_{|v|\ge A\sqrt{n}} \mathbf{E} f^2(v, v+\xi_1)\,dv$$

in the previous inequality and using (5.21), (5.23), we obtain

$$\mathbf{P}\Big(\sup_{\{|x|>2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} |R_n(t,x)| > \varepsilon\Big) \le \rho + \frac{4A}{\pi\delta\varepsilon^2} \int\limits_{|v|\ge A\sqrt{n}} \mathbf{E} f^2(v, v+\xi_1)\,dv.$$

Hence, in view of the arbitrariness of ρ and the condition (7.21), this implies (7.33). □

We define the processes $\hat{R}_n(t,x)$ and $\check{R}_n(t,x)$ analogously to how it was done in § 6. The summation is carried out from $k = 2$. We consider the analog of Proposition 6.7.

Proposition 7.5. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{\{|z|\le 2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} \big|\check{R}_n(t,z)\big| > \varepsilon\Big) = 0. \tag{7.34}$$

We provide only the final estimate, which implies (7.34), because the main aspects of the proof were described in the proof of Proposition 7.4. The estimate has the form

$$\mathbf{P}\Big(\sup_{\{|x|\le 2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} \big|\check{R}_n(t,x)\big| > \varepsilon\Big) \le \frac{4A}{\pi\delta\varepsilon^2} \int_{-\infty}^{\infty} \mathbf{E}\big\{f^2(v, v+\xi_1)\mathbb{1}_{\{|f(v,v+\xi_1)|>n^{1/4}\}}\big\}dv.$$

The analog of Proposition 6.8 is the following assertion.

Proposition 7.6. *For any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{\{|x|\le 2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} \big|\hat{R}_n(t,x)\big| > \varepsilon\Big) = 0. \tag{7.35}$$

Proof. Using Doob's inequality for martingales (see (5.8), $p = 4$, Ch. I), we get

$$\mathbf{P}\Big(\sup_{\{|z|\le 2A\}\bigcap \mathbb{Z}_n^\delta} \sup_{0\le t\le 1} \big|\hat{R}_n(t,z)\big| > \varepsilon\Big) \le \varepsilon^{-4} \sum_{|\varkappa|\le 2A\sqrt{n}/\delta} \mathbf{E}\hat{R}_n^4(1, \varkappa\delta/\sqrt{n}). \tag{7.36}$$

To estimate the fourth moment of $\hat{R}_n(1, \varkappa\delta/\sqrt{n})$ we use the relations (5.19)–(5.30) and the notations introduced there. Set $\zeta_n(k,v) := \hat{r}_n(v, v+\xi_k)$. In this case we have a special situation, due to the absence of the first term. Therefore,

$$\mathbf{E}\hat{R}_n^q(m/n, \varkappa\delta/\sqrt{n}) = \widetilde{Z}_n^{(q)}(m, \varkappa\delta), \qquad \varkappa \in \mathbb{Z},$$

where $\widetilde{Z}_n^{(q)}$ is defined by (5.9) with the summation from $k=2$. Then from (5.28) and (5.29) we have

$$\widetilde{Z}_n^{(4)}(m,\varkappa\delta) \le \sum_{j=1}^{4} \frac{4!}{j!(4-j)!}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=2}^{m}\bigl|\mathfrak{Z}_n^{j,4-j}(m-k,u)\bigr||\varphi(u)|^{k-1}\,du. \qquad (7.37)$$

The goal now is to estimate the variables $\bigl|\mathfrak{Z}_n^{j,4-j}(m-k,u)\bigr|$, $j=1,2,3,4$. To this end we are forced to use the variables $Z_n^{(r)}$ instead of the variables $\widetilde{Z}_n^{(r)}$, $r=0,1,2,3$, because in this case the first term in the sum is significant.

Analogously to the corresponding part of the proof of Proposition 6.8, we get that $Z_n^{(1)}(m,\varkappa\delta)=0$. Then $\mathfrak{z}_n^{j,1}\equiv 0$, $\mathfrak{Z}_n^{j,1}\equiv 0$, $j=1,2,3,4$.

For $j\ge 2$

$$\bigl|\mathfrak{Z}_n^{j,0}(m-k,u)\bigr| \le \frac{16}{n^{j/2}}\int_{-\infty}^{\infty}\mathbf{E}\bigl\{|f(v,v+\xi_1)|^j\,\mathbb{I}_{\{|f(v,v+\xi_1)|\le n^{1/4}\}}\,dv \le \frac{C}{n^{j/4}\sqrt{n}}. \qquad (7.38)$$

For $m\le n$ we have

$$Z_n^{(2)}(m,\varkappa\delta) = \frac{1}{n}(\hat{h}_n^{(2)}(-\varkappa\delta)-\hat{h}_n^2(-\varkappa\delta)) + \frac{1}{n}\sum_{k=2}^{m}\mathbf{E}(\hat{h}_n^{(2)}(\nu_{k-1}-\varkappa\delta)-\hat{h}_n^2(\nu_{k-1}-\varkappa\delta)).$$

Then, using (5.19), (5.21) and (5.23), we get

$$Z_n^{(2)}(m,\varkappa\delta) \le \frac{1}{\sqrt{n}} + \frac{1}{2\pi n}\int_{-\infty}^{\infty}\mathbf{E}f^2(v,v+\xi_1)\,dv\int_{-\infty}^{\infty}|\varphi(u)|\Bigl|\frac{1-\varphi^{m-1}(u)}{1-\varphi(u)}\Bigr|du \le \frac{C}{\sqrt{n}}.$$

Therefore, the estimate (6.48) is valid. In addition, the estimate (6.49) holds, where

$$\alpha_n(\theta) := \int_{-\infty}^{\infty}\mathbf{E}\bigl\{|f(v,v+\xi_1)|\mathbb{I}_{\{\theta\sqrt{n}<|\xi_1|\}}\bigr\}\,dv \to 0.$$

Using the equalities $\mathfrak{z}_n^{1,1}\equiv 0$, $\mathfrak{Z}_n^{1,1}\equiv 0$, (5.30) with $r=2$, the estimate (7.38) for $j=2$, (6.49), and (5.22)–(5.24), we get

$$\bigl|\mathfrak{Z}_n^{1,2}(m-k,u)\bigr| \le \int_{-\infty}^{\infty}\mathbf{E}\bigl|\zeta_n(1,v)\mathfrak{z}_n^{2,0}(m-k-1,\xi_1+v)\bigr|dv$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}|G_n(u,s)|\sum_{l=2}^{m-k}\bigl|\mathfrak{Z}_n^{2,0}(m-k-l,s)\bigr||\varphi(s)|^{l-1}\,ds \le \frac{2}{nn^{1/4}}\int_{-\infty}^{\infty}\mathbf{E}f^2(v,v+\xi_1)\,dv$$

$$+\frac{C}{n^{3/2}}\int_{-\infty}^{\infty}((1\wedge|s|\theta\sqrt{n})+\alpha_n(\theta))|\varphi(s)|\frac{1-|\varphi^{n-1}(s)|}{1-|\varphi(s)|}ds \le \frac{C}{n}\bigl(\sqrt{\theta}+\alpha_n(\theta)\bigr).$$

Substituting this in (5.30), $r = 3$, we obtain

$$\big|\mathfrak{Z}_n^{1,3}(m-k,u)\big| \le \int_{-\infty}^{\infty} \mathbf{E}\big|\zeta_n(1,v)\big(3\mathfrak{z}_n^{1,2}(m-k-1,\xi_1+v)+\mathfrak{z}_n^{3,0}(m-k-1,\xi_1+v)\big)\big|dv$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_n(u,s)| \sum_{l=2}^{m-k} \big(3\big|\mathfrak{Z}_n^{1,2}(m-k-l,s)\big| + \big|\mathfrak{Z}_n^{3,0}(m-k-l,s)\big|\big)|\varphi^{l-1}(s)|\,ds$$

$$\le \frac{C}{nn^{1/4}} + \frac{C}{n\sqrt{n}} \int_{-\infty}^{\infty} \big((1\wedge|s|\theta\sqrt{n})+\alpha_n(\theta)\big)\big(\alpha_n(\theta)+\sqrt{\theta}\big)|\varphi(s)|\big(n\wedge\big(1+\tfrac{1}{s^2}\big)\big)ds$$

$$\le \frac{C}{n}\big(\alpha_n(\theta)+\sqrt{\theta}\big)^2.$$

We now have all estimates needed for the right-hand side of the inequality (7.37) and we can estimate $\widetilde{R}_n^4(n,\varkappa\delta)$. We have

$$\widetilde{Z}_n^{(4)}(n,\varkappa\delta) \le \frac{C}{n}\big(\alpha_n(\theta)+\sqrt{\theta}\big)\int_{-\infty}^{\infty} |\varphi(u)|\big(n\wedge\big(1+\tfrac{1}{u^2}\big)\big)\,du \le \frac{C}{\sqrt{n}}\big(\alpha_n(\theta)+\sqrt{\theta}\big).$$

Substituting this estimate in the right-hand side of (7.36), we get

$$\mathbf{P}\bigg(\sup_{\{|z|\le 2A\}\bigcap \mathbb{Z}_n^{\delta}} \sup_{0\le t\le 1} \big|\hat{R}_n(t,x)\big| > \varepsilon\bigg) \le \frac{4AC}{\varepsilon^4\delta}\big(\alpha_n(\theta)+\sqrt{\theta}\big).$$

Proposition 7.6 and therefore Lemma 7.3 are proved. □

This, in turn, completes the proof of Theorem 7.3. □

§ 8. Strong invariance principle for local times

Under the name of a *strong invariance principle for local times* we combine results that do not simply establish weak convergence of processes to the Brownian local time, but also give an explicit form for the order of the rate of convergence. In this section, we derive an a.s. estimate of the rate of convergence of the difference $q_n(t,x) - h\ell(t,x)$ to zero under the condition that the 8th moment of the random walk ν_k exists.

Theorem 8.1. *Assume that the condition* (D) *holds,* $\mathbf{E}|\xi_1|^8 < \infty$,

$$\sum_{v=-\infty}^{\infty} \mathbf{E}f^2(v,v+\xi_1) < \infty \tag{8.1}$$

and

$$\sum_{v=-\infty}^{\infty} \mathbf{E}\big[\big(|v|+|\xi_1|\big)|f(v,v+\xi_1)|\big] < \infty. \tag{8.2}$$

Suppose that

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{1\le k\le n} \sup_{v} |f(v, v+\xi_k)| > n^{1/4}\Big) < \infty. \tag{8.3}$$

Then a.s.

$$\limsup_{n\to\infty} \frac{n^{1/4}}{\ln n} \sup_{0\le t\le 1} |W_n(t) - W(t)| < \infty, \tag{8.4}$$

$$\limsup_{n\to\infty} \frac{n^{1/4}}{\ln n} \sup_{(t,x)\in[0,1]\times\mathbf{R}} |q_n(t,x) - h\ell(t,x)| < \infty. \tag{8.5}$$

Remark 8.1. The condition

$$\mathbf{E} \sup_{v} f^8(v, v+\xi_1) < \infty$$

suffices for (8.3).

Indeed, the left-hand side of (8.3) can be estimated as follows:

$$\sum_{n=1}^{\infty} n\mathbf{P}\big(\sup_{v} f^4(v, v+\xi_1) \ge n\big)$$

$$\le \sum_{n=1}^{\infty} \mathbf{E}\Big\{ \sup_{v} f^4(v, v+\xi_1); \sup_{v} f^4(v, v+\xi_1) \ge n\Big\} \le \mathbf{E} \sup_{v} f^8(v, v+\xi_1). \tag{8.6}$$

Remark 8.2. In view of the expression for the exact modulus of continuity of the Brownian local time (Theorem 11.1 Ch. V) the oscillation of the function $\ell(t,x)$ has order $n^{-1/4}\sqrt{\ln n}$ for x varying in an interval of length $\frac{1}{\sqrt{n}}$. Since the function $q_n(t,x)$ is not varying for $x \in \Big[\frac{k}{\sqrt{n}}, \frac{k+1}{\sqrt{n}}\Big)$, $k\in\mathbb{Z}$, the rate of convergence determined by (8.5) is optimal up to the factor $\sqrt{\ln n}$.

Theorem 8.2. *Assume that the conditions* (C), (7.19), (7.23)–(7.25), *and* (8.3) *are satisfied. Suppose that* $\mathbf{E}|\xi_1|^8 < \infty$, *the function* $\mathbf{E} f^2(y, y+\xi_1)$ *is bounded,*

$$\int_{-\infty}^{\infty} \mathbf{E} f^2(v, v+\xi_1)\, dv < \infty, \tag{8.7}$$

and

$$\int_{-\infty}^{\infty} \mathbf{E}\big\{\big(1+|v|+|\xi_1|\big)|f(v, v+\xi_1)|\big\} dv < \infty. \tag{8.8}$$

Then the relations (8.4) *and* (8.5) *hold* a.s.

Our main goal is to prove (8.5). The relation (8.4) was proved in § 3 (formula (3.8)).

Remark 8.3. In both cases (D) and (C) we derive (8.5) with the help of the first part of the Borel–Cantelli lemma (see § 1 Ch. I) from the following relation: for a nonrandom constant $K > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |q_n(t,x) - h\ell(t,x)| > Kn^{-1/4}\ln n \Big) < \infty. \tag{8.9}$$

Remark 8.4. It follows from (8.9) that for some constant $K > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{1\le k\le n} \sup_{v\in\mathbf{R}} |f(v, v+\xi_k)| > Kn^{1/4}\ln n \Big) < \infty,$$

i.e., the condition (8.3) is close to unimprovable one.

Indeed,

$$\frac{1}{\sqrt{n}} \sup_{1\le k\le n} \sup_{v\in\mathbf{R}} |f(v, v+\xi_k^n)| \le \sup_{1\le k\le n} \sup_{x\in\mathbf{R}} \Big| q_n\Big(\frac{k}{n}, x\Big) - h\ell\Big(\frac{k}{n}, x\Big) \Big|$$
$$+ \sup_{1\le k\le n} \sup_{x\in\mathbf{R}} \Big| q_n\Big(\frac{k-1}{n}, x\Big) - h\ell\Big(\frac{k-1}{n}, x\Big) \Big| + \sup_{1\le k\le n} \sup_{x\in\mathbf{R}} h\Big| \ell\Big(\frac{k}{n}, x\Big) - \ell\Big(\frac{k-1}{n}, x\Big) \Big|.$$

Now we can use (8.9) and Theorem 10.1 of Ch. V.

Let $\widetilde{q}_n(t,x)$, $(t,x) \in [0,1]\times\mathbf{R}$, be the process defined in Theorem 6.3. Then it follows from the scaling property described before Theorem 6.3 and from (8.9) that the following assertion is true.

Proposition 8.1. *Under the conditions of Theorem* 8.1

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\widetilde{q}_n(t,x) - h\ell(nt, x\sqrt{n})| > Kn^{1/4}\ln n \Big) < \infty \tag{8.10}$$

for some nonrandom constant $K > 0$.

By the first part of the Borel–Cantelli lemma (see § 1 Ch. I), this result implies the next assertion.

Proposition 8.2. *Under the conditions of Theorem* 8.1

$$\limsup_{n\to\infty} \frac{1}{n^{1/4}\ln n} \sup_{(t,x)\in[0,1]\times\mathbf{R}} |\widetilde{q}_n(t,x) - h\ell(nt, x\sqrt{n})| < \infty \qquad \text{a.s.} \tag{8.11}$$

Proof of Theorem 8.1. By the construction of the random walk ν_k^n (see § 2), $W_n(t) = W(\tau_n(t))$, where $\tau_n(t) = \sum_{l=1}^{[nt]} H_n^{(l)}$.

Set

$$\zeta_n(k,v) := n^{1/4}\big\{ n^{-1/2} f(v, v+\xi_k^n)\mathbb{1}_{\{|f(v,v+\xi_k^n)|\le n^{1/4}\}} - h\ell^{(k)}(\widetilde{H}_n^{(k)}, -v/\sqrt{n}) \big\},$$

$$V_n(t,x) = \sum_{k=1}^{[nt]} \zeta_n(k, \nu_{k-1}^n - [x\sqrt{n}]),$$

where the moments $\widetilde{H}_n^{(k)}$ and the variables $\ell^{(k)}(\widetilde{H}_n^{(k)}, -v/\sqrt{n})$ were defined at the beginning of the proof of Theorem 6.1. Here let $\theta = n^{-1/4}$.

Lemma 8.1. *For any* $(t,x)\in[0,1]\times\mathbf{R}$ *and some* $\lambda>0$

$$\mathbf{E}\exp(\lambda|V_n(t,x)|)\le 2. \tag{8.12}$$

Proof. In order to prove (8.12), it suffices to prove that for all $0\le t\le 1$ and $p=1,2,\dots$

$$|\mathbf{E}V_n^p(t,x)|\le L^p p! \tag{8.13}$$

for some $L>0$. The inequality (8.12) is derived from (8.13) in the same way as (3.14) was obtained from (3.15). We prove (8.13) by induction with respect to p. For this we use the notations of § 5 and the relations (5.9)–(5.17). Let $m=[nt]$, $\varkappa=[x\sqrt{n}]$.

Further we assume for simplicity that $\mathbf{P}(\xi_1=0)=0$, i.e., $p_0=1$.

Proposition 8.3. *For any* $u,y\in\mathbf{R}$

$$|G_n(u,y)|\le\frac{B}{n^{1/4}}\Big((1\wedge|y|)+(1\wedge|u|)+\frac{1}{n^{1/4}}\Big), \tag{8.14}$$

where B *is a constant.*

Proof. Since, in view of (8.1),

$$\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{|f(v,v+\xi_1)|\mathbb{1}_{\{|f(v,v+\xi_1)|>n^{1/4}\}}\big\}\le\frac{C}{n^{1/4}},$$

by analogy with (6.12) and (6.13) ($\theta=n^{-1/4}$) we obtain

$$n^{1/4}G_n(0,0)=\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{f(v,v+\xi_1)\mathbb{1}_{\{|f(v,v+\xi_1)|\le n^{1/4}\}}\big\}$$

$$-\frac{4h}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{\mu+\eta}{\bar\mu+\bar\eta}\Big(\bar\mu\sum_{v=0}^{\bar\eta}(\bar\eta-v)+\bar\eta\sum_{v=-\mu}^{-1}(\bar\mu+v)\Big)\mathbb{1}_{\{\mu>0,\eta>0\}}\Big\}=h+O(n^{-1/4})$$

$$-\frac{4h}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\bar\mu\bar\eta\mathbb{1}_{\{\mu>0,\eta>0\}}\big\}=O(n^{-1/4})+O\big(\mathbf{E}\{\xi_1^2;|\xi_1|>n^{1/4}\}\big)=O(n^{-1/4}).$$

Analogously to (6.14) we get

$$n^{1/4}|G_n(u,y)-G_n(0,0)|\le\sum_{v=-\infty}^{\infty}\mathbf{E}\Big\{\big|e^{iv(yu)+iy\xi_1}-1\big|\big|f(v,v+\xi_1)\big|\mathbb{1}_{\{|f(v,v+\xi_1)|\le n^{1/4}\}}\Big\}$$

$$+\frac{4}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{\frac{2\bar\mu\bar\eta}{\bar\mu+\bar\eta}\mathbb{1}_{\{\eta>0,\mu>0\}}\sum_{v=-\bar\mu}^{\bar\eta}\Delta_v\Big\},$$

where for $-\bar\mu\le v\le\bar\eta$ the estimate

$$|\Delta(v)|\le 2(\eta-v)\big((1\wedge(|vu|+|y|(\mu+v)))\big)+2(\mu+v)\big((1\wedge(|vu|+|y|(\eta-v)))\big)$$

$$\le (\mu+\eta)\big((1\wedge(|vu|+|y|(\mu+\eta)))\big) \le (\mu+\eta)^2\big((1\wedge(|u|+|y|)\big)$$

holds. Then, using the inequality $\big|e^{ix}-1\big| \le 2(1\wedge|x|)$, we have

$$n^{1/4}|G_n(u,y)-G_n(0,0)| \le 2\bigg(C\wedge\sum_{v=-\infty}^{\infty}\mathbf{E}\big\{\big(|v|(|y|+|u|)+|y||\xi_1|\big)|f(v,v+\xi_1)|\bigg)$$

$$+\big((1\wedge(|u|+|y|)\big)\frac{8}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{\bar\mu\bar\eta(\mu+\eta)^2\big\} \le C\big\{(1\wedge|y|)+(1\wedge|u|)\big\}.$$

The estimates for $|G_n(0,0)|$ and $|G_n(u,y)-G_n(0,0)|$ yield (8.14). □

We start to prove (8.13) by induction on p. According to our notation, $\mathbf{E}V_n^p(t,x) = Z_n^{(p)}(m,\varkappa)$. The induction hypothesis is

$$|\mathfrak{Z}_n^{1,q-2}(k,z)| \le BL^{q-2}(q-2)!\frac{1}{n^{1/4}}\Big((1\wedge|z|)+\frac{1}{n^{1/4}}\Big), \qquad q\ge 2, \tag{8.15}$$

$$|Z_n^{(p)}(m,\varkappa)| \le L^p p!, \qquad 1\le m\le n, \qquad \varkappa\in\mathbb{Z}, \qquad p\le q-1, \tag{8.16}$$

where B is the constant from (8.14) and L is some constant, $L>B$. We must prove (8.15) for $q+1$ instead of q and (8.16) for $p=q$. Note that $\mathfrak{Z}_n^{1,0}(k,z)=G_n(z,0)$. Moreover, in view of (5.3) with $\beta=1$ and (5.4)–(5.7), we have

$$|Z_n^{(1)}(m,\varkappa)| = \frac{1}{2\pi}\bigg|\int_{-\pi}^{\pi} e^{-iz\varkappa}G_n(z,0)\frac{1-\varphi^m(z)}{1-\varphi(z)}\,dz\bigg|$$

$$\le \frac{C}{n^{1/4}}\int_{-\pi}^{\pi}((1\wedge|z|)+n^{-1/4})(m\wedge z^{-2})\,dz \le \frac{C}{n^{1/4}}\Big(9m^{1/4}+\frac{4}{n^{1/4}}\sqrt{m}\Big)\le L.$$

Therefore, when $q=2$ the induction hypothesis holds.

Obviously $\widetilde H_n^{(1)}=\overline H_n^{(1)}$. Then taking into account (2.2), (2.5) and (2.9), we get analogously to (6.9) that for $j\ge 2$

$$n^{j/2}\mathbf{E}\ell^j\big(\widetilde H_n^{(1)},\frac{v}{\sqrt n}\big)$$

$$=\frac{2^{j+1}j!}{\mathbf{E}|\xi_1|}\mathbf{E}\Big\{(\mu+\eta)\frac{(\bar\mu+v)^j(\bar\eta-v)^j}{(\bar\mu+\bar\eta)^j}\Big(\frac{\bar\mu}{\bar\mu+v}\mathbb{1}_{\{0\le v\le\bar\eta,\bar\mu>0\}}+\frac{\bar\eta}{\bar\eta-v}\mathbb{1}_{\{-\bar\mu\le v<0,\bar\eta>0\}}\Big)\Big\}$$

$$\le\frac{2^{j+1}j!}{\mathbf{E}|\xi_1|}\mathbf{E}\big\{(\mu+\eta)\{\bar\mu\bar\eta^{j-1}\mathbb{1}_{\{0\le v\le\bar\eta,\bar\mu>0\}}+\bar\eta\bar\mu^{j-1}\mathbb{1}_{-\bar\mu\le v<0,\bar\eta>0\}}\}\Big\}.$$

Consequently,

$$\sum_{v=-\infty}^{\infty}\mathbf{E}\ell^j\big(\widetilde H_n^{(1)},-\frac{v}{\sqrt n}\big)\le\frac{2^{j+1}j!}{\mathbf{E}|\xi_1|n^{j/2}}\mathbf{E}\Big\{(\mu+\eta)(\bar\mu\bar\eta^j+\bar\eta\bar\mu^j)\mathbb{1}_{\{\mu>0,\eta>0\}}\Big\}$$

$$\le\frac{2^{j+1}j!\mathbf{E}|\xi_1|^3}{n^{(j+2)/4}\mathbf{E}|\xi_1|}.$$

For $j \geq 2$, $r \leq q-1$,

$$|\mathfrak{Z}_n^{j,r}(k,z)| \leq \sum_{v=-\infty}^{\infty} \mathbf{E}|\zeta_n(k,v)|^j\, L^r r!$$

$$\leq \frac{C4^j j! L^r r!}{\sqrt{n}}\Big(\sum_{v=-\infty}^{\infty} \mathbf{E} f^2(v, v+\xi_1) + \mathbf{E}|\xi_1|^3\Big) \leq \frac{L^r\, B^j\, r! j!}{\sqrt{n}},$$

where we can suppose that B is the constant figuring in (8.14).

