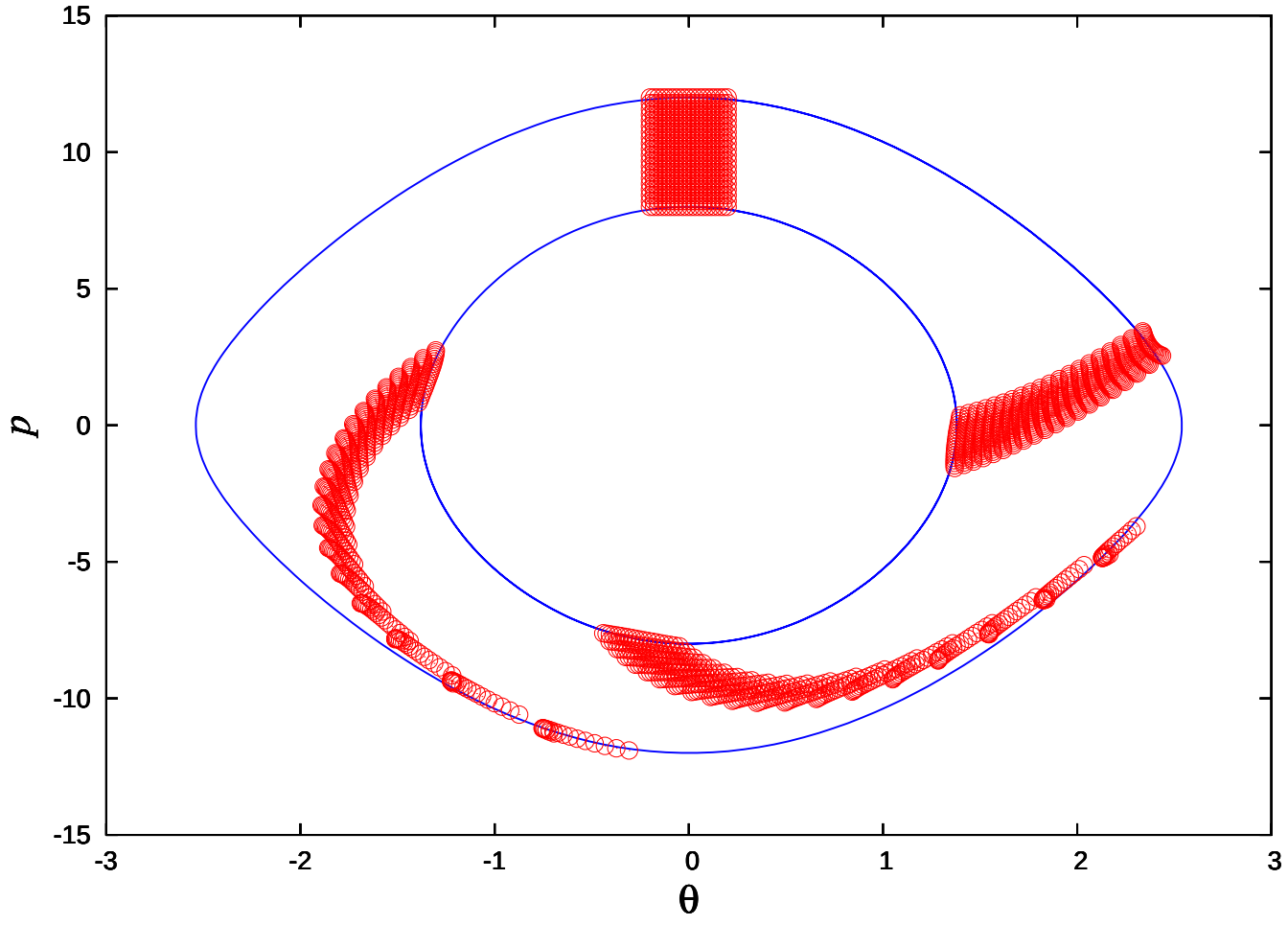
Advanced mechanics
PHYS*3400

Lecture notes (January 2008)



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Chapter 1
Newtonian mechanics

1.1 Reference frames

An important aspect of the fundamental law of Newtonian mechanics,

$$\boldsymbol{F} = m\boldsymbol{a}, \tag{1.1.1}$$

is that it is formulated in a reference frame which is either at rest or moving with a uniform velocity (the velocity must be constant both in magnitude and in direction). Such frames are called *inertial frames*. A *reference frame* is a set of three axes attached to a point O called the *origin*. The position of the origin in space is arbitrary, but some specific choices are sometimes convenient. For example, when describing a system of N bodies it is usually a good idea to place the origin at the centre of mass (which will be introduced below). The origin of an inertial frame is either fixed or moving uniformly relative to another inertial frame. The orientation of the axes is also arbitrary, but some specific choices can again simplify the description. For example, when studying the motion of a particle in a gravitational field it is convenient to align one of the coordinate axes with the direction of the gravitational force.

The coordinate axes define a set of basis vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. (These are sometimes denoted $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.) These vectors point in the directions of increasing x, y, and z, respectively, and they all have a unit norm: $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1$; this property is indicated by the "hat" notation. Relative to a choice of origin O, a particle has a position vector $\boldsymbol{r}(t)$ at time t. This is decomposed in the basis as

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}} + z(t)\hat{\boldsymbol{z}}. \tag{1.1.2}$$

The functions $x(t)$, $y(t)$, and $z(t)$ are the particle's *coordinates* relative to the reference frame. The coordinates change as t varies, and the particle traces a *trajectory* in three-dimensional space. The central goal of Newtonian mechanics is to determine this trajectory, assuming that the force $\boldsymbol{F}$ acting on the particle is known at all times.

The particle's velocity vector is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \dot{x}(t)\hat{\boldsymbol{x}} + \dot{y}(t)\hat{\boldsymbol{y}} + \dot{z}(t)\hat{\boldsymbol{z}}, \tag{1.1.3}$$

where we have introduced the notation $\dot{x} = dx/dt = v_x$; we shall also use $\dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$ as an alternative notation for the vector $\boldsymbol{v}$. The particle's momentum vector is defined by

$$\boldsymbol{p} = m\boldsymbol{v}, \tag{1.1.4}$$

where m is the particle's mass. The particle's acceleration vector is

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \ddot{x}(t)\hat{\boldsymbol{x}} + \ddot{y}(t)\hat{\boldsymbol{y}} + \ddot{z}(t)\hat{\boldsymbol{z}}, \tag{1.1.5}$$

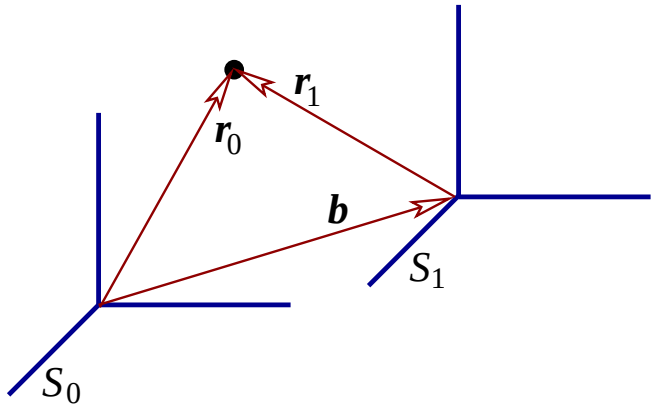


Figure 1.1: Two reference frames, S_0 and S_1, separated by a displacement $\boldsymbol{b}$.

with the notation $\ddot{x} = d^2x/dt^2 = \dot{v}_x = a_x$. Newton's equation, $m\boldsymbol{a} = \boldsymbol{F}$, has the mathematical structure of a system of second-order differential equations for the coordinates $x(t)$, $y(t)$, and $z(t)$. To describe the particle's trajectory, knowing the force, it is necessary to integrate these differential equations.

Suppose that we have two reference frames, S_0 and S_1, separated by a displacement $\boldsymbol{b}$ (see Fig. 1.1). Relative to S_1 the position vector of a particle is $\boldsymbol{r}_1$; relative to S_0 it is $\boldsymbol{r}_0$. The transformation between the two position vectors is clearly $\boldsymbol{r}_0 = \boldsymbol{b} + \boldsymbol{r}_1$, or

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 - \boldsymbol{b}. \tag{1.1.6}$$

Suppose now that S_1 moves relative to S_0, so that the vector $\boldsymbol{b}$ depends on time. Since the position vectors also depend on time, Eq. (1.1.6) should be written as $\boldsymbol{r}_1(t) = \boldsymbol{r}_0(t) - \boldsymbol{b}(t)$. Taking a time derivative produces the transformation between the velocity vectors:

$$\boldsymbol{v}_1 = \boldsymbol{v}_0 - \dot{\boldsymbol{b}}. \tag{1.1.7}$$

Taking a second time derivative gives us the transformation between the acceleration vectors:

$$\boldsymbol{a}_1 = \boldsymbol{a}_0 - \ddot{\boldsymbol{b}}. \tag{1.1.8}$$

If S_0 is an inertial frame, then the equations of motion for the particle as viewed in S_0 are $m\boldsymbol{a}_0 = \boldsymbol{F}$. In S_1 the equations are instead

$$m\boldsymbol{a}_1 = \boldsymbol{F} - m\ddot{\boldsymbol{b}}. \tag{1.1.9}$$

We see that Newton's equation is preserved only if $\ddot{\boldsymbol{b}} = \boldsymbol{0}$, that is, if $\dot{\boldsymbol{b}}$ is a constant vector. In this case S_1 moves relative to S_0 with a constant velocity, and it is also an inertial frame. When, however, S_1 is not inertial, the equations of motion do not take the Newtonian form. We have instead Eq. (1.1.9), which can be rewritten as

$$m\boldsymbol{a}_1 = \boldsymbol{F} + \boldsymbol{F}_{\text{fictitious}},$$

with $\boldsymbol{F}_{\text{fictitious}} = -m\ddot{\boldsymbol{b}}$. The second term on the right can be thought of as a fictitious force that arises from the fact that the reference frame is not inertial. A well-known example is the centrifugal force, which arises in a rotating (and therefore non-inertial) frame of reference.

