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Understanding radiation impedance through animations

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It is very common for students to struggle to grasp a conceptual understanding of radiation impedance. Often graduate students need exposure to impedance concepts in more than one course before they start to understand its importance and meaning. Instructors commonly suggest that impedance can be thought of as a resistance, ignoring the imaginary part of the impedance (the reactance), and the word "sloshing" is sometimes used to describe what the radiation reactance represents. This paper will describe animations that have been developed in MATLAB to help students visualize the motion of a spherical source and a baffled circular piston along with the corresponding radiated sound pressure near these surfaces. These animations utilize the relevant physical equations to observe the phase relationship of the pressure to the surface velocity. Thinking of radiation impedance as the ratio of a potential quantity to a flow quantity can be helpful, with this ratio having an in phase component and a component where the two quantities are 90 degree phase shifted. Quotes from several acoustics textbooks on this subject are collected in this paper as well.

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1. INTRODUCTION

Radiation impedance in acoustics is a challenging topic for students to understand. Before taking acoustics courses that cover radiation impedance topics, often students have only been exposed to electrical impedance, and sometimes that exposure has been limited to thinking of impedance as a purely real quantity – resistance. The author has found it helpful to think of acoustic radiation impedance in two ways. The first as simply the ratio of two quantities, the radiated acoustic pressure adjacent to a vibrating surface divided by the particle velocity of the fluid right next to the vibrating surface. The second helpful definition is to think of radiation impedance as a useful way to determine the efficiency in which vibration of a surface is translated into radiated sound away from that surface.

Impedance quantities, in general, consist of a real part (the resistance) and an imaginary part (the reactance). For example the impedance

$$Z = R + jX, \tag{1}$$

where R is the resistance, X is the reactance, and $j = \sqrt{-1}$ (the imaginary number). Within the acoustics discipline impedance may be defined within several physical domains in which the impedance in that domain is defined as a ratio of the potential quantity (voltage like and usually related to a force) to a flow quantity (current like and usually related to a velocity). Electrical impedance, Z_E, is the ratio of voltage, e, to current, i, mechanical impedance, Z_M, is the ratio of force, F, to velocity, v, and acoustical impedance, Z_A, is the ratio of acoustic pressure, p, to volume velocity, U (volume velocity is the particle velocity times a characteristic area). Often another impedance quantity is often defined within the acoustic domain called specific acoustic impedance, z_{SA}, which is the ratio of acoustic pressure to particle velocity, u.

In addition to the impedances in different physical domains, impedance may be used to describe the ratio of quantities either for special situations or quantities at specific locations. The characteristic impedance is a quantity that describes a special situation, generally where plane waves travel outward but are not reflected back. The mechanical characteristic impedance is defined as $\rho_0 cS$, where ρ_0 is the mass density in the medium, c is the wave speed in the medium, and S is a characteristic surface area for the medium. The specific acoustic characteristic impedance is defined as $\rho_0 c$. The acoustical characteristic impedance is defined as $\rho_0 c/S$. One location specific impedance quantity that is commonly defined is the input impedance. This is often used for example in musical acoustics applications where the input impedance is defined as the ratio of the potential quantity to the flow quantity at the input to the musical resonator, such as at the mouthpiece of a clarinet. A second location specific impedance quantity is the radiation impedance. This is defined as the ratio of the potential and flow quantities at the face of a vibrating surface, such as a radially oscillating sphere or a baffled circular piston. This later quantity, the radiation impedance, will be the focus of this paper.

This paper will describe some animations that the author has developed to allow the viewer to visualize the phase relationship between the potential and flow quantities on the surface of a monopole radiator and on the surface of a baffled circular piston in two dimensions (links to videos of the animations will be given and if all else fails, the reader is welcome to contact the author via electronic mail to obtain the latest link for the videos – bea@byu.edu). It is important to note that the theory and animations presented are only valid in the steady state, after all initial transients have died out. Before showing static images of these animations, the paper will describe radiation impedance and present a collection of quotes from several acoustics textbooks on radiation impedance. A brief theoretical development of the time dependent displacement and pressure equations that have been animated will be given, and finally low and high frequency approximations of radiation impedance will be given and the implications of these expressions will guide the visual understanding that we are seeking as we view the images from the animations.

2. DEFINITION OF RADIATION IMPEDANCE

A. Radiation Impedance and its Real and Imaginary Parts

As mentioned previously, in the acoustic domain, the radiation impedance is defined as the ratio of the pressure to the volume velocity at the face of a radiator. Technically the pressure is obtained from the spatial average of the pressure over the entire vibrating surface. The particle velocity of the fluid immediately adjacent to the vibrating surface is equal to the mechanical velocity of the surface (the condition of continuity) and thus the volume velocity is the particle velocity times the surface area of the vibrating object. Throughout this paper we will assume that every part of the object that is vibrating is vibrating in phase with the same amplitude, which is sometimes called pistonic motion or breathing motion. The entire surface of the object does not need to be vibrating but each part that is must vibrate together (in phase).

The theoretical expressions for the radiated pressure and volume velocity quantities at the surface of the vibrating object can include imaginary number components, and thus the ratio of these quantities (the radiation impedance) is therefore a complex quantity in general. The real part of the radiation impedance (the radiation resistance) can be thought of as governing the energy loss from the radiator in the form of sound radiation away from the radiator. The imaginary part of the radiation impedance (the radiation reactance) can be thought of as governing the energy stored in the fluid that continually reacts with the vibrating surface and affects or impedes its motion. This stored energy does not travel away from the radiator. If the application of interest requires the efficient generation of sound, such as for a listener from a loudspeaker, then a high radiation resistance and a low radiation reactance is desired. If the application of interest is to reduce the noise generated by some vibrating object then a low radiation resistance and a high radiation reactance is desired.