Applying (5.17) and using the estimates obtained above for the variables $\mathfrak{Z}_n^{j,q-1-j}$, $j = 1, \dots, q-1$, and (5.5)–(5.8), we get

$$|\mathfrak{Z}_n^{1,q-1}(k,z)| \leq L^{q-1}(q-1)! \sum_{j=1}^{q-1} \Big(\frac{B}{L}\Big)^j \frac{B}{2\pi\sqrt{n}} \int_{-\pi}^{\pi} \Big((1 \wedge |z|) + (1 \wedge |s|) + \frac{1}{n^{1/4}}\Big)$$

$$\times \Big((1 \wedge |s|) + \frac{1}{n^{1/4}}\Big) \frac{1 - |\varphi(s)|^k}{1 - |\varphi(s)|} ds \leq \frac{L^{q-1}(q-1)! B^2}{(L-B) n^{1/4}} \Big((1 \wedge |z|) + \frac{1}{n^{1/4}}\Big).$$

For $L \geq 2B$ this estimate takes the desired form (8.15). Substituting the estimates for $\mathfrak{Z}_n^{j,q-j}$, $j = 1, \dots, q$, in (5.16), we obtain

$$\big|Z_n^{(q)}(m, \varkappa)\big| \leq \sum_{j=1}^{q} q! L^{q-j} \frac{B^{j+1}}{2\pi n^{1/4}} \int_{-\pi}^{\pi} \Big((1 \wedge |z|) + \frac{1}{n^{1/4}}\Big) \frac{1 - |\varphi(z)|^m}{1 - |\varphi(z)|}\, dz$$

$$\leq B L^q q! \sum_{j=1}^{q} \Big(\frac{B}{L}\Big)^j \leq L^q q! \frac{B^2}{L-B}.$$

We choose L such that $L \geq B(1+B)$. Therefore, the estimate (8.16) holds for $p = q$. This completes the proof by induction and, consequently, proves (8.13). Lemma 8.1 is proved. □

We proceed to the proof of (8.9). Set

$$\widetilde{\Omega}_n := \Omega_n \bigcap \Big(\sup_{1 \leq k \leq n} \sup_v |f(v, v+\xi_k^n)| \leq n^{1/4}\Big).$$

Recall that $\Omega_n = \{\bar{\nu}_k^n = \nu_k^n, k = 1, 2, \dots, n\}$ (see § 3), where $\theta = n^{-1/4}$. We use the fact that $|\nu_k^n| \leq n^{5/4}$, $k = 1, 2, \dots, n$, on the set Ω_n and therefore,

$$\ell^{(k)}\big(\widetilde{H}_n^{(k)}, ([x\sqrt{n}] - \nu_{k-1}^n)/\sqrt{n}\big) = 0$$

on this set for $|[x\sqrt{n}]| \geq n^2$. Let $V_n(t,x)$ be the process defined at the beginning of the proof of Theorem 8.1. We set

$$\widetilde{V}_n(t,x) = n^{1/4}\big(q_n(t,x) - h\ell(\tau_n(t), [x\sqrt{n}]/\sqrt{n})\big),$$

where $\tau_n(t) := \sum_{l=1}^{[nt]} H_n^{(l)}$, $t \in [0,1]$. On the set $\widetilde{\Omega}_n$ the processes $\widetilde{V}_n$ and V_n coincide. Therefore, for any $K > 0$,

$$\mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\widetilde{V}_n(t,x)| > K\ln n\Big) \le \mathbf{P}(\widetilde{\Omega}_n^{\mathrm{c}}) + \mathbf{P}\Big(\sup_{|x|\le n^{3/2}} \sup_{0\le t\le 1} |V_n(t,x)| > K\ln n\Big)$$

$$+\mathbf{P}\Big(\sup_{|x|>n^{3/2}} \sum_{k=1}^{n} |f(\nu_{k-1}^n - [x\sqrt{n}], \nu_k^n - [x\sqrt{n}])| > Kn^{1/4}\ln n, \Omega_n\Big) =: \sum_{j=1}^{3} p_{j,n}. \tag{8.17}$$

It follows from (3.11) and (8.3) that

$$\sum_{n=1}^{\infty} p_{1,n} < \infty. \tag{8.18}$$

Applying (8.12), we get that for $K \ge 5/\lambda$

$$p_{2,n} \le \sum_{k=1}^{n} \sum_{|v|\le n^2} \mathbf{P}(|V_n(k/n, v/\sqrt{n})| > K\ln n) \le 4n^3 e^{-\lambda K\ln n} \le \frac{4}{n^2}. \tag{8.19}$$

From (8.2) we get

$$p_{3,n} \le \frac{1}{Kn^{1/4}\ln n} \sum_{k=1}^{n} \mathbf{E} \sup_{|v|\ge n^2/2} |f(v, v+\xi_k)|$$

$$\le \frac{2n}{n^2 Kn^{1/4}\ln n} \sum_{|v|\ge n^2/2} |v|\mathbf{E}|f(v, v+\xi_1)| \le \frac{C}{n^{5/4}\ln n}. \tag{8.20}$$

In view of (8.17)–(8.20) and the equality

$$n^{1/4}(q_n(t,x) - h\ell(t,x)) = \widetilde{V}_n(t,x) + h\big(\ell(\tau_n(t), x_n) - \ell(t, x_n)\big) + h\big(\ell(t,x_n) - \ell(t,x))\big),$$

where $x_n = [x\sqrt{n}]/\sqrt{n}$, we have that to prove (8.9) it suffices to verify that for some constant $K > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\ell(\tau_n(t), x) - \ell(t,x)| > Kn^{-1/4}\ln n\Big) < \infty, \tag{8.21}$$

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} \sup_{|x-y|\le 1/\sqrt{n}} |\ell(t,y) - \ell(t,x)| > Kn^{-1/4}\ln n\Big) < \infty. \tag{8.22}$$

Using the monotonicity of $\ell(t,x)$ with respect to t, applying (3.18), and the estimate (5.27) of Ch. V together with the Remark 5.3 of Ch. V, we get

$$\mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}} |\ell(\tau_n(t), x) - \ell(t,x)| > \frac{K\ln n}{n^{1/4}}\Big) \le \mathbf{P}(\Omega_n^{\mathrm{c}}) + \frac{2}{n^{1+\rho}}$$

$$+\mathbf{P}\Big(\sup_{(t,x)\in[0,1]\times\mathbf{R}}\Big|\ell\Big(t+\frac{K_1\ln n}{\sqrt{n}},x\Big)-\ell\Big(t-\frac{K_1\ln n}{\sqrt{n}},x\Big)\Big|>\frac{K\ln n}{n^{1/4}}\Big)$$

$$\le \mathbf{P}(\Omega_n^c)+\frac{2}{n^{1+\rho}}+\frac{\sqrt{n}}{K_1\ln n}\mathbf{P}\Big(\sup_{x\in\mathbf{R}}\ell\Big(\frac{3K_1\ln n}{\sqrt{n}},x\Big)>\frac{K\ln n}{n^{1/4}}\Big)$$

$$\le \mathbf{P}(\Omega_n^c)+\frac{2}{n^{1+\rho}}+\frac{LK^2\sqrt{n}}{3K_1^2}\exp\Big(-\frac{K^2\ln n}{6K_1}\Big).$$

For the validity of (8.21), it is sufficient to choose $K^2>9K_1$.

We prove (8.22). Set $\|\cdot\|:=\sup_{t\in[0,1]}|\cdot|$. Let $0<\Delta\le 1$, $r\in[0,\infty)$. Note that $W(0)=0$. It is not hard to derive from the estimate (8.4) of Ch. V that

$$\mathbf{P}(\|\ell(t,r+\Delta)-\ell(t,r)\|>h)\le 4\exp\Big(-\frac{r}{2}-\frac{3h}{8\sqrt{\Delta}}\Big) \qquad (8.23)$$

for $\sqrt{\Delta}\le h\le \frac{1}{\Delta}$. Now, using this estimate, we prove that

$$\mathbf{P}\Big(\sup_{r<x\le r+\Delta}\|\ell(t,x)-\ell(t,r)\|>h\Big)\le 8\exp\Big(-\frac{r}{2}-\frac{3\alpha h}{8\sqrt{\Delta}}\Big) \qquad (8.24)$$

for $B\sqrt{\Delta}\le h\le\frac{1}{\Delta}$ with some $\alpha\in(0,1)$ and $B\ge 1$.

Let m be an arbitrary positive integer. Since the process $\ell(t,x)$ is continuous, it suffices to prove the estimate (8.24) for the case when the supremum is taken over the set $[r,r+\Delta]\cap D_{2^m}$, where $D_{2^m}=\{j/2^m|j\in\mathbb{Z}\}$, $\Delta\in D_{2^m}$, $r\in D_{2^m}$, $\Delta\ge 1/2^m$. The proof is carried out by induction with respect to Δ. For $\Delta=1/2^m$ the estimate (8.24) follows from (8.23). Assume that (8.24) holds for all Δ' strictly less than Δ and prove (8.24). Set $\Delta_0:=[\Delta 2^{m-1}]/2^m$, $h_1:=\alpha h\sqrt{\Delta_0/\Delta}$, $h_2:=h(1-\alpha\sqrt{\Delta_0/\Delta})$, where α will be chosen later. Then

$$\mathbf{P}\Big(\sup_{\substack{r<x\le r+\Delta\\ x\in D_{2^m}}}\|\ell(t,x)-\ell(t,r)\|>h\Big)\le \mathbf{P}\Big(\sup_{\substack{r<x\le r+\Delta_0\\ x\in D_{2^m}}}\|\ell(t,x)-\ell(t,r)\|>h\Big)$$

$$+\mathbf{P}(\|\ell(t,r+\Delta_0)-\ell(t,r)\|>h_1)+\mathbf{P}\Big(\sup_{\substack{r+\Delta_0<x\le r+\Delta\\ x\in D_{2^m}}}\|\ell(t,x)-\ell(t,r+\Delta_0)\|>h_2\Big)$$

$$\le 4e^{-r/2}\Big\{2\exp\Big(-\frac{3\alpha h}{8\sqrt{\Delta_0}}\Big)+\exp\Big(-\frac{3\alpha h}{8\sqrt{\Delta}}\Big)+2\exp\Big(-\frac{\alpha h(1-\alpha\sqrt{\Delta_0/\Delta})}{8\sqrt{\Delta-\Delta_0}}\Big)\Big\}. \quad (8.25)$$

We must not only prove (8.24) but, using the induction hypothesis, also satisfy the inequalities $h_1\ge\sqrt{\Delta_0}$ and $h_2\ge B\sqrt{\Delta-\Delta_0}$. They will be valid if $1/\alpha\le B$ and $\sqrt{\Delta}-\alpha\sqrt{\Delta_0}\ge\sqrt{\Delta-\Delta_0}$. We choose α small enough, then choose B so large that in addition to these inequalities we have for all Δ and m the estimates

$$\exp\Big(-\frac{3\alpha B\sqrt{\Delta}}{8}\Big(\frac{1}{\sqrt{\Delta_0}}-\frac{1}{\sqrt{\Delta}}\Big)\Big)\le\frac{1}{4},$$

$$\exp\Big(-\frac{3\alpha B\sqrt{\Delta}}{8}\Big(\frac{1-\alpha\sqrt{\Delta_0/\Delta}}{\sqrt{\Delta-\Delta_0}}-\frac{1}{\sqrt{\Delta}}\Big)\Big)\le\frac14.$$

This is possible, because $1/3\le\Delta_0/\Delta\le 1/2$. Then (8.25) implies (8.24).

We now prove with the help of (8.24) that (8.22) holds for sufficiently large K. Using the symmetry property of the Brownian motion, we get

$$\mathbf{P}\Big(\sup_{|x-y|\le\frac{1}{\sqrt{n}}}\|\ell(t,y)-\ell(t,x)\|>\frac{K\ln n}{n^{1/4}}\Big)$$

$$\le 2\sum_{k=0}^{\infty}\mathbf{P}\Big(\sup_{\frac{k}{\sqrt{n}}<x\le\frac{k+1}{\sqrt{n}}}\|\ell(t,x)-\ell(t,k/\sqrt{n})\|>\frac{K\ln n}{3n^{1/4}}\Big)$$

$$\le 16\exp\Big(-\frac{\alpha K\ln n}{8}\Big)\sum_{k=0}^{\infty}e^{-k/(2\sqrt{n})}=\frac{16n^{-\alpha K/8}}{1-e^{-1/(2\sqrt{n})}}\le 64\sqrt{n}\,n^{-\alpha K/8}.$$

For $K>12/\alpha$ the right-hand side of this inequality is the element of a convergent series, which proves (8.22). Theorem 8.1 is proved. □

Proof of Theorem 8.2. We can assume without loss of generality that $r=1$, $\alpha_1=\beta_1=0$ in condition (7.19). Set $\delta:=\frac{1}{\sqrt{n}}$, $\mathbb{Z}_n:=\big\{\frac{j}{n}\big|j\in\mathbb{Z}\big\}$. As in the case of the weak invariance principle, the condition (7.19) gives us the opportunity to consider in (8.9) the supremum only over $x\in\mathbb{Z}_n$ instead of the supremum over all real values x.

From (7.19) and (7.23) it follows analogously to (7.29) that

$$\sup_{t\in[0,1]}\sup_{z\in\mathbb{Z}_n}\sup_{z\le x<z+1/n}|q_n(t,x)-q_n(t,z)|$$

$$\le\sup_{z\in\mathbb{Z}_n}C_n(z)+2Q\sup_{z\in\mathbb{Z}_n}\ell_n^{(1/\sqrt{n})}(1,z)+2Q\sup_{z\in\mathbb{Z}_n}\ell_n^{(-1/\sqrt{n})}(1,z)+\frac{2Q}{n},\tag{8.26}$$

where

$$C_n(z):=\frac1n\sum_{k=2}^{n}C(\nu_{k-1}^n-z\sqrt{n},\nu_k^n-z\sqrt{n}),$$

and the constant Q is taken from (7.23). Here, in view of (7.23), we estimated the term corresponding to $k=1$ by the value $\frac{Q}{n}$.

In view of (8.3), (8.22) and (8.26), it suffices to prove the following assertions.

Proposition 8.4. *For some nonrandom constant* $K>0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\Big(\sup_{(t,z)\in[0,1]\times\mathbb{Z}_n}|q_n(t,z)-h\ell(t,z)|>Kn^{-1/4}\ln n\Big)<\infty.\tag{8.27}$$

Proposition 8.5. *The following series of probabilities converges*:

$$\sum_{n=1}^{\infty}\mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n}C_n(z)>n^{-1/4}\ln n\Big)<\infty.\tag{8.28}$$

Proposition 8.6. *The following series of probabilities converges*:

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n} \ell_n^{(\pm 1/\sqrt{n})}(1,z) > n^{-1/4}\ln n\Big) < \infty. \tag{8.29}$$

Proof of Proposition 8.4. Since the supremum with respect to z is taken over a discrete lattice, the proof of (8.27) is analogous to that of (8.9) for integer random walk. The fact that here the lattice has the array spacing $1/n$ does not lead to essential changes in the proof. Therefore we mention only a few details.

Like in the integer random walk case one shows that the process $V_n(t,x)$ satisfies (8.12). In the definition of the process $V_n(t,x)$ we should replace $[x\sqrt{n}]$ by $x\sqrt{n}$. The estimate (8.13) is proved by induction. The difference in this case is only that some terms must be estimated separately. The induction hypothesis consists of the inequalities (8.15) and (8.16). The estimate (8.14) holds. We have $\mathfrak{Z}_n^{1,0}(k,z) = G_n(z,0)$ and, in view of (5.28), (5.29), and (5.22)–(5.24), we get

$$|Z_n^{(1)}(m,\varkappa)| = \Big|\mathfrak{z}_n^{1,0}(m-1,-\varkappa) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iz\varkappa}G_n(z,0)\varphi(z)\frac{1-\varphi^{m-1}(z)}{1-\varphi(z)}\,dz\Big|$$

$$\le Cn^{-1/4} + Cn^{-1/4}\int_{-\infty}^{\infty} ((1\wedge|z|)+n^{-1/4})(m\wedge(1+z^{-2}))|\varphi(z)|\,dz \le L.$$

Here for the first term we used the estimate (6.9). Therefore, when $q = 2$ the induction hypothesis holds.

Applying (5.30) and using the estimates for the variables $\mathfrak{Z}_n^{j,q-1-j}$, $j\ge 2$, analogous to those obtained in the proof of Lemma 8.1, we get

$$|\mathfrak{Z}_n^{1,q-1}(k,z)| \le L^{q-1}(q-1)!\sum_{j=1}^{q-1}\Big(\frac{B}{L}\Big)^j\Big(\frac{B}{n^{1/4}} + \frac{B}{2\pi\sqrt{n}}\int_{-\infty}^{\infty}\Big((1\wedge|z|)+(1\wedge|s|)+\frac{1}{n^{1/4}}\Big)$$

$$\times\Big((1\wedge|s|)+\frac{1}{n^{1/4}}\Big)|\varphi(s)|\frac{1-|\varphi(s)|^{k-1}}{1-|\varphi(s)|}\,ds\Big) \le \frac{L^{q-1}(q-1)!B^2}{(L-B)n^{1/4}}\Big((1\wedge|z|)+\frac{1}{n^{1/4}}\Big).$$

For $L\ge 2B$ this estimate takes the required form (8.15). Substituting into (5.29) the estimates for $\mathfrak{Z}_n^{j,q-j}, j = 1,\dots,q$, we get

$$\big|I_n^{j,q-j}(m,\varkappa)\big| \le L^{q-j}(q-j)!j!\frac{B^{j+1}}{2\pi n^{1/4}}\int_{-\infty}^{\infty}\Big((1\wedge|z|)+\frac{1}{n^{1/4}}\Big)|\varphi(z)|\frac{1-|\varphi(z)|^{m-1}}{1-|\varphi(z)|}\,dz$$

$$\le L^{q-j}(q-j)!j!B^{j+1}.$$

It is easy to derive the estimate

$$|\mathfrak{z}_n^{j,q-j}(m-1,-\varkappa)| \le B^j j! L^{q-j}(q-j)!.$$

We now choose L such that $2B\sum_{j=1}^{\infty}(B/L)^j \le 1$, i.e., $L \ge B(1+2B)$. Then from (5.28) we deduce (8.16) for $p = q$. Thus the estimate (8.13) holds and therefore the estimate (8.12) is proved.

Next we use the analogue of (8.17) in which the term $p_{1,n}$ has the same form, the supremum in $p_{2,n}$ is taken over the set $\{\{|z| \le n^3\}\bigcap\{z \in \mathbb{Z}_n\}\}$ instead of the set $\{|x| \le n^{3/2}\}$, and

$$p_{3,n} := \mathbf{P}\biggl(\sup_{\substack{|z|>n^3\\ z\in\mathbb{Z}_n}}\sum_{k=1}^{n}|f(\nu_{k-1}^n - z\sqrt{n}, \nu_k^n - z\sqrt{n})| > Kn^{1/4}\ln n, \Omega_n\biggr).$$

In view of (3.11) and condition (8.3), we have (8.18). When estimating $p_{2,n}$ the constant K must be chosen greater or equal than $7/\lambda$, because in this case the lattice array spacing equals $1/n$, in contrast to the array spacing $1/\sqrt{n}$ in the discrete case.

We estimate the probability $p_{3,n}$. The estimation of this probability for continuous random walks differs from that proposed in (8.20). The term corresponding to $k = 1$, is estimated with the help of the condition (8.3). Excluding this term and using (5.19) with $\beta = 1$, (5.21), and (5.23), we get

$$\widetilde{p}_{3,n} := \mathbf{P}\biggl(\sup_{\substack{|z|>n^3\\ z\in\mathbb{Z}_n}}\sum_{k=2}^{n}|f(\nu_{k-1} - z\sqrt{n}, \nu_k - z\sqrt{n})| > Kn^{1/4}\ln n, \Omega_n\biggr)$$

$$\le \frac{1}{Kn^{1/4}\ln n}\sum_{\substack{|z|>n^3\\ z\in\mathbb{Z}_n}}\mathbf{E}\sum_{k=2}^{n}\bigl(|f(\nu_{k-1} - z\sqrt{n}, \nu_k - z\sqrt{n})|\mathbb{1}_{\{|\nu_{k-1}|\le n^{5/4}\}}\bigr)$$

$$\le \frac{Cn^{1/4}}{K\ln n}\sum_{\substack{|z|>n^3\\ z\in\mathbb{Z}_n}}\int_{-n^{5/4}}^{n^{5/4}}\mathbf{E}|f(y - z\sqrt{n}, y - z\sqrt{n} + \xi_1)|\,dy$$

$$\le \frac{Cn^{1/4}}{K\ln n}\sum_{|j|>n^4}\int_{-n^{5/4}-j/\sqrt{n}}^{n^{5/4}-j/\sqrt{n}}\mathbf{E}|f(v, v + \xi_1)|\,dv.$$

Since $\mathbb{Z}_n$ is the lattice with the array spacing n^{-1}, there are less than $2n^{7/4}$ points of the form $z\sqrt{n}$ in the interval $(-n^{5/4}, n^{5/4})$. Therefore, using (8.8), we have

$$\widetilde{p}_{3,n} \le \frac{2Cn^{7/4}n^{1/4}}{K\ln n}\int_{|v|>n^{7/2}}\mathbf{E}|f(v, v + \xi_1)|\,dv$$

$$\le \frac{4Cn^2}{Kn^{7/2}\ln n}\int_{-\infty}^{\infty}(1 + |v|)\mathbf{E}|f(v, v + \xi_1)|\,dv \le \frac{4C}{n\sqrt{n}\ln n}. \qquad (8.30)$$

Proposition 8.4 is proved. □

Proof of Proposition 8.5. The following arguments are analogous to the derivation of the estimate (8.30). Using (5.19) with $\beta = 1$, (5.21), (5.23) and (7.24), we get

$$\mathbf{P}\Big(\sup_{\substack{|z|>n^5\\ z\in\mathbb{Z}_n}} C_n(z) > n^{-1/4}\ln n\Big) \le \mathbf{P}(\Omega_n^{\mathrm{c}})$$

$$+\frac{1}{n^{3/4}\ln n}\sum_{\substack{|z|>n^5\\ z\in\mathbb{Z}_n}}\mathbf{E}\sum_{k=2}^{n}\Big(C(\nu_{k-1}-z\sqrt{n},\nu_k-z\sqrt{n})\mathbb{1}_{\{|\nu_{k-1}|\le n^{5/4}\}}\Big)$$

$$\le \mathbf{P}(\Omega_n^{\mathrm{c}}) + \frac{C}{n^{1/4}\ln n}\sum_{\substack{|z|>n^5\\ z\in\mathbb{Z}_n}}\int_{-n^{5/4}}^{n^{5/4}}\mathbf{E}C(y-z\sqrt{n},y-z\sqrt{n}+\xi_1)\,dy$$

$$\le \mathbf{P}(\Omega_n^{\mathrm{c}}) + \frac{C}{n^{1/4}\ln n}\sum_{|j|>n^6}\int_{-n^{5/4}-j/\sqrt{n}}^{n^{5/4}-j/\sqrt{n}}\mathbf{E}C(v,v+\xi_1)\,dv \le \mathbf{P}(\Omega_n^{\mathrm{c}})$$

$$+\frac{2Cn^{7/4}}{n^{1/4}\ln n}\int_{|v|>n^{11/2}/2}\mathbf{E}C(v,v+\xi_1)\,dv \le \mathbf{P}(\Omega_n^{\mathrm{c}})$$

$$+\frac{2\sqrt{2}C}{n^{5/4}\ln n}\int_{-\infty}^{\infty}\big(1+\sqrt{|v|}\big)\mathbf{E}C(v,v+\xi_1)\,dv \le \mathbf{P}(\Omega_n^{\mathrm{c}}) + \frac{C}{n^{5/4}\ln n}.$$

Taking into account conditions (7.23), (7.25) and using relations (5.28), (5.29), we can prove by induction as in (7.8) that for $p = 1, 2, \dots$

$$\mathbf{E}\Big(\frac{1}{\sqrt{n}}\sum_{k=2}^{n}C(\nu_{k-1}-z\sqrt{n},\nu_k-z\sqrt{n})\Big)^p \le L^p p!.$$

This implies the estimate $\mathbf{E}C_n^{28}(z) \le \dfrac{C}{n^{14}}$. Then

$$\mathbf{P}\Big(\sup_{\substack{|z|\le n^5\\ z\in\mathbb{Z}_n}} C_n(z) > n^{-1/4}\ln n\Big) \le \sum_{\substack{|z|\le n^5\\ z\in\mathbb{Z}_n}}\frac{n^7}{\ln^{28}n}\mathbf{E}C_n^{28}(z) \le \frac{2Cn^6}{n^7\ln^{28}n} = \frac{2C}{n\ln^{28}n}.$$

The required result (8.28) follows from these estimates. □

Proof of Proposition 8.6. Since

$$\ell_n^{(1/\sqrt{n})}(1,z) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{1}_{[0,1/\sqrt{n})}\Big(\nu_{k-1}^n - z\sqrt{n}\Big) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{1}_{[0,1/n)}\Big(\frac{\nu_{k-1}^n}{\sqrt{n}} - z\Big),$$

and on the set Ω_n one has the estimates $|\nu_{k-1}^n| \le n^{5/4}$, $k = 1, 2, \dots, n$, we have

$$\mathbf{P}\Big(\sup_{z\in\mathbb{Z}_n} \ell_n^{(1/\sqrt{n})}(1,z) > \frac{\ln n}{n^{1/4}}, \Omega_n\Big) = \mathbf{P}\Bigg(\sup_{\substack{|z|<2n^{3/4}\\ z\in\mathbb{Z}_n}} \ell_n^{(1/\sqrt{n})}(1,z) > \frac{\ln n}{n^{1/4}}, \Omega_n\Bigg)$$

$$\le \sum_{|j|\le 2n^{7/4}} \mathbf{P}\Big(\ell_n^{(1/\sqrt{n})}(1, j/\sqrt{n}) > \frac{\ln n}{n^{1/4}}\Big). \tag{8.31}$$

We now prove that for any integer $p \ge 1$

$$\mathbf{E}(\ell_n^{(1/\sqrt{n})}(1,z))^p \le \frac{L^p p!}{n^{p/2}}. \tag{8.32}$$

We use (5.9) and (5.26)–(5.29). Set

$$\zeta_n(k,v) := \mathbb{I}_{[0,1/\sqrt{n})}(v), \qquad \varkappa = z\sqrt{n}.$$

Then

$$Z_n^{(p)}(m,\varkappa) = \mathbf{E}\Big(\sum_{k=1}^{n} \mathbb{I}_{[0,1/\sqrt{n})}\big(\nu_{k-1} - \varkappa\big)\Big)^p.$$

Applying (5.28) and (5.29), we prove by induction that for any integer $p \ge 1$

$$Z_n^{(p)}(m,\varkappa) \le L^p p!. \tag{8.33}$$

By (5.26),

$$G_n(u,y) = \int_{-\infty}^{\infty} e^{i(y-u)v}\mathbf{E}e^{iy\xi_1}\,\mathbb{I}_{[0,1/\sqrt{n})}(v)\,dv.$$

Therefore, $|G_n(u,y)| \le 1/\sqrt{n}$. In addition, $\mathfrak{Z}_n^{1,0}(k,z) = G_n(z,0)$ and, in view of (5.28), (5.29) and (5.22)–(5.24),

$$|Z_n^{(1)}(m,\varkappa)| = \Big|\mathfrak{z}_n^{1,0}(m-1,-\varkappa) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iz\varkappa}G_n(z,0)\varphi(z)\frac{1-\varphi^{m-1}(z)}{1-\varphi(z)}\,dz\Big|$$

$$\le \mathbb{I}_{[0,1/n)}(v) + \frac{1}{\sqrt{n}}\int_{-\infty}^{\infty} (m \wedge (1+z^{-2}))|\varphi(z)|\,dz \le C.$$

Consequently, for $p = 1$ and $L \ge C$ the induction hypothesis holds. We assume that (8.33) holds for all $p \le q-1$ and prove it for $p = q$.