We now consider a situation in which S_1 and S_0 are both inertial. We assume, in fact, that they share a common origin O, but that they differ in the orientation of the coordinate axes. A concrete example (see Fig. 1.2) is one in which S_1 is obtained from S_0 by a rotation around the z axis. In this case the basis vectors $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{y}}_1$ differ in direction from $\hat{\boldsymbol{x}}_0$ and $\hat{\boldsymbol{y}}_0$. Similarly, the particle's coordinates $x_1(t)$

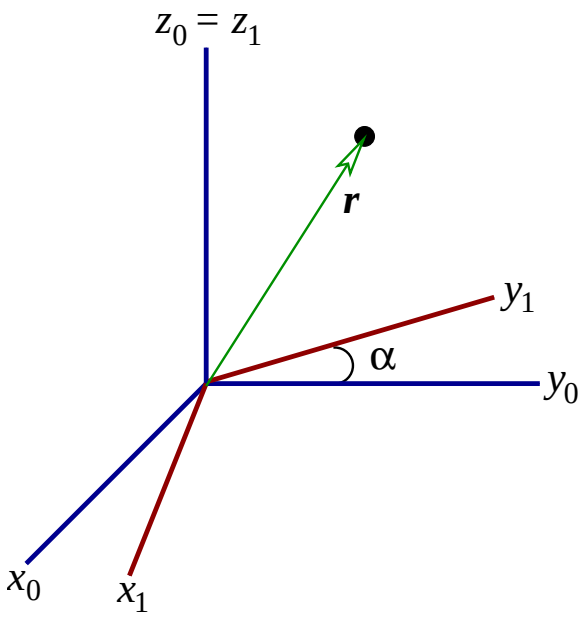


Figure 1.2: The frame S_1 is obtained from S_0 by a rotation around their common z axis.

and $y_1(t)$ differ from $x_0(t)$ and $y_0(t)$. But it is an important fact that the position vector $\boldsymbol{r}(t)$ is not affected by the rotation:

$$\begin{aligned} \boldsymbol{r}_1 &= x_1\hat{\boldsymbol{x}}_1 + y_1\hat{\boldsymbol{y}}_1 + z_1\hat{\boldsymbol{z}}_1 \\ &= x_0\hat{\boldsymbol{x}}_0 + y_0\hat{\boldsymbol{y}}_0 + z_0\hat{\boldsymbol{z}}_0 \\ &= \boldsymbol{r}_0. \end{aligned}$$

This conclusion follows simply from the fact that $\boldsymbol{r} = \boldsymbol{r}_1 = \boldsymbol{r}_0$ is a vector which points from O to the particle, independently of the orientation of the reference frame. So although the basis vectors and the coordinates all change separately under a rotation of the frame, the position vector is invariant. From this observation it follows that $\boldsymbol{v}_1 = \boldsymbol{v}_0 = \boldsymbol{v}$ and $\boldsymbol{a}_1 = \boldsymbol{a}_0 = \boldsymbol{a}$: the velocity and acceleration vectors also are invariant under a rotation of the reference frame. Similar considerations reveal that the vector $\boldsymbol{F}$ is invariant, and we conclude that the form of Newton's equation $\boldsymbol{F} = m\boldsymbol{a}$ is not affected by a rotation of the reference frame. (These invariance properties are exactly what motived the formulation of Newton's mechanics in terms of vectorial quantities.)

Exercise 1.1. Determine how the coordinates x and y, as well as the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, change under a rotation around the z axis by an angle α. Then show mathematically that $\boldsymbol{r}$ is invariant under the transformation.

1.2 Alternative coordinate systems

The discussion of the previous section will have made it clear that the Cartesian coordinates (x, y, z) play an important role in Newtonian mechanics. We might even say that they have a preferred status. The same can be said of the associated set of basis vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. We are aware, however, of situations in which it may be advantageous not to use the Cartesian coordinates, but to switch to another, more convenient system. What happens then to the formulation of our fundamental law,

$\boldsymbol{F} = m\boldsymbol{a}$? The answer, as we shall see in this section, is that while the law itself does not change, its concrete mathematical form may actually look very different.

To keep things specific we choose here to work in the x-y plane (we set $z = 0$) and to consider a specific example of an alternative coordinate system, the *polar coordinates* r and ϕ. These are defined by

$$x = r\cos\phi, \qquad y = r\sin\phi; \tag{1.2.1}$$

the radial coordinate r measures the distance from the origin to the particle, and ϕ is the angle relative to the x axis. In terms of the new coordinates the position vector is

$$\boldsymbol{r} = (r\cos\phi)\hat{\boldsymbol{x}} + (r\sin\phi)\hat{\boldsymbol{y}}, \tag{1.2.2}$$

and it is now a function of r and ϕ. We may express this as $\boldsymbol{r} = \boldsymbol{r}(r,\phi)$, and the vector $\boldsymbol{r}$ points to the position identified by the coordinates (r,ϕ). Notice that r is the magnitude of the position vector: $\boldsymbol{r}\cdot\boldsymbol{r} = r^2$.

As the particle moves in the plane its coordinates r and ϕ vary with time, and the particle's velocity vector is $\boldsymbol{v} = \dot{\boldsymbol{r}}$, or

$$\boldsymbol{v} = (\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\hat{\boldsymbol{x}} + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)\hat{\boldsymbol{y}}. \tag{1.2.3}$$

Notice that the magnitude of the velocity vector is *not equal* to $\dot{r}$; instead $\boldsymbol{v}\cdot\boldsymbol{v} = \dot{r}^2 + r^2\dot{\phi}^2$. The acceleration vector is then $\boldsymbol{a} = \dot{\boldsymbol{v}}$, or

$$\begin{aligned}\boldsymbol{a} \;=\; & (\ddot{r}\cos\phi - 2\dot{r}\dot{\phi}\sin\phi - r\dot{\phi}^2\cos\phi - r\ddot{\phi}\sin\phi)\hat{\boldsymbol{x}} \\ & + (\ddot{r}\sin\phi + 2\dot{r}\dot{\phi}\cos\phi - r\dot{\phi}^2\sin\phi + r\ddot{\phi}\cos\phi)\hat{\boldsymbol{y}}.\end{aligned} \tag{1.2.4}$$

As presented here, these vectors are resolved in the Cartesian basis $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. It is more convenient to resolve them instead in the polar basis $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$, where

$$\hat{\boldsymbol{r}} = \text{unit vector pointing in the direction of increasing } r \tag{1.2.5}$$

and

$$\hat{\boldsymbol{\phi}} = \text{unit vector pointing in the direction of increasing } \phi. \tag{1.2.6}$$

It is important to note that *these new basis vectors*, unlike $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, *are not constant vectors*: their directions change as we move from point to point in the plane.

To find an expression for $\hat{\boldsymbol{r}}$ we observe that by construction, the infinitesimal vector

$$\delta\boldsymbol{r} \equiv \boldsymbol{r}(r+\delta r,\phi) - \boldsymbol{r}(r,\phi) = \frac{\partial\boldsymbol{r}}{\partial r}\,\delta r$$

points in the direction of increasing r. This means that $\hat{\boldsymbol{r}}$ must be proportional to $\partial\boldsymbol{r}/\partial r$. Looking back at Eq. (1.2.2), we see that this is given by $\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$, and we find that this vector already has a unit norm: $(\partial\boldsymbol{r}/\partial r)\cdot(\partial\boldsymbol{r}/\partial r) = \cos^2\phi + \sin^2\phi = 1$. We conclude that

$$\hat{\boldsymbol{r}} = \frac{\partial\boldsymbol{r}}{\partial r} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}} \tag{1.2.7}$$

is the desired basis vector. We proceed similarly to find an expression for $\hat{\boldsymbol{\phi}}$. We observe that the infinitesimal vector

$$\delta\boldsymbol{r} \equiv \boldsymbol{r}(r,\phi+\delta\phi) - \boldsymbol{r}(r,\phi) = \frac{\partial\boldsymbol{r}}{\partial\phi}\,\delta\phi$$

points in the direction of increasing ϕ. (Be careful: this is a different $\delta\boldsymbol{r}$ from the one considered before!) This means that $\hat{\boldsymbol{\phi}}$ must be proportional to $\partial\boldsymbol{r}/\partial\phi$, which is given by $-r\sin\phi\,\hat{\boldsymbol{x}} + r\cos\phi\,\hat{\boldsymbol{y}}$. The squared norm of this vector is $(\partial\boldsymbol{r}/\partial\phi)\cdot$

$(\partial \boldsymbol{r}/\partial\phi) = r^2\sin^2\phi + r^2\cos^2\phi = r^2$, and to get a unit vector we must divide $\partial\boldsymbol{r}/\partial\phi$ by r. We conclude that

$$\hat{\boldsymbol{\phi}} = \frac{1}{r}\frac{\partial \boldsymbol{r}}{\partial\phi} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \tag{1.2.8}$$

is the desired basis vector.