B. Phase Relationships between Pressure and Particle Motion

For many vibrating objects, at low frequencies such that the wavelength of sound is much larger than the vibrating source dimensions, the mechanical radiation impedance, Z_{MR}, is dominated by the radiation reactance, X_{MR}, and specifically a mass like radiation reactance of the form

$$Z_{MR} \approx X_{MR} = \omega M_{MR}, \tag{2}$$

where ω is the angular frequency and M_{MR} is the acoustical radiation mass loading. M_{MR} represents an effective mass of fluid that adds to the inertia of the vibrating structure (a fluid loading) and thereby impedes the motion. As frequency increases, the reactance linearly increases, further impeding the motion. Another important feature of the radiation reactance at low frequencies is that the vibrating object's velocity lags behind the pressure of the fluid next to it by approximately 90°. For systems vibrating sinusoidally, the displacement amplitude always lags behind the velocity by 90°. Thus the displacement and pressure are essentially 180° out of phase at low frequencies. When the displacement is a maximum positive value, the pressure is a negative maximum value and vice versa. Keep in mind that these relationships hold only at the surface of the object.

Additionally, for many objects vibrating at high frequencies such that the wavelength of sound is much smaller than the vibrating source dimensions, the radiation impedance is dominated by the radiation resistance, R_{MR}. The radiation resistance equals the characteristic impedance at high frequencies. Now that the radiation impedance is dominated by a resistance, the surface pressure and the object's velocity are in phase. Thus the displacement lags behind the pressure by 90° at the surface.

C. Definitions and Usefulness of Radiation Impedance

I will now reproduce various definitions of radiation impedance, and then later definitions of other quantities of interest, as given in a few different acoustics textbooks. Morse defines radiation impedance as "the reaction of the air back on the piston".[1] Randall defines radiation impedance as "a measure of the reaction on the source due to the radiation".[2] Beranek defines it as "a quantitative statement of the manner in which the medium reacts against the motion of a vibrating surface".[3] Skudrzyk defines it as "the ratio of the force per unit area to the piston velocity".[4] Kinsler and Frey *et al.* state that it is the "impedance …

associated with the radiation of sound".[5] Fahy discusses radiation impedance extensively and often refers to "fluid loading" as being related "to the forces a fluid exerts on a vibrating structure with which it is in contact, in reaction to that vibration".[6] Finally, in a different book, Fahy states that it is "employed to relate the vibration velocity of a rigid body to the associated fluid reaction force" and "the impedance presented by the fluid in reaction to disturbance by this wave determines both the fluid pressure on the boundary and the effectiveness with which the wave radiates sound energy".[7]

In regards to the usefulness of radiation impedance, Randall said that this impedance "will determine the speaker's effectiveness as a source of sound".[2] Skudrzyk said that it is used to "characterize the properties of the reflecting medium".[4] Kinsler and Frey *et al.* said that it "is used in calculating the coupling between acoustic waves and a driving source or driven load".[5] Fahy discussed how fluid loading, particularly for dense fluids, "can change both their natural frequencies and dampings-by radiation- and hence can influence the vibration response to excitation forces".[6] Pierce said that it is "useful in the prediction of the effects of the surrounding fluid on the vibration of a solid", that it is more important for underwater applications than for air applications, and that "it is useful in the analysis of the efficiency with which a vibration can generate sound".[8] Blackstock said that "an analysis of the impedance presented to the source by the medium leads to useful conclusions about efficiency of spherical radiators".[9] Garrett states that "this is critical to the design of sources that operate in dense fluids (i.e., liquids rather than gases) since the source has to provide sufficient power to accelerate and decelerate the surrounding fluid as it pulsates".[10]

D. Radiation Resistance and Mass-Loading

Now on the subject of radiation resistance, Rayleigh states that "the reaction of the air is represented by a frictional force".[11] Randall states that "the resistive component" is "the only part involved in radiation of real sound energy".[2] Beranek, in referring to the radiated sound energy related to the radiation resistance stated that "this radiated energy is useful and represents the power output of the loudspeaker".[3] Olson stated that "this is the part responsible for the dissipation of energy".[12] Junger and Feit described it as stemming from "a resistive or damping force which embodies energy lost by the pulsating sphere in the form of radiated acoustic energy".[13] Fahy stated that it "is associated with energy radiation, which produces radiation damping".[6] Finally, Pierce stated that it is the "impedance associated with sound generation by fluid motion".[8] Garrett stated that it "is an accounting loss rather than an irreversible increase in entropy. For radiation, the energy propagates away; it is not absorbed, it is expelled".[10]

On the subject of the radiation reactance and the mass loading it represents at low frequencies, Rayleigh stated that "this part of the reaction of the air is therefore represented by supposing the vibrating plate to carry with it a mass of air".[11] Randall, in referring to the sound power used up by radiation reactance stated that it "is 'wattless' power, involving energy which surges out from the source and then back towards the source, without ever being radiated as sound waves" and that it involves "the mass or inertial property of the air that is involved".[2] Beranek, in discussing the energy required to move a vibrating loudspeaker said that some of the energy "is stored (reactive) energy that is returned to the generator".[3] Skudrzyk said that it is "the mass reaction of the medium to the vibrating sphere", the "additional apparent mass of the sphere", and "accession to inertia".[4] Junger and Feit stated that it is "a term proportional to the surface particle acceleration, embodying the inertia force associated with the accession to inertia or entrained mass of fluid set into motion by the pulsating surface of the spherical source".[13] Fahy stated that "this is associated with the kinetic energy of fluid motion in the proximity of the piston".[6] Pierce described it as the "fluid entrained by the piston".[8] Finally, Garrett states that this reactance is because "the fluid surrounding the source is behaving like an effective mass".[10] It should be noted that accession refers to the new attainment of something, in this case inertia. The fluid mass is additional mass that the vibrating surface must now move. To entrain means to draw along with, meaning that the mass moves along with the vibrating surface.