By (8.33), with $j = 1, 2, \dots q$,

$$|\mathfrak{Z}_n^{j,q-j}(k,z)| = \Big|\int_{-\infty}^{\infty} e^{-iuv}\mathfrak{z}_n^{j,q-j}(k,z)\,dv\Big| \le \frac{C}{\sqrt{n}}L^{q-j}(q-j)!.$$

Therefore, by (5.29),

$$|I_n^{j,q-j}(m,\varkappa)| = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} e^{-iu\varkappa}\sum_{k=2}^{m} \mathfrak{Z}_n^{j,q-j}(m-k,u)\,\varphi^{k-1}(u)\,du\right| \le CL^{q-j}(q-j)!.$$

Substituting these estimates and estimates $|\mathfrak{z}_n^{j,q-j}(m-1,-\varkappa)| \le L^{q-j}(q-j)!$ into (5.28), we get that (8.33) holds for $p=q$ and $L \ge C+2$. Thus (8.33) is proved, and, consequently, so is (8.32). Applying (8.31) and (8.32) with $p=11$, we get

$$\mathbf{P}\Bigl(\sup_{\substack{|z|<2n^{3/4}\\ z\in\mathbb{Z}_n}} \ell_n^{(1/\sqrt{n})}(1,z) > n^{-1/4}\ln n\Bigr) \le \frac{L^{11}11!n^{11/4}}{\ln^{11/2} n}\,\frac{2n^{7/4}}{n^{11/2}} = \frac{2L^{11}11!}{n\ln^{11/2} n}.$$

Similar arguments work for the process $\ell_n^{(-1/\sqrt{n})}(1,z)$, $z\in\mathbb{Z}_n$. Proposition 8.6 is proved. This completes the proof of Theorem 8.2. □

§ 9. Applications of invariance principle

This section is devoted to various applications of the invariance principle for local times. The proofs of these results clearly illustrate how the concept of weak convergence of a process, introduced in § 1, is much more effective than the classical definition.

In practical applications of the concept of weak convergence of processes it should be kept in mind that in view of the Proposition 1.1 of Ch. I, the validity of (1.1) for any $\varepsilon > 0$ is equivalent to the following assertion: for any sequence of positive integers tending to infinity there exists a subsequence n_k such that

$$\lim_{n\to\infty}\sup_{s\in\Sigma}|X'_{n_k}(s) - X'_\infty(s)| = 0 \qquad \text{a.s.}$$

Throughout this section, $\nu_k = \sum_{l=1}^{k}\xi_l$, $k=0,1,2,\dots$, is an integer recurrent random walk with unit variance satisfying the condition (D). Let $\ell_n(t,x)$, $(t,x)\in[0,\infty)\times\mathbf{R}$, be the process defined in (6.1), $W(t)$, $t\ge 0$, be a Brownian motion and $\ell(t,x)$, $(t,x)\in[0,\infty)\times\mathbf{R}$, be its local time.

We consider the following examples.

1. Let $\sigma_n(k)$ and $f_n(l,k)$, $(l,k)\in\mathbb{Z}^2$, be sequences of functions defined at points with integer coordinates. Consider the processes

$$\eta_n(t) := \sum_{k=1}^{[nt]}\sigma_n(k)\,f_n(\nu_{k-1},\nu_k), \qquad t\in[0,1]. \tag{9.1}$$

Here we restrict ourselves to the case when the functions f_n depend only on two successive steps of the random walk. It is also possible to consider dependence on a greater number of steps, but this does not lead to any fundamental changes.

Set $g_n(l) := \mathbf{E}f_n(l,l+\xi_1)$, $g_n^{(2)}(l) := \mathbf{E}f_n^2(l,l+\xi_1)$,

$$G_n(x) := \sqrt{n}\,\mathrm{sign}\,x \sum_{(x-|x|)\sqrt{n}/2\le l<(x+|x|)\sqrt{n}/2} g_n(l).$$

Theorem 9.1. *Let*

$$\lim_{n\to\infty} \sup_{0\le s\le 1} |\sigma_n([ns]) - \sigma(s)| = 0, \tag{9.2}$$

where $\sigma(s)$, $s \in [0,1]$, *is a bounded function. Suppose that for every* $x \in \mathbf{R}$

$$G_n(x) \to G(x), \tag{9.3}$$

and that for any $A > 0$

$$\sup_n \sqrt{n} \sum_{|l|\le A\sqrt{n}} |g_n(l)| < \infty, \tag{9.4}$$

$$\lim_{n\to\infty} \sqrt{n} \sum_{|l|\le A\sqrt{n}} \Big(g_n^{(2)}(l) - g_n^2(l)\Big) = 0. \tag{9.5}$$

Then the processes $\eta_n(t)$, $t \in [0,1]$, *converge weakly as* $n \to \infty$ *to the process*

$$\eta(t) := \int_{-\infty}^{\infty} \int_0^t \sigma(s)\, \ell(ds, x)\, dG(x), \qquad t \in [0,1]. \tag{9.6}$$

Remark 9.1. From (9.4) it follows that the function G has bounded variation on any finite interval and the integral (9.6) is finite, because the Brownian local time $\ell(t,x)$ is continuous in (t,x) a.s. and for any $t > 0$ it is nonzero only on the set $\Big(\inf\limits_{0\le s\le t} W(s), \sup\limits_{0\le s\le t} W(s)\Big)$.

Proof of Theorem 9.1. For each n we take instead of the random walk ν_k the random walk ν_k^n, constructed from the Brownian motion W in §2. After this replacement the process $\eta_n(t)$ is denoted by $\eta_n'(t)$. This replacement corresponds to the reconstruction of processes, involved in the definition of weak convergence. By (3.4), for any $\rho > 0$ one can choose $A = A(\rho)$ such that

$$\mathbf{P}\Big(\sup_{0\le l\le n} |\nu_l^n| > A\sqrt{n}\Big) < \rho, \qquad\qquad \mathbf{P}\Big(\sup_{0\le s\le 1} |W(s)| > A\Big) < \rho. \tag{9.7}$$

Set

$$\chi_k := 1\!\!1_{\left\{\sup\limits_{0\le l\le k} |\nu_l^n| \le A\sqrt{n}\right\}}, \qquad \bar\eta_n(t) := \sum_{k=1}^{[nt]} \sigma_n(k) g_n(\nu_{k-1}^n),$$

$$s_n(l) := \sum_{k=1}^{l} \chi_{k-1}\sigma_n(k)\,(f_n(\nu_{k-1}^n, \nu_k^n) - g_n(\nu_{k-1}^n)), \qquad l = 1, 2, \dots, n.$$

The variables $s_n(l)$, $l = 1, 2, \dots, n$, form a martingale with respect to the family of σ-algebras $\mathcal{F}_l$ generated by the random walk ν_k^n up to the time l. Therefore, using Doob's inequality for martingales (see (5.8), $p = 2$, Ch. I), we get

$$\mathbf{P}\Big(\sup_{0\le l\le n} |s_n(l)| > \varepsilon\Big) \le \frac{1}{\varepsilon^2}\mathbf{E} s_n^2(n) \le \frac{C}{\varepsilon^2}\mathbf{E}\sum_{k=1}^{n} \chi_{k-1}\big(g_n^{(2)}(\nu_{k-1}) - g_n^2(\nu_{k-1})\big)$$

$$\le \frac{C}{\varepsilon^2} \sum_{j\le A\sqrt{n}} \big(g_n^{(2)}(j) - g_n^2(j)\big)\mathbf{E}\sum_{k=1}^{n} \mathbb{I}_{\{j\}}(\nu_{k-1})$$

$$\le \frac{C\sqrt{n}}{\varepsilon^2} \sum_{|j|\le A\sqrt{n}} (g_n^{(2)}(j) - g_n^2(j)) \underset{n\to\infty}{\longrightarrow} 0. \tag{9.8}$$

Here we used the estimate

$$\frac{1}{\sqrt{n}}\mathbf{E}\sum_{k=1}^{n} \mathbb{I}_{\{j\}}(\nu_{k-1}) \le C,$$

which follows from (5.3) with $\beta = 1$ and from the estimates (5.4), (5.6). By (9.5), (9.7), (9.8) and the estimate

$$\mathbf{P}\Big(\sup_{0\le t\le 1} |\eta_n'(t) - \bar\eta_n(t)| > \varepsilon\Big) \le \mathbf{P}\Big(\sup_{0\le l\le n} |\nu_l^n| > A\sqrt{n}\Big) + \mathbf{P}\Big(\sup_{0\le l\le n} |s_n(l)| > \varepsilon\Big)$$

it follows that to prove the theorem it suffices to establish the uniform convergence of $\bar\eta_n(t)$ to $\eta(t)$, $t \in [0,1]$, in probability.

The process $\bar\eta_n(t)$ is represented (see for the comparison (6.2)) in the form

$$\bar\eta_n(t) = \sum_{l=-\infty}^{\infty} \sqrt{n}\, g_n(l) \sum_{k=1}^{[nt]} \sigma_n(k)\frac{1}{\sqrt{n}}\mathbb{I}_{\{l\}}(\nu_{k-1}^n)$$

$$= \int_{-\infty}^{\infty} dG_n(x) \sum_{k=1}^{[nt]} \sigma_n(k)\frac{1}{\sqrt{n}}\mathbb{I}_{\{0\}}(\nu_{k-1}^n - [x\sqrt{n}]) = \int_{-\infty}^{\infty}\int_0^t \sigma_n([ns])\,\ell_n(ds,x)\,dG_n(x),$$

where $dG_n(x)$ is taken to be nonzero only at the jump points of the function $G_n(x)$ and is equal there to the size of the jump. We set

$$\Omega_{1,n} := \Big\{\sup_{0\le l\le n} |\nu_l^n| \le A\sqrt{n}\Big\} \bigcap \Big\{\sup_{0\le s\le 1} |W(s)| \le A\Big\},$$

$$U_n(t,x) := \int_0^t \sigma_n([ns])\,\ell_n(ds,x) - \int_0^t \sigma(s)\,\ell(ds,x).$$

By (9.7), $\mathbf{P}\big(\Omega_{1,n}^{c}\big) \le 2\rho$. On the set $\Omega_{1,n}$ for all $1 \le k \le n$ we have

$$\bar\eta_n(t) - \eta(t) = \int_{-A}^{A} (U_n(t,x) - U_k(t,x))\,dG_n(x)$$

$$+ \int_{-A}^{A}\int_0^t \sigma_k([ks])\,\ell_k(ds,x)(dG_n(x) - dG(x)) + \int_{-A}^{A} U_k(t,x)\,dG(x). \tag{9.9}$$

In view of (9.3), the second integral in (9.9) converges a.s. to zero as $n \to \infty$ uniformly in $t \in [0,1]$.

For $0 < m < n$ we use the estimate

$$\sup_{t\in[0,1]} |U_n(t,x)| \le \int_0^1 |\sigma_n([ns]) - \sigma_m([ms])|\, \ell_n(ds,x)$$

$$+ \sup_{t\in[0,1]} \left| \int_0^t \sigma_m([ms])\,(\ell_n(ds,x) - \ell(ds,x)) \right| + \int_0^1 |\sigma_m([ms]) - \sigma(s)|\, \ell(ds,x).$$

By (9.2), the function $\sigma_m([ms])$, $s \in [0,1]$, is uniformly bounded, therefore the second term on the right-hand side of this estimate does not exceed

$$K \sup_{0\le s\le 1} \bigl|\ell_n(s,x) - \ell(s,x)\bigr|,$$

where K is a constant. Now from (9.2) and (6.4) we deduce, by letting first n and then m tend to infinity, that

$$\sup_{(t,x)\in[0,1]\times\mathbf{R}} U_n(t,x) \to 0$$

in probability. This together with (9.4) implies that the first and third integrals in (9.9) tend to zero in probability as $n \to \infty$ and $k \to \infty$, uniformly in $t \in [0,1]$. $\square$

2. We consider the limit behavior of the process

$$\mu_n(s,t) := \sum_{k=1}^{[ns]} \sum_{l=1}^{[nt]} f_n(\nu_{k-1}, \nu_{l-1}), \qquad s,t \in [0,1].$$

Set

$$F_n(x,y) := n \operatorname{sign} x \sum_{(x-|x|)\sqrt{n}/2 \le p < (x+|x|)\sqrt{n}/2} \operatorname{sign} y \sum_{(y-|y|)\sqrt{n}/2 \le q < (y+|y|)\sqrt{n}/2} f_n(p,q).$$

Theorem 9.2. *Suppose that $F_n(x,y) \to F(x,y)$ for all $(x,y) \in \mathbf{R}^2$ and for any $A > 0$*

$$\sup_n n \sum_{|p|\le A\sqrt{n}} \sum_{|q|\le A\sqrt{n}} |f_n(p,q)| < \infty. \tag{9.10}$$

Then the processes $\mu_n(s,t)$, $(s,t) \in [0,1]^2$, converge weakly as $n \to \infty$ to the process

$$\mu(s,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ell(s,x)\ell(t,y)\, F(dx,dy), \qquad (s,t) \in [0,1]^2.$$

Remark 9.2. It follows from (9.10) that F has bounded variation on any rectangle in $\mathbf{R}^2$ and hence the process $\mu(s,t)$, $(s,t) \in [0,1]^2$, is well defined.

To prove Theorem 9.2 it suffices only to observe that after the replacement of the random walk ν_k by the random walk ν_k^n in the definition of the process $\mu_n(s,t)$ we can denote it by $\mu_n'(s,t)$ and get the following representation:

$$\mu_n'(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ell_n(s,x)\ell_n(t,x)\, F_n(dx,dy).$$

The subsequent arguments are analogous to the corresponding arguments in the proof of Theorem 9.1.

3. We consider the number $L(l,m,r)$ of pairs (i,j) for which $\nu_i - \nu_j = r$ for $i<l$ and $j<m$. The quantity $L(m,m,0)$ can be interpreted as twice the number of self-crossings of the random walk ν_k up to the time m. As is not hard to show, the two-parameter process $V(s,t) := W(s) - W(t)$, $(s,t) \in [0,\infty)^2$, has a local time, i.e., the following limit exists a.s.:

$$l(s,t,z) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^s du \int_0^t dv\, \mathbb{1}_{[z,z+\varepsilon)}(V(u,v)), \qquad (s,t,z) \in [0,\infty)^2 \times \mathbf{R}.$$

Using (1.2) of Ch. V, it is not hard to see that

$$l(s,t,z) = \int_0^t \ell(s, z+W(v))\, dv = \int_{-\infty}^{\infty} \ell(s,z+y)\ell(t,y)\, dy.$$

Theorem 9.3. *The processes* $n^{-3/2}L([ns],[nt],[z\sqrt{n}])$ *converge weakly to the process* $l(s,t,z)$, $(s,t,z) \in [0,1]^2 \times \mathbf{R}$.

Using the definition of the process $\ell_n(t,x)$ (see § 6), we have

$$n^{-3/2}L'([ns],[nt],[z\sqrt{n}]) := \frac{1}{n^{3/2}} \sum_{i=1}^{[ns]}\sum_{j=1}^{[nt]} \mathbb{1}_{\{[z\sqrt{n}]\}}(\nu_{i-1}^n - \nu_{j-1}^n)$$

$$= \sum_{l=-\infty}^{\infty} \frac{1}{n^{3/2}} \sum_{i=1}^{[ns]} \mathbb{1}_{\{[z\sqrt{n}]+l\}}(\nu_{i-1}^n) \sum_{j=1}^{[nt]} \mathbb{1}_{\{l\}}(\nu_{j-1}^n) = \int_{-\infty}^{\infty} \ell_n\Big(s, \frac{[z\sqrt{n}]+[y\sqrt{n}]}{\sqrt{n}}\Big)\ell_n(t,y)\, dy.$$

Now Theorem 9.3 is an obvious consequence of this representation and (6.4), (9.7).

Remark 9.3. For any fixed z the weak convergence of $n^{-3/4}L([ns],[nt],[z\sqrt{n}])$ as a processes of the variables $(s,t) \in [0,1]^2$ follows from Theorem 9.2 with $f_n(m,q) = \frac{1}{n^{3/2}}\mathbb{1}_{\{r\}}(m-q)$, $r = [z\sqrt{n}]$.

4. We consider the limit behavior of sums of independent random variables defined on the random walk ν_k, $k = 0,\dots,n$. Let $\{X_j\}_{j=-\infty}^{\infty}$ be a sequence of independent random variables and let

$$S_n := \sum_{k=1}^{n} X_{\nu_{k-1}}.$$

Assume that the variables X_j, $j \in \mathbb{Z}$, do not depend on the random walk ν_k and are identically distributed with zero mean and unit variance.

We define the processes $B_n(t)$, $t \in [0,1]$, by

$$B_n(t) := n^{-3/4}\, S_{[nt]}.$$

Let $\overline{W}(x)$ and $\overline{W}(-x)$, $x \ge 0$, be independent Brownian motions with zero initial value. Moreover, we assume that the processes $\overline{W}(x)$, $x \in \mathbf{R}$, and $\ell(t,z)$, $(t,z) \in [0,1] \times \mathbf{R}$, are independent.

Theorem 9.4. *The processes* $B_n(t)$, $t \in [0,1]$, *converge weakly as* $n \to \infty$ *to the process*

$$B(t) := \int_{-\infty}^{\infty} \ell(t,x)\, d\overline{W}(x), \qquad t \in [0,1]. \tag{9.11}$$

Remark 9.4. The stochastic integral in (9.11) is actually taken over the random interval $\Big(\inf_{s\in[0,t]} W(s), \sup_{s\in[0,t]} W(s)\Big)$, which is the support of the local time $\ell(t,x)$, $x \in \mathbf{R}$, of the Brownian motion W.

Proof of Theorem 9.4. Set

$$\overline{W}_n(x) := n^{-1/4} \sum_{j=0}^{[x\sqrt{n}]} X_j, \quad x \ge 0, \qquad \overline{W}_n(x) := -n^{-1/4} \sum_{j=[x\sqrt{n}]+1}^{-1} X_j, \quad x < 0.$$

According to the invariance principle for random walks (1.2), one can construct from the Brownian motions $\overline{W}(x)$, $x \ge 0$ and $\overline{W}(x)$, $x \le 0$, (the Skorohod embedding scheme) the processes $\overline{W}'_n(x)$, $n = 1, 2, \dots$, such that for each n their finite-dimensional distributions coincide with those of the process $\overline{W}_n(x)$, and for any $A > 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{x\in[-A,A]} |\overline{W}'_n(x) - \overline{W}(x)| > \varepsilon\Big) = 0. \tag{9.12}$$

Due to this construction, for each n the variables

$$X^n_j := n^{1/4}\Big(\overline{W}'_n\big(\tfrac{j}{\sqrt{n}}\big) - \overline{W}'_n\big(\tfrac{j-1}{\sqrt{n}}\big)\Big), \qquad j \in \mathbb{Z},$$

are independent and identically distributed with the variables X_j.

For each n we take instead of ν_k the random walk ν^n_k constructed from the Brownian motion process W in § 2. The Brownian motion W is independent of the process $\overline{W}$.

We represent the process $B'_n(t)$ constructed from ν^n_k and $\overline{W}'_n(x)$ as follows:

$$B'_n(t) := \frac{1}{n^{3/4}} \sum_{k=1}^{[nt]} X^n_{\nu^n_{k-1}} = \frac{1}{n^{3/4}} \sum_{j=-\infty}^{\infty} X^n_j \sum_{k=1}^{[nt]} \mathbb{1}_{\{0\}}(\nu^n_{k-1} - j)$$

$$= \frac{1}{n^{1/4}} \sum_{j=-\infty}^{\infty} X_j^n \, \ell_n\big(t, \tfrac{j}{\sqrt{n}}\big) = \int_{-\infty}^{\infty} \ell_n(t,x)\, d\overline{W}'_n(x).$$

The differential $d\overline{W}'_n(x)$ is taken to be nonzero only at the points of jumps of the process $\overline{W}'_n(x)$ and is equal to the size of the jump. The finite-dimensional distributions of the processes $B_n(t)$ and $B'_n(t)$, $t \in [0,1]$, coincide.

We first prove that

$$B'_n(t) \to B(t) \tag{9.13}$$

in probability for any $t \in [0,1]$.

By (3.4), for any $\rho > 0$ we can choose $A > 0$ such that (9.7) holds. Set as before

$$\Omega_{1,n} := \Big\{ \sup_{0 \le l \le n} |\nu_l^n| \le A\sqrt{n} \Big\} \bigcap \Big\{ \sup_{0 \le s \le 1} |W(s)| \le A \Big\},$$

In view of (9.7),

$$\mathbf{P}(\Omega_{1,n}^{c}) \le 2\rho. \tag{9.14}$$

On the set $\Omega_{1,n}$ we have

$$\{x : \ell_n(1,x) > 0\} \cup \{x : \ell(1,x) > 0\} \subset [-A, A] \tag{9.15}$$

and, consequently,

$$B'_n(t) - B(t) = \int_{-A}^{A} (\ell_n(t,x) - \ell_m(t,x))\, d\overline{W}'_n(x) + \int_{-A}^{A} \ell_m(t,x)\big(d\overline{W}'_n(x) - d\overline{W}(x)\big)$$

$$+ \int_{-A}^{A} (\ell_m(t,x) - \ell(t,x))\, d\overline{W}(x), \qquad 0 < m < n. \tag{9.16}$$

Letting first n and then m tend to ∞, we get that the second integral in (9.16) tends to zero by virtue of the convergence $\overline{W}'_n(x) \to \overline{W}(x)$ in probability, and the third tends to zero in view of (6.4) and (3.6) Ch. II. For the first integral in (9.16) we have

$$\mathbf{E}\Big| \int_{-A}^{A} (\ell_n(t,x) - \ell_m(t,x))\, d\overline{W}'_n(x) \Big|^2 \le \int_{-A}^{A} \mathbf{E}\big(\ell_n(t,x) - \ell_m(t, [x\sqrt{n}]/\sqrt{n})\big)^2 dx,$$

which in conjunction with (6.4), (6.31) implies that the first integral in (9.16) tends to zero in mean square. Now (9.13) is a consequence of (9.14) and the convergence to zero of the integrals in (9.16).