Exercise 1.2. Check that $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\phi}} = 0$.

Let us now work out the components of the vectors $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ in the basis $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}})$. According to Eqs. (1.2.2) and (1.2.7) we have

$$\begin{aligned} \boldsymbol{r}\cdot\hat{\boldsymbol{r}} &= \big[(r\cos\phi)\hat{\boldsymbol{x}} + (r\sin\phi)\hat{\boldsymbol{y}}\big]\cdot\big[\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\big] \\ &= r\cos^2\phi + r\sin^2\phi \\ &= r. \end{aligned}$$

Similarly, Eqs. (1.2.2) and (1.2.8) give

$$\begin{aligned} \boldsymbol{r}\cdot\hat{\boldsymbol{\phi}} &= \big[(r\cos\phi)\hat{\boldsymbol{x}} + (r\sin\phi)\hat{\boldsymbol{y}}\big]\cdot\big[-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}\big] \\ &= -r\sin\phi\cos\phi + r\sin\phi\cos\phi \\ &= 0. \end{aligned}$$

From these results we infer that

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}, \tag{1.2.9}$$

and this expression should not come as a surprise, given the meaning of the quantities involved. Proceeding similarly with the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$, we find that they are decomposed as

$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}} \tag{1.2.10}$$

and

$$\boldsymbol{a} = \big(\ddot{r} - r\dot{\phi}^2\big)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\big(r^2\dot{\phi}\big)\hat{\boldsymbol{\phi}} \tag{1.2.11}$$

in the new basis. As we have pointed out, the components of $\boldsymbol{r}$ in the polar basis are obvious, and the components of $\boldsymbol{v}$ also can be understood easily: The radial component of the velocity vector must clearly be $v_r = \dot{r}$, and the tangential component must be $v_\phi = r\dot{\phi}$ because the factor of r converts the angular velocity $\dot{\phi}$ into a linear velocity.

The components of the acceleration vector are not so easy to interpret. It is important to notice that the radial component of the acceleration vector is not simply $a_r = \ddot{r}$, and the angular component is not simply $a_\phi = \ddot{\phi}$. It is a general observation that *the components of the acceleration vector are not simple in nonCartesian coordinate systems.* It should be observed that the radial component of the acceleration vector contains both a radial part $\ddot{r}$ and a centrifugal part $-r\dot{\phi}^2 = -v_\phi^2/r$.

Exercise 1.3. Verify by explicit calculation that Eqs. (1.2.10) and (1.2.11) are correct.

Suppose now that the force $\boldsymbol{F}$ has been resolved in the polar basis $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}})$. We have

$$\boldsymbol{F} = F_r\hat{\boldsymbol{r}} + F_\phi\hat{\boldsymbol{\phi}}, \tag{1.2.12}$$

and Newton's law $\boldsymbol{F} = m\boldsymbol{a}$ breaks down into two separate equations, the radial component

$$\ddot{r} - r\dot{\phi}^2 = \frac{F_r}{m} \tag{1.2.13}$$

and the angular component

$$\frac{d}{dt}\big(r^2\dot{\phi}\big) = \frac{rF_\phi}{m}. \tag{1.2.14}$$

These are the equations of motion for a particle subjected to a force $\boldsymbol{F}$, expressed in polar coordinates (r, ϕ). When, for example, $F_\phi = 0$ and the force is purely radial, then according to Eq. (1.2.14), $r^2\dot{\phi} = rv_\phi$ is a constant of the motion. When, in addition, $\ddot{r} = 0$ and the particle travels on a circle $r =$ constant, then Eq. (1.2.13) reduces to $r\dot{\phi}^2 = v_\phi^2/r = -F_r/m$; this is the familiar equality between the centrifugal acceleration v_ϕ^2/r and (minus) the radial component of the force (divided by the mass).

Exercise 1.4. Consider the spherical coordinates (r, θ, ϕ) defined by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. Show that in this alternative coordinate system, the basis vectors are given by

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}},\\
\hat{\boldsymbol{\theta}} &= \frac{1}{r}\frac{\partial \boldsymbol{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}},\\
\hat{\boldsymbol{\phi}} &= \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{r}}{\partial \phi} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}.
\end{aligned}$$

Verify that these vectors are all orthogonal to each other.

1.3 Mechanics of a single body

In this section we explore some consequences of the law $\boldsymbol{F} = m\boldsymbol{a}$ when it applies to a single particle.

1.3.1 Line integrals

We begin with a review of some relevant mathematics. Let $\boldsymbol{A}$ be a vector field in three-dimensional space. (A *vector field* is a vector that is defined in a region of space and which may vary from position to position in that region.) Let C be a curve in three-dimensional space, and let $d\boldsymbol{s}$ be the displacement vector along the curve. The displacement vector is defined so that $d\boldsymbol{s}$ is everywhere tangent to the curve, and such that its norm $ds = |d\boldsymbol{s}|$ is equal to the distance between two neighbouring points on the curve; the total length of the curve is the integral $\int_C ds$. Now introduce

$$\int_1^2 \boldsymbol{A}\cdot d\boldsymbol{s},$$

the *line integral* of the vector field $\boldsymbol{A}$ between point 1 and point 2 on the curve C. Such integrals occur often in physics. In the present context the force $\boldsymbol{F}$ will play the role of the vector field $\boldsymbol{A}$, and the particle's trajectory will play the role of the curve C; we then have $d\boldsymbol{s} = d\boldsymbol{r} = \boldsymbol{v}dt$ and the line integral will be the work done by the force as the particle moves from point 1 to point 2.

It is a fundamental theorem of vector calculus that if a line integral between two fixed points in space does *not* depend on the curve joining the points, then the vector field $\boldsymbol{A}$ must be the gradient $\boldsymbol{\nabla} f$ of some scalar function f. This theorem is essentially a consequence of the identity

$$\int_1^2 \boldsymbol{\nabla} f\cdot d\boldsymbol{s} = \int_1^2 \frac{df}{ds}\,ds = f(2) - f(1) \quad \text{independently of the curve,}$$

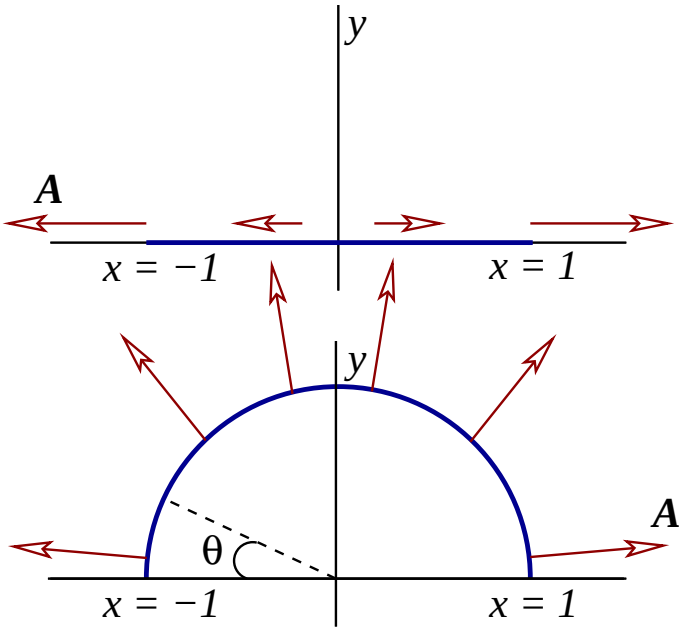


Figure 1.3: Line integrals of a vector field $\boldsymbol{A}$.

which is a generalization of the statement $\int_a^b (df/dx)\,dx = f(b) - f(a)$ from ordinary calculus. Another way of presenting this result is to say that if $\boldsymbol{A} = \boldsymbol{\nabla} f$, then $\oint \boldsymbol{A} \cdot d\boldsymbol{s} = 0$ for any closed curve C in three-dimensional space. This last statement follows because if the curve C is closed, point 2 is identified with point 1, and $\oint \boldsymbol{\nabla} f \cdot d\boldsymbol{s} = f(1) - f(1) = 0$.

To illustrate these notions let us work through a concrete example. Consider the vector field $\boldsymbol{A} = (x, y)$ in two-dimensional space. We wish first to evaluate the line integral of $\boldsymbol{A}$ along the x axis, from $x = -1$ to $x = +1$ (see Fig. 1.3). The safest way to proceed is to first obtain a *parametric description* of the curve C, which in this case is the line segment that links the points $x = \mp 1$. We may describe this curve in the following way:

$$x(u) = -1 + 2u, \qquad y(u) = 0,$$

where the parameter u is restricted to the interval $0 \le u \le 1$. (The choice of parameterization is arbitrary; we might just as well have chosen x as the parameter, but it is generally a good idea to keep the parameter distinct from the coordinates.) From these equations it follows that the displacement vector on C has the components $dx = 2\,du$ and $dy = 0$, so that $d\boldsymbol{s} = (2\,du, 0)$. The vector field evaluated on C is $\boldsymbol{A} = (-1 + 2u, 0)$, and we have $\boldsymbol{A} \cdot d\boldsymbol{s} = 2(-1 + 2u)\,du$. The line integral is then

$$\int_C \boldsymbol{A} \cdot d\boldsymbol{s} = \int_0^1 2(-1 + 2u)\,du.$$

Evaluating this ordinary integral is straightforward, and the result is zero. We therefore have

$$\int_C \boldsymbol{A} \cdot d\boldsymbol{s} = 0$$

for this choice of curve linking the points $(x = -1, y = 0)$ and $(x = 1, y = 0)$.