3. SPHERICAL SOURCE RADIATION IMPEDANCE

Consider a sphere of radius a that is vibrating in the steady state with a velocity

$$v(t) = u_0 e^{j\omega t}, \tag{3}$$

where u_0 is the velocity amplitude and t is time. A radially oscillating sphere would naturally generate spherical sound waves. Thus we assume the radiated pressure has the form

$$p(r,t) = \frac{A}{r} e^{j(\omega t - \quad)}, \tag{4}$$

where A is the pressure amplitude, r is the radial distance from the origin that is in the center of the sphere, and $k = \omega/c$ is the acoustic wavenumber. The particle velocity on the surface of the sphere must equal the velocity of the sphere to ensure continuity of the fluid,

$$u_r(a,t) = u_0 e^{j\omega t}. \tag{5}$$

To solve for A we need to relate the particle velocity to the pressure on the surface of the sphere. The linearized Euler equation can be used to relate these two quantities:

$$\rho_0 \frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r}. \tag{6}$$

Substitution of the pressure expression from Eq. (4) and the particle velocity expression from Eq. (5) into Euler's equation we evaluate the partial derivatives (using the product rule for the r derivative) to obtain

$$j\omega\rho_0 u_0 e^{j\omega t} = -\left[-jk\frac{A}{r}e^{j(\omega t - kr)} - \frac{A}{r^2}e^{j(\omega t - kr)}\right]_{r=a}. \tag{7}$$

We then factor out the original pressure expression from the terms on the right hand side and on the left hand side we multiply by $\frac{c}{c}$ to arrive at

$$jk\rho_0 c u_0 e^{j\omega t} = \frac{A}{a} e^{j(\omega t - ka)}\left[jk + \frac{1}{a}\right] \tag{8}$$

We now recognize that the $u_0 e^{j\omega t}$ term is $u_r(a,t)$ and the $\frac{A}{r}e^{j(\omega t - kr)}$ term is $p(a,t)$ yielding

$$jk\rho_0 c u_r(a,t) = p(a,t)\left[jk + \frac{1}{a}\right]. \tag{9}$$

We can now solve for the mechanical radiation impedance,

$$Z_{MR} = \frac{S \cdot p(a,t)}{u_r(a,t)} = \frac{jk\rho_0 cS}{\left[jk + \frac{1}{a}\right]} = \frac{\rho_0 cS}{\frac{1}{jk}\left[jk + \frac{1}{a}\right]} = \frac{\rho_0 cS}{\left[1 - \frac{j}{ka}\right]}, \tag{10}$$

where S is the surface area of the sphere, $S = 4\pi a^2$. Finally we rationalize the denominator, multiply by $\frac{(ka)^2}{(ka)^2}$, and separate into real and imaginary parts,

$$Z_{MR} = \frac{\rho_0 cS\left[1 + \frac{j}{ka}\right]}{\left[1 - \frac{j}{ka}\right]\left[1 + \frac{j}{ka}\right]} = \frac{\rho_0 cS\left[1 + \frac{j}{ka}\right]}{\left[1 + \frac{1}{(ka)^2}\right]} = \frac{\rho_0 cS(ka)^2}{[1 + (ka)^2]} + j\frac{\rho_0 cS(ka)}{[1 + (ka)^2]}. \tag{11}$$

Figure 1 displays plots of Z_{MR} from Eq. (11) as a function of ka.

At low frequencies, $ka \ll 1$, and the mechanical radiation impedance simplifies to

$$Z_{MR}(ka \ll 1) = \rho_0 cS(ka)^2 + j\rho_0 cS(ka) \approx j\rho_0 cS(ka) = j\omega(\rho_0 Sa). \tag{12}$$

The product Sa represents a volume that is three times larger than the volume of the sphere, V, at equilibrium, and is the volume of the fluid displaced by the sphere.

$$Z_{MR}(ka \ll 1) = j\omega(3\rho_0 V) = j\omega(3m), \tag{13}$$

where m is the mass of the fluid displaced by the sphere.[10] In this case the particle velocity lags behind the pressure by 90°.

At high frequencies, $ka \gg 1$, and the mechanical radiation impedance simplifies to

$$Z_{MR}(ka \gg 1) = \rho_0 cS + j\frac{\rho_0 cS}{(ka)} \approx \rho_0 cS, \tag{14}$$

which represents a purely resistive radiation impedance. In this case the pressure and the particle velocity are in phase.

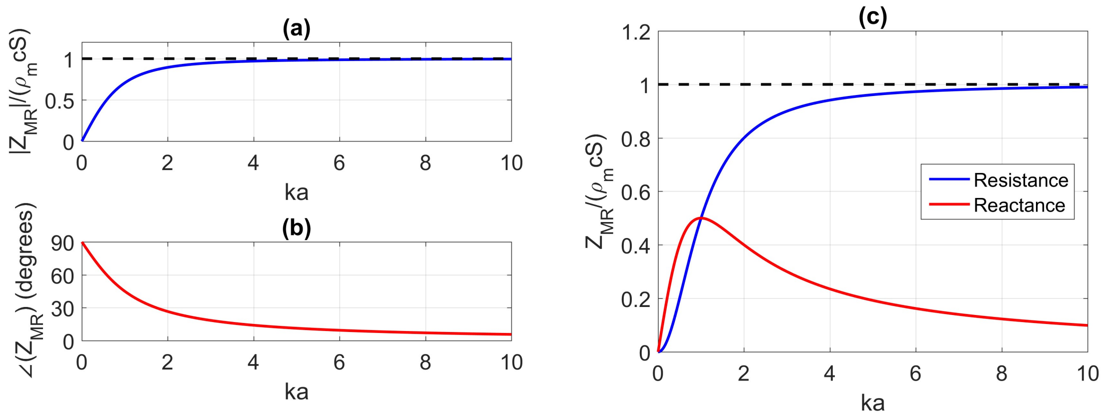

Figure 1. Radiation impedance versus ka ***for a radially oscillating sphere: (a) magnitude, (b) phase, and (c) real and imaginary parts.***