In view of (9.13), to establish the uniform convergence in probability of the processes $B'_n(t)$ to the process $B(t)$ it suffices to verify the weak compactness condition: for any $\varepsilon > 0$

$$\lim_{h \to 0} \limsup_{n \to \infty} \mathbf{P}\Big(\sup_{|t-s|<h} |B'_n(t) - B'_n(s)| > \varepsilon \Big) = 0. \tag{9.17}$$

Indeed, this condition implies that for any $\varepsilon > 0$ and $\rho > 0$ there exist $h = h(\varepsilon, \rho)$ and $n_0 = n_0(\varepsilon, \rho, h)$ such that for all $n > n_0$

$$\mathbf{P}\Big(\sup_{T(h)} |B'_n(t) - B'_n(s)| > \varepsilon\Big) < \rho,$$

where $T(h) := \{(s,t) : |t-s| \le h; s, t \in [0,1]\}$. In view of the uniform continuity of $B(t)$ in probability which, in turn, is the consequence (see (3.6) Ch. II) of (9.14) and the continuity of the Brownian local time $\ell(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}$, we have

$$\mathbf{P}\Big(\sup_{T(h)} |B(t) - B(s)| > \varepsilon\Big) < \rho$$

for all sufficiently small h. In $[0,1]$ we take the lattice $\Sigma = \{t_i\}_{i=0}^N$ with the array spacing $h = 1/N$, where N is large enough to guarantee the uniform closeness of $B'_n(t)$ and $B(t)$ on the lattice Σ. Using (9.13), we choose $n_1 = n_1(\varepsilon, \rho, h)$ such that for all $n > n_1$

$$\mathbf{P}\Big(\sup_{t_i \in \Sigma} |B'_n(t_i) - B(t_i)| > \varepsilon\Big) < \rho.$$

These estimates imply that for $n > n_0 \vee n_1$

$$\mathbf{P}\Big(\sup_{t \in [0,1]} |B'_n(t) - B(t)| > 3\varepsilon\Big) < 3\rho,$$

and this means that the processes $B_n(t)$, $t \in [0,1]$, converge weakly to the process $B(t)$, because ε and ρ are arbitrary (see (1.1)).

We now prove (9.17). Since

$$\int_{-\infty}^{\infty} \ell_n(t,x)\,dx = [nt]/n,$$

to prove (9.17) it suffices, in view of (9.14), to verify that

$$\lim_{h \to 0} \limsup_{n \to \infty} \mathbf{P}\Big(\sup_{|t-s|<h} \Big| \int_{-A}^{A} (\ell_n(t,x) - \ell_n(s,x))\,d\overline{W}'_n(x)\Big| > \varepsilon\Big) = 0.$$

It is not hard to get from Proposition 4.1 that for this it suffices to verify the following estimate: for all $\varepsilon > 0$, t, s, $|t-s| \ge 1/n$

$$\mathbf{P}\Big(\Big| \int_{-A}^{A} (\ell_n(t,x) - \ell_n(s,x))\,d\overline{W}'_n(x)\Big| > \varepsilon\Big) \le \frac{C}{\varepsilon^2}\,|t-s|^{3/2}. \tag{9.18}$$

We first prove the estimate

$$\mathbf{P}(\nu_n = 0) \le \frac{C}{\sqrt{n}}. \tag{9.19}$$

Since $\varphi(u) = 1 - u^2/2 + o(u^2)$ as $u \to 0$, in view of condition (D) (see § 5), we have $1 - |\varphi(u)| \geq \widetilde{\rho} u^2$, $u \in [-\pi, \pi]$, for some $\widetilde{\rho} > 0$. Then for some $\widetilde{\rho} \leq 1/\pi^2$,

$$0 \leq |\varphi(u)| \leq 1 - \widetilde{\rho} u^2 \leq e^{-\widetilde{\rho} u^2}, \qquad u \in [-\pi, \pi].$$

By (5.2),

$$\mathbf{P}(\nu_n = 0) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \bigl|\varphi^n(u)\bigr| \, du \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-n\widetilde{\rho} u^2} \, du = \frac{1}{2\pi\sqrt{n}} \int_{-\pi n}^{\pi n} e^{-\widetilde{\rho} v^2} \, dv.$$

This implies (9.19).

Now, using the Markov property of the random walk and (9.19), we obtain

$$\mathbf{E} \Bigl| \int_{-A}^{A} (\ell_n(t,x) - \ell_n(s,x)) \, d\overline{W}_n(x) \Bigr|^2 \leq \mathbf{E} \int_{-\infty}^{\infty} \ell_n^2 \Bigl(\frac{[nt] - [ns]}{n}, x \Bigr) dx$$

$$= \mathbf{E} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{n}} \Bigl(\frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-[ns]} \mathbb{I}_{\{\nu_j = k\}} \Bigr)^2 = \frac{1}{n^{3/2}} \mathbf{E} \sum_{k=-\infty}^{\infty} \sum_{i=0}^{[nt]-[ns]} \sum_{j=0}^{[nt]-[ns]} \mathbb{I}_{\{\nu_j = \nu_i\}} \mathbb{I}_{\{\nu_j = k\}}$$

$$= \frac{1}{n^{3/2}} \sum_{i=0}^{[nt]-[ns]} \sum_{j=0}^{[nt]-[ns]} \mathbf{P}\bigl(\nu_i - \nu_j = 0\bigr) \leq \frac{2C}{n^{3/2}} \sum_{i=0}^{[nt]-[ns]} \sum_{j=1}^{i} \frac{1}{\sqrt{j}} \leq C_1 |t-s|^{3/2}$$

for $|t - s| \geq 1/n$, and, hence (9.18) holds. □

5. The *range* R_n *of the random walk* ν_k at time n is defined to be the number of points of the random set $\{0, \nu_1, \dots, \nu_n\}$.

Let $R(t) = \sup_{0 \leq s \leq t} W(s) - \inf_{0 \leq s \leq t} W(s)$ be the *range of the Brownian motion* W at time t.

Theorem 9.5. *The processes* $\frac{1}{\sqrt{n}} R_{[nt]}$, $t \in [0,1]$, *converge weakly as* $n \to \infty$ *to the process* $R(t)$, $t \in [0,1]$.

Remark 9.5. From the invariance principle for random walks (3.4) it follows that the processes $\sup_{0 \leq s \leq t} W_n(s) - \inf_{0 \leq s \leq t} W_n(s)$, $t \in [0,1]$ converge weakly as $n \to \infty$ to the process $R(t)$, $t \in [0,1]$. Theorem 9.5 states that in the random interval $\Bigl(\inf_{0 \leq s \leq t} W_n(s), \sup_{0 \leq s \leq t} W_n(s) \Bigr)$ there are only $o(\sqrt{n})$ points not visited by the random walk.

Proof of Theorem 9.5. By (6.4), there is a sequence ε_n such that $\varepsilon_n \to 0$ and the probability of the set

$$\Omega_{2,n} := \Bigl\{ \sup_{(t,x) \in [0,1] \times \mathbf{R}} |\ell_n(t,x) - \ell(t,x)| \leq \varepsilon_n \Bigr\}$$

tends to 1. We set

$$\check{W}(t) := \sup_{0\le s\le t} W(s), \qquad \hat{W}(t) := \inf_{0\le s\le t} W(s),$$

$$\check{W}_n(t) := \sup_{0\le s\le t} W_n(s), \qquad \hat{W}_n(t) := \inf_{0\le s\le t} W(s),$$

$$\Delta_n(t) := \int_{-\infty}^{\infty} \mathbb{1}_{(0,\varepsilon_n]}(\ell(t,x))\,dx.$$

The quantity $\Delta_n(t)$ is the Lebesgue measure of the set of points x, in which the strictly positive local time at the moment t does not exceed ε_n. We prove that

$$\sup_{0<t\le 1} \Delta_n(t) \to 0, \qquad \text{for} \quad n\to\infty, \quad \text{a.s.} \tag{9.20}$$

Taking into account the continuity of the process ℓ and that a.s.

$$\{(t,x) : \ell(t,x) > 0, 0 < t \le 1\} = \{(t,x) : \hat{W}(t) < x < \check{W}(t), 0 < t \le 1\}, \tag{9.21}$$

we get that $\Delta_n(t) \to 0$ a.s. for every $t \in (0,1]$. For any $\rho > 0$ one can choose $h > 0$ and $n_0 = n_0(h)$ such that for every $0 \le l \le 1/h$

$$\check{W}((l+1)h) - \check{W}(lh) < \rho/4, \qquad \hat{W}(lh) - \hat{W}((l+1)h) < \rho/4,$$

$$\sup_{1\le l\le 1/h} \sup_{n>n_0} \Delta_n(lh) < \rho/2.$$

Since $\ell(lh,x) \le \ell(t,x)$ for $t \in [lh,(l+1)h)$, we have

$$\Delta_n(t) \le \Delta_n(lh) + \check{W}((l+1)h) - \check{W}(lh) + \hat{W}(lh) - \hat{W}((l+1)h) < \rho,$$

for $n > n_0$ that proves (9.20).

Let R'_n be the range of the random walk ν_k^n, $k = 1,\dots,n$, i.e., the number of points of the random set $\{0,\nu_1^n,\dots,\nu_n^n\}$. We use the relations

$$\frac{1}{\sqrt{n}} R'_{[nt]} = \int_{-\infty}^{\infty} \mathbb{1}_{(0,\infty)}(\ell_n(t,x))\,dx, \qquad R(t) = \int_{-\infty}^{\infty} \mathbb{1}_{(0,\infty)}(\ell(t,x))\,dx.$$

Then

$$\frac{1}{\sqrt{n}} R'_{[nt]} - R(t) = \int_{-\infty}^{\infty} \big(\mathbb{1}_{(0,\infty)}(\ell_n(t,x)) - \mathbb{1}_{(0,\infty)}(\ell(t,x))\big)\,dx.$$

Consequently,

$$\Big|\frac{1}{\sqrt{n}} R'_{[nt]} - R(t)\Big| \le \Delta_n(t) + \int_{-\infty}^{\infty} \mathbb{1}_{(\varepsilon_n,\infty)}(\ell(t,x))\big|\mathbb{1}_{(0,\infty)}(\ell_n(t,x)) - \mathbb{1}_{(0,\infty)}(\ell(t,x))\big|\,dx.$$

On the set $\Omega_{2,n}$ the second term on the right-hand side of this relation is equal to zero. Thus for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]}\Big|\frac{1}{\sqrt{n}}R'_{[nt]} - R(t)\Big| > \varepsilon\Big) \le \lim_{n\to\infty} \mathbf{P}(\Omega^{c}_{2,n}) + \lim_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,1]} \Delta_n(t) > \varepsilon\Big).$$

Since the probability of the complement of $\Omega_{2,n}$ tends to zero, (9.20) shows that Theorem 9.5 is valid. □

6. We consider an example of application of the strong invariance principle. Let

$$\widetilde{q}_n(t,x) := \sum_{k=1}^{[nt]} f\big(\nu^1_{k-1} - [x\sqrt{n}], \nu^1_k - [x\sqrt{n}]\big), \qquad (t,x) \in [0,1]\times \mathbf{R}.$$

The next result shows that the rate of growth of the maximum values of the processes $\widetilde{q}_n(1,x)$, $x \in \mathbf{R}$, is asymptotically bounded by the quantity $h\sqrt{2n\ln\ln n}$, where

$$h := \sum_{l=-\infty}^{\infty} \mathbf{E} f(l, l+\xi_1).$$

Theorem 9.6. *Under the conditions of Theorem* 8.1

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\ln\ln n}} \sup_{x\in\mathbf{R}} \widetilde{q}_n(1,x) = h \qquad \text{a.s.} \tag{9.22}$$

Proof. Indeed,

$$\frac{1}{\sqrt{2n\ln\ln n}} \sup_{x\in\mathbf{R}} \widetilde{q}_n(1,x) = \frac{1}{\sqrt{2n\ln\ln n}} \sup_{x\in\mathbf{R}} \big\{\widetilde{q}_n(1,x) - h\ell(n, x\sqrt{n}) + h\ell(n, x\sqrt{n})\big\}.$$

According to Proposition 8.2,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n\ln\ln n}} \sup_{x\in\mathbf{R}} \big|\widetilde{q}_n(1,x) - h\ell(n, x\sqrt{n})\big| = 0 \qquad \text{a.s.,}$$

and by Theorem 9.1 of Ch. V,

$$\limsup_{n\to\infty} \frac{\ell(n,0)}{\sqrt{2n\ln\ln n}} = \limsup_{n\to\infty} \frac{1}{\sqrt{2n\ln\ln n}} \sup_{x\in\mathbf{R}} \ell(n,x) = 1 \qquad \text{a.s.}$$

From these relations we get (9.22). □

HEAT TRANSFER PROBLEM

In 1827 the British botanist Robert Brown observed that a grain of pollen located on the surface of a liquid performs a random motion. This motion of a very light grain of pollen (a particle) is the result of collisions with molecules of the liquid. The collisions occur within a short period of time and are uniformly distributed in all directions. This leads to instantaneous changes in the direction of the motion. The particle wanders on the surface, executing rectilinear motions (steps) and changing its direction randomly.

From the mathematical point of view, we can describe this motion by a random walk on a plane. If we are interested in the description of only one of the coordinates of a two-dimensional random walk, then we deal with a one-dimensional random walk. A model of a simple one-dimensional random walk can be described as follows. Suppose that a particle moves along the vertical axis of the coordinate plane. The particle is moving abruptly up and down. Making a step in one of these directions, the particle has approximately equal chances to make the next step, either up or down, regardless of the directions of the previous steps. The values of these steps are sufficiently small, they are comparable with the intermolecular distances. For simplicity, one can consider them to be equal. Denote the step size by h. Each step is carried out almost instantly. This means that the particle performs the step h up or down during a time of second order of smallness compared with the step size. It appears that most models, accurately representing the motion of light particles in a liquid, are based on a random walk with steps h executed during a time of order h^2. For simplicity, we assume that this time is exactly equal to h^2.

As a result of the previous assumptions, we arrive at the following model of a simple random walk. Let the motion start from a point x and take a step of length h up or down with probability $1/2$ during the time h^2. Assume that each step is independent of all previous steps. This random walk is represented in the form $x + h\nu_k$, where $\nu_k = \sum_{l=1}^{k} \xi_l$, $\nu_0 = 0$, and ξ_l, $l = 1, 2, \dots$, are independent identically distributed random variables with $\mathbf{P}(\xi_1 = 1) = 1/2$, $\mathbf{P}(\xi_1 = -1) = 1/2$.

Denote by $W_h(s)$, $s \in [0, T]$, a random process given by the linear interpolation (random polygonal line) constructed from the points $(kh^2, x + h\nu_k)$, $k = 0, 1, 2, \dots$. This process is easier to depict graphically compared to the analogous step process $\widetilde{W}_h(s) := x + h\nu_{[s/h^2]}$, where $[u]$ is the largest integer not exceeding u.

The asymptotic behavior as $h \to 0$ of these processes is the same, because

$$\sup_{0 \le s \le T} |W_h(s) - \widetilde{W}_h(s)| \le h. \tag{1.1}$$

From the weak invariance principle for random walks (see § 3 Ch. VII) with $h^2 = 1/n$, $n = 1, 2, \dots$, it follows that the processes $W_h(s)$, $s \in [0, T]$, converge weakly as $h \to 0$ to a Brownian motion $W(s)$, $s \in [0, T]$, $W(0) = x$.

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8

This means that for a broad class of continuous functionals of $W_h(s)$, $s \in [0, T]$, the distributions of every functional of these processes converge to the corresponding distribution of the same functional of $W(s)$, $s \in [0, T]$.

Thus, when h decreases the process W_h becomes increasingly similar to the Brownian motion W, i.e., the continuous process with independent increments having the normal distribution,

$$\mathbf{P}\big(W(v) - W(u) \in [y, y + dy)\big) = \frac{e^{-y^2/2(v-u)}}{\sqrt{2\pi(v-u)}}\, dy. \tag{1.2}$$

Mathematics is a extremely formalized science. This enables us to carry out very rigorous proofs. However, in a formal approach it is difficult to clarify essential details of the problem under investigation and to see possible applications. We will focus on an informal, although quite rigorous, description of the heat transfer problem. For this we make certain assumptions. It is important that we represent the physical phenomena of the heat transfer problem by means of exchange of energy (via collisions) between molecules of a substance.

We have already considered the representation of the movement of a light particle in a liquid under the influence of collisions with molecules as a Brownian motion. An analogous interpretation can be given for heat transfer. We consider the physical problem of heat transfer in a thin homogeneous rod of infinite length and solve this problem by probabilistic methods. We place the rod vertically so that the vertical axis corresponds to the coordinate specifying the position of a molecule on the rod. The horizontal axis describes the time. We assume that the molecules are disposed in horizontal layers (levels) one above the other. Denote the distance between the layers by h.

We consider an arbitrary horizontal layer. The temperature of this layer is defined to be the total energy of its molecules. Since the rod is thin, one can neglect the differences in the energy between molecules in different parts of any selected layer. In this regard, we assume that at the initial moment the energy of molecules inside the layer is the same. Denote by $\Phi(y)$, where y is the coordinate of a layer, the initial value of the temperature at $t = 0$. Let $\Phi(y)$, $y \in \mathbf{R}$, be a continuous function. We assume also that there is no energy exchange between the rod and the external environment. The question is what will be the temperature at the time t on the layer x? Let each layer consists of m molecules, where m is sufficiently large. It is clear that m is comparable with $1/h^2$, because h characterizes the intermolecular distance in the rod of a fixed thickness. The energy of any molecule in the layer y at the initial moment is equal to $\frac{1}{m}\,\Phi(y)$. Assume that during the time h^2 each molecule of the layer y exchanges its energy by elastic collision with probability $1/2$ with the molecule located above it in the layer $y + h$ and with probability $1/2$ with the molecule below it in the layer $y - h$.

Assume for simplicity that along the same layer no energy transfer occurs. Let us consider a particular molecule in the layer y. We are interested in how the energy of this molecule is transferred along the rod by means of elastic collisions between molecules.

By the above assumptions, the selected value $\frac{1}{m}\,\Phi(y)$ of the energy of a particular molecule is transferred along the rod according to a random walk. One can even imagine that the molecule itself wanders and carries its energy. However, the molecules are rigidly attached to the lattice sites.

Our mathematical model of the heat transfer, involves two parameters, m and h. Since intermolecular distances are very small, in what follows we consider the limiting case as $m \to \infty$ and $h \to 0$. This case serves as a good enough approximation of the real heat transfer process. In our model we compute the temperature $u_{m,h}(t, x)$ of the layer with the coordinate x at the time t. We can assume that the number of molecules in the layer does not change in time, so at any moment there are m molecules in every layer. We ask, from which points of the initial state the molecules got their energy? To answer this question we choose the paths of random walks, along which the initial energies were transferred to the molecules in the layer with the coordinate x at the time t. Then we consider these sample paths in the inverse time direction, i.e., we simply invert the time on these paths.

The simple random walk considered in inverse time does not change its probabilistic properties, i.e., it is a simple random walk. The same is true for a Brownian motion. Consider the layer with the coordinate x (briefly the layer x). Denote by $W_h^{(l)}(s)$, $s \in [0, t]$, $l = 1, 2, \ldots, m$, the paths along which the molecules of this layer got their energy, where l is the index assigned to every molecule in the layer $x = W_h^{(l)}(0)$. In this case $u_{m,h}(t, x)$ is the temperature of the layer x at time t and $W_h^{(l)}(t)$ is the initial position from which the energy of the molecule was obtained. Since the different molecules transfer the energy independently, the processes $W_h^{(l)}(s)$, $s \in [0, t]$, are independent for different l. They are identically distributed as well as the process $W_h(s)$ defined above by the random walk ν_k.

Several of these paths are shown in the figure below, where the left and the right rods are represented at the initial and final moments, respectively.

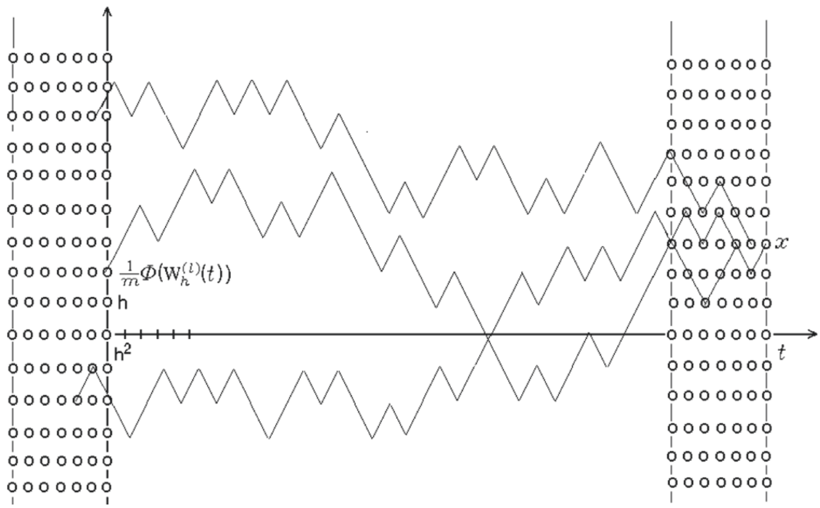

Fig.1

The process of energy transfer of a particular molecule is represented here in the inverse time. It starts at x on the right-hand side and ends at the location of the rod at the initial time. Such a representation reflects our interest in those paths, along which the energy was acquired by molecules in the layer x at time t. In a real situation, the transfer of heat energy flows from left to right.

Along the path $W_h^{(l)}(s)$, $s \in [0,t]$, the energy $\frac{1}{m}\,\Phi\big(W_h^{(l)}(t)\big)$ is transferred to the lth molecule in the layer x at time t, because for the heat transfer process this path starts at the layer with the coordinate $W_h^{(l)}(t)$ and the molecule at this point has the energy $\frac{1}{m}\,\Phi\big(W_h^{(l)}(t)\big)$. The temperature in the layer x is the sum of the energies of individual molecules, i.e.,

$$u_{m,h}(t,x) = \frac{1}{m}\sum_{l=1}^{m}\Phi\big(W_h^{(l)}(t)\big). \tag{1.3}$$

Since the processes $W_h^{(l)}(s)$, $s \in [0,t]$, $l = 1,2,\dots,m$, are independent, application of the strong law of large numbers yields

$$u_{m,h}(t,x) \to \mathbf{E}_x\Phi(W_h(t)) \qquad \text{as} \quad m \to \infty. \tag{1.4}$$

Here and in what follows the subscript x means that the expectation is computed with respect to the process starting from the point x. Since the processes $W_h(s)$, $s \in [0,t]$, converge weakly as $h \to 0$ (see § 3 Ch. VII) to the Brownian motion $W(s)$, $s \in [0,t]$, we have

$$\mathbf{E}_x\Phi(W_h(t)) \to \mathbf{E}_x\Phi(W(t)). \tag{1.5}$$

Thus for large m and small h,

$$u_{m,h}(t,x) \approx u(t,x) := \mathbf{E}_x\Phi(W(t)).$$

This enables us to consider $u(t,x)$ as the approximate value of the temperature in the rod in the layer x at time t.

By (1.2),

$$u(t,x) = \mathbf{E}_x\Phi(W(t)) = \mathbf{E}\Phi(W(t) - W(0) + x) = \int_{-\infty}^{\infty}\Phi(y)\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}\,dy.$$

By direct differentiation, it can easily be verified that this function satisfies the *heat equation*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x), \qquad t \in (0,\infty), \quad x \in \mathbf{R}, \tag{1.6}$$

with the initial condition

$$u(0,x) = \Phi(x), \qquad x \in \mathbf{R}. \tag{1.7}$$

Thus we come to the well-known solution of the heat transfer problem in a thin homogeneous isolated rod of infinite length.

We now complicate the problem. Assume that a thin layer (crack) divides the rod at some level q, through which the energy is transferred with loss. For example, part of the energy is taken by air molecules, filling the crack, and then is removed from the rod. The question is how the energy is distributed along the rod at the time t if the initial temperature is given by the function $\Phi(y)$? To answer this question we will consider the transfer of the energy $\frac{1}{m}\Phi(y)$ of a single molecule from the layer y. As already explained, the transfer of energy along the rod is described by a simple random walk. However, in our case some amount of the energy is lost when transferred through the layer q. We assume that this amount is proportional to h multiplied by some coefficient β. Thus after crossing the layer q, the energy $\frac{1}{m}\Phi(y)$ is reduced to the energy $(1-\beta h)\frac{1}{m}\Phi(y)$. Denote by $\phi_h(t,q)$ the number of crossings of the level q by the process $W_h(s)$, $s \in [0,t]$. Then according to our mathematical model, the amount of energy transferred to the molecule at the time t in the layer x along the path $W_h(\cdot)$ equals $\frac{1}{m}\Phi(W_h(t))(1-\beta h)^{\phi_h(t,q)}$. The temperature $u_{m,h}(t,x)$ in the layer x at the time t is the sum of such energies for individual molecules. By analogy with (1.3) and (1.4), we have

$$u_{m,h}(t,x) \to \mathbf{E}_x\big[\Phi(W_h(t))(1-\beta h)^{\phi_h(t,q)}\big] \qquad \text{as} \quad m \to \infty. \tag{1.8}$$

Since the number of different points of the lattice $kh, k = 0, \pm 1, \pm 2, \ldots$, that are visited by the process W_h during the time t is proportional to $1/h$ and there are t/h^2 steps, the number of crossings of a level is also proportional to $1/h$, i.e., $\phi_h(t,q) \asymp 1/h$. The number $\phi_h(t,q)$ of crossings is a functional of sample paths of the random walk $W_h(kh^2)$, $k = 0, 1, \ldots, [t/h^2]$. Since the processes $W_h(s)$, $s \in [0,t]$, converge weakly as $h \to 0$ to the Brownian motion $W(s)$, $s \in [0,t]$, the processes $\ell_h(t,q) := h\phi_h(t,q)$, $(t,q) \in [0,\infty) \times \mathbf{R}$, converge weakly to the Brownian local time process $\ell(t,q)$ (see § 6 Ch. VII). Here we again come to the same situation as for a random walk and a Brownian motion (see Preface). The process $\ell_h(t,q)$ is easily described constructively: it is the number of crossings of the level q by the process $W_h(s)$ up to the time t, multiplied by h. At the same time, explicit formulas for the finite-dimensional distributions of the process $\ell_h(t,q)$, $(t,q) \in [0,\infty) \times \mathbf{R}$, apparently cannot be obtained. The limit process $\ell(t,q)$ is impossible to visualize, but it admits a probabilistic description in terms of finite-dimensional distributions. Moreover, we can write out some other characteristics (see Ch. V).