Let us now evaluate the line integral of $\boldsymbol{A}$ along a different curve C' which joins the same two endpoints (refer again to Fig. 1.3); we choose for C' a semi-circle of unit radius, which we describe by the parametric relations

$$x(\theta) = -\cos\theta, \qquad y(\theta) = \sin\theta,$$

with a parameter θ running from $\theta = 0$ to $\theta = \pi$. Now we have $dx = \sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$, and the displacement vector on C' is $d\boldsymbol{s} = (\sin\theta\, d\theta, \cos\theta\, d\theta)$. The vector field evaluated on C' is $\boldsymbol{A} = (-\cos\theta, \sin\theta)$, and we have $\boldsymbol{A} \cdot d\boldsymbol{s} = 0$. The line integral is obviously

$$\int_{C'} \boldsymbol{A} \cdot d\boldsymbol{s} = 0$$

for this choice of curve also. You might experiment with other curves, and invariably you will find that $\int \boldsymbol{A} \cdot d\boldsymbol{s} = 0$ for *all* curves C that link the points $(-1, 0)$ and $(1, 0)$ in the x-y plane.

Exercise 1.5. Evaluate the line integral $\int_{C''} \boldsymbol{A} \cdot d\boldsymbol{s}$ for the vector field $\boldsymbol{A} = (x, y)$, for a curve C'' that consists of a line segment that goes from $(-1, 0)$ to $(0, -1)$ and another line segment that goes from $(0, -1)$ to $(1, 0)$.

Because the line integral is independent of the path, $\boldsymbol{A}$ must be the gradient of a scalar function f. We must have $A_x = \partial f/\partial x = x$ and $A_y = \partial f/\partial y = y$. Integrating the first equation gives

$$f = \frac{1}{2}x^2 + \text{unknown function of } y,$$

where we indicate that the "constant of integration" can in fact depend on y, which is held fixed during integration with respect to x. Integrating instead the second equation gives

$$f = \frac{1}{2}y^2 + \text{unknown function of } x.$$

These results are compatible only if the unknown function of y is in fact $\frac{1}{2}y^2$, and the unknown function of x is $\frac{1}{2}x^2$. We may still add a true constant to the result, and we find that the function f must be given by

$$f = \frac{1}{2}\big(x^2 + y^2\big) + f_0,$$

where $f_0 =$ constant. It is then easy to verify that $\boldsymbol{\nabla} f = \boldsymbol{A}$. It now becomes clear why the line integral had to be zero for any path linking the points $(-1, 0)$ and $(1, 0)$: Irrespective of the path the integral has to be equal to $f(1, 0) - f(-1, 0) = (\frac{1}{2} + f_0) - (\frac{1}{2} + f_0) = 0$, as we have found for C and C'.

1.3.2 Conservation of linear momentum

We now proceed with our exploration of the consequences of the dynamical law $\boldsymbol{F} = m\boldsymbol{a}$. The first main consequence follows immediately from Newton's equation: In the absence of a force acting on the particle, the linear momentum $\boldsymbol{p} = m\boldsymbol{v}$ is a constant vector. This follows from the alternative expression of Newton's law,

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}; \tag{1.3.1}$$

if $\boldsymbol{F} = \boldsymbol{0}$ then $d\boldsymbol{p}/dt = \boldsymbol{0}$ and the vector $\boldsymbol{p}$ must be constant. We therefore have conservation of (linear) momentum in the absence of an applied force.

1.3.3 Conservation of angular momentum

Relative to a choice of origin O, the *angular momentum* of a particle at position $\boldsymbol{r}$ is defined by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}. \tag{1.3.2}$$

The angular-momentum vector changes if the origin of the reference frame is shifted to a different point in space. The *torque* acting on the particle is defined by

$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}. \tag{1.3.3}$$

(This is also called the moment of force.) We have, as a consequence of Newton's equation, $d\boldsymbol{L}/dt = m(\boldsymbol{v}\times\boldsymbol{v}+\boldsymbol{r}\times\boldsymbol{a}) = \boldsymbol{r}\times\boldsymbol{F}$, since the first term obviously vanishes. This gives

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}, \tag{1.3.4}$$

and we obtain a statement of angular-momentum conservation: In the absence of a torque acting on the particle, the angular momentum $\boldsymbol{L}$ is a constant vector. It is clear that $\boldsymbol{N} = \boldsymbol{0}$ when $\boldsymbol{F} = \boldsymbol{0}$, but it is possible to have a vanishing torque even when $\boldsymbol{F} \neq \boldsymbol{0}$; this occurs when $\boldsymbol{F}$ always points in the direction of $\boldsymbol{r}$.

1.3.4 Conservation of energy

The statements of conservation of linear and angular momenta were easy to formulate and prove, but these statements hold only in very rare circumstances: $\boldsymbol{F}$ must vanish for $\boldsymbol{p}$ to be constant, and $\boldsymbol{N}$ must vanish for $\boldsymbol{L}$ to be constant. As we shall see, the statement of conservation of energy is more difficult to make, but it holds much more widely.

Let a particle move from point 1 to point 2 under the action of a force $\boldsymbol{F}$. The total *work done* on the particle by the force, as it moves from 1 to 2, is by definition the line integral

$$W_{12} = \int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}, \tag{1.3.5}$$

where $d\boldsymbol{r} = \boldsymbol{v}\,dt$ is the displacement vector along the particle's trajectory. As we shall now infer, the line integral is equal to the total change in the particle's *kinetic energy*,

$$T = \frac{1}{2}mv^2 = \text{kinetic energy}, \tag{1.3.6}$$

as it moves from 1 to 2. We have introduced the notation $v^2 = \boldsymbol{v}\cdot\boldsymbol{v} = |\boldsymbol{v}|^2$. The statement of the *work-energy theorem* is thus

$$W_{12} = T(2) - T(1). \tag{1.3.7}$$

To prove this we substitute $\boldsymbol{F} = m d\boldsymbol{v}/dt$ and $d\boldsymbol{r} = \boldsymbol{v}\,dt$ inside the line integral of Eq. (1.3.5). We get

$$W_{12} = m\int_1^2 \frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}\,dt.$$

The integrand is

$$\begin{aligned}
\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v} &= \frac{dv_x}{dt}v_x + \frac{dv_y}{dt}v_y + \frac{dv_x}{dt}v_y \\
&= \frac{1}{2}\frac{d}{dt}v_x^2 + \frac{1}{2}\frac{d}{dt}v_y^2 + \frac{1}{2}\frac{d}{dt}v_z^2 \\
&= \frac{1}{2}\frac{d}{dt}\Big(v_x^2 + v_y^2 + v_z^2\Big) \\
&= \frac{d}{dt}\Big(\frac{1}{2}v^2\Big),
\end{aligned}$$

and the line integral becomes

$$W_{12} = \int_1^2 \frac{d}{dt}\Big(\frac{1}{2}mv^2\Big)\,dt = \int_1^2 \frac{dT}{dt}\,dt = \int_1^2 dT = T(2) - T(1).$$

This is the same statement as in Eq. (1.3.7), and we have established the work-energy theorem.

In very many situations the line integral $\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}$ is actually *independent* of the trajectory adopted by the particle to go from point 1 to point 2. In these situations we must have that $\boldsymbol{F}$ is the gradient of some scalar function $f(\boldsymbol{r})$. We write $f=-V$, inserting a minus sign for reasons of convention, and express the force as

$$\boldsymbol{F}=-\boldsymbol{\nabla}V(\boldsymbol{r}). \tag{1.3.8}$$

The scalar function V is known as the *potential energy* of the particle. When $\boldsymbol{F}$ is expressed as in Eq. (1.3.8) the line integral of Eq. (1.3.5) becomes

$$W_{12}=-\int_1^2 \boldsymbol{\nabla}V\cdot d\boldsymbol{r}=-\big[V(2)-V(1)\big],$$

and this is clearly independent of the particle's trajectory: The total work done is equal to the difference $V(1)-V(2)$ no matter how the particle moves from 1 to 2. Equation (1.3.7) then becomes $V(1)-V(2)=T(2)-T(1)$, or $T(1)+V(1)=T(2)+V(2)$. This tells us that the quantity $T+V$ stays constant as the particle moves from point 1 to point 2. We therefore have obtained the statement of conservation of *total mechanical energy*

$$E=T+V=\frac{1}{2}mv^2+V(\boldsymbol{r}) \tag{1.3.9}$$

for a particle moving under the action of a force $\boldsymbol{F}$ that derives from a potential V.