4. BAFFLED CIRCULAR PISTON RADIATION IMPEDANCE

The steady state sound pressure radiated from the baffled circular piston cannot be specified exactly in three dimensions in the near field because it has no closed form solution for 3D radiation in this region. Typically textbooks derive the on axis sound pressure from the piston face out into the far field, which may have pressure nulls along the axis at high frequencies, and/or they derive the far field angular dependence of the sound pressure to analyze the directivity of the baffled circular piston. Morse and Ingard[14] provide the mechanical radiation impedance of a baffled circular piston of radius a,

$$Z_{MR} = \rho_0 cS\left[1 - \frac{2J_1(2ka)}{(2ka)}\right] + j\rho_0 cS\left[\frac{2H_1(2ka)}{(2ka)}\right], \tag{15}$$

where J_1 is the first-order Bessel function of the first kind and H_1 is the first-order Struve function. Garrett[10] gives the series approximation of the resistance and reactance terms,

$$R_{MR} = \rho_0 cS\left[1 - \frac{2J_1(2ka)}{(2ka)}\right] \cong \rho_0 cS\left[\frac{(2ka)^2}{2\cdot 4} - \frac{(2ka)^4}{2\cdot 4^2\cdot 6} + \frac{(2ka)^6}{2\cdot 4^2\cdot 6^2\cdot 8} + (-1)^{n+1}\frac{(2ka)^{2n}}{\frac{1}{4(n+1)}\prod_{i=1}^{n}(2i)^2}\dots\right], \tag{16}$$

$$X_{MR} = \rho_0 cS\left[\frac{2H_1(2ka)}{(2ka)}\right] \cong \rho_0 cS\frac{4}{\pi}\left[\frac{(2ka)}{3} - \frac{(2ka)^3}{3^2\cdot 5} + \frac{(2ka)^5}{3^2\cdot 5^2\cdot 7} + (-1)^{n+1}\frac{(2ka)^{2n-1}}{\frac{1}{2n+1}\prod_{i=1}^{n}(2i+1)^2}\dots\right]. \tag{17}$$

Figure 2 displays plots of Z_{MR} from Eq. (15) as a function of ka.

At low frequencies, $ka \ll 1$, the R_{MR} and X_{MR} functions simplify to the leading, first order terms of Eqs. (16) and (17)

$$R_{MR}(ka \ll 1) \cong \rho_0 cS\left[\frac{(2ka)^2}{2\cdot 4}\right] = \frac{\rho_0 cS}{2}(ka)^2, \tag{18}$$

$$X_{MR}(ka \ll 1) \cong \rho_0 cS\frac{4}{\pi}\left[\frac{(2ka)}{3}\right] = \frac{8\rho_0 cS}{3\pi}(ka). \tag{19}$$

Note that $X_{MR} \gg R_{MR}$ since R_{MR} is second order in ka and $X_{MR}$is first order in ka, thus

$$Z_{MR}(ka \ll 1) \cong j\frac{8\rho_0 cS}{3\pi}(ka) = j\omega\left(\frac{8}{3\pi}\rho_0 Sa\right). \tag{20}$$

Again this impedance is mass like just like the expression in Eq. (13) is mass like. Again the particle velocity lags behind the pressure by 90°.

At high frequencies, $ka \gg 1$, the R_{MR} and X_{MR} functions simplify to

$$R_{MR}(ka \gg 1) \cong \rho_0 cS, \tag{21}$$

$$X_{MR}(ka \gg 1) \cong \frac{\rho_0 cS}{ka} \approx 0. \tag{22}$$

Thus

$$Z_{MR}(ka \gg 1) \approx \rho_0 cS. \tag{23}$$

This impedance is purely resistive at high frequencies, just like the expression in Eq. (14). Again the pressure and the particle velocity are in phase.

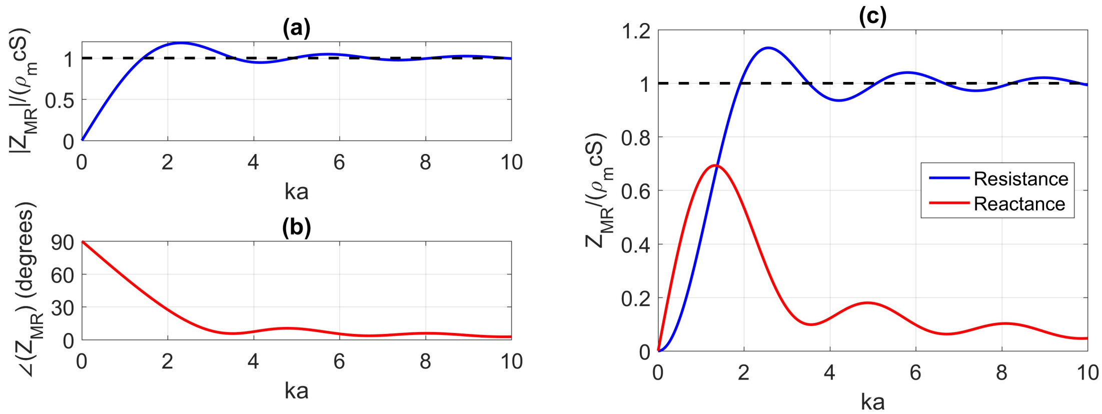

Figure 2. Radiation impedance versus ka for a baffled circular piston: (a) magnitude, (b) phase, and (c) real and imaginary parts.

5. SPHERICAL SOURCE RADIATION ANIMATIONS

To animate a moving spherical source along with the radiated pressure in the steady state, a two-dimensional animation is created. The spherical source is represented by a circle that expands and contracts in radius. The velocity of the circle is represented in Eq. (3), $v(t) = u_0 e^{j\omega t}$. The displacement of the circle would then be

$$x(t) = \frac{\partial v}{\partial t} = j\omega u_0 e^{j\omega t} = j\omega v. \tag{24}$$

Thus the displacement lags behind the velocity by 90°. The real part of the displacement is what should be seen in an animation. In the animations, we will consider the circle to be moving with a displacement that may be described by a sine wave. To create the animation, the circle is plotted at its rest radius, plus a time varying sine wave to cause it to move outward and then inward.