Passing in (1.8) to the limit as $h \to 0$, we get

$$\mathbf{E}_x\big[\Phi(W_h(t))(1-\beta h)^{\phi_h(t,q)}\big] \to \mathbf{E}_x\big[\Phi(W(t))e^{-\beta\ell(t,q)}\big].$$

Consequently, the temperature in the layer x at time t in the rod divided by the crack is determined with a high degree of accuracy by the function

$$u(t,x) = \mathbf{E}_x\big[\Phi(W(t))e^{-\beta\ell(t,q)}\big]. \tag{1.9}$$

This function is the solution of the corresponding differential problem (see § 3 Ch. III). We find an expression for $u(t,x)$ in the case $\Phi(y) \equiv 1$, i.e., when at the initial moment the temperature is distributed along the rod uniformly. According to (3.13) of Ch. III, the explicit form of the Laplace transform $M(x) := \lambda\int_0^\infty e^{-\lambda t}u(t,x)\,dt$ of this function with respect to time is given by

$$M(x) = 1 - \frac{\beta}{\beta+\sqrt{2\lambda}}\, e^{-|x-q|\sqrt{2\lambda}}.$$

Dividing this expression by λ and inverting the Laplace transform with respect to λ (see Appendix 3, formula 12), we obtain

$$u(t,x) = \frac{\sqrt{2}}{\sqrt{\pi t}}\int_0^{|x-q|} e^{-v^2/2t}dv + \frac{\sqrt{2}}{\sqrt{\pi t}}e^{(2|x-q|\beta t+\beta^2t^2)/2t}\int_{|x-q|+\beta t}^{\infty} e^{-v^2/2t}dv. \tag{1.10}$$

This shows that the rod is cooling most rapidly in the neighborhood of q. It is interesting that (see (10.7) Ch. I)

$$u(t,x) \approx \frac{\sqrt{2}}{\sqrt{\pi t}}\int_0^{|x-q|} e^{-v^2/2t}\,dv + \frac{\sqrt{2}}{\sqrt{\pi t}}\,\frac{e^{-(x-q)^2/2t}}{(\beta+|x-q|/t)}, \qquad u_x'(t,q+0) \approx \frac{\sqrt{2}}{\sqrt{\pi t}} \quad \text{as } \beta\to\infty.$$

This formula implies that if all the energy transferring through the layer q (through the crack) gets lost ($\beta\to\infty$), then the temperature in the rod at time t is given by the formula

$$u(t,x) = \frac{\sqrt{2}}{\sqrt{\pi t}}\int_0^{|x-q|} e^{-v^2/2t}\,dv.$$

Further we focus on the problem of heat transfer with a loss of energy along the rod. Assume, for example, that the rod is made of a material with pores, through which heat is transported outside the rod. In particular, the material can be impregnated with a liquid that vaporizes when heated. Suppose that in any layer y the part of energy lost by a single molecule during the transfer is proportional to $h^2f(y)$, where f is a continuous nonnegative function. Then in the layer y the available energy transferred by collision is multiplied by the factor $1-h^2f(y)$. Therefore, along the path $W_h(s)$, $s\in[0,t]$, a molecule in the layer x receives the initial energy $\frac{1}{m}\Phi(W_h(t))$ multiplied by $\prod_{j=0}^{t/h^2}(1-h^2f(W_h(t-jh^2)))$. Thus at the time t, the amount of energy transferred to the lth molecule in the layer x, $l=1,2,\dots,m$, along the path $W_h^{(l)}(s)$ is equal to

$$\frac{1}{m}\Phi(W_h^{(l)}(t))\prod_{j=0}^{t/h^2}\bigl(1-h^2f\bigl(W_h^{(l)}(t-jh^2)\bigr)\bigr).$$

The total energy (the temperature in the layer x at time t) is

$$u_{m,h}(t,x) = \frac{1}{m}\sum_{l=1}^{m} \varPhi(W_h^{(l)}(t)) \prod_{j=0}^{t/h^2} \big(1 - h^2 f\big(W_h^{(l)}(jh^2)\big)\big),$$

after the substitution $t - jh^2$ by jh^2.

Since the processes $W_h^{(l)}(s)$, $s \in [0,t]$, describing the paths along which the lth molecule acquires its energy, are independent and identically distributed, the temperature in the layer x satisfies the relation

$$u_{m,h}(t,x) \to \mathbf{E}_x\Big[\varPhi(W_h(t)) \prod_{j=0}^{t/h^2} \Big(1 - h^2 f(W_h(jh^2))\Big)\Big] \qquad \text{as } m\to\infty, \tag{1.11}$$

thanks to the strong law of large numbers. Passing to the limit as $h \to 0$ with the help of the weak invariance principle (see § 3 Ch. VII for $h^2 = 1/n$, $n = 1,2,\dots$), we obtain

$$\begin{aligned}
&\mathbf{E}_x\Big[\varPhi(W_h(t)) \prod_{j=0}^{t/h^2} \big(1 - h^2 f(W_h(jh^2))\big)\Big] \\
&\quad\approx \mathbf{E}_x\Big[\varPhi(W_h(t)) \exp\Big(-\sum_{j=0}^{t/h^2} f(W_h(jh^2))h^2\Big)\Big] \\
&\quad\approx \mathbf{E}_x\Big[\varPhi(W_h(t)) \exp\Big(-\int_0^t f(W_h(s))\,ds\Big)\Big] \\
&\quad\approx \mathbf{E}_x\Big[\varPhi(W(t)) \exp\Big(-\int_0^t f(W(s))\,ds\Big)\Big] =: u(t,x).
\end{aligned}$$

Thus for heat transfer with a loss of energy along the rod, this limit determines with a high accuracy the temperature in the rod in the layer x at time t.

In § 13 Ch. II (Theorem 13.2 with $b(x) \equiv 1$ and $a(x) \equiv 0$) it is proved that the function $u(t,x)$ is the solution of the *heat equation*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) - f(x)u(t,x), \qquad (t,x) \in (0,\infty)\times\mathbf{R}, \tag{1.12}$$

with the initial condition

$$u(0,x) = \varPhi(x), \qquad x \in \mathbf{R}. \tag{1.13}$$

It is quite natural to assume that the initial temperature is bounded. In this case the function $\varPhi(x)$, $x \in \mathbf{R}$, is bounded and the problem (1.12), (1.13) is reduced by means of the Laplace transform

$$U(x) = \lambda\int_0^\infty e^{-\lambda t}u(t,x)dt, \qquad \lambda > 0,$$

(see § 12 Ch. II, Theorem 12.4) to the ordinary differential equation

$$\frac{1}{2}U''(x) - (\lambda + f(x))U(x) = -\lambda \Phi(x), \qquad x \in \mathbf{R}, \tag{1.14}$$

which has a unique bounded solution on the real line.

Of course, in order to get the temperature at time t we need to solve the problem of inverting the Laplace transform with respect to the parameter λ.

The heat transfer problem with a loss of energy can also be considered for a rod of finite length, for example, when a constant temperature is imposed at the ends of the rod.

First we consider the case when there is no heat loss. Suppose that the rod is located along the vertical axis. Let the coordinates of the bottom and top endpoints be a and b, respectively. In this situation a peculiarity of the heat transfer is the following. Let the initial energy be transferred along a path of a random walk that ends at time t at the point x. If the path reaches the coordinate b, then the energy at this moment is set to be $\frac{1}{m}\Phi(b)$ no matter what it was before. This new value of energy is passed along the path to the corresponding molecule in the layer x if the path does not reach the coordinate a till time t. If the lower coordinate a is reached, then the energy is set to be the new value $\frac{1}{m}\Phi(a)$.

In order to understand what amount of energy each of the molecules in the layer x gets, we invert the time of the random path. As usual, we denote the path in inverse time by $W_h^{(l)}(s)$, $s \in [0,t]$, $l = 1, 2, \dots, m$. In this case, $W_h^{(l)}(0) = x$, where x is the coordinate of the layer, the temperature of which we are interested in. Along the path $W_h^{(l)}(s)$, $s \in [0,t]$, the lth molecule gets the energy $\frac{1}{m}\Phi(W_h^{(l)}(t))$ if the path does not reach the boundary of the interval (a,b) at time t and the energy $\frac{1}{m}\Phi(a)$ or $\frac{1}{m}\Phi(b)$ if the path considered in direct time reaches the corresponding endpoint during the latest visit before the time t. After inverting the time the last visiting moment becomes the first one. Let $H_{a,b}^{h,l} := \min\{s : W_h^{(l)}(s) \notin (a,b)\}$. As a result of what we said above, the energy $\frac{1}{m}\Phi\big(W_h^{(l)}\big(t \wedge H_{a,b}^{h,l}\big)\big)$ is transferred along the path $W_h^{(l)}(s)$, $s \in [0,t]$, to the lth molecule in the layer x. Recall that the symbol $\wedge$ denotes the minimum of the values under consideration. The temperature in the layer x equals the sum of the energies of the individual molecules, i.e.,

$$u_{m,h}(t,x) = \frac{1}{m}\sum_{l=1}^{m} \Phi(W_h^{(l)}(t \wedge H_{a,b}^{h,l})). \tag{1.15}$$

For different values of l the processes $W_h^{(l)}(s)$, $s \in [0,t]$, are independent and distributed as the process $W_h(s)$, $s \in [0,t]$. Therefore, applying the strong law of large numbers, we find that

$$u_{m,h}(t,x) \to \mathbf{E}_x \Phi\big(W_h\big(t \wedge H_{a,b}^h\big)\big) \qquad \text{as} \quad m \to \infty, \tag{1.16}$$

where $H_{a,b}^h$ is the time of the first exit to the boundary of the interval (a,b) for the process $W_h(s)$, $s \geq 0$, $W_h(0) = x$.

Since for any $t > 0$ the processes $W_h(s)$, $s \in [0,t]$, converge weakly as $h \to 0$ (see § 3 Ch. VII) to the Brownian motion $W(t)$, $s \in [0,t]$, we conclude that $H^h_{a,b} \to H_{a,b}$, where $H_{a,b} := \min\{s : W(s) \notin (a,b)\}$ is the first exit time of the Brownian motion $W(s)$, $s \ge 0$, $W(0) = x$, from the interval (a,b). Moreover,

$$\mathbf{E}_x \Phi\big(W_h\big(t \wedge H^h_{a,b}\big)\big) \to \mathbf{E}_x \Phi(W(t \wedge H_{a,b})). \tag{1.17}$$

It is natural to treat $u(t,x) := \mathbf{E}_x \Phi(W(t \wedge H_{a,b}))$ as an approximate value of the temperature in the layer x at time t in a rod of finite length and with fixed temperature at its ends.

Let us return to the problem of the heat transfer with a loss of energy. In contrast to the previous case, during each collision at a layer y a molecule gets the energy available to this moment multiplied by the factor $1 - h^2 f(y)$, which reflects the loss of energy. However, when a path reaches the boundary of the interval (a,b), the molecule gets either the energy $\frac{1}{m}\Phi(a)$, or the energy $\frac{1}{m}\Phi(b)$, depending on whether a or b is reached. During the further transfer of energy, produced by collisions at any layer y, it decreases each time proportionally to the factor $1 - h^2 f(y)$. The key point here is the last time before t when the path hits the boundary. After this moment the boundary temperatures do not affect the process of energy transfer.

Thus the energy transferred along the path $W^{(l)}_h(s)$, $s \in [0,t]$, to the lth molecule in the layer x is equal to

$$\frac{1}{m}\Phi\big(W^{(l)}_h\big(t \wedge H^{h,l}_{a,b}\big)\big) \prod_{j=0}^{(t \wedge H^{h,l}_{a,b})/h^2} \big(1 - h^2 f\big(W^{(l)}_h(jh^2)\big)\big), \qquad W^{(l)}_h(0) = x.$$

The temperature in the layer x is the sum of the energies of individual molecules, i.e.,

$$u_{m,h}(t,x) = \frac{1}{m}\sum_{l=1}^{m} \Phi\big(W^{(l)}_h\big(t \wedge H^{h,l}_{a,b}\big)\big) \prod_{j=0}^{(t \wedge H^{h,l}_{a,b})/h^2} \big(1 - h^2 f\big(W^{(l)}_h(jh^2)\big)\big). \tag{1.18}$$

Applying the strong law of large numbers, we get

$$u_{m,h}(t,x) \to \mathbf{E}_x \bigg\{\Phi(W_h(t \wedge H^h_{a,b})) \prod_{j=0}^{(t \wedge H^h_{a,b})/h^2} \big(1 - h^2 f\big(W_h(jh^2)\big)\big)\bigg\} \tag{1.19}$$

as $m \to \infty$. Passing to the limit as $h \to 0$, we obtain

$$\mathbf{E}_x \bigg[\Phi(W_h(t \wedge H^h_{a,b})) \prod_{j=0}^{(t \wedge H^h_{a,b})/h^2} \big(1 - h^2 f\big(W_h(jh^2)\big)\big)\bigg]$$

$$\to \mathbf{E}_x \bigg[\Phi(W(t \wedge H_{a,b})) \exp\bigg(-\int_0^{t \wedge H_{a,b}} f(W(s))\,ds\bigg)\bigg] =: u(t,x).$$

This limit is taken as the temperature in the layer x at time t of the rod of a finite length during the heat transfer with a loss of energy. As was proved in § 11 Ch. II (Theorem 11.1, $\sigma(x) \equiv 1$, $\mu(x) \equiv 0$), the function $u(t,x)$, $(t,x) \in [0,\infty) \times [a,b]$, is the solution of the *heat equation*

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x) - f(x) u(t,x), \qquad x \in (a,b), \tag{1.20}$$

with the boundary conditions

$$u(0,x) = \Phi(x), \tag{1.21}$$
$$u(t,a) = \Phi(a), \qquad u(t,b) = \Phi(b). \tag{1.22}$$

In conclusion, we note that in the case of a rod made from an inhomogeneous material, the energy exchange between a molecule in the layer y and molecules of the upper or lower layers is realized with unequal probabilities. These probabilities may depend on y. The energy transfer goes along paths, which no longer constitute a simple random walk. Such paths converge as $h \to 0$ to a diffusion process $X(s)$, $s \in [0,t]$. Let the drift coefficient of this process be $\mu(x)$, $x \in \mathbf{R}$, and the diffusion coefficient be $\sigma^2(x)$, $x \in \mathbf{R}$. Then these coefficients completely determine the heat transfer in the inhomogeneous rod. As it was explained above, to derive the heat equation we must invert the time of the paths along which the energy is transferred. It is important that the *time reversed diffusion* $X^*(s) = X(t-s)$, $s \in [0,t]$, has a generator conjugate to the generator of the diffusion in the direct time. Then the temperature of the inhomogeneous rod is defined by the function

$$\mathbf{E}_x \biggl[\Phi(X^*(t \wedge H^*_{a,b})) \exp\biggl(- \int_0^{t \wedge H^*_{a,b}} f(X^*(s)) ds \biggr) \biggr] =: u^*(t,x),$$

where $H^*_{a,b} := \min\{s : X^*(s) \notin (a,b)\}$. According to Theorem 11.1 Ch. II, this function is the solution of the Cauchy problem (11.4)–(11.6) Ch. II, in which equation (11.4) is replaced by the conjugate one.

SPECIAL FUNCTIONS

1. Hyperbolic functions

$$\operatorname{sh} x := \frac{1}{2}(e^x - e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\operatorname{ch} x := \frac{1}{2}(e^x + e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2k!}$$

$$\operatorname{th} x := \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

$$\operatorname{cth} x := \frac{\operatorname{ch} x}{\operatorname{sh} x}$$

2. Gamma function

$$\Gamma(x) := \int_0^{\infty} u^{x-1} e^{-u}\, du, \quad \operatorname{Re} x > 0$$

$$\Gamma(x+1) = x\Gamma(x), \qquad \frac{1}{\Gamma(x)} \simeq x \quad \text{as } x \to 0$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\Bigl(x + \frac{1}{2}\Bigr)$$

$$\Gamma(ax+b) \simeq \sqrt{2\pi} e^{-ax} (ax)^{ax+b-1/2} \quad \text{as } x \to \infty, \quad a > 0$$

$$\Gamma(n+1) = n!, \qquad \Gamma(n+1/2) = 1 \cdot 3 \cdots (2n-1) 2^{-n} \sqrt{\pi}, \quad n = 1, 2, \dots$$

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(3/2) = \sqrt{\pi}/2, \qquad \Gamma(-1/2) = -2\sqrt{\pi}$$

3. Bessel functions

$$J_\nu(x) := \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}$$

$$Y_\nu(x) := \frac{1}{\sin(\nu\pi)} \bigl(J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\bigr)$$

$$Y_\nu'(x) J_\nu(x) - Y_\nu(x) J_\nu'(x) = Y_\nu(x) J_{\nu+1}(x) - Y_{\nu+1}(x) J_\nu(x) = 2/(\pi x)$$

$0 < j_{\nu,1} < j_{\nu,2} < \cdots$ – positive zeros of $J_\nu(x)$ for $\nu \geq 0$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8

4. Modified Bessel functions

$$I_\nu(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) := \frac{\pi}{2\sin(\nu\pi)}\big(I_{-\nu}(x) - I_\nu(x)\big)$$

are linearly independent solutions of the Bessel equation

$$x^2 Y''(x) + xY'(x) - (x^2+\nu^2)Y(x) = 0$$

Integral representations: for $x > 0$

$$I_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} e^{xt}\,dt, \qquad \operatorname{Re}\nu > -1/2$$

$$K_\nu(x) = \frac{\sqrt{\pi}(x/2)^\nu}{\Gamma(\nu+1/2)} \int_{1}^{\infty} (t^2-1)^{\nu-1/2} e^{-xt}\,dt, \qquad \operatorname{Re}\nu > -1/2$$

Properties:

$$I_{-\nu}(x) = I_\nu(x), \qquad \nu = 1, 2, \ldots$$

$$K_{-\nu}(x) = K_\nu(x)$$

$$I'_\nu(x)K_\nu(x) - I_\nu(x)K'_\nu(x) = I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = 1/x$$

$$\big(x^{-\nu} I_\nu(x)\big)' = x^{-\nu} I_{\nu+1}(x), \qquad \big(x^{\nu} I_\nu(x)\big)' = x^{\nu} I_{\nu-1}(x)$$

$$\big(x^{-\nu} K_\nu(x)\big)' = -x^{-\nu} K_{\nu+1}(x), \qquad \big(x^{\nu} K_\nu(x)\big)' = -x^{\nu} K_{\nu-1}(x)$$

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x), \qquad K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x)$$

$$I_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \quad \text{as } x \to 0, \quad \nu \neq -1, -2, \ldots$$

$$K_\nu(x) \simeq \frac{\Gamma(\nu)}{2}\left(\frac{x}{2}\right)^{-\nu}, \quad \nu > 0 \quad \text{as } x \to 0$$

$K_0(x) \simeq -\ln(x/2) - \gamma$ as $x \to 0$, where γ is the Euler constant

$$I_\nu(x) \simeq \frac{1}{\sqrt{2\pi x}} e^x \quad \text{as } x \to \infty$$

$$K_\nu(x) \simeq \frac{\sqrt{\pi}}{\sqrt{2x}} e^{-x} \quad \text{as } x \to \infty$$

$$I_{\nu\sqrt{\lambda}}\big(\nu\sqrt{\gamma}\,e^{x/\nu}\big) \simeq \frac{1}{\sqrt{2\pi}(\lambda+\gamma)^{1/4}\sqrt{\nu}}\left(\frac{\sqrt{\gamma}}{\sqrt{\lambda}+\sqrt{\lambda+\gamma}}\right)^{\nu\sqrt{\lambda}} e^{(\nu+x)\sqrt{\lambda+\gamma}} \quad \text{as } \nu\to\infty$$

$$K_{\nu\sqrt{\lambda}}\big(\nu\sqrt{\gamma}\,e^{x/\nu}\big) \simeq \frac{\sqrt{\pi}}{(\lambda+\gamma)^{1/4}\sqrt{2\nu}}\left(\frac{\sqrt{\gamma}}{\sqrt{\lambda}+\sqrt{\lambda+\gamma}}\right)^{-\nu\sqrt{\lambda}} e^{-(\nu+x)\sqrt{\lambda+\gamma}} \quad \text{as } \nu\to\infty$$

Special cases:

$$I_{\frac{1}{2}}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\operatorname{sh} x; \quad K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{\sqrt{2x}}e^{-x}; \quad I_{-\frac{1}{2}}(x) = \frac{\sqrt{2}}{\sqrt{\pi x}}\operatorname{ch} x;$$

$$I_{\frac{3}{2}}(x) = \frac{\sqrt{2}}{\sqrt{\pi}x^{3/2}}(x\operatorname{ch} x - \operatorname{sh} x); \qquad K_{\frac{3}{2}}(x) = \frac{\sqrt{\pi}}{\sqrt{2}x^{3/2}}e^{-x}(x+1)$$

5. Airy function

$$\operatorname{Ai}(x) := \frac{1}{3}\sqrt{x}\left(I_{-1/3}\big(\tfrac{2}{3}x^{3/2}\big) - I_{1/3}\big(\tfrac{2}{3}x^{3/2}\big)\right) = \frac{\sqrt{x}}{\pi\sqrt{3}}K_{1/3}\big(\tfrac{2}{3}x^{3/2}\big)$$

$\cdots\alpha_k < \cdots < \alpha_2 < \alpha_1 < 0$ – zeros of the Airy function $\operatorname{Ai}(x)$

$\cdots\alpha'_k < \cdots < \alpha'_2 < \alpha'_1 < 0$ – zeros of the derivative of the Airy function $\operatorname{Ai}'(x)$

6. Hermite polynomials and related functions

$$\operatorname{He}_n(x) := (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\big(e^{-x^2/2}\big) = \sum_{0\le k\le n/2}\frac{(-1)^k 2^{-k} n!}{k!(n-2k)!}x^{n-2k}$$

$$\operatorname{h}_y(n,v) := \mathcal{L}_\gamma^{-1}\big((2\gamma)^{n/2-1/2}e^{-v\sqrt{2\gamma}}\big) = \frac{1}{\sqrt{2\pi}y^{(n+1)/2}}e^{-v^2/2y}\operatorname{He}_n\Big(\frac{v}{\sqrt{y}}\Big), \quad 0<v$$

7. Binomial series

$$(1+x)^\mu = \sum_{k=0}^{\infty}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\,k!}x^k = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(-\mu+k)}{\Gamma(-\mu)\,k!}x^k, \qquad |x|<1$$

8. Error functions

$$\operatorname{Erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}\,dv = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(2k+1)} = \frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{k=0}^{\infty}\frac{2^k x^{2k+1}}{1\cdot 3\cdot 5\cdots(2k+1)}$$

$$\operatorname{Erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-v^2}\,dv = 1-\operatorname{Erf}(x)$$

$$\operatorname{Erfi}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{v^2}\,dv = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{k!(2k+1)}$$

$$\operatorname{Erfid}(x,y) := \operatorname{Erfi}\Big(\frac{x}{\sqrt{2}}\Big) - \operatorname{Erfi}\Big(\frac{y}{\sqrt{2}}\Big)$$

$$\operatorname{Erfc}(x) \simeq \frac{1}{\sqrt{\pi}x}e^{-x^2} \quad \text{as } x \to \infty$$

9. Parabolic cylinder functions

$$D_{-\nu}(x) := e^{-x^2/4}2^{-\nu/2}\sqrt{\pi}\Big\{\frac{1}{\Gamma((\nu+1)/2)}\Big(1+\sum_{k=1}^{\infty}\frac{\nu(\nu+2)\cdots(\nu+2k-2)}{(2k)!}x^{2k}\Big)$$
$$-\frac{x\sqrt{2}}{\Gamma(\nu/2)}\Big(1+\sum_{k=1}^{\infty}\frac{(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{(2k+1)!}x^{2k}\Big)\Big\}$$

and $D_{-\nu}(-x)$ are linearly independent solutions of the differential equation

$$Y''(x) - \Big(\frac{x^2}{4}+\frac{2\nu-1}{2}\Big)Y(x) = 0, \qquad x \in \mathbf{R}$$