We can verify directly from Eq. (1.3.9) that the total energy is a constant of the motion. We have

$$\frac{dE}{dt}=\frac{1}{2}m\frac{dv^2}{dt}+\frac{dV}{dt}.$$

As we have seen,

$$\frac{dv^2}{dt}=2\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}.$$

The potential energy V depends on time only through the changing position of the particle: $V=V(\boldsymbol{r}(t))=V(x(t),y(t),z(t))$. We therefore have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}+\frac{\partial V}{\partial z}\frac{dz}{dt}\\ &= \boldsymbol{\nabla}V\cdot\boldsymbol{v}.\end{aligned}$$

All of this gives

$$\begin{aligned}\frac{dE}{dt} &= m\boldsymbol{a}\cdot\boldsymbol{v}+\boldsymbol{\nabla}V\cdot\boldsymbol{v}\\ &= \boldsymbol{F}\cdot\boldsymbol{v}-\boldsymbol{F}\cdot\boldsymbol{v}\\ &= 0,\end{aligned}$$

as expected.

An example of a force that derives from a potential is gravity: The force

$$\boldsymbol{F}_{\text{gravity}}=m\boldsymbol{g}=mg(0,0,-1) \tag{1.3.10}$$

is the negative gradient of

$$V_{\text{gravity}}=mgz. \tag{1.3.11}$$

We have indicated that the vector $\boldsymbol{g}$ points in the negative z direction (down, that is); its magnitude is the gravitational acceleration $g\simeq 9.8\ \text{m/s}^2$. The total

mechanical energy E is conserved when a particle moves under the action of the gravitational force.

An example of a force that does *not* derive from a potential is the frictional force

$$\boldsymbol{F}_{\text{friction}} = -k\boldsymbol{v}, \tag{1.3.12}$$

where $k > 0$ is the coefficient of friction; this force acts in the direction opposite to the particle's motion and exerts a drag. It is indeed easy to see that $\boldsymbol{F}_{\text{friction}}$ cannot be expressed as the gradient of a function of $\boldsymbol{r}$. (The expression $V_{\text{friction}} = k\boldsymbol{v}\cdot\boldsymbol{r}$ might seem to work, but this potential depends on both $\boldsymbol{r}$ and $\boldsymbol{v}$, and this is not allowed.) This implies that in the presence of a frictional force, the total mechanical energy of a particle is *not conserved*. The reason is that the friction produces heat, which is rapidly dissipated away; because this heat comes at the expense of the particle's mechanical energy, E cannot be conserved. Energy conservation as a whole, of course, applies: the amount by which E decreases matches the amount of heat dissipated into the environment.

It is important to understand that the work-energy theorem of Eq. (1.3.7) is always true, whether or not the force $\boldsymbol{F}$ derives from a potential. But whether E is conserved or not depends on this last property: When $\boldsymbol{F} = -\boldsymbol{\nabla}V$ we have $dE/dt = 0$ and the total mechanical energy is conserved; but E is not in general conserved when the force does not derive from a potential.

1.3.5 Case study #1: Particle in a gravitational field

To illustrate the formalism presented in the preceding subsections we now review the problem of determining the motion of a particle in a gravitational field. The force is given by Eq. (1.3.10), $\boldsymbol{F} = m\boldsymbol{g} = mg(0,0,-1)$, and the potential by Eq. (1.3.11), $V = mgz$. The equations of motion are

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = -g. \tag{1.3.13}$$

These are easily integrated:

$$x(t) = x(0) + v_x(0)t, \qquad y(t) = y(0) + v_y(0)t, \qquad z(t) = z(0) + v_z(0)t - \frac{1}{2}gt^2. \tag{1.3.14}$$

These equations describe parabolic motion. Here $x(0), y(0), z(0)$ are the positions at time $t = 0$, and $v_x(0)$, $v_y(0)$, and $v_z(0)$ are the components of the velocity vector at $t = 0$; these quantities are the *initial conditions* that must be specified in order for the motion to be uniquely known at all times. The velocity vector at time t is obtained by differentiating Eqs. (1.3.14); we get

$$v_x(t) = v_x(0), \qquad v_y(t) = v_y(0), \qquad v_z(t) = v_z(0) - gt. \tag{1.3.15}$$

With Eqs. (1.3.14) and (1.3.15) we have sufficient information to compute the total mechanical energy $E = T + V$ of the particle. After some simple algebra we obtain

$$E = \frac{1}{2}m\Big[v_x(0)^2 + v_y(0)^2 + v_z(0)^2\Big] + mgz(0) \tag{1.3.16}$$

for all times t; this is clearly a constant of the motion.

Exercise 1.6. Verify that Eqs. (1.3.14) really give the solution to the equations of motion $\ddot{\boldsymbol{r}} = \boldsymbol{g}$. Then compute E and make sure that your result agrees with Eq. (1.3.16).

1.3.6 Case study #2: Particle in a gravitational field subjected to air resistance

We now suppose that the particle is subjected to both a gravitational force $m\boldsymbol{g}$ and a frictional force $-k\boldsymbol{v}$ supplied by the ambient air. For convenience we set $k = m/\tau$, thereby defining the quantity τ, and the total applied force is

$$\boldsymbol{F} = m(\boldsymbol{g} - \boldsymbol{v}/\tau). \tag{1.3.17}$$

The equations of motion are $m\boldsymbol{a} = \boldsymbol{F}$, or $\boldsymbol{a} = \boldsymbol{g} - \boldsymbol{v}/\tau$, or again

$$\ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}}/\tau = \boldsymbol{g}. \tag{1.3.18}$$

We assume that the particle is released from a height h with a zero initial velocity. The initial conditions are therefore $z(0) = h$ and $\dot{z}(0) = 0$. We assume also, for simplicity, that there is no motion in the x and y directions. The only relevant component of Eq. (1.3.18) is therefore

$$\dot{v} + v/\tau = -g, \tag{1.3.19}$$

where we have set $v = \dot{z}$. To arrive at Eq. (1.3.19) we have used the fact that $\boldsymbol{g} = g(0, 0, -1)$.

Our task is to solve the first-order differential equation of Eq. (1.3.19). We use the method of variation of parameters. Suppose first that $g = 0$. In this case the equation becomes $dv/dt = -v/\tau$ or $dv/v = -dt/\tau$. This is easily integrated, and we get $\ln(v/c) = -t/\tau$, or $v = c\,e^{-t/\tau}$. This is the solution for $g = 0$, and the constant of integration c is the solution's parameter. To handle the case $g \neq 0$ we allow c to depend on time — we vary the parameter — and we substitute the trial solution

$$v(t) = c(t)e^{-t/\tau}$$

into Eq. (1.3.19). We have $\dot{v} = \dot{c}e^{-t/\tau} - v/\tau$ and $-g = \dot{v} + v/\tau = \dot{c}e^{-t/\tau}$. The differential equation for $c(t)$ is therefore

$$\dot{c} = -ge^{t/\tau},$$

so that

$$c(t) = -g\tau e^{t/\tau} + c_0,$$

where c_0 is a true constant of integration. The result for $v(t)$ is then

$$v(t) = -g\tau + c_0 e^{-t/\tau}.$$

To determine c_0 we invoke the initial condition $v(0) = 0$. Because $v(0) = -g\tau + c_0$ we have that $c_0 = g\tau$. Our final answer is therefore

$$v(t) = -g\tau\big[1 - e^{-t/\tau}\big]. \tag{1.3.20}$$

This is $\dot{z}$, the z component of the particle's velocity vector. Integrating Eq. (1.3.20) gives $z(t)$, the position of the particle as a function of time.

Exercise 1.7. Integrate Eq. (1.3.20) and obtain $z(t)$. Make sure to impose the initial condition $z(0) = h$.

Equation (1.3.20) simplifies when t is much smaller than $\tau = m/k$. At such early times, when $t/\tau \ll 1$, the exponential is well approximated by $e^{-t/\tau} \simeq 1 - t/\tau$ and Eq. (1.3.20) becomes

$$v(t) \simeq -gt,$$

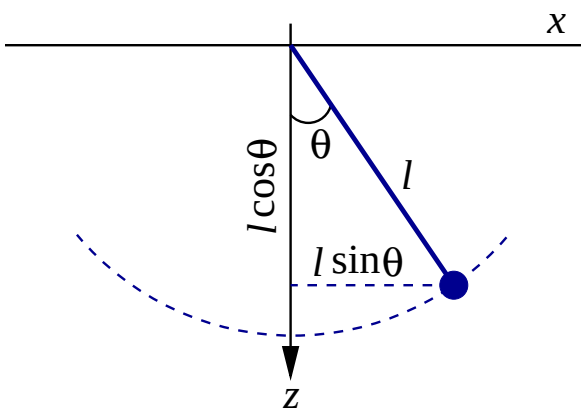


Figure 1.4: Motion of a pendulum. For convenience the z axis is taken to point downward. The angle between the position of the pendulum and the vertical is denoted $\theta(t)$.