We can return to Eq. (8) and solve for the constant A. The $e^{j\omega t}$ term cancels and we can move the terms from the right hand side of the equation over to the left to solve for A,

$$A = \frac{jk\rho_0 c u_0 a e^{jka}}{\left[jk + \frac{1}{a}\right]}. \tag{25}$$

If we multiply the numerator and the denominator by πa and with some rearranging of the numerator we arrive at

$$A = \frac{j\rho_0 cku_0 \pi a^2 e^{jka}}{\pi[1 + jka]}. \tag{26}$$

If we define a source strength, $Q = 4\pi a^2 u_0$, that represents the surface area of the sphere times the velocity magnitude, then we arrive at

$$A = \frac{j\rho_0 ckQe^{jka}}{4\pi[1 + jka]}. \tag{27}$$

If we assume that any frequency radiated by this simple source has a wavelength, λ, much larger than a then we can say that $\lambda \gg a$ or $1 \gg a/\lambda$, which is the definition of an acoustic simple source. If we multiply both sides by 2π and recognize that $k = 2\pi/\lambda$ then $2\pi \gg ka$. We could also say that ka is small enough such that $1 \gg ka$. This allows us to make a couple simplifications to Eq. (17), including $[1 + jka] = 1$ and $e^{jka} = 1$, such that Eq. (17) becomes

$$A = \frac{j\rho_0 ckQ}{4\pi}. \tag{28}$$

We finally arrive at an expression for the radiated sound pressure from a simple source

$$p(r,t) = \frac{j\rho_0 ckQ}{4\pi} e^{j(\omega t - kr)}. \tag{29}$$

The radiated pressure to animate is the pressure expressed in Eq. (29) evaluated at all points greater than the rest radius away from the origin. The pressure at each (x, y) position in a Cartesian axis animation at a specific snapshot in time then is

$$p(x, y, t) = Re\left\{\frac{j\rho_0 ckQ}{4\pi\sqrt{x^2+y^2}} e^{j\left(\omega t - k\sqrt{x^2+y^2}\right)}\right\}. \quad (30)$$

The time may then be varied between $t = 0$ and $t = T$, where T is the period of the selected frequency. The pulsating sphere can be represented by a circle that is expanding and contracting a displacement that varies in time with a $\sin(\omega t)$ dependence, with perhaps an exaggerated amplitude in order to clearly see its motion. For a good video animation it is suggested that you create around 50 time snapshot frames or more within that period. The animation can then be converted to an animated .gif file or a regular video file and played on loop.

First we will consider snapshots from an animation of the behavior at low frequencies. At low frequencies, as noted in Section 3, the particle velocity lags behind the radiated surface pressure by 90°. This means that the displacement and the radiated pressure on the surface of the circle will be exactly out of phase, 180°, with respect to one another. When the circle is expanded out to its maximum extent (maximum positive displacement) then the radiated pressure on its surface is in a state of maximum expansion, or rarefaction. When the circle contracts inward to its maximum extent (maximum negative displacement) then the radiated pressure on its surface is in a state of maximum compression. To illustrate what happens at low frequencies we will assume a frequency value such that $ka = 0.1$ (this means that $a = \lambda/(20\pi)$. Figure 3 displays some snapshots in time of the progression of the animation at several fractions of T. The next snapshot would be back to $t = 0$ again. Notice how the pressure at $t = 0$ and $t = 0.5T$ are nearly zero and the velocity is maximum in the outward direction and maximum in the inward direction, respectively (the pressures would be zero at extreme low frequencies). The pressure at $t = 0.25T$ is a maximum rarefaction, showing that the displacement and pressure are exactly out of phase. The pressure at $t = 0.75T$ is a maximum compression. Note that the displacement of the circle has been exaggerated for visualization purposes.

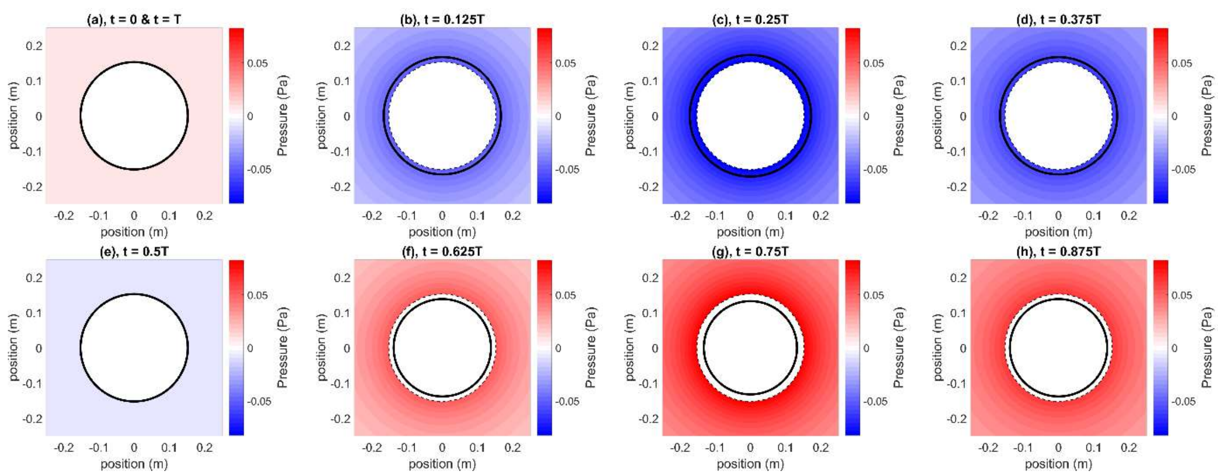

Figure 3. Snapshots in time of the steady-state, spherical source, represented by the solid black circle, emitting a low frequency ($ka = 0.1$) as it moves outward and then inward and the corresponding radiated pressure. The solid black circle represents the actual radius of the spherical source as it expands and contracts (with exaggerated motion); the dashed circle represents the equilibrium radius. The sphere initially expands outward and then contracts inward.