Integral representation: for $x \in \mathbf{R}$

$$D_{-\nu}(x) = \frac{1}{\Gamma(\nu)}e^{-x^2/4}\int_0^{\infty} t^{\nu-1}e^{-xt-t^2/2}dt, \qquad \operatorname{Re}\nu > 0$$

Properties:

$$D'_{-\nu}(x) = -\frac{x}{2}D_{-\nu}(x) - \nu D_{-\nu-1}(x) = \frac{x}{2}D_{-\nu}(x) - D_{-\nu+1}(x)$$

$$\big(e^{x^2/4}D_{-\nu}(x)\big)' = -\nu e^{x^2/4}D_{-\nu-1}(x), \qquad \big(e^{-x^2/4}D_{-\nu}(x)\big)' = -e^{-x^2/4}D_{-\nu+1}(x)$$

$$\big(x^{1-\nu}e^{-x^2/4}D_{-\nu}(x)\big)' = -x^{-\nu}e^{-x^2/4}D_{-\nu+2}(x)$$

$$D'_{-\nu}(-x)D_{-\nu}(x) + D_{-\nu}(-x)D'_{-\nu}(x) = -\frac{\sqrt{2\pi}}{\Gamma(\nu)}$$

$$\lim_{\theta\downarrow 0} 2^{\alpha/4\theta}\Gamma\big(\frac{\alpha}{4\theta}+\frac{1}{2}\big)D_{-\alpha/2\theta}(x\sqrt{2\theta}) = \sqrt{\pi}e^{-x\sqrt{\alpha}}, \quad x \in \mathbf{R}$$

Special cases:

$$D_n(x) = e^{-x^2/4}\operatorname{He}_n(x), \quad D_{-n-1}(x) = \frac{\sqrt{\pi}e^{-x^2/4}}{(-1)^n\sqrt{2}n!}\frac{d^n}{dx^n}\big(e^{x^2/2}\operatorname{Erfc}\big(\frac{x}{\sqrt{2}}\big)\big), \ n = 0,1,\dots$$

$$D_{-1/2}(x) = \frac{\sqrt{x}}{\sqrt{2\pi}}K_{1/4}\big(\frac{x^2}{4}\big), \qquad D_{-1/2}(-x) - D_{-1/2}(x) = \sqrt{\pi x}I_{1/4}\big(\frac{x^2}{4}\big), \quad x \ge 0$$

$$D_{-1/2}(-x) + D_{-1/2}(x) = \sqrt{\pi x}I_{-1/4}\big(\frac{x^2}{4}\big), \quad x \ge 0$$

10. Kummer and Whittaker functions

$$M(a,b,x) := 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)x^k}{b(b+1)\cdots(b+k-1)k!}$$

$$U(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a,b,x) + x^{1-b}\frac{\Gamma(b-1)}{\Gamma(a)} M(1+a-b,2-b,x)$$

are linearly independent solutions of the Kummer equation

$$xY''(x) + (b-x)Y'(x) - aY(x) = 0, \qquad x > 0$$

Kummer transformations:

$$M(a,b,x) = e^x M(b-a,b,-x), \qquad U(a,b,x) = x^{1-b}U(1+a-b,2-b,x)$$

$$x^{1-b}M(1+a-b,2-b,x) = x^{1-b}e^x M(1-a,2-b,-x)$$

$$e^{-x}U(b-a,b,x) = x^{1-b}e^{-x}U(1-a,2-b,x), \qquad x > 0$$

Whittaker functions:

$$M_{n,m}(x) := x^{m+1/2}e^{-x/2}M(m-n+1/2,2m+1,x)$$

$$W_{n,m}(x) := x^{m+1/2}e^{-x/2}U(m-n+1/2,2m+1,x)$$

Integral representations:

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}dt, \ \operatorname{Re} b > \operatorname{Re} a > 0$$

$$U(a,b,x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt}t^{a-1}(1+t)^{b-a-1}dt, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} x > 0$$

Properties:

$$W_{n,m}(x) = \frac{\Gamma(-2m)}{\Gamma(1/2-m-n)} M_{n,m}(x) + \frac{\Gamma(2m)}{\Gamma(1/2+m-n)} M_{n,-m}(x)$$

$$\frac{\partial}{\partial x} M(a,b,x)\, U(a,b,x) - M(a,b,x)\, \frac{\partial}{\partial x} U(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)} x^{-b}e^x$$

$$M'_{n,m}(x)W_{n,m}(x) - M_{n,m}(x)W'_{n,m}(x) = \frac{\Gamma(2m+1)}{\Gamma(m-n+1/2)}$$

$$\frac{\partial}{\partial x} M(a,b,x) = \frac{a}{b} M(a+1,b+1,x), \qquad \frac{\partial}{\partial x} U(a,b,x) = -aU(a+1,b+1,x)$$

$$\frac{\partial}{\partial x}(x^b M(a,b+1,x)) = b\,x^{b-1}M(a,b,x)$$

$$\frac{\partial}{\partial x}(x^b U(a,b+1,x)) = (b-a)x^{b-1}U(a,b,x)$$

$$\frac{\partial}{\partial x}(e^{-x}M(a,b,x)) = \frac{a-b}{b}e^{-x}M(a,b+1,x)$$

$$\frac{\partial}{\partial x}(e^{-x}U(a,b,x)) = -e^{-x}U(a,b+1,x)$$

$$\frac{\partial}{\partial x}\big(x^b e^{-x}M(a+1,b+1,x)\big) = bx^{b-1}e^{-x}M(a,b,x)$$

$$\frac{\partial}{\partial x}\big(x^b e^{-x}U(a+1,b+1,x)\big) = -x^{b-1}e^{-x}U(a,b,x)$$

$$\lim_{\theta\downarrow 0} M\Big(\frac{a}{4\theta},b+1,\theta x\Big) = 2^b\Gamma(b+1)(xa)^{-b/2}I_b(\sqrt{xa})$$

$$\lim_{\theta\downarrow 0} \theta^b\Gamma\Big(\frac{a}{4\theta}\Big)U\Big(\frac{a}{4\theta},b+1,\theta x\Big) = 2^{1-b}(a/x)^{b/2}K_b(\sqrt{xa})$$

Special cases:

$$M_{0,m}(2x) = 2^{2m}\Gamma(m+1)\sqrt{2x}I_m(x), \quad x \ge 0$$

$$W_{0,m}(2x) = \sqrt{2x/\pi}K_m(x), \quad x \ge 0$$

$$M_{1/4-\nu/2,1/4}(x^2/2) = \frac{\Gamma(\nu/2)}{4\sqrt{\pi}}2^{\nu/2-1/4}\sqrt{x}\big(D_{-\nu}(-x) - D_{-\nu}(x)\big), \quad x \ge 0$$

$$M_{1/4-\nu/2,-1/4}(x^2/2) = \frac{\Gamma((\nu+1)/2)}{2\sqrt{\pi}}2^{\nu/2-1/4}\sqrt{x}\big(D_{-\nu}(-x) + D_{-\nu}(x)\big), \quad x \ge 0$$

$$W_{1/4-\nu/2,1/4}(x^2/2) = W_{1/4-\nu/2,-1/4}(x^2/2) = 2^{\nu/2-1/4}\sqrt{x}D_{-\nu}(x), \quad x \ge 0$$

$$M_{-1/4,1/4}(x^2) = \frac{\sqrt{\pi x}}{2}\,e^{x^2/2}\,\mathrm{Erf}(x), \qquad M_{1/4,1/4}(x^2) = \frac{\sqrt{\pi x}}{2}\,e^{-x^2/2}\,\mathrm{Erfi}(x)$$

$$W_{-1/4,1/4}(x^2) = \sqrt{\pi x}\,e^{x^2/2}\,\mathrm{Erfc}(x), \qquad W_{1/4,1/4}(x^2) = \sqrt{x}\,e^{-x^2/2}, \quad x \ge 0$$

11. Hypergeometric functions

For $-1 < x < 1$

$$F(\alpha,\beta,\gamma,x) := 1 + \sum_{k=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\beta(\beta+1)\cdots(\beta+k-1)x^k}{\gamma(\gamma+1)\cdots(\gamma+k-1)k!},$$

for $0 < x < 2$

$$G(\alpha,\beta,\gamma,x) := \frac{1}{\Gamma(\alpha+\beta+1-\gamma)} F(\alpha,\beta,\alpha+\beta+1-\gamma,1-x),\ \operatorname{Re}(\alpha+\beta+1-\gamma)>0$$

are linearly independent solutions of the hypergeometric differential equation

$$x(1-x)Y''(x)+(\gamma-(\alpha+\beta+1)x)Y'(x)-\alpha\beta Y(x)=0, \qquad 0<x<1$$

Integral representations: for $-1<x<1$

$$F(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}dt, \quad \operatorname{Re}\gamma>\operatorname{Re}\alpha>0$$

for $0<x<\infty$

$$G(\alpha,\beta,\gamma,x)=\frac{1}{\Gamma(\alpha)\Gamma(\beta+1-\gamma)}\int_0^\infty t^{\alpha-1}(1+t)^{\gamma-\alpha-1}(1+tx)^{-\beta}dt, \quad \operatorname{Re}(\beta+1-\gamma)>0, \quad \operatorname{Re}\alpha>0$$

Properties:

$$\frac{\partial}{\partial x}F(\alpha,\beta,\gamma,x)\,G(\alpha,\beta,\gamma,x)-F(\alpha,\beta,\gamma,x)\,\frac{\partial}{\partial x}G(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)x^{-\gamma}(1-x)^{\gamma-\alpha-\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\frac{\partial}{\partial x}F(\alpha,\beta,\gamma,x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x)$$

$$\frac{\partial}{\partial x}G(\alpha,\beta,\gamma,x)=-\alpha\beta G(\alpha+1,\beta+1,\gamma+1,x)$$

$$\frac{\partial}{\partial x}\big((1-x)^\alpha F(\alpha,\beta,\gamma,x)\big)=\frac{\alpha(\beta-\gamma)}{\gamma}(1-x)^{\alpha-1}F(\alpha+1,\beta,\gamma+1,x)$$

$$\frac{\partial}{\partial x}\big((1-x)^\alpha G(\alpha,\beta,\gamma,x)\big)=-\alpha(1-x)^{\alpha-1}G(\alpha+1,\beta,\gamma+1,x)$$

$$\frac{\partial}{\partial x}\big(x^\gamma(1-x)^{1+\alpha+\beta-\gamma}F(\alpha+1,\beta+1,\gamma+1,x)\big)=\gamma x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma,x)$$

$$\frac{\partial}{\partial x}\big(x^\gamma(1-x)^{1+\alpha+\beta-\gamma}G(a+1,b+1,\gamma+1,x)\big)=-\frac{x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}}{1+\alpha+\beta-\gamma}G(\alpha,\beta,\gamma,x)$$

$$F(\alpha,\beta,\gamma,x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,x), \qquad 0\le x<1$$

$$G(\alpha,\beta,\gamma,x)=x^{1-\gamma}G(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x), \qquad 0<x\le 1$$

$$F(\alpha,\beta,\gamma,1/x)=\frac{x^\beta}{\Gamma(\beta)}\int_0^\infty e^{-xt}t^{\beta-1}M(\alpha,\gamma,t)\,dt, \qquad 1\le x, \quad \operatorname{Re}\beta>0$$

$$G(\alpha,\beta,\gamma,1/x)=\frac{x^\beta}{\Gamma(\beta)\Gamma(\beta+1-\gamma)}\int_0^\infty e^{-xt}t^{\beta-1}U(\alpha,\gamma,t)\,dt, \qquad 0\le x, \quad \operatorname{Re}\beta>0$$

$$G(\alpha,\beta,\gamma,x)=\frac{\Gamma(1-\gamma)F(\alpha,\beta,\gamma,x)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)}+\frac{\Gamma(\gamma-1)x^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,x)}{\Gamma(\alpha)\Gamma(\beta)},\qquad 0<x<1$$

$$\lim_{\beta\to\infty}F\Big(\alpha,\beta,\gamma,\frac{x}{\beta}\Big)=M(\alpha,\gamma,x),\qquad \lim_{\beta\to\infty}\Gamma(\beta+1-\gamma)G\Big(\alpha,\beta,\gamma,\frac{x}{\beta}\Big)=U(\alpha,\gamma,x)$$

Special cases:

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},\quad F(\alpha,\beta,\beta,x)=(1-x)^{-\alpha},\quad G(\alpha,\beta,\beta+1,x)=\frac{x^{-\beta}}{\Gamma(\alpha)}$$

$$G(\alpha,\beta,\gamma,0)=\frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)},\quad G(\alpha,\beta,\gamma,1)=\frac{1}{\Gamma(\alpha+\beta+1-\gamma)}$$

12. Legendre functions

Legendre function of the first kind
for $-1<x<1$,

$$\widetilde{P}_p^q(x):=\frac{1}{\Gamma(1-q)}\Big(\frac{1+x}{1-x}\Big)^{q/2}F\Big(-p,1+p,1-q,\frac{1-x}{2}\Big),\qquad \operatorname{Re}q<1,$$

for $1<x$

$$P_p^q(x):=\frac{2^q x^{q-p-1}}{(x^2-1)^{q/2}\Gamma(1-q)}F\Big(\frac{p-q+1}{2},\frac{p-q+2}{2},1-q,1-\frac{1}{x^2}\Big),\qquad \operatorname{Re}q<1,$$

Legendre function of the second kind, divided by $e^{iq\pi}$, for $1<x$

$$\widetilde{Q}_p^q(x):=\frac{\sqrt{\pi}\Gamma(p+q+1)(x^2-1)^{q/2}}{2^{p+1}\Gamma(p+3/2)\,x^{p+q+1}}F\Big(\frac{p+q+1}{2},\frac{p+q+2}{2},p+\frac{3}{2},\frac{1}{x^2}\Big)$$

are linearly independent solutions of the Legendre equation

$$(1-x^2)Y''(x)-2xY'(x)+\Big(p(1+p)-\frac{q^2}{1-x^2}\Big)Y(x)=0$$

Integral representations: for $-1<x<1$

$$\widetilde{P}_p^q(x)=\frac{2^{-p}(1-x^2)^{-q/2}}{\Gamma(-p-q)\Gamma(1+p)}\int_0^\infty(x+\operatorname{ch}t)^{q-p-1}\operatorname{sh}^{1+2p}t\,dt,\qquad \operatorname{Re}q<\operatorname{Re}(-p)<1$$

for $1<x$

$$\widetilde{Q}_p^q(x)=\frac{2^{-q}(x^2-1)^{q/2}\sqrt{\pi}\Gamma(p+q+1)}{\Gamma(p-q+1)\Gamma(q+1/2)}\int_0^\infty(x+\sqrt{x^2-1}\operatorname{ch}t)^{-q-p-1}\operatorname{sh}^{2q}t\,dt\qquad \operatorname{Re}(p\pm q+1)>0$$

Properties:

$$\frac{d}{dx}P_p^q(x)\widetilde{Q}_p^q(x)-P_p^q(x)\frac{d}{dx}\widetilde{Q}_p^q(x)=\frac{\Gamma(1+p+q)}{\Gamma(1+p-q)(x^2-1)},\qquad 1<x$$

$$\frac{d}{dx}\widetilde{P}_p^q(-x)\widetilde{P}_p^q(x)-\widetilde{P}_p^q(-x)\frac{d}{dx}\widetilde{P}_p^q(x)=\frac{2}{\Gamma(-p-q)\Gamma(1+p-q)(1-x^2)},\qquad -1<x<1$$

$$\widetilde{Q}_p^{-q}(x)=\frac{\Gamma(p-q+1)}{\Gamma(p+q+1)}\widetilde{Q}_p^q(x),\qquad P_p^q(x)=P_{-p-1}^q(x),\qquad \widetilde{P}_p^q(x)=\widetilde{P}_{-p-1}^q(x)$$

$$\lim_{x\downarrow 1}(x^2-1)^{q/2}P_p^q(x)=\frac{2^q}{\Gamma(1-q)},\qquad \widetilde{P}_p^{-p}(x)=\frac{(1-x^2)^{p/2}}{2^p\Gamma(p+1)},\qquad -1<x<1$$

$$\widetilde{P}_{-p}^{-p}(-x)+\widetilde{P}_{-p}^{-p}(x)=\frac{2^p\Gamma(p)}{\Gamma(2p)(1-x^2)^{p/2}},\qquad \widetilde{P}_{1-p}^{-p}(-x)-\widetilde{P}_{1-p}^{-p}(x)=\frac{2^p\Gamma(p)\,x}{\Gamma(2p-1)(1-x^2)^{p/2}}$$

$$\frac{d}{dx}\Big(\frac{P_p^q(x)}{(x^2-1)^{q/2}}\Big)=\frac{P_p^{q+1}(x)}{(x^2-1)^{(q+1)/2}},\qquad \frac{d}{dx}\Big(\frac{\widetilde{Q}_p^q(x)}{(x^2-1)^{q/2}}\Big)=-\frac{\widetilde{Q}_p^{q+1}(x)}{(x^2-1)^{(q+1)/2}},\qquad 1<x$$

$$\frac{d}{dx}\big((x^2-1)^{q/2}P_p^q(x)\big)=(p+q)(p-q+1)(x^2-1)^{(q-1)/2}P_p^{q-1}(x),\qquad 1<x$$

$$\frac{d}{dx}\big((1-x^2)^{q/2}\widetilde{P}_p^q(x)\big)=(p+q)(p-q+1)(1-x^2)^{(q-1)/2}\widetilde{P}_p^{q-1}(x),\qquad -1<x<1$$

$$\frac{d}{dx}\big((1-x^2)^{(p+1)/2}\widetilde{P}_p^q(x)\big)=(q-p-1)(1-x^2)^{(p-1)/2}\widetilde{P}_{p+1}^q(x),\qquad -1<x<1$$

$$\frac{d}{dx}\big((x^2-1)^{q/2}\widetilde{Q}_p^q(x)\big)=-(p+q)(p-q+1)(x^2-1)^{(q-1)/2}\widetilde{Q}_p^{q-1}(x),\qquad 1<x$$

$$\lim_{p\to\infty}p^qP_p^{-q}\big(\operatorname{ch}\big(\tfrac{z}{p}\big)\big)=I_q(z),\qquad \lim_{p\to\infty}p^{-q}\widetilde{Q}_p^q\big(\operatorname{ch}\big(\tfrac{z}{p}\big)\big)=K_q(z)$$

Special cases:

$$P_p^{-1/2}(\operatorname{ch}x)=\frac{2\sqrt{2}\operatorname{sh}(x(p+1/2))}{\sqrt{\pi}(2p+1)\operatorname{sh}^{1/2}x},\qquad P_p^{1/2}(\operatorname{ch}x)=\frac{\sqrt{2}\operatorname{ch}(x(p+1/2))}{\sqrt{\pi}\operatorname{sh}^{1/2}x}$$

$$\widetilde{Q}_p^{-1/2}(\operatorname{ch}x)=\frac{\sqrt{2\pi}e^{-x(p+1/2)}}{(2p+1)\operatorname{sh}^{1/2}x},\qquad \widetilde{Q}_p^{1/2}(\operatorname{ch}x)=\frac{\sqrt{\pi}e^{-x(p+1/2)}}{\sqrt{2}\operatorname{sh}^{1/2}x}$$

13. Theta functions of imaginary argument

$$\operatorname{cs}_y(u,v):=\mathcal{L}_\gamma^{-1}\Big(\frac{\operatorname{ch}(u\sqrt{2\gamma})}{\sqrt{2\gamma}\operatorname{sh}(v\sqrt{2\gamma})}\Big)=\frac{1}{\sqrt{2\pi y}}\sum_{k=-\infty}^{\infty}e^{-(u+v+2kv)^2/2y}$$

$$\operatorname{sc}_y(u,v):=\mathcal{L}_\gamma^{-1}\Big(\frac{\operatorname{sh}(u\sqrt{2\gamma})}{\sqrt{2\gamma}\operatorname{ch}(v\sqrt{2\gamma})}\Big)=\frac{1}{\sqrt{2\pi y}}\sum_{k=-\infty}^{\infty}(-1)^ke^{-(v-u+2kv)^2/2y}$$

$$\operatorname{ss}_y(u,v):=\mathcal{L}_\gamma^{-1}\Big(\frac{\operatorname{sh}(u\sqrt{2\gamma})}{\operatorname{sh}(v\sqrt{2\gamma})}\Big)=\sum_{k=-\infty}^{\infty}\frac{v-u+2kv}{\sqrt{2\pi}y^{3/2}}e^{-(v-u+2kv)^2/2y},\qquad u<v$$

$$\widetilde{\operatorname{ss}}_y(u,v):=\mathcal{L}_\gamma^{-1}\Big(\frac{\operatorname{sh}(u\sqrt{2\gamma})}{\gamma\operatorname{sh}(v\sqrt{2\gamma})}\Big)=\sum_{k=-\infty}^{\infty}\operatorname{sign}(v-u+2kv)\operatorname{Erfc}\Big(\frac{|v-u+2kv|}{\sqrt{2y}}\Big)$$

$$\mathrm{cc}_y(u,v) := \mathcal{L}_\gamma^{-1}\Big(\frac{\mathrm{ch}(u\sqrt{2\gamma})}{\mathrm{ch}(v\sqrt{2\gamma})}\Big) = \sum_{k=-\infty}^{\infty}(-1)^k\frac{u+v+2kv}{\sqrt{2\pi}y^{3/2}}e^{-(u+v+2kv)^2/2y}, \quad u<v$$

$$\widetilde{\mathrm{cc}}_y(u,v) := \mathcal{L}_\gamma^{-1}\Big(\frac{\mathrm{ch}(u\sqrt{2\gamma})}{\gamma\,\mathrm{ch}(v\sqrt{2\gamma})}\Big) = \sum_{k=-\infty}^{\infty}(-1)^k\,\mathrm{sign}(u+v+2kv)\,\mathrm{Erfc}\Big(\frac{|u+v+2kv|}{\sqrt{2y}}\Big)$$

$$\mathrm{s}_y(v) := L_\gamma^{-1}\Big(\frac{\sqrt{2\gamma}}{\mathrm{sh}(v\sqrt{2\gamma})}\Big) = \frac{\sqrt{2}}{\sqrt{\pi}y^{5/2}}\sum_{k=0}^{\infty}((2k+1)^2v^2-y)e^{-(2k+1)^2v^2/2y}, \quad 0<v$$

14. Two parameter functions connected to the Bessel functions

$$S_\nu(x,y) := (xy)^{-\nu}\big(I_\nu(x)K_\nu(y)-K_\nu(x)I_\nu(y)\big)$$

$$C_\nu(x,y) := (xy)^{-\nu}\big(I_{\nu+1}(x)K_\nu(y)+K_{\nu+1}(x)I_\nu(y)\big)$$

$$F_\nu(x,z) := (xz)^{-\nu}I_\nu((x+z-|x-z|)/2)K_\nu((x+z+|x-z|)/2)$$
$$= \begin{cases} (xz)^{-\nu}I_\nu(x)K_\nu(z), & x\le z \\ (xz)^{-\nu}K_\nu(x)I_\nu(z), & z\le x \end{cases}$$

Properties:

$C_\nu(x,y)$ as a function of y and $S_\nu(x,y)$, $F_\nu(x,z)$ as functions of both variables satisfy the Bessel equation

$$Z''(x)+\frac{2\nu+1}{x}Z'(x)-Z(x)=0. \tag{B}$$

$$S_\nu(x,x)=0, \quad C_\nu(x,x)=x^{-1-2\nu}, \quad S_\nu(x,y)=-S_\nu(y,x), \quad F_\nu(x,z)=F_\nu(z,x)$$

$$\frac{\partial}{\partial x}S_\nu(x,y)=C_\nu(x,y), \qquad \frac{\partial}{\partial y}S_\nu(x,y)=-C_\nu(y,x), \quad \frac{\partial}{\partial y}C_\nu(x,y)=-xyS_{\nu+1}(x,y)$$

$$\frac{\partial}{\partial x}\big(x^{2\nu+1}C_\nu(x,y)\big)=x^{2\nu+1}S_\nu(x,y), \quad \frac{\partial}{\partial x}F_\nu(z+0,z)-\frac{\partial}{\partial x}F_\nu(z-0,z)=-z^{-1-2\nu}$$

$$S_\nu(q,p)S_\nu(r,z)+S_\nu(q,r)S_\nu(z,p)=S_\nu(q,z)S_\nu(r,p)$$

$$S_\nu(q,r)C_\nu(r,p)+C_\nu(r,q)S_\nu(r,p)=r^{-1-2\nu}S_\nu(q,p)$$

$$\lim_{\theta\downarrow 0} S_0(x\theta,y\theta)=\ln(x/y)$$

$$\lim_{\theta\downarrow 0} \theta^{2\nu}S_\nu(x\theta,y\theta)=\frac{1}{2\nu}(y^{-2\nu}-x^{-2\nu}), \qquad \nu\ne 0$$