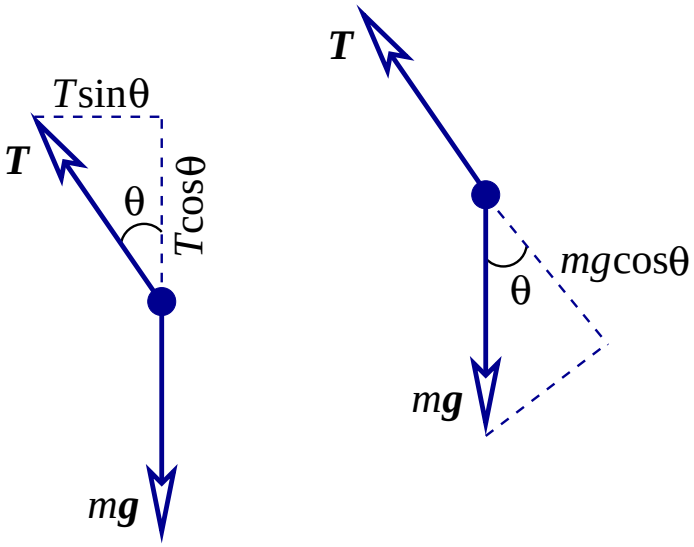


Figure 1.5: Forces acting on the pendulum.

in agreement with Eq. (1.3.15). At such early times the velocity is low, and the frictional force is so weak that it has no noticeable effect on the motion. As v increases the frictional force becomes more important and it starts to dominate over gravity. At late times, when t is much larger than τ, the exponential term in Eq. (1.3.20) is very small, and the velocity is now approximated by

$$v(t) \simeq -g\tau.$$

At such late times the velocity is constant: The particle has reached its *terminal velocity* given by $v_{\text{terminal}} = g\tau = gm/k$.

1.3.7 Case study #3: Motion of a pendulum

We now examine the motion of a pendulum, which consists of an object of mass m attached to a massless, but rigid, rod of length ℓ. The geometry of the problem is illustrated in Fig. 1.4; we shall describe the motion of the pendulum in terms of the swing angle θ.

As shown in Fig. 1.5, there are two forces acting on the pendulum. The first is gravity, pulling down, and the second is the tension within the rod, which always

pulls in the rod's direction. The geometry of the problem suggests that it might be a good idea to involve the polar coordinates introduced in Sec. 1.2. Adapting the notation somewhat, we express the Cartesian coordinates x and z of the mass m in terms of the new coordinates r and θ; the relationship is

$$x = r\sin\theta, \qquad z = r\cos\theta. \tag{1.3.21}$$

At a later stage of the calculation we will incorporate the fact that the distance r between m and the origin of the coordinate system is constant: $r = \ell$. For the moment, however, we shall pretend that r is free to change with time.

The polar coordinates (r, θ) come with the basis of unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, with

$$\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}}$$

and

$$\hat{\boldsymbol{\theta}} = \frac{1}{r}\frac{\partial \boldsymbol{r}}{\partial \theta} = \cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}},$$

where $\boldsymbol{r}(r,\theta) = r\sin\theta\,\hat{\boldsymbol{x}} + r\cos\theta\,\hat{\boldsymbol{z}}$ is the position vector expressed in terms of the polar coordinates. The unit vector $\hat{\boldsymbol{r}}$ points in the direction of increasing r (always away from the origin), while the unit vector $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ.

As we have seen in Sec. 1.2, the acceleration vector of the mass m can be expressed in the polar coordinates and resolved in the new basis vectors. Repeating the calculations carried out there, we find

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{\boldsymbol{\theta}}. \tag{1.3.22}$$

The net force acting on the mass m is $\boldsymbol{F} = \boldsymbol{T} + m\boldsymbol{g}$, the vectorial sum of the tension and gravitational forces, respectively. Because the tension is directed along the rod, we have $\boldsymbol{T} = -T\hat{\boldsymbol{r}}$, with T denoting the magnitude of the tension. The force of gravity, on the other hand, is directed along the z direction, and we have $m\boldsymbol{g} = mg\hat{\boldsymbol{z}}$. Resolving this in the new basis (Fig. 1.5), we have $m\boldsymbol{g} = mg\cos\theta\,\hat{\boldsymbol{r}} - mg\sin\theta\,\hat{\boldsymbol{\theta}}$, and the net force is

$$\boldsymbol{F} = (-T + mg\cos\theta)\,\hat{\boldsymbol{r}} - mg\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{1.3.23}$$

Equating this to $m\boldsymbol{a}$ produces

$$m(\ddot{r} - r\dot{\theta}^2) = -T + mg\cos\theta, \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -g\sin\theta,$$

the equations of motion for the pendulum.

These equations simplify considerably when we finally incorporate the fact that $r = \ell$ and does not change with time (so that $\dot{r} = \ddot{r} = 0$). The first equation gives us an expression for the tension: $T = m(\ell\dot{\theta}^2 + g\cos\theta)$. The second equation reduces to $\ell\ddot{\theta} = -g\sin\theta$, or

$$\ddot{\theta} + \omega^2\sin\theta = 0, \tag{1.3.24}$$

where

$$\omega = \sqrt{g/\ell} \tag{1.3.25}$$

has the dimensions of inverse time (or frequency).

Exercise 1.8. Make sure that you can reproduce all the algebra that goes into the derivation of Eqs. (1.3.24) and (1.3.25).

Exercise 1.9. Equation (1.3.24) can also be derived on the basis of Eq. (1.3.4), $d\boldsymbol{L}/dt = \boldsymbol{N}$, where $\boldsymbol{L} = m\boldsymbol{r}\times\boldsymbol{v}$ is the pendulum's angular momentum and $\boldsymbol{N} = \boldsymbol{r}\times\boldsymbol{F}$

the net torque acting on it. Work through the details and verify that this equation does indeed lead to Eq. (1.3.24). This method of derivation does not require the new basis of unit vectors; all calculations can be carried out in the Cartesian basis.

The second-order differential equation of Eq. (1.3.24) determines the motion of the pendulum. It can immediately be integrated once with respect to time. The trick is to multiply Eq. (1.3.24) by $\dot{\theta}$; this gives

$$\ddot{\theta}\dot{\theta} + (\omega^2 \sin\theta)\dot{\theta} = 0.$$

Now note that

$$\ddot{\theta}\dot{\theta} = \frac{1}{2}\frac{d}{dt}\dot{\theta}^2$$

and

$$(\sin\theta)\dot{\theta} = -\frac{d}{dt}\cos\theta.$$

We therefore have

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right) = 0,$$

or

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \varepsilon = \text{constant}. \tag{1.3.26}$$

This is a first-order differential equation for $\theta(t)$.

It seems intuitively plausible that the conserved quantity ε should have something to do with the pendulum's total energy E. This is indeed the case. The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$, according to our previous results. The potential energy associated with the gravitational force is $V = -mgz = -mg\ell\cos\theta = -m\ell^2\omega^2\cos\theta$, where we have used Eq. (1.3.25). The potential energy associated with the rod's tension is zero: The tension always acts in the rod's direction, which is always perpendicular to the direction of the motion; the tension does no work on the pendulum. We finally have $E = T + V = m\ell^2(\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta)$, or

$$E = m\ell^2\varepsilon. \tag{1.3.27}$$

We shall call ε the pendulum's *reduced energy*. Similarly, we shall call $\frac{1}{2}\dot{\theta}^2$ the *reduced kinetic energy* and $\nu(\theta) \equiv -\omega^2\cos\theta$ the *reduced potential energy*.

The qualitative features of the pendulum's motion can be understood without further calculation, purely on the basis of the following graphical construction. We draw an *energy diagram*, a plot of the reduced potential energy $\nu(\theta) = -\omega^2\cos\theta$ as a function of θ, together with the constant value of the reduced energy ε (see Fig. 1.6). According to Eq. (1.3.26), which we rewrite as

$$\frac{1}{2}\dot{\theta}^2 = \varepsilon - \nu(\theta), \qquad \nu(\theta) = -\omega^2\cos\theta, \tag{1.3.28}$$

the difference between ε and $\nu(\theta)$ is equal to the reduced kinetic energy $\frac{1}{2}\dot{\theta}^2$. For motion to take place this difference must be positive, and a quick examination of the diagram reveals immediately the regions for which $\varepsilon - \nu(\theta) \leq 0$. Motion is possible within these regions, and impossible outside.

For example, when $\varepsilon < \omega^2$ we see that the motion of the pendulum takes place between the two well-defined limits $\theta = \pm\theta_0$; motion is impossible beyond these points. This situation corresponds to ordinary pendulum motion: The weight oscillates back and forth around the horizontal axis ($\theta = 0$), with an amplitude θ_0. The diagram reveals that the angular velocity $|\dot{\theta}|$ is maximum when the weight crosses

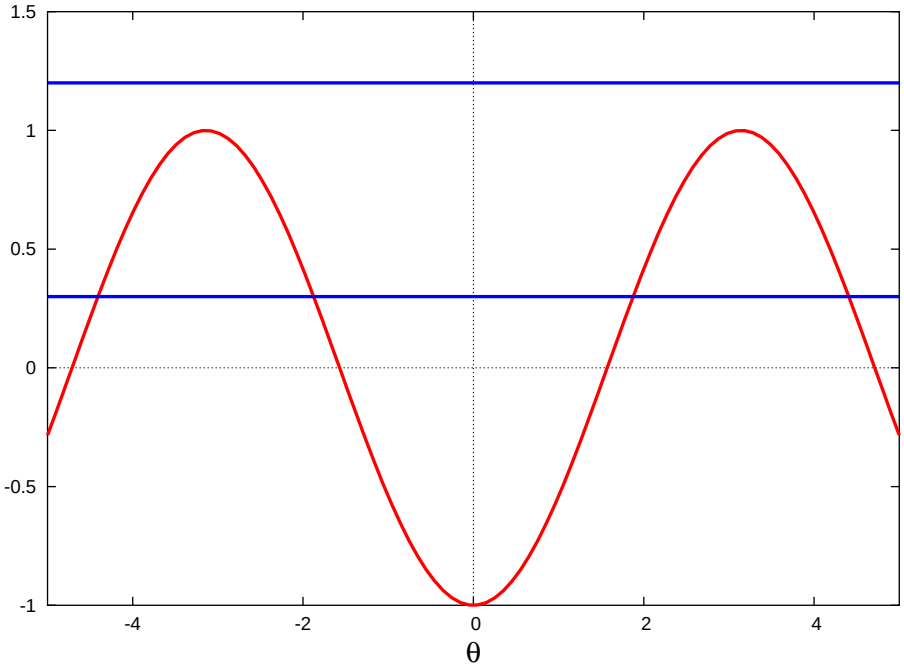


Figure 1.6: Energy diagram for the pendulum. The difference between the line $\varepsilon =$ constant and the curve $\nu(\theta) = -\omega^2 \cos\theta$ is the reduced kinetic energy $\frac{1}{2}\dot{\theta}^2$, which must be positive for motion to take place. The lower value of ε is such that $\varepsilon < \omega^2$. The higher value is such that $\varepsilon > \omega^2$. In the plot ω^2 is set equal to 1.