I will now attempt to explain the physical reason as to why the displacement and the pressure are out of phase for low frequencies. As the circle moves inward, it starts to pull inward on the nearby fluid molecules, accelerating the fluid inward, but the fluid doesn't immediately respond to this motion due to its inertia. Then when the circle is maximally contracted it momentarily stops moving but the fluid is still moving inward. This causes the fluid to become maximally compressed as the surface starts to move outward, opposite to the direction in which the fluid is moving. Eventually the fluid begins to move back outward, pushed along by the surface. Then when the surface reaches its maximum extent it momentarily stops moving. However, the fluid continues to move outward, and thus the fluid is expanded as some of the fluid stays with the surface (the continuity condition) but much of it is moving outward. The cycle then repeats. This description fits well with the idea that the fluid is being drawn along with the surface. The nearby fluid behaves as an added mass because when you apply a force (which is proportional to pressure) on a lumped mass, the mass doesn't immediately move, rather there is some inertia involved. The mass has to be accelerated into motion. When a mass is moving and you try to apply a force in the opposite direction, the mass first has to slow down to a stop and then it can move in the opposite direction. The kinetic energy of the circle, when the circle is moving at its fastest speed, is not in phase with the potential energy of the fluid, when the sound pressure is maximum. Thus at low frequencies there is not an efficient transfer of energy from the moving spherical source to radiated sound pressure.

Next we will consider an animation of the behavior at high frequencies. At high frequencies, as noted in Section 3, the particle velocity is in phase with the radiated surface pressure. This means that the displacement will lag behind the radiated pressure by 90°. When the circle is expanded out to its maximum extent (maximum positive displacement) then the radiated pressure on its surface will be zero. When the circle contracts inward to its maximum extent (maximum negative displacement) then the radiated pressure on its surface will again be zero. To illustrate what happens at high frequencies we will assume a frequency value such that $ka = 10$ (this means that $a = 10\lambda/(2\pi)$. Figure 4 displays some snapshots in time of the progression of the animation at several fractions of T. The next snapshot would be back to $t = 0$ again. Notice how the surface pressure at $t = 0$ and $t = 0.5T$ are in a state of maximum compression and rarefaction, respectively, and in these cases the velocity is maximum in the outward and then inward directions, respectively, as well. When the circle is expanded ($t = 0.25T$) or contracted ($t = 0.75T$) to its full extent, the pressure is zero and the velocity of the circle is also zero.

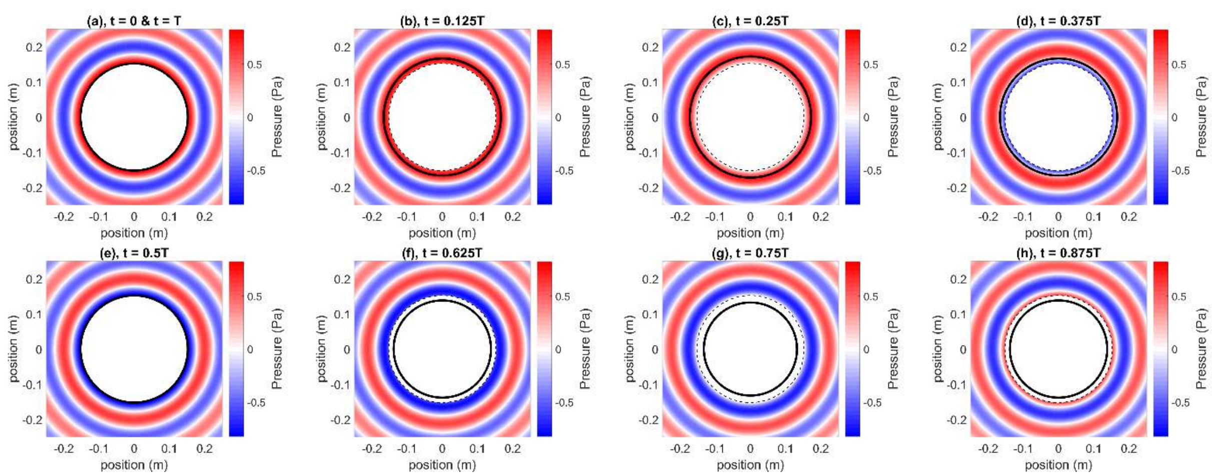

Figure 4. Snapshots in time of the steady-state, spherical source, represented by the black circle, emitting a low frequency ($\mathbf{ka} = \mathbf{10}$) as it moves outward and then inward and the corresponding radiated pressure. The solid black circle represents the actual radius of the spherical source as it expands and contracts (with exaggerated motion); the dashed circle represents the equilibrium radius. The sphere initially expands outward and then contracts inward.

At high frequencies, the radiation impedance is purely real, meaning that the kinetic energy of the circle is efficiently converted into radiated sound pressure (potential energy). When the circle moves between its point of maximal contraction to the point of maximal extension, the fluid is compressed at the exact moment the circle is moving outward at its fastest speed. Conversely when the circle moves between its point of maximal extent to the point of maximal contraction, the fluid is expanded at the exact moment the circle is moving inward at its fastest speed. It's like the fluid is responding instantly to the motion of the circle, as if the surface doesn't have to do any work on he fluid. The point of maximum speed is when the kinetic energy of the circle is at its maximum, which is the same instant as when the potential energy of the fluid is maximum since the potential energy is proportional to the squared pressure.

6. BAFFLED CIRCULAR PISTON RADIATION ANIMATIONS

A two-dimensional animation of the steady state motion of a baffled circular piston, represented as a line in two-dimensional space, and the corresponding radiated sound pressure will now be presented. The piston will be a vertical line that moves to the right (positive displacement) and to the left (negative displacement) at a single frequency. Again the displacement of the piston will lag behind the velocity of the piston by 90°. At low frequencies we again expect the displacement of the piston to be 180° out of phase with the radiated sound pressure on the surface of the piston. At high frequencies we also expect the displacement of the piston to lag behind the radiated surface pressure by 90°, just as with the spherical source.