Special cases:

$$S_{-1/2}(x,y)=\mathrm{sh}(x-y), \qquad C_{-1/2}(x,y)=\mathrm{ch}(x-y)$$

$$S_{1/2}(x,y)=\frac{1}{xy}\,\mathrm{sh}(x-y), \qquad C_{1/2}(x,y)=\frac{1}{x^2y}(x\,\mathrm{ch}(x-y)-\mathrm{sh}(x-y))$$

$$F_{-1/2}(x,z)=\frac{1}{2}(e^{-|x-z|}+e^{-(x+z)}), \qquad F_{1/2}(x,z)=\frac{1}{2xz}(e^{-|x-z|}-e^{-(x+z)})$$

INVERSE LAPLACE TRANSFORMS

General formulas

$$\mathcal{L}_\gamma^{-1}\big(F(\gamma)\big) =: f(y), \quad \text{where} \quad F(\gamma) = \int_0^\infty e^{-\gamma y} f(y)dy, \qquad \operatorname{Re}\gamma \ge \sigma \ge 0$$

0. $$f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\gamma y}F(\gamma)d\gamma, \qquad c > \sigma$$

a. $$\mathcal{L}_\gamma^{-1}\big(F(\alpha\gamma+\beta)\big) = \frac{1}{\alpha}e^{-\beta y/\alpha}f\Big(\frac{y}{\alpha}\Big), \qquad \alpha > 0$$

b. $$\mathcal{L}_\gamma^{-1}\big(e^{-\beta\gamma}F(\gamma)\big) = f(y-\beta)\mathbb{1}_{[\beta,\infty)}(y), \qquad \beta > 0$$

c. $$\mathcal{L}_\gamma^{-1}\bigg(\int_\gamma^\infty F(x)dx\bigg) = \frac{1}{y}f(y)$$

d. $$\mathcal{L}_\gamma^{-1}\Big(\frac{F(\gamma)}{\gamma}\Big) = \int_0^y f(x)dx$$

e. $$\mathcal{L}_\gamma^{-1}\big(\gamma F(\gamma) - f(+0)\big) = f'(y)$$

f. $$\mathcal{L}_\gamma^{-1}\big(F_1(\gamma)F_2(\gamma)\big) = \int_0^y f_1(x)f_2(y-x)dx = \int_0^y f_1(y-x)f_2(x)dx = f_1(y)*f_2(y)$$

g. $$\mathcal{L}_\gamma^{-1}\big(F(\sqrt{\gamma})\big) = \frac{1}{2\sqrt{\pi}y^{3/2}}\int_0^\infty xe^{-x^2/4y}f(x)dx$$

i. $$\mathcal{L}_\gamma^{-1}\big(\gamma^\mu F(\sqrt{\gamma})\big) = \frac{\sqrt{2}}{\sqrt{\pi}(2y)^{\mu+1}}\int_0^\infty e^{-x^2/8y}D_{2\mu+1}\Big(\frac{x}{\sqrt{2y}}\Big)f(x)dx$$

j. $$\mathcal{L}_\gamma^{-1}\mathcal{L}_\eta^{-1}\big(F_1(\gamma+\eta)F_2(\eta)\big) = f_1(y)f_2(g-y)\mathbb{1}_{[0,g]}(y)$$

k. $$\mathcal{L}_\gamma^{-1}\mathcal{L}_\eta^{-1}\big(F_1(p\gamma+\eta)F_2(q\gamma+\eta)\big) = f_1\Big(\frac{|y-qg|}{|p-q|}\Big)f_2\Big(\frac{|y-pg|}{|q-p|}\Big)\frac{\mathbb{1}_{[(p\wedge q)g,(p\vee q)g]}(y)}{|p-q|}$$

l. $$\mathcal{L}_\gamma^{-1}\mathcal{L}_\eta^{-1}\mathcal{L}_\lambda^{-1}\big(F_1(\gamma+\eta+\lambda)F_2(\eta+\lambda)F_3(\lambda)\big) = f_1(y)f_2(g-y)f_3(t-g)\mathbb{1}_{[0,g]}(y)\mathbb{1}_{[y,t]}(g)$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8

1. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\gamma+\beta}\Big) = e^{-\beta y}$

2. $\mathcal{L}_\gamma^{-1}\big(e^{-\sqrt{\alpha}\sqrt{\gamma}}\big) = \frac{\sqrt{\alpha}}{2\sqrt{\pi}y^{3/2}}e^{-\alpha/4y},$ $\alpha > 0$

3. $\mathcal{L}_\gamma^{-1}\big(\sqrt{\gamma}e^{-\sqrt{\alpha}\sqrt{\gamma}}\big) = \frac{1}{\sqrt{\pi}y^{5/2}}\Big(\frac{\alpha}{4} - \frac{y}{2}\Big)e^{-\alpha/4y},$ $\alpha > 0$

4. $\mathcal{L}_\gamma^{-1}\big(\gamma e^{-\sqrt{\alpha}\sqrt{\gamma}}\big) = \frac{\sqrt{\alpha}}{4\sqrt{\pi}y^{5/2}}\Big(\frac{\alpha}{2y} - 3\Big)e^{-\alpha/4y},$ $\alpha > 0$

5. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\sqrt{\gamma}}e^{-\sqrt{\alpha}\sqrt{\gamma}}\Big) = \frac{1}{\sqrt{\pi y}}e^{-\alpha/4y},$ $\alpha \geq 0$

6. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\gamma}e^{-\sqrt{\alpha}\sqrt{\gamma}}\Big) = \operatorname{Erfc}\Big(\frac{\sqrt{\alpha}}{2\sqrt{y}}\Big),$ $\alpha \geq 0$

7. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\gamma^{3/2}}e^{-\sqrt{\alpha}\sqrt{\gamma}}\Big) = \frac{2\sqrt{y}}{\sqrt{\pi}}e^{-\alpha/4y} - \sqrt{\alpha}\operatorname{Erfc}\Big(\frac{\sqrt{\alpha}}{2\sqrt{y}}\Big),$ $\alpha \geq 0$

8. $\mathcal{L}_\gamma^{-1}\Big(\gamma^{\mu/2}e^{-\sqrt{\alpha}\sqrt{\gamma}}\Big) = \frac{\sqrt{2}}{\sqrt{\pi}(2y)^{\mu/2+1}}e^{-\alpha/8y}D_{\mu+1}\Big(\frac{\sqrt{\alpha}}{\sqrt{2y}}\Big),$ $\alpha > 0$

9. $\mathcal{L}_\gamma^{-1}\Big(\frac{e^{-\sqrt{\alpha}\sqrt{\gamma}}}{\gamma-\beta}\Big) = \frac{e^{\beta y+\sqrt{\alpha\beta}}}{2}\operatorname{Erfc}\Big(\frac{\sqrt{\alpha}}{2\sqrt{y}} + \sqrt{\beta y}\Big) + \frac{e^{\beta y-\sqrt{\alpha\beta}}}{2}\operatorname{Erfc}\Big(\frac{\sqrt{\alpha}}{2\sqrt{y}} - \sqrt{\beta y}\Big)$

10. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\sqrt{\gamma}+\beta}e^{-\alpha\sqrt{\gamma}}\Big) = \frac{1}{\sqrt{\pi y}}e^{-\alpha^2/4y} - \beta e^{\alpha\beta+\beta^2 y}\operatorname{Erfc}\Big(\frac{\alpha}{2\sqrt{y}} + \beta\sqrt{y}\Big),$ $\alpha \geq 0$

11. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\sqrt{\gamma}(\sqrt{\gamma}+\beta)}e^{-\alpha\sqrt{\gamma}}\Big) = e^{\alpha\beta+\beta^2 y}\operatorname{Erfc}\Big(\frac{\alpha}{2\sqrt{y}} + \beta\sqrt{y}\Big),$ $\alpha \geq 0$

12. $\mathcal{L}_\gamma^{-1}\Big(\frac{\beta}{\gamma(\sqrt{\gamma}+\beta)}e^{-\alpha\sqrt{\gamma}}\Big) = \operatorname{Erfc}\Big(\frac{\alpha}{2\sqrt{y}}\Big) - e^{\alpha\beta+\beta^2 y}\operatorname{Erfc}\Big(\frac{\alpha}{2\sqrt{y}} + \beta\sqrt{y}\Big),$ $\alpha \geq 0$

13. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\sqrt{\gamma+\alpha}+\sqrt{\gamma+\beta}}\Big) = \frac{e^{-\beta y} - e^{-\alpha y}}{2\sqrt{\pi}(\alpha-\beta)y^{3/2}}$

14. $\mathcal{L}_\gamma^{-1}\Big(\frac{\sqrt{\gamma+\alpha}}{\sqrt{\gamma-\alpha}} - 1\Big) = \alpha\big(I_0(\alpha y) + I_1(\alpha y)\big),$ $\alpha > 0$

15. $\mathcal{L}_\gamma^{-1}\big(e^{\alpha/\gamma} - 1\big) = \frac{\sqrt{\alpha}}{\sqrt{y}}I_1(2\sqrt{\alpha y}),$ $\alpha > 0$

16. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\gamma^{\mu+1}}e^{\alpha/\gamma}\Big) = \big(\frac{y}{\alpha}\big)^{\mu/2}I_\mu(2\sqrt{\alpha y}),$ $\alpha > 0 \quad \mu > -1$

17. $\mathcal{L}_\gamma^{-1}\Big(\frac{\beta^\mu e^{-\alpha\sqrt{\gamma^2-\beta^2}}}{\sqrt{\gamma^2-\beta^2}\big(\gamma+\sqrt{\gamma^2-\beta^2}\big)^\mu}\Big) = \Big(\frac{y-\alpha}{y+\alpha}\Big)^{\mu/2}I_\mu(\beta\sqrt{y^2-\alpha^2})\mathbb{1}_{(\alpha,\infty)}(y)$

18. $\mathcal{L}_\gamma^{-1}\big(D_{-2\nu}(2\sqrt{\alpha\gamma})\big) = \frac{2^{1/2-\nu}\sqrt{\alpha}(y-\alpha)^{\nu-1}}{\Gamma(\nu)(y+\alpha)^{\nu+1/2}}\mathbb{1}_{(\alpha,\infty)}(y),$ $\alpha > 0,\ \nu > 0$

19. $\mathcal{L}_\gamma^{-1}\big(\operatorname{Erfc}(\sqrt{\alpha\gamma})\big) = \frac{\sqrt{\alpha}}{\pi y\sqrt{y-\alpha}}\mathbb{1}_{(\alpha,\infty)}(y),$ $\alpha > 0$

20. $\mathcal{L}_\gamma^{-1}\big(e^{-\alpha\gamma}-\sqrt{\pi\alpha\gamma}\,\mathrm{Erfc}\big(\sqrt{\alpha\gamma}\big)\big)=\frac{\sqrt{\alpha}}{2y^{3/2}}\mathbb{I}_{(\alpha,\infty)}(y),$ $\alpha>0$

21. $\mathcal{L}_\gamma^{-1}\big(\sqrt{\gamma}e^{-2\alpha\sqrt{\gamma}}\,\mathrm{Erfc}\big(\beta+\sqrt{\gamma}\big)\big)$ $\alpha+\beta>0$

$$=\Big\{\Big(\frac{\alpha+\beta}{\pi y(y-1)^{3/2}}+\frac{\alpha}{\pi y^2\sqrt{y-1}}\Big)e^{-\beta^2-(\alpha+\beta)^2/(y-1)}$$

$$+\Big(\frac{\alpha^2}{\sqrt{\pi}y^{5/2}}-\frac{1}{2\sqrt{\pi}y^{3/2}}\Big)e^{-\alpha^2/y}\,\mathrm{Erfc}\Big(\frac{\alpha}{\sqrt{y(y-1)}}+\frac{\beta\sqrt{y}}{\sqrt{y-1}}\Big)\Big\}\mathbb{I}_{[1,\infty)}(y)$$

22. $\mathcal{L}_\gamma^{-1}\big(\gamma^{\mu-1}K_{2\nu}(2\sqrt{\alpha\gamma})\big)=\frac{1}{2\sqrt{\alpha}}y^{1/2-\mu}e^{-\alpha/2y}W_{\mu-1/2,\nu}(\alpha/y),$ $\alpha>0$

23. $\mathcal{L}_\gamma^{-1}\big(\gamma^{-1/2}K_{2\nu}(2\sqrt{\alpha\gamma})\big)=\frac{1}{2\sqrt{\pi y}}e^{-\alpha/2y}K_\nu(\alpha/2y),$ $\alpha>0$

24. $\mathcal{L}_\gamma^{-1}\big(\gamma^{\nu/2}K_\nu(\sqrt{\alpha\gamma})\big)=\frac{\alpha^{\nu/2}}{(2y)^{\nu+1}}e^{-\alpha/4y},$ $\alpha>0,\ \nu\ge 0$

25. $\mathcal{L}_\gamma^{-1}\Big(\frac{e^{\alpha/2\gamma}}{\gamma^\mu}M_{-\mu,\nu}\big(\frac{\alpha}{\gamma}\big)\Big)=\frac{\sqrt{\alpha}\Gamma(2\nu+1)}{\Gamma(\mu+\nu+1/2)}y^{\mu-1/2}I_{2\nu}(2\sqrt{\alpha y}),$ $\mu+\nu>-1/2$

26. $\mathcal{L}_\gamma^{-1}\big(\Gamma(\gamma-\beta)W_{-\gamma,\beta+1/2}(4\alpha)\big)=\alpha^{-\beta}(\mathrm{sh}(y/2))^{2\beta}e^{-2\alpha\,\mathrm{cth}(y/2)},$ $\alpha>0\ge\beta$

27. $\mathcal{L}_\gamma^{-1}\Big(\frac{1}{\gamma}\exp\big(\frac{\alpha+\beta}{2\gamma}\big)I_\nu\big(\frac{\sqrt{\alpha\beta}}{\gamma}\big)\Big)=I_\nu(\sqrt{2\alpha y})I_\nu(\sqrt{2\beta y}),$ $\nu>-1$

28. $\mathcal{L}_\gamma^{-1}\big(I_\nu(x\sqrt{2\gamma})K_\nu(z\sqrt{2\gamma})\big)=\frac{1}{2y}\exp\Big(-\frac{x^2+z^2}{2y}\Big)I_\nu\Big(\frac{xz}{y}\Big),$ $0\vee x\le z$

29. $\mathcal{L}_\gamma^{-1}\big(\Gamma(\gamma)D_{-\gamma}(x)D_{-\gamma}(z)\big)=\frac{e^{y/2}}{\sqrt{2\,\mathrm{sh}\,y}}\exp\Big(-\frac{(x^2+z^2)\,\mathrm{ch}\,y+2xz}{4\,\mathrm{sh}\,y}\Big)$

30. $\mathcal{L}_\gamma^{-1}\big(\gamma^{-1}2^{\gamma/2}\Gamma\big(\frac{\gamma}{2}+\frac{1}{2}\big)D_{-\gamma}(x)\big)=\sqrt{\pi}\,e^{-x^2/4}\,\mathrm{Erfc}\Big(\frac{x}{\sqrt{2(e^{2y}-1)}}\Big),$ $x\ge 0$

31. $\mathcal{L}_\gamma^{-1}\Big(\frac{D_{-\gamma}(x)}{D_{-\gamma}(0)}\Big)=\frac{xe^{y/2}}{2\sqrt{\pi}\,\mathrm{sh}^{3/2}y}\exp\Big(-\frac{x^2\,\mathrm{ch}\,y}{4\,\mathrm{sh}\,y}\Big),$ $x\ge 0$

32. $\mathcal{L}_\gamma^{-1}\big(\Gamma(1/2+\nu+\gamma)M_{-\gamma,\nu}(x^2)W_{-\gamma,\nu}(z^2)\big)$ $0\le x\le z$

$$=\frac{\Gamma(2\nu+1)xz}{2\,\mathrm{sh}(y/2)}\exp\Big(-\frac{(x^2+z^2)\,\mathrm{ch}(y/2)}{2\,\mathrm{sh}(y/2)}\Big)I_{2\nu}\Big(\frac{xz}{\mathrm{sh}(y/2)}\Big),$$ $\nu>-1/2$

33. $\mathcal{L}_\gamma^{-1}\big(\Gamma(\gamma)M(\gamma,\nu+1,x^2)U(\gamma,\nu+1,z^2)\big)$ $0\le x\le z$

$$=\frac{\Gamma(\nu+1)e^{(\nu+1)y/2}}{2x^\nu z^\nu\,\mathrm{sh}(y/2)}\exp\Big(\frac{x^2+z^2}{2}-\frac{(x^2+z^2)\,\mathrm{ch}(y/2)}{2\,\mathrm{sh}(y/2)}\Big)I_\nu\Big(\frac{xz}{\mathrm{sh}(y/2)}\Big),$$ $\nu>-1$

34. $\mathcal{L}_\gamma^{-1}\mathcal{L}_\eta^{-1}\Big(\frac{1}{\gamma\eta+a\gamma+b\eta+c}\Big)=e^{-by-ag}I_0(2\sqrt{(ab-c)yg})$

35. $\mathcal{L}_\gamma^{-1}\mathcal{L}_\eta^{-1}\Big(\frac{\gamma+b}{\gamma\eta+a\gamma+b\eta+c}-\frac{1}{\eta+a}\Big)=e^{-by-ag}\sqrt{(ab-c)g/y}\,I_1(2\sqrt{(ab-c)yg})$

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

Below ψ and φ denote a nonnegative increasing and decreasing, respectively, solution of the following differential equations, $w = \psi'\varphi - \psi\varphi'$ is the Wronskian, λ and q are given positive parameters:

1. $\frac{1}{2}Y''(x) - \lambda Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = e^{x\sqrt{2\lambda}}, \qquad \varphi(x) = e^{-x\sqrt{2\lambda}}, \qquad w = 2\sqrt{2\lambda}$$

2. $\frac{1}{2}Y''(x) - (\lambda + \gamma x)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \sqrt{\lambda + \gamma x}\, I_{1/3}\Big(\frac{2\sqrt{2}}{3\gamma}(\lambda + \gamma x)^{3/2}\Big), \qquad w = 3\gamma/2, \qquad \gamma > 0,$$

$$\varphi(x) = \sqrt{\lambda + \gamma x}\, K_{1/3}\Big(\frac{2\sqrt{2}}{3\gamma}(\lambda + \gamma x)^{3/2}\Big) = \pi\sqrt{3}\,\mathrm{Ai}\big(2^{1/3}\gamma^{-2/3}(\lambda + \gamma x)\big)$$

3. $\frac{1}{2}Y''(x) - \big(\lambda + \frac{\gamma^2}{2}x^2\big)Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = D_{-1/2-\lambda/\gamma}(-x\sqrt{2\gamma}), \qquad \varphi(x) = D_{-1/2-\lambda/\gamma}(x\sqrt{2\gamma}),$$

$$w = \frac{2\sqrt{\pi\gamma}}{\Gamma(1/2 + \lambda/\gamma)}, \qquad \gamma > 0$$

4. $\frac{1}{2}Y''(x) - \big(\lambda + \frac{\gamma^2 - 2^{-2}}{2x^2}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \sqrt{x}\, I_\gamma\big(x\sqrt{2\lambda}\big), \qquad \varphi(x) = \sqrt{x}\, K_\gamma\big(x\sqrt{2\lambda}\big), \qquad w = 1, \qquad \gamma \ge \frac{1}{2}$$

a. $\frac{1}{2}Y''(x) - \frac{\gamma^2 - 2^{-2}}{2x^2}Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{1/2+\gamma}, \qquad \varphi(x) = x^{1/2-\gamma}, \qquad w = 2\gamma, \qquad \gamma \ge \frac{1}{2}$$

5. $\frac{1}{2}Y''(x) - \big(\lambda + \frac{p^2 - 2^{-2}}{2x^2} + \frac{q^2x^2}{2}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-1/2}M_{-\lambda/2q,p/2}(qx^2), \qquad \varphi(x) = x^{-1/2}W_{-\lambda/2q,p/2}(qx^2),$$

A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8

$$w = \frac{2q\Gamma(1+p)}{\Gamma((1+p+\lambda/q)/2)}, \qquad p \ge \frac{1}{2}$$

a. $\frac{1}{2}Y''(x) - \big(\frac{p^2 - 2^{-2}}{2x^2} + \frac{q^2x^2}{2}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \sqrt{x}I_{p/2}(qx^2/2), \qquad \varphi(x) = \sqrt{x}K_{p/2}(qx^2/2), \qquad w = 2, \qquad p \ge \frac{1}{2}$$

6. $\frac{1}{2}Y''(x) - \big(\lambda + \frac{p^2 - 2^{-2}}{2x^2} + \frac{q}{x}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = M_{-q/\sqrt{2\lambda},p}(2x\sqrt{2\lambda}), \qquad \varphi(x) = W_{-q/\sqrt{2\lambda},p}(2x\sqrt{2\lambda}),$$

$$w = \frac{2\sqrt{2\lambda}\Gamma(2p+1)}{\Gamma(p+1/2+q/\sqrt{2\lambda})}, \qquad p \ge \frac{1}{2}$$

a. $\frac{1}{2}Y''(x) - \big(\frac{p^2 - 2^{-2}}{2x^2} + \frac{q}{x}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \sqrt{x}I_{2p}(2\sqrt{2qx}), \qquad \varphi(x) = \sqrt{x}K_{2p}(2\sqrt{2qx}), \qquad w = \frac{1}{2}, \qquad p \ge \frac{1}{2}$$

7. $\frac{1}{2}Y''(x) - (\lambda + \gamma e^{2\beta x})Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = I_{\sqrt{2\lambda}/|\beta|}\Big(\frac{\sqrt{2\gamma}}{|\beta|}e^{\beta x}\Big), \quad \varphi(x) = K_{\sqrt{2\lambda}/|\beta|}\Big(\frac{\sqrt{2\gamma}}{|\beta|}e^{\beta x}\Big), \quad w = \beta, \quad \gamma > 0$$

8. $\frac{1}{2}Y''(x) - \big(\lambda + \frac{p^2}{2}e^{2\beta x} + qe^{\beta x}\big)Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = e^{-\beta x/2}M_{-q/p|\beta|,\sqrt{2\lambda}/|\beta|}\big(\tfrac{2p}{|\beta|}e^{\beta x}\big), \qquad w = \frac{2p\Gamma(2\sqrt{2\lambda}/|\beta|+1)}{\Gamma(\sqrt{2\lambda}/|\beta|+1/2+q/p|\beta|)},$$

$$\varphi(x) = e^{-\beta x/2}W_{-q/p|\beta|,\sqrt{2\lambda}/|\beta|}\big(\tfrac{2p}{|\beta|}e^{\beta x}\big), \qquad p > 0$$

9. $\frac{1}{2}Y''(x) + \mu Y'(x) - (\lambda + f(x))Y(x) = 0, \qquad x \in \mathbf{R}$
the solution $Y_{\mu,\lambda}(x)$ of this equation satisfies the relation

$$Y_{\mu,\lambda}(x) = e^{-\mu x}Y_{0,\lambda+\mu^2/2}(x)$$

10. $\frac{1}{2}Y''(x) + \frac{1}{x}Y'(x) - (\lambda + f(x))Y(x) = 0, \qquad x > 0$
the solution of this equation is $x^{-1}Y_{0,\lambda}(x)$, where $Y_{0,\lambda}(x)$ is the solution of Equation 9 for $\mu = 0$

11. $Y''(x) + \frac{1}{x}Y'(x) - \Big(1 + \frac{\nu^2}{x^2}\Big)Y(x) = 0, \qquad x > 0$

$\psi(x) = I_\nu(x), \qquad \varphi(x) = K_\nu(x), \qquad w = \frac{1}{x}, \qquad \nu > 0$

12. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \lambda Y(x) = 0, \qquad x > 0$

$\psi(x) = x^{-\nu}I_\nu(x\sqrt{2\lambda}), \qquad \varphi(x) = x^{-\nu}K_\nu(x\sqrt{2\lambda}), \quad w = \frac{1}{x^{2\nu+1}}, \quad \nu > -1$

a. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) = 0, \qquad x > 0$

$\psi(x) = 1, \qquad \varphi(x) = x^{-2\nu}, \qquad w = \frac{2\nu}{x^{2\nu+1}}, \qquad \nu > 0$

b. $\frac{1}{2}Y''(x) + \frac{1}{2x}Y'(x) = 0, \qquad x > 1$

$\psi(x) = \ln x, \qquad \varphi(x) = 1, \qquad w = \frac{1}{x}$

c. $\frac{1}{2}Y''(x) + \frac{1}{x}Y'(x) - \lambda Y(x) = 0, \qquad x > 0$

$\psi(x) = \frac{1}{x}\,\mathrm{sh}(x\sqrt{2\lambda}), \qquad \varphi(x) = \frac{1}{x}e^{-x\sqrt{2\lambda}}, \qquad w = \frac{2\sqrt{2\lambda}}{x^2}$

13. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \gamma x Y(x) = 0, \qquad x > 0$

