Práctico 3: Integrales estocásticas[1][2]

1. Let $0 = t_0 < t_1 < \cdots < t_n = T$ and s_j is an arbitrary point in $[t_j, t_{j+1}]$ for each j. The value of the Riemann integral does not depend on the choice of the points s_j in $[t_j, t_{j+1}]$. In the stochastic case the approximating sums will have the form

$$\sum_{j=0}^{n-1} f(s_j)(W(t_{j+1}) - W(t_j)).$$

It turns out that the limit of such approximations *does* depend on the choice of the intermediate points s_j in $[t_j, t_{j+1}]$. In the next exercise we take $f(t) = W(t)$ and consider two different choices of intermediate points.

Let $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ with $t_j^n = \frac{Tj}{n}$ be a partition of the interval $[0, T]$ into n equal parts. Find the following limits in mean square L^2:

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_j)(W(t_{j+1}) - W(t_j))$$

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_{j+1})(W(t_{j+1}) - W(t_j))$$

Hint: Apply the quadratic variation limit

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T, \quad \text{in } L^2.$$

You will need to transform the sums to make this possible. The identities

$$a(b-a) = \frac{1}{2}(b-a)^2 - \frac{1}{2}(a-b)^2,$$

$$b(b-a) = \frac{1}{2}(b-a)^2 + \frac{1}{2}(a-b)^2,$$

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[2] Algunos ejercicios están tomados del libro *Basic Stochastic Processes* de Zdzisław Brzeźniak and Tomasz Zastawniak, Springer 1975, otros del libro *Stochastic Processes* de A. Borodin, Birkhauser, 2017

may be of help.

2. (a) Demostrar que los procesos y $\{W(t)\}_{0\le t\le T}$ y $\{W(t)^2\}_{0\le t\le T}$ pertenecen a la clase de integrandos $H_2[0,T]$.
(b) Sea $f\colon \mathbf{R}\to\mathbf{R}$ tal que $|f(x)|\le A|x|^n+B$ para algún n natural. Demostrar que $\{f(W(t))\}_{0\le t\le T}$ está también en $H_2[0,T]$.

3. Denote

$$I(f)=\int_0^T f(s)\,dW(s).$$

The isometry property proved in class reads

$$\mathbf{E}\left(I(f)^2\right)=\mathbf{E}\int_0^T f(s)^2\,ds.$$

for any simple process. Verify that for any simple processes f, g

$$\mathbf{E}\left(I(f)I(g)\right)=\mathbf{E}\int_0^T f(s)g(s)\,ds.$$

Hint 1: Try to adapt the proof of the isometry property. Use a common partition $0=t_0<t_1<\cdots<t_n$ in which to represent both f and g.
Hint 2: Alternatively, use the polarization identity (valid for any space with norm $\|x\|=\sqrt{\langle x,x\rangle}$):

$$\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right).$$

4. (a) Calcular la distribución de

$$\int_0^t f(s)\,dW(s).$$

(b) Calcular la distribución de

$$\int_0^t s\,dW(s),$$

condicional a $W(t)=z$.

5. Fórmula de partes. Proof departing from the definition of stochastic integral that

$$\int_0^t s\,dW(s) = tW(t) - \int_0^t W(s)\,ds.$$

The integral on the right-hand side is understood as a Riemann integral defined pathwise, i.e. separately for each $\omega \in \Omega$.)

Hint: You may want to use the partition of $[0, T]$ into n equal parts. The sums approximating the stochastic integral can be transformed with the aid of the identity

$$c(b-a) = (db - ca) - b(d-c).$$

6. Obtain directly from the definition of Itô integral:

$$2\int_0^t W(s)\,dW(s) = W(t)^2 - t.$$

Hint: It is convenient to use a partition of the interval $[0, T]$ into n equal parts. The limit of the sums approximating the integral has been found in Exercise 1.

7. Verify the equality

$$\int_0^T W(t)^2\,dW(t) = \frac{1}{3}W(T)^3 - \int_0^T W(t)\,dt,$$

where the integral on the right-hand side is a Riemann integral.