In order to plot the radiated sound pressure at any spatial position, particularly in the near field of the piston, it was determined that a two-dimensional Rayleigh integral approach would be simple and accurate enough to illustrate how the piston moves in relation to the radiated sound pressure. The Rayleigh integral equation allows the vibrating surface to be broken up into any number of simple sources within an infinite baffle surrounding the vibrating surface. The two-dimensional Rayleigh integral is

$$p(x,y) = 2\int_{-a}^{a} \frac{j\rho_0 ck(dQ)}{4\pi R} e^{-jkR} = \frac{j\rho_0 \omega u_0}{2\pi} \int_{-a}^{a} \frac{e^{-jk}}{R} dS, \tag{31}$$

where $R = \sqrt{(x-x_s)^2 + (y-y_s)^2}$, with $x_s = 0$ because every part of the piston is on the y-axis, $dQ = u_0 dS$ is the source strength of the small simple source segment vibrating with a velocity amplitude of u_0 and of surface area dS, and a is the radius of the piston. Figure 5 displays a drawing of the geometry of the two-dimensional baffled piston. The piston is denoted by the thick, solid, vertical black line. The ith simple source segment being evaluated is located at position $(x_s = 0, y_s)$ and the position (x, y) where the pressure is to be evaluated, is a distance R away, denoted by the thinner solid line arrow. Technically a two-dimensional representation of the baffled circular piston is not modeling a circular piston in a two-dimensional baffle but rather a line in a one-dimensional baffle. However, the motion of the piston and the phase of the radiated pressure on the piston surface should still be accurately modeled by this two-dimensional representation. The Rayleigh integral is converted to a discrete summation in order to compute the radiated pressure at discrete $(x > 0, y)$ positions in a two-dimensional space

$$p(x,y) = \frac{j\rho_0 \omega u_0}{2\pi} dS \sum_{i=1}^{N} \frac{e^{-jk\sqrt{x^2+\left(y-y_{s,i}\right)^2}}}{\sqrt{x^2+\left(y-y_{s,i}\right)^2}}, \tag{32}$$

where N is the number of discretized pieces of the linear piston, dS is the area of the discrete pieces but since the pieces are lines, dS can just be the length of each line piece squared (the actual absolute pressure values are not very important for the animations, and $y_{s,i}$ is the y-position of the ith piece of the baffle between $-a$ and a. Equation (32) gives the complex radiated pressure at each $(x > 0, y)$ field point and then to animate this sound field over time we multiply the $p(x, y)$ of Eq. (32) by $e^{j\omega t}$ and take the real part. The piston can be represented by a vertical line moving to the right and left with a displacement of $d(x, t) = x_o \sin(\omega t)$, where x_o is the desired displacement amplitude of the piston, which might be wise to exaggerate in order to clearly see its motion. Again, for a good video animation it is suggested that you create around 50 time snapshot frames between $t = 0$ and $t = T$.

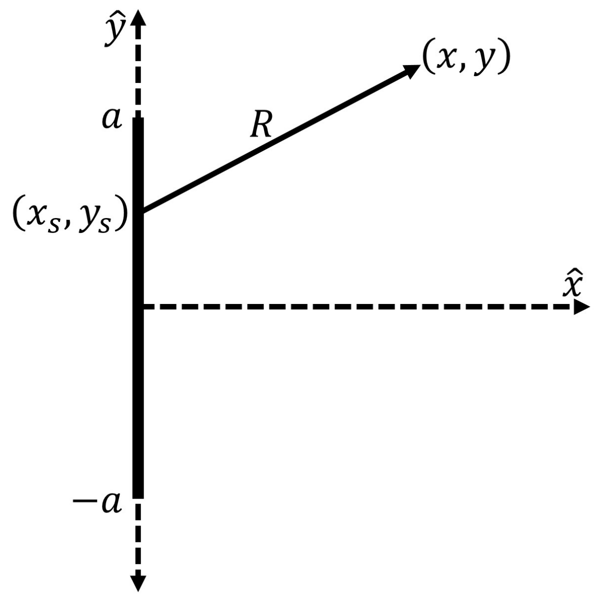

Figure 5. Drawing of the geometry involved in the Rayleigh integral formulation to solve for the radiated pressure from the baffled piston. The piston is broken up into many small discrete sized pieces. The summation of the radiated pressure from each of the discrete pieces is determined at each (x, y) position.

Figure 6 plots several snapshots in time of the animation of the baffled circular piston vibrating at a low frequency and the corresponding radiated sound pressure. Similar conclusions as for the spherical source can be made with the animation of a low frequency ($ka = 0.1$) baffled circular piston and the corresponding radiated sound pressure on the piston's surface. As the piston moves to the left it draws the fluid along with it, with a delay. As the piston stops at the maximum negative extent, the fluid is still moving to the left. The piston starts to accelerate to the right while the fluid is moving left and the fluid thus becomes maximally compressed. The fluid then starts to move to the right as the piston pushes it to the right. When the piston stops at its maximum positive extent the fluid keeps moving to the right. Some fluid remains in contact with the piston while the fluid is moving to the right so the fluid expands and a maximum rarefaction occurs. The process then repeats.