$\psi(x) = x^{-\nu}I_{2\nu/3}\big(\frac{2}{3}x^{3/2}\sqrt{2\gamma}\big), \qquad \varphi(x) = x^{-\nu}K_{2\nu/3}\big(\frac{2}{3}x^{3/2}\sqrt{2\gamma}\big),$

$w = \frac{3}{2}x^{-2\nu-1}, \qquad \nu > -1, \qquad \gamma > 0$

14. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \big(\lambda + \frac{\gamma^2}{2x^2}\big)Y(x) = 0, \qquad x > 0$

$\psi(x) = x^{-\nu}I_{\sqrt{\nu^2+\gamma^2}}\big(x\sqrt{2\lambda}\big), \qquad \varphi(x) = x^{-\nu}K_{\sqrt{\nu^2+\gamma^2}}\big(x\sqrt{2\lambda}\big),$

$w = x^{-2\nu-1}, \qquad \nu \ge 0$

a. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \frac{\gamma^2}{2x^2}Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{\sqrt{\nu^2+\gamma^2}-\nu}, \qquad \varphi(x) = x^{-\sqrt{\nu^2+\gamma^2}-\nu}, \qquad w = \frac{2\sqrt{\nu^2+\gamma^2}}{x^{2\nu+1}}, \qquad \nu \geq 0$$

15. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \big(\lambda + \frac{p^2}{2x^2} + \frac{q^2x^2}{2}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-\nu-1}M_{-\lambda/2q,\sqrt{p^2+\nu^2}/2}(qx^2), \qquad w = \frac{2q\Gamma(1+\sqrt{p^2+\nu^2})x^{-2\nu-1}}{\Gamma((1+\sqrt{p^2+\nu^2}+\lambda/q)/2)},$$

$$\varphi(x) = x^{-\nu-1}W_{-\lambda/2q,\sqrt{p^2+\nu^2}/2}(qx^2), \qquad \nu \geq 0$$

a. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \big(\frac{p^2}{2x^2} + \frac{q^2x^2}{2}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-\nu}I_{\sqrt{\nu^2+p^2}/2}(qx^2/2), \qquad w = 2x^{-2\nu-1},$$

$$\varphi(x) = x^{-\nu}K_{\sqrt{\nu^2+p^2}/2}(qx^2/2), \qquad \nu \geq 0$$

16. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \big(\lambda + \frac{p^2}{2x^2} + \frac{q}{x}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \frac{1}{x^{\nu+1/2}}M_{-q/\sqrt{2\lambda},\sqrt{\nu^2+p^2}}(2x\sqrt{2\lambda}), \qquad w = \frac{2x^{1-2\nu}\sqrt{2\lambda}\Gamma(2\sqrt{\nu^2+p^2}+1)}{\Gamma(\sqrt{\nu^2+p^2}+1/2+q/\sqrt{2\lambda})},$$

$$\varphi(x) = \frac{1}{x^{\nu+1/2}}W_{-q/\sqrt{2\lambda},\sqrt{\nu^2+p^2}}(2x\sqrt{2\lambda}), \qquad \nu \geq 0$$

a. $\frac{1}{2}Y''(x) + \frac{2\nu+1}{2x}Y'(x) - \big(\frac{p^2}{2x^2} + \frac{q}{x}\big)Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-\nu}I_{2\sqrt{\nu^2+p^2}}(2\sqrt{2qx}), \qquad w = 2^{-1}x^{-2\nu-1},$$

$$\varphi(x) = x^{-\nu}K_{2\sqrt{\nu^2+p^2}}(2\sqrt{2qx}), \qquad \nu \geq 0$$

17. $Y''(x) - \left(\frac{x^2}{4} + \frac{2\nu-1}{2}\right)Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = D_{-\nu}(-x), \qquad \varphi(x) = D_{-\nu}(x), \qquad w = \frac{\sqrt{2\pi}}{\Gamma(\nu)} \qquad \nu > 0$$

18. $\sigma^2Y''(x) - xY'(x) - \left(\lambda + \frac{(\gamma^2-1)x^2}{4\sigma^2}\right)Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = e^{x^2/4\sigma^2}D_{-\frac{1}{2}+\frac{1}{2\gamma}-\frac{\lambda}{\gamma}}\big(-\frac{x\sqrt{\gamma}}{\sigma}\big), \qquad \varphi(x) = e^{x^2/4\sigma^2}D_{-\frac{1}{2}+\frac{1}{2\gamma}-\frac{\lambda}{\gamma}}\big(\frac{x\sqrt{\gamma}}{\sigma}\big),$$

$$w = \frac{\sqrt{2\pi\gamma}}{\sigma\Gamma\left(\frac{1}{2} - \frac{1}{2\gamma} + \frac{\lambda}{\gamma}\right)} e^{x^2/2\sigma^2}, \qquad \sigma > 0, \qquad \gamma \geq 1$$

a. $\sigma^2 Y''(x) - xY'(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\sigma\sqrt{2}} e^{v^2} dv, \qquad \varphi(x) = 1, \qquad w = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} e^{x^2/2\sigma^2}, \qquad \sigma > 0$$

19. $\sigma^2 Y''(x) - xY'(x) - \left(\lambda + \frac{\gamma x}{\sigma}\right) Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = e^{x^2/4\sigma^2} D_{\gamma^2-\lambda}\left(-\frac{x}{\sigma} - 2\gamma\right), \qquad \varphi(x) = e^{x^2/4\sigma^2} D_{\gamma^2-\lambda}\left(\frac{x}{\sigma} + 2\gamma\right),$$

$$w = \frac{\sqrt{2\pi}}{\sigma\Gamma(\lambda - \gamma^2)} e^{x^2/2\sigma^2}, \qquad \sigma > 0, \qquad \sqrt{\lambda} > \gamma \geq 0$$

20. $\sigma^2 Y''(x) - xY'(x) - \left(\lambda + \frac{(p^2 - 2^{-2})\sigma^2}{x^2} + \frac{(q^2-1)x^2}{4\sigma^2}\right) Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-1/2} e^{x^2/4\sigma^2} M_{\frac{1-2\lambda}{4q}, \frac{p}{2}}\left(\frac{qx^2}{2\sigma^2}\right), \qquad w = \frac{q\Gamma(p+1) e^{x^2/2\sigma^2}}{\sigma^2 \Gamma\left(\frac{p+1}{2} + \frac{2\lambda - 1}{4q}\right)},$$

$$\varphi(x) = x^{-1/2} e^{x^2/4\sigma^2} W_{\frac{1-2\lambda}{4q}, \frac{p}{2}}\left(\frac{qx^2}{2\sigma^2}\right), \qquad p \geq \frac{1}{2}, \qquad q \geq 1$$

21. $\sigma^2 Y''(x) + \left(\frac{\sigma^2(2\nu+1)}{x} - x\right) Y'(x) - \left(\lambda + \frac{p^2\sigma^2}{x^2} + \frac{(q^2-1)}{4\sigma^2} x^2\right) Y(x) = 0, \qquad x > 0$

$$\psi(x) = x^{-\nu-1} e^{x^2/4\sigma^2} M_{\frac{\nu+1-\lambda}{2q}, \frac{1}{2}\sqrt{\nu^2+p^2}}\left(\frac{qx^2}{2\sigma^2}\right),$$

$$\varphi(x) = x^{-\nu-1} e^{x^2/4\sigma^2} W_{\frac{\nu+1-\lambda}{2q}, \frac{1}{2}\sqrt{\nu^2+p^2}}\left(\frac{qx^2}{2\sigma^2}\right),$$

$$w = \frac{q e^{x^2/2\sigma^2} \Gamma\left(\sqrt{\nu^2+p^2} + 1\right)}{\sigma^2 x^{2\nu+1} \Gamma\left(\frac{1}{2}\sqrt{\nu^2+p^2} + \frac{1}{2} + \frac{\lambda-\nu-1}{2q}\right)}, \qquad q \geq 1, \qquad \nu \geq 0$$

22. $\frac{1}{2} Y''(x) + \left(\frac{2\nu+1}{2x} - \gamma x\right) Y'(x) - \lambda Y(x) = 0, \qquad x > 0$

for $\gamma > 0$, $\nu > -1$

$$\psi(x) = M\left(\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\right), \qquad \varphi(x) = U\left(\frac{\lambda}{2\gamma}, \nu+1, \gamma x^2\right)$$

$$w = \frac{2\Gamma(\nu+1)x^{-2\nu-1}}{\Gamma(\lambda/2\gamma)\gamma^{\nu}} e^{\gamma x^2}$$

for $\gamma > 0$, $\nu < 0$

$$\psi(x) = x^{-2\nu} M\big(\tfrac{\lambda}{2\gamma} - \nu, 1-\nu, \gamma x^2\big), \qquad \varphi(x) = x^{-2\nu} U\big(\tfrac{\lambda}{2\gamma} - \nu, 1-\nu, \gamma x^2\big)$$

$$w = \frac{2\Gamma(1-\nu)x^{-2\nu-1}}{\Gamma(\lambda/2\gamma - \nu)\gamma^{\nu}} e^{\gamma x^2}$$

for $\gamma < 0$, $\nu > -1$

$$\psi(x) = e^{\gamma x^2} M\big(\nu+1-\tfrac{\lambda}{2\gamma}, \nu+1, -\gamma x^2\big), \quad \varphi(x) = e^{\gamma x^2} U\big(\nu+1-\tfrac{\lambda}{2\gamma}, \nu+1, -\gamma x^2\big)$$

$$w = \frac{2\Gamma(\nu+1)x^{-2\nu-1}}{\Gamma(\nu+1-\lambda/2\gamma)|\gamma|^{\nu}} e^{-\gamma x^2}$$

for $\gamma < 0$, $\nu < 0$

$$\psi(x) = x^{-2\nu} e^{\gamma x^2} M\big(1-\tfrac{\lambda}{2\gamma}, 1-\nu, -\gamma x^2\big), \ \varphi(x) = x^{-2\nu} e^{\gamma x^2} U\big(1-\tfrac{\lambda}{2\gamma}, 1-\nu, -\gamma x^2\big)$$

$$w = \frac{2\Gamma(1-\nu)x^{-2\nu-1}}{\Gamma(1-\lambda/2\gamma)|\gamma|^{\nu}} e^{-\gamma x^2}$$

23. $\frac{1}{2}Y''(x) + \rho \operatorname{cth}(x) Y'(x) - \frac{1}{2}(\mu^2 - \rho^2)Y(x) = 0, \quad x > 0$

$$\psi(x) = \frac{P_{\mu-1/2}^{-\rho+1/2}(\operatorname{ch} x)}{\operatorname{sh}^{\rho-1/2} x}, \qquad \varphi(x) = \frac{\widetilde{Q}_{\mu-1/2}^{\rho-1/2}(\operatorname{ch} x)}{\operatorname{sh}^{\rho-1/2} x}, \quad \omega = \frac{1}{\operatorname{sh}^{2\rho} x}, \quad \mu \ge \rho > -\frac{1}{2}$$

24. $\frac{1}{2}Y''(x) - \rho \operatorname{th}(x) Y'(x) - \frac{1}{2}(\mu^2 - \rho^2)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \operatorname{ch}^{\rho} x \big(\widetilde{P}_{\rho}^{\mu}(-\operatorname{th} x) + \widetilde{P}_{\rho}^{\mu}(\operatorname{th} x)\big), \qquad \mu - \rho < 0, \ \mu + \rho < 0$$

$$\varphi(x) = \operatorname{ch}^{\rho} x \, \widetilde{P}_{\rho}^{\mu}(\operatorname{th} x), \qquad \omega = \frac{2 \operatorname{ch}^{2\rho} x}{\Gamma(-\mu-\rho)\Gamma(1-\mu+\rho)}$$

25. $\frac{1}{2}Y''(x) + \big(\operatorname{cth} x - \rho \operatorname{th} x\big) Y'(x) - \frac{1}{2}(\mu^2 - (\rho-1)^2)Y(x) = 0, \qquad x > 0$

$$\psi(x) = \frac{\operatorname{ch}^{\rho} x}{\operatorname{sh} x}\big(\widetilde{P}_{\rho}^{\mu}(-\operatorname{th} x) - \widetilde{P}_{\rho}^{\mu}(\operatorname{th} x)\big), \qquad \mu - \rho + 1 < 0, \ \mu + \rho < 1$$

$$\varphi(x) = \frac{\operatorname{ch}^{\rho} x}{\operatorname{sh} x} \widetilde{P}_{\rho}^{\mu}(\operatorname{th} x), \qquad \omega = \frac{2 \operatorname{ch}^{2\rho} x}{\operatorname{sh}^2 x \, \Gamma(-\mu-\rho)\Gamma(1-\mu+\rho)}$$

26. $\frac{1}{2}Y''(x) + \big(\big(\gamma - \frac{1}{2}\big) \operatorname{cth} x - \big(\beta - \alpha - \frac{1}{2}\big) \operatorname{th} x\big) Y'(x) - 2\alpha(\beta - \gamma)Y(x) = 0$

$$\psi(x) = \frac{1}{\operatorname{ch}^{2\alpha} x} F\big(\alpha, \beta, \gamma, \operatorname{th}^2 x\big), \qquad \varphi(x) = \frac{1}{\operatorname{ch}^{2\alpha} x} G\big(\alpha, \beta, \gamma, \operatorname{th}^2 x\big),$$

$$w = \frac{2\Gamma(\gamma) \operatorname{ch}^{2(\beta-\alpha-1/2)} x}{\Gamma(\alpha)\Gamma(\beta) \operatorname{sh}^{2\gamma-1} x}, \qquad \alpha > 0,\ \beta \geq \gamma > 0$$

For the differential equations in 27–29 it is assumed that $\alpha\beta > 0$, $\alpha+\beta \geq 0$, $\alpha+\beta+1 > \gamma > 0$, and α, β can be complex conjugate.

27. $x(1-x)Y''(x) + (\gamma - (\alpha+\beta+1)x)Y'(x) - \alpha\beta Y(x) = 0, \qquad 0 < x < 1$

$$\psi(x) = F(\alpha, \beta, \gamma, x), \qquad \varphi(x) = G(\alpha, \beta, \gamma, x), \qquad \omega = \frac{\Gamma(\gamma)(1-x)^{\gamma-\alpha-\beta-1}}{x^{\gamma}\Gamma(\alpha)\Gamma(\beta)}$$

28. $Y''(x) + \Big(1 - \gamma + \dfrac{\alpha+\beta-1}{e^x+1}\Big)Y'(x) - \dfrac{\alpha\beta e^x}{(e^x+1)^2} Y(x) = 0, \qquad x \in \mathbf{R}$

$$\psi(x) = G\Big(\alpha, \beta, \gamma, \frac{1}{e^x+1}\Big), \qquad \varphi(x) = F\Big(\alpha, \beta, \gamma, \frac{1}{e^x+1}\Big),$$

$$w = \frac{\Gamma(\gamma)\,(e^x+1)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)\,e^{x(\alpha+\beta-\gamma)}}$$

29. $Y''(x) + \Big(\dfrac{\gamma}{1-e^{-x}} - \alpha - \beta\Big)Y'(x) - \dfrac{\alpha\beta e^{-x}}{1-e^{-x}} Y(x) = 0, \qquad x > 0$

$$\psi(x) = F\big(\alpha, \beta, \gamma, 1-e^{-x}\big), \qquad \varphi(x) = G\big(\alpha, \beta, \gamma, 1-e^{-x}\big),$$

$$w = \frac{\Gamma(\gamma)\,e^{x(\alpha+\beta-\gamma)}}{\Gamma(\alpha)\Gamma(\beta)\,(1-e^{-x})^{\gamma}}$$

EXAMPLES OF TRANSFORMATIONS OF MEASURES ASSOCIATED WITH DIFFUSION PROCESSES

Let $\wp(\cdot)$ be a continuous bounded functional of the processes of § 16 Ch. IV. For brevity the time parameter of a process is placed in subscript.

$$\mathbf{E}_x\wp(W_s^{(\mu)},0\le s\le t)=e^{-\mu x-\mu^2t/2}\mathbf{E}_x\Big\{\wp(W_s,0\le s\le t)e^{\mu W_t}\Big\}$$

$$\mathbf{E}_x\Big\{\wp(W_s^{(\mu)},0\le s\le t);W_t^{(\mu)}\in dz\Big\}=e^{\mu(z-x)-\mu^2t/2}\mathbf{E}_x\{\wp(W_s,0\le s\le t);W_t\in dz\}$$

$$\mathbf{E}_x\Big\{\wp(W_s+\mu s+\eta s^2,0\le s\le t);W_t+\mu t+\eta t^2\in dz\Big\}$$
$$=e^{\mu(z-x)-2\eta^2t^3/3-\mu\eta t^2-\mu^2t/2+2\eta tz}\mathbf{E}_x\Big\{\wp(W_s,0\le s\le t)\exp\Big(-2\eta\int_0^t W_s\,ds\Big);W_t\in dz\Big\}$$

$$\mathbf{E}_x\wp(R_s^{(3)},0\le s\le t)=x^{-1}\mathbf{E}_x\{W_t\wp(W_s,0\le s\le t);\ 0<\inf_{0\le s\le t}W_s\}$$

$$\mathbf{E}_x\Big\{\wp(R_s^{(3)},0\le s\le t);R_t^{(3)}\in dz\Big\}=\frac{z}{x}\mathbf{E}_x\{\wp(W_s,0\le s\le t);0<\inf_{0\le s\le t}W_s,W_t\in dz\}$$

$$\mathbf{E}_x\wp(R_s^{(n)},0\le s\le t)$$
$$=x^{(1-n)/2}\mathbf{E}_x\Big\{W_t^{(n-1)/2}\wp(W_s,0\le s\le t)\exp\Big(-\frac{(n-1)(n-3)}{8}\int_0^t\frac{ds}{W_s^2}\Big);0<\inf_{0\le s\le t}W_s\Big\}$$

$$\mathbf{E}_x\Big\{\wp(R_s^{(n)},0\le s\le t);R_t^{(n)}\in dz\Big\}$$
$$=\frac{z^{(n-1)/2}}{x^{(n-1)/2}}\mathbf{E}_x\Big\{\wp(W_s,0\le s\le t)\exp\Big(-\frac{(n-1)(n-3)}{8}\int_0^t\frac{ds}{W_s^2}\Big);0<\inf_{0\le s\le t}W_s,W_t\in dz\Big\}$$

$$\mathbf{E}_x\wp(Q_s^{(3)},0\le s\le t)=\frac{e^{\gamma t}}{x}\mathbf{E}_x\{U_t\wp(U_s,0\le s\le t);\ 0<\inf_{0\le s\le t}U_s\}$$

$$\mathbf{E}_x\Big\{\wp(Q_s^{(3)},0\le s\le t);Q_t^{(3)}\in dz\Big\}=\frac{ze^{\gamma t}}{x}\mathbf{E}_x\{\wp(U_s,0\le s\le t);0<\inf_{0\le s\le t}U_s,U_t\in dz\}$$

$$\mathbf{E}_x\wp(Q_s^{(n)},0\le s\le t)x^{(n-1)/2}e^{-\gamma(n-1)t/2}$$
$$=\mathbf{E}_x\Big\{U_t^{(n-1)/2}\wp(U_s,0\le s\le t)\exp\Big(-\frac{(n-1)(n-3)\sigma^2}{8}\int_0^t\frac{ds}{U_s^2}\Big);0<\inf_{0\le s\le t}U_s\Big\}$$

$$\mathbf{E}_x\Big\{\wp(Q_s^{(n)},0\le s\le t);Q_t^{(n)}\in dz\Big\}e^{-\gamma(n-1)t/2}$$
$$=\frac{z^{(n-1)/2}}{x^{(n-1)/2}}\mathbf{E}_x\Big\{\wp(U_s,0\le s\le t)\exp\Big(-\frac{(n-1)(n-3)\sigma^2}{8}\int_0^t\frac{ds}{U_s^2}\Big);0<\inf_{0\le s\le t}U_s,U_t\in dz\Big\}$$

$$\mathbf{E}_x\wp(U_s,0\le s\le t)$$
$$=e^{x^2\gamma/2\sigma^2+\gamma t/2}\mathbf{E}_{x/\sigma}\Big\{e^{-\gamma W_t^2/2}\wp(\sigma W_s,0\le s\le t)\exp\Big(-\frac{\gamma^2}{2}\int_0^t W_s^2ds\Big)\Big\}$$

$$\mathbf{E}_x\Big\{\wp(U_s,0\le s\le t);U_t\in dz\Big\}$$
$$=e^{(x^2-z^2)\gamma/2\sigma^2+\gamma t/2}\mathbf{E}_{x/\sigma}\Big\{\wp(\sigma W_s,0\le s\le t)\exp\Big(-\frac{\gamma^2}{2}\int_0^t W_s^2\,ds\Big);W_t\in\frac{dz}{\sigma}\Big\}$$

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n-FOLD DIFFERENTIATION FORMULAS

Let $f^{(n)}(x) := \frac{d^n}{dx^n} f(x)$, $n = 1, 2, 3, \dots$, $\quad f^{(0)}(x) := f(x)$, $\quad 0! = 1$.

1. Leibniz's formula for the nth derivative of the product of two functions:

$$\big(f(x)g(x)\big)^{(n)} = \sum_{k=0}^{n} \frac{n\,!}{k\,!\,(n-k)\,!}\; f^{(k)}(x)\, g^{(n-k)}(x).$$

2. de Bruno's formula for the nth derivative of the composition of two functions:

$$\big(F(f(x))\big)^{(n)} = n\,! \sum_{m=1}^{n} F^{(m)}(f(x)) \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1+k_2+\dots+k_n=m}} \prod_{j=1}^{n} \frac{1}{k_j\,!}\Big(\frac{f^{(j)}(x)}{j\,!}\Big)^{k_j}.$$

3. The nth derivative of the ratio of two functions:

$$\Big(\frac{f(x)}{g(x)}\Big)^{(n)} = \frac{f^{(n)}(x)}{g(x)} + \sum_{1\le m\le l\le n} \frac{(-1)^m m! n! f^{(n-l)}(x)}{(n-l)! g^{m+1}(x)} \sum_{\substack{k_1+2k_2+\dots+lk_l=l \\ k_1+k_2+\dots+k_l=m}} \prod_{j=1}^{l} \frac{1}{k_j\,!}\Big(\frac{g^{(j)}(x)}{j!}\Big)^{k_j}.$$

4. The $(n+1)$th derivative of the inverse function (see Bödewadt (1942)). Let F be a given smooth function with the inverse f. Then

$$f^{(n+1)}(x) = \sum_{m=1}^{n} \frac{(-1)^m (n+m)!}{(F'(f(x)))^{n+m+1}} \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1+k_2+\dots+k_n=m}} \prod_{j=1}^{n} \frac{1}{k_j!}\Big(\frac{F^{(j+1)}(f(x))}{(j+1)!}\Big)^{k_j}.$$

The second sum in formulas 2–4 is taken over all combinations of nonnegative integers $k_1, k_2, \dots$ such that the indicated equalities hold.

5. The nth derivative of the composition of two functions (see Adams (1947)):

$$\big(F(f(x))\big)^{(n)} = \sum_{m=1}^{n} (-1)^m F^{(m)}(f(x)) \sum_{l=1}^{m} \frac{(-1)^l}{l!(m-l)!} \big(f^l(x)\big)^{(n)} f^{m-l}(x).$$

Consequences of formula 5:

$$\big(F\big(\tfrac{1}{x}\big)\big)^{(n)} = \sum_{k=0}^{n-1} \frac{(-1)^n (n-1)! n! F^{(n-k)}(1/x)}{k!(n-1-k)!(n-k)! x^{2n-k}}; \qquad \big(e^{a/x}\big)^{(n)} = \sum_{k=0}^{n-1} \frac{(-1)^n (n-1)! n! a^{n-k} e^{a/x}}{k!(n-1-k)!(n-k)! x^{2n-k}};$$

$$\big(F(x^2)\big)^{(n)} = \sum_{0\le k\le n/2} \frac{n! F^{(n-k)}(x^2)}{k!(n-2k)!(2x)^{2k-n}}; \qquad \big(e^{ax^2}\big)^{(n)} = e^{ax^2} \sum_{0\le k\le n/2} \frac{n! a^{n-k} (2x)^{n-2k}}{k!(n-2k)!};$$

$$\big(F(\sqrt{x})\big)^{(n)} = \sum_{k=0}^{n-1} \frac{(-1)^k (n-1+k)! F^{(n-k)}(\sqrt{x})}{k!(n-1-k)!(2\sqrt{x})^{n+k}}; \qquad \Big(\frac{c+dx}{a+bx}\Big)^{(n)} = \frac{(-1)^n n! (cb-ad) b^{n-1}}{(a+bx)^{n+1}};$$

$$\Big(\frac{c+d\sqrt{x}}{a+b\sqrt{x}}\Big)^{(n)} = \sum_{k=0}^{n-1} \frac{(-1)^n (n-1+k)!(n-k)(cb-ad) b^{n-k-1}}{k!(a+b\sqrt{x})^{n-k+1}(2\sqrt{x})^{n+k}}.$$

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A. N. Borodin, *Stochastic Processes*, Probability and Its Applications, https://doi.org/10.1007/978-3-319-62310-8