$\theta = 0$, and that the pendulum comes to a momentary rest ($\dot{\theta} = 0$) when $\theta = \pm\theta_0$. This amplitude is determined by setting $\dot{\theta} = 0$ in Eq. (1.3.28); we have

$$\varepsilon = \nu(\theta_0) = -\omega^2 \cos\theta_0. \tag{1.3.29}$$

This equation can be solved for θ_0 whenever $\varepsilon < \omega^2$; there are no solutions otherwise. When $\varepsilon > \omega^2$ the diagram reveals that there are no intersections between the line $\varepsilon =$ constant and the curve $\nu(\theta)$. There are no points at which $\frac{1}{2}\dot{\theta}^2 = 0$, θ is allowed to increase without bound, and the motion is not limited. This high-energy situation corresponds to the weight doing complete revolutions around the pivot point.

Points in the energy diagram at which the line $\varepsilon =$ constant meets the curve $\nu(\theta)$ are called *turning points*. At these points the reduced kinetic energy $\frac{1}{2}\dot{\theta}^2$ drops to zero and $\dot{\theta}$ changes sign, either from the positive to the negative (if θ was increasing toward θ_0), or from the negative to the positive (if θ was decreasing toward $-\theta_0$). These are the points at which the pendulum reaches its maximum angle and turns around.

Combining Eqs. (1.3.28) and (1.3.29) gives

$$\frac{1}{2}\dot{\theta}^2 = \omega^2(\cos\theta - \cos\theta_0), \tag{1.3.30}$$

and this is a first-order differential equation for $\theta(t)$. This equation, unfortunately, cannot be solved in closed form, unless θ_0 is assumed to be very small (we shall deal separately with this simple case at the end of this subsection). The best we can do is to express t in terms of an integral involving θ. First we take the square root of Eq. (1.3.30),

$$\dot{\theta} = \pm\sqrt{2}\omega\sqrt{\cos\theta - \cos\theta_0},$$

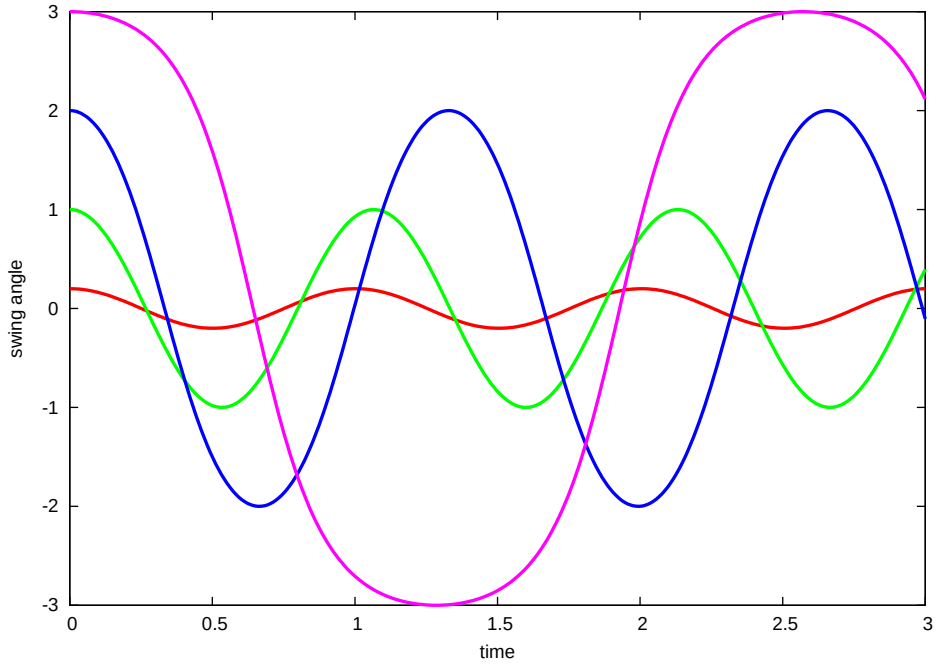


Figure 1.7: Motion of a pendulum with four different amplitudes. We plot the swing angle θ (in radians) as a function of time. The unit of time is $2\pi/\omega$. Notice that the period of oscillation increases with the amplitude. Notice also that for high amplitude, the curve differs significantly from a sinusoid.

and we solve for dt. After integration we get

$$t = \pm\frac{1}{\sqrt{2}\omega}\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} + \text{constant}. \tag{1.3.31}$$

This integral must be evaluated numerically, and the result $t(\theta)$ must be inverted to give $\theta(t)$; the inversion must also be done numerically. To obtain these details requires some labour, and this will not be pursued here. The results of a numerical integration are presented in Fig. 1.7.

The motion of the pendulum is clearly periodic, and Eq. (1.3.31) allows us to calculate the period P, the time required for the pendulum to complete a full cycle of oscillation (θ going from $-\theta_0$ to $+\theta_0$ and then back to $-\theta_0$.) This is twice the time required to go from $-\theta_0$ to $+\theta_0$, or four times the time required to go from $\theta = 0$ to $\theta = \theta_0$. So the period is given by

$$P = \frac{4}{\sqrt{2}\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

To put this integral in standard form we change the variable of integration to

$$z = \frac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta_0}$$

and introduce the parameter

$$s = \sin\tfrac{1}{2}\theta_0. \tag{1.3.32}$$

Simple manipulations reveal that

$$\frac{dz}{d\theta} = \frac{\sqrt{1 - s^2z^2}}{2s}, \qquad \sqrt{\cos\theta - \cos\theta_0} = \sqrt{2}s\sqrt{1 - z^2},$$

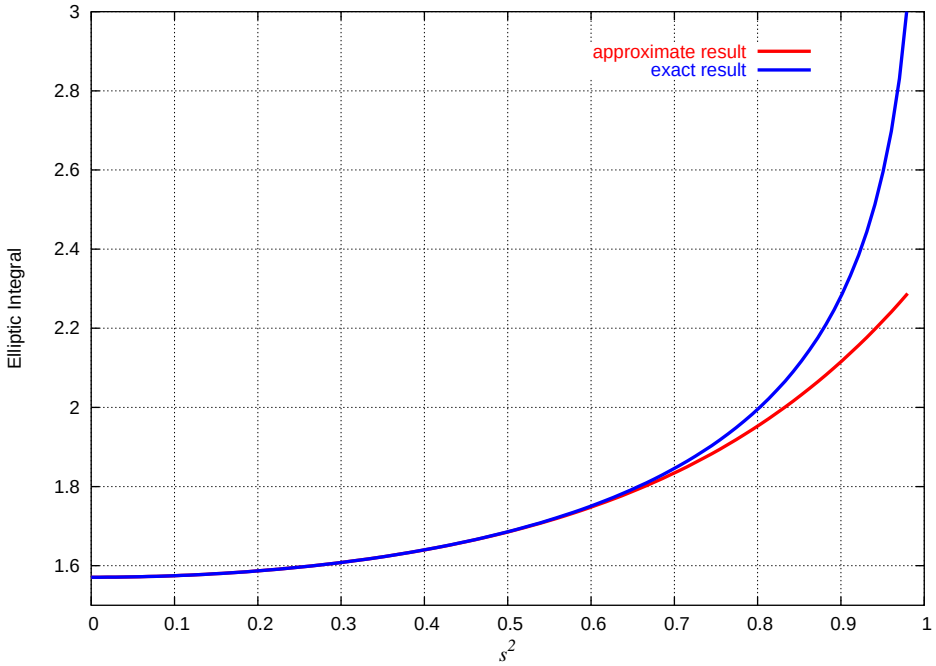


Figure 1.8: A plot of $K(s^2)$, the complete elliptic integral of the first kind, as a function of s^2 in the interval $0 \leq s^2 < 1$. The function diverges as s^2 approaches 1. Also plotted is the approximation given in the text.

and the expression for P becomes

$$P = \frac{4}{\omega} K(s^2), \tag{1.3.33}$$

where

$$K(s^2) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-s^2z^2)}} \tag{1.3.34}$$

is a special function known as the *complete elliptic integral of the first kind*. A plot of this function is shown in Fig. 1.8. While this result is perhaps not too revealing, it allows us to conclude that the period increases with the amplitude of the motion. This follows because P depends on $s^2 = \sin^2 \frac{1}{2}\theta_0$ through the elliptic integral.