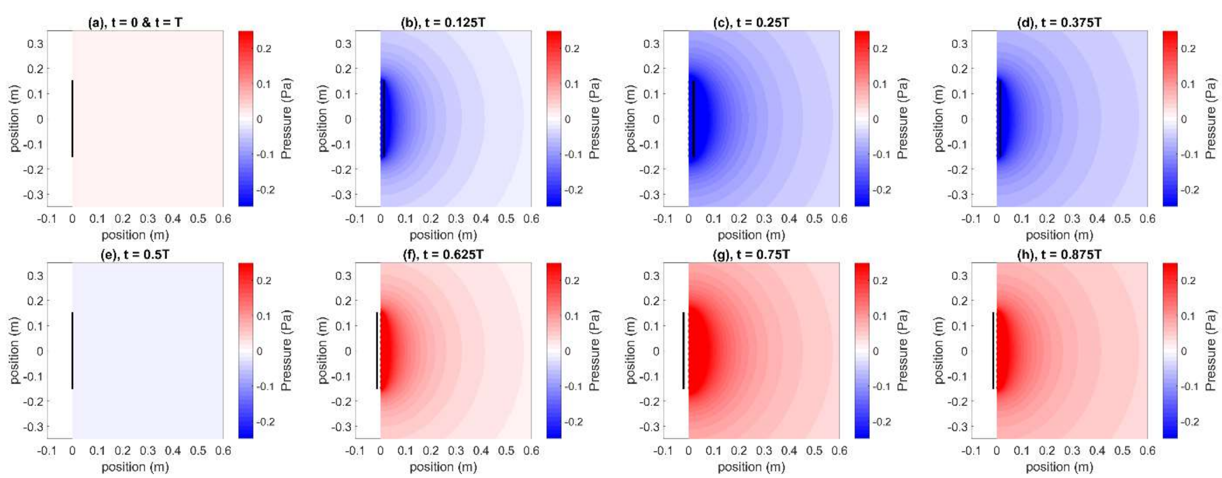

Figure 6. Snapshots in time of the steady-state, baffled circular piston source, represented by the black line, emitting a low frequency ($ka = 0.1$) as it moves to the right and then to the left and the corresponding radiated pressure. The solid black line represents the piston's displacement (exaggerated motion) as it moves initially to the right and then left, while the dashed line indicates the piston's equilibrium position.

Figure 7 plots several snapshots in time of the animation of the baffled circular piston vibrating at a high frequency and the corresponding radiated sound pressure. Similar conclusions as for the spherical source can be made with the animation of a high frequency ($ka = 10$) baffled circular piston and the corresponding spatially-averaged, radiated sound pressure on the piston's surface. As the piston moves to the right from its negative maximum position on the left to its positive maximum position on the right, the fluid is maximally compressed (maximum pressure compression) at the moment of maximum positive velocity. Conversely as the piston retreats from its maximum positive displacement position on the right back to its maximum negative displacement position on the left, the fluid is maximally expanded (maximum pressure rarefaction) at the moment of maximum negative velocity. The fluid is compressing at the same instant that the velocity is maximal, again meaning that the kinetic energy of the piston is being optimally transferred to potential energy of the fluid.

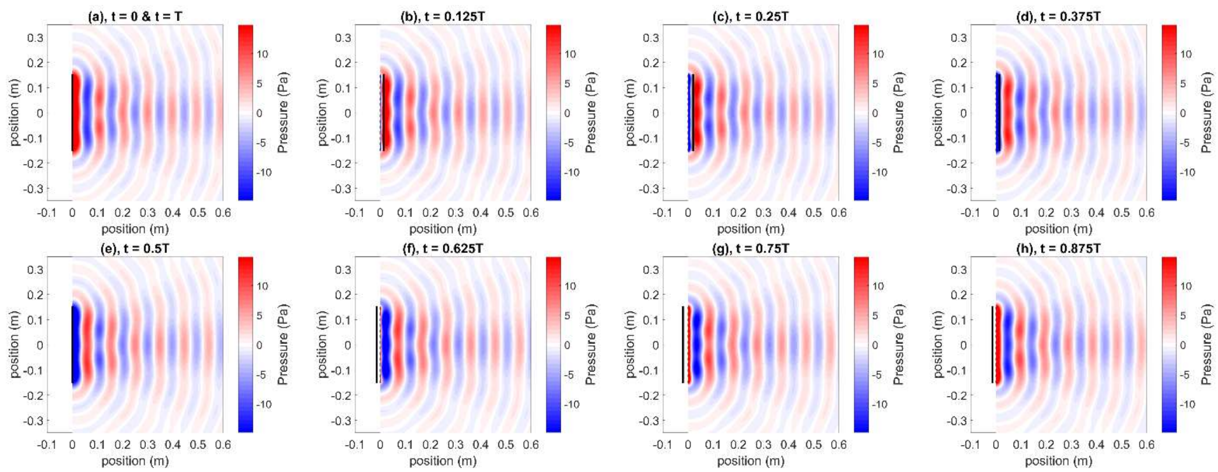

Figure 7. Snapshots in time of the steady-state, baffled circular piston source, represented by the black line, emitting a low frequency ($ka = 10$) as it moves to the right and then to the left and the corresponding radiated pressure. The solid black line represents the piston's displacement (exaggerated motion) as it moves initially to the right and then left, while the dashed line indicates the piston's equilibrium position.

7. CONCLUSION

Radiation impedance is often a difficult concept for students to grasp and it may be helpful to think of it simply as the ratio of the radiated pressure on the surface of the vibrating object to the velocity of the object. The radiation impedance allows the phase relationship between the surface pressure and the object velocity to be quantified. At low frequencies these two quantities are generally not in phase, with the velocity lagging behind the surface pressure by 90°. At high frequencies these two quantities are generally in phase. The phase relationship between these two quantities is very useful in understanding the sound radiation efficiency from the vibrating object. When these quantities are in phase there is an optimal transfer of energy from the vibrating object to radiated sound pressure. When these quantities are not in phase then the energy transfer is not efficient and little sound is radiated by the object.

This paper has developed the equations and described animations that may be used to simultaneously visualize the object's vibration velocity and the corresponding radiated sound pressure near the object's surface for a spherical source radiator (operating in a breathing mode) and for a baffled circular piston. Animations similar to the static images displayed in Figs. 3-4 and 6-7 for the sphere and the baffled piston, respectively, may be found in Ref. 15. It is hoped that the collection of statements from various books on radiation impedance will be useful for those trying to obtain a better conceptual understanding of radiation impedance.

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