Hint: As in the exercises above, it is convenient to use the partition of $[0, T]$ into n equal parts. You may also need the following identity:

$$(a^2 - b^2)^2 = (a-b)^4 + 4(a-b)^3 b + 4(a-b)^2 b^2,$$

8. *Wiener integral.* (a) Use Itô's formula to prove that

$$\int_0^T e^{W(t)-t/2} dW(t) = e^{W(T)-T/2} - 1.$$

(b) Use the Euler scheme to simulate the integral

$$I = \int_0^1 e^{W(t)-t/2} dW(t).$$

Plot a histogram and estimate the expectation and the variance of I.

(c) Consider the random variable

$$J = e^{W(1)-1/2} - 1.$$

Plot a histogram, compute the expectation and variance of J.

(d) Plot the two histograms in the same figure and comment the results.

9. We consider the stochastic integral

$$I = \int_0^1 e^t dW(t).$$

(a) Compute $\mu = \mathbf{E}(I)$ and $\sigma^2 = \mathbf{var}(I)$.

(b) Simulate values of I using the Euler scheme for stochastic integrals and estimate approximately μ and σ^2.

(c) Plot in the same figure a histogram of the sample for I with the corresponding normal density.

10. *Itô isometry.* We check the isometry property in an example using simulation. Consider the process $\{h(t) = e^{W(t)} \colon 0 \leq t \leq 1\}$. The property states

$$\mathbf{E}\left(\int_0^1 e^{W(t)} dW(t)\right)^2 = \int_0^1 \mathbf{E}[(e^{W(t)})^2] dt = \int_0^T \mathbf{E}(e^{2W(t)}) dt.$$

(a) First, using that $\mathbf{E}(e^{\mathcal{N}(\mu,\sigma^2)}) = e^{\mu+\sigma^2/2}$, compute $\int_0^1 \mathbf{E}(e^{2W(s)}) ds$.

(b) Compute by simulation

$$\mathbf{E}\left(\int_0^1 e^{W(s)} dW(s)\right)^2, \quad 0 \leq t \leq 1,$$

with the corresponding error, and check that the numbers coincide.

11. We want to check numerically that

$$\int_0^1 W(t) dW(t) = \frac{1}{2}(W(1)^2 - 1).$$

Write then a code to compute a Brownian trajectory, compute both the integral and the result. Repeat the previous experiment a reasonable number of times, and plot the results in an (x, y) plot.

12. *Hermite polynomial of degree three.* (a) Write a code to simulate the integral

$$I = \int_0^1 (W(t)^2 - t)dW(t).$$

Plot a histogram and compute the expectation and the variance of I.

(b) Consider the random variable

$$J = \frac{1}{3}W(1)^3 - W(1).$$

Plot a histogram, compute the expectation and variance of J.

(c) Use Itô formula with the function[3] $H_3(t, x) = \frac{1}{3}x^3 - tx$ to prove that in fact $I = J$.

13. *Hermite polynomial of degree four.* Define

$$H_4(t, x) = x^4 - 6x^2t + 3t^2.$$

Use Itô's formula to prove that

$$H_4(t, W(t)) = 12 \int_0^t H_3(s, W(s))dW(s)$$

Conclude that $\mathbf{E}H_4(t, W(t)) = 0$.

14. *Brownian Bridge.* (a) Let $\{W(t) \colon 0 \le t \le 1\}$ be a Brownian motion. Prove that the process

$$R(t) = (1 - t) \int_0^t \frac{1}{1 - r} dW(r), \quad 0 \le t \le 1,$$

is a Brownian bridge.

(b) We are interested in the random variable

$$A = \int_0^1 R(t)dt.$$

[3] $H_3(t, x)$ is the Hermite polynomial of degree 3.

Prove that $\mathbf{E}(A) = 0$, and device a simulation scheme using the representation of part (a) to estimate $\mathbf{var}(A)$ (True value $1/12$).