Exercise 1.10. Make sure that you can reproduce all the algebra that goes into the derivation of Eqs. (1.3.33) and (1.3.34).

We can be more explicit when $s = \sin \frac{1}{2}\theta_0$ is fairly small compared with 1. In this situation it is known that the elliptic integral can be approximated by

$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 s^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 s^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 s^6 + \cdots\right].$$

Substituting this into Eq. (1.3.33) gives

$$P = \frac{2\pi}{\omega}\left[1 + \frac{1}{4}s^2 + \frac{9}{64}s^4 + \frac{25}{256}s^6 + \cdots\right]. \tag{1.3.35}$$

When the oscillations are very small, that is when $\theta_0 \ll 1$, we have that $s^2 \ll 1$ and the period is well approximated by the leading term in the power expansion,

$P \simeq 2\pi/\omega$. In this limit the period becomes independent of the motion's amplitude.

Exercise 1.11. It is not too difficult to derive the preceding approximation to the elliptic integral. When s^2 is small the factor $(1-s^2z^2)^{-1/2}$ inside the integral of Eq. (1.3.34) can be expressed as a Taylor series about $s = 0$. Show that this gives

$$(1 - s^2z^2)^{-1/2} = 1 + \frac{1}{2}s^2z^2 + \frac{3}{8}s^4z^4 + \frac{5}{16}s^6z^6 + \cdots.$$

With this expansion the elliptic integral becomes

$$K = \int_0^1 \frac{dz}{\sqrt{1-z^2}} + \frac{1}{2}s^2 \int_0^1 \frac{z^2\,dz}{\sqrt{1-z^2}} + \frac{3}{8}s^4 \int_0^1 \frac{z^4\,dz}{\sqrt{1-z^2}} + \frac{5}{16}s^6 \int_0^1 \frac{z^6\,dz}{\sqrt{1-z^2}} + \cdots.$$

Evaluate these integrals and verify that your result agrees with the expression quoted in the text.

The case of small oscillations is particularly simple to deal with. Go back to Eq. (1.3.24), $\ddot{\theta} + \omega^2 \sin\theta = 0$, and assume that θ is so small that $\sin\theta$ is well approximated by θ. The equation simplifies to

$$\ddot{\theta} + \omega^2\theta = 0, \tag{1.3.36}$$

and we have simple harmonic motion. The general solution to this equation is

$$\theta(t) = \theta_0 \cos(\omega t + \delta), \tag{1.3.37}$$

where θ_0 is the amplitude and δ the initial phase. The solution reveals that the period of the motion is $P = 2\pi/\omega$, in complete agreement with our previous results.

1.4 Mechanics of a system of bodies

1.4.1 Equations of motion

Generalizing the discussion of the preceding section, we now consider a system of N bodies subjected to their mutual forces. For simplicity we assume that there are no external forces acting on the particles; these would originate from outside the system. Each particle in the system is labeled by a number $A = 1, 2, 3, \cdots, N$. The motion of body A is governed by the equation

$$m_A \boldsymbol{a}_A = \boldsymbol{F}_A, \tag{1.4.1}$$

where m_A is the mass of the body, $\boldsymbol{a}_A$ its acceleration, and $\boldsymbol{F}_A$ is the force acting on the body due to all other bodies. Relative to an arbitrary choice of origin O, the position vector of body A is $\boldsymbol{r}_A(t)$, its velocity is $\boldsymbol{v}_A(t) = \dot{\boldsymbol{r}}_A$, and its acceleration is $\boldsymbol{a}_A(t) = \dot{\boldsymbol{v}}_A = \ddot{\boldsymbol{r}}_A$.

The force acting on body A can be expressed as a sum of individual forces exerted by each other body. We write

$$\boldsymbol{F}_A = \sum_{B \neq A} \boldsymbol{F}_{AB}. \tag{1.4.2}$$

Here, $\boldsymbol{F}_{AB}$ is the force exerted on A by B; the sum over B obviously excludes A because a body does not exert a force on itself. We assume Newton's *third law*, which states that

$$\boldsymbol{F}_{BA} = -\boldsymbol{F}_{AB}. \tag{1.4.3}$$

In words, the force exerted on B by A is equal in magnitude and opposite in direction to the force exerted on A by B. Suppose, for example, that the force exerted on A by B is repulsive; then the force exerted on B by A will also be repulsive, and it will point in the opposite direction.

1.4.2 Centre of mass

The *centre of mass* of a system of N bodies is at a position $\boldsymbol{R}$ which is defined by

$$\boldsymbol{R} = \frac{1}{M}\sum_A m_A \boldsymbol{r}_A, \tag{1.4.4}$$

where

$$M = \sum_A m_A \tag{1.4.5}$$

is the total mass of the system.

The centre of mass moves in accordance with Newton's law, which implies

$$\begin{aligned} M\ddot{\boldsymbol{R}} &= \sum_A m_A \boldsymbol{a}_A \\ &= \sum_A \boldsymbol{F}_A \\ &= \sum_{A,B,A\neq B} \boldsymbol{F}_{AB}, \end{aligned}$$

where we have used Eq. (1.4.2). In the last line we sum over both A and B (both from 1 to N), but we make sure to exclude all terms for which $A = B$. Let us examine the double sum in the special case of three particles. We have

$$\begin{aligned} \sum_{A,B,A\neq B} \boldsymbol{F}_{AB} &= \sum_{A=1}^{N}\sum_{B=1}^{N} \boldsymbol{F}_{AB} \\ &= \sum_{A=1}^{N} \big(\boldsymbol{F}_{A1} + \boldsymbol{F}_{A2} + \boldsymbol{F}_{A3}\big) \\ &= \big(\boldsymbol{F}_{21} + \boldsymbol{F}_{31}\big) + \big(\boldsymbol{F}_{12} + \boldsymbol{F}_{32}\big) + \big(\boldsymbol{F}_{13} + \boldsymbol{F}_{23}\big) \\ &= \big(\boldsymbol{F}_{21} + \boldsymbol{F}_{12}\big) + \big(\boldsymbol{F}_{31} + \boldsymbol{F}_{13}\big) + \big(\boldsymbol{F}_{32} + \boldsymbol{F}_{23}\big) \\ &= \boldsymbol{0}. \end{aligned}$$

The double sum vanishes by virtue of Newton's third law, and this property remains true for arbitrary values of N. We therefore have

$$\ddot{\boldsymbol{R}} = \boldsymbol{0}, \qquad \Rightarrow \qquad \boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)t. \tag{1.4.6}$$

The centre of mass moves with a uniform velocity, and it therefore defines the origin of another inertial frame.

It is usually convenient to shift the origin of the reference frame to the centre of mass, by defining new positions vectors $\boldsymbol{r}'_A(t)$ according to

$$\boldsymbol{r}'_A = \boldsymbol{r}_A - \boldsymbol{R}. \tag{1.4.7}$$

It should be kept in mind that the centre of mass defines the origin of an inertial frame only when there are no external forces acting on the particles. When external forces are present each particle moves according to $m_A\boldsymbol{a}_A = \boldsymbol{F}_A^{\text{internal}} + \boldsymbol{F}_A^{\text{external}}$, where the first term represents the internally-produced force acting on A, and the second term represents the external force. It is then easy to show that the centre of mass will move according to $M\ddot{\boldsymbol{R}} = \sum_A \boldsymbol{F}_A^{\text{external}}$; it is accelerated by the net sum of all the external forces.

Exercise 1.12. Prove the preceding statement.

1.4.3 Total linear and angular momenta

The total linear momentum of the system of N bodies is defined by

$$\boldsymbol{P} = \sum_A \boldsymbol{p}_A = \sum_A m_A \boldsymbol{v}_A, \tag{1.4.8}$$

where $\boldsymbol{p}_A$ are the individual momenta. We have

$$\boldsymbol{P} = \frac{d}{dt} \sum_A m_A \boldsymbol{r}_A,$$

or, according to Eq. (1.4.4),

$$\boldsymbol{P} = M\dot{\boldsymbol{R}}. \tag{1.4.9}$$

The total momentum therefore follows the motion of the centre of mass. Because $\dot{\boldsymbol{R}}(t) = \dot{\boldsymbol{R}}(0)$ according to Eq. (1.4.6), we have the important statement that *the total linear momentum is a constant vector*. If the origin of the inertial frame is at the centre of mass, then $\boldsymbol{R} = \boldsymbol{0}$ and $\dot{\boldsymbol{R}} = \boldsymbol{0}$; this means that $\boldsymbol{P} = \boldsymbol{0}$. In this centre-of-mass frame, the total momentum of the system of particles is zero.

The total angular momentum of the system is

$$\boldsymbol{L} = \sum_A \boldsymbol{r}_A \times \boldsymbol{p}_A = \sum_A m_A \boldsymbol{r}_A \times \boldsymbol{v}_A. \tag{1.4.10}$$

Its rate of change is calculated as

$$\begin{aligned} \dot{\boldsymbol{L}} &= \sum_A m_A \big(\boldsymbol{v}_A \times \boldsymbol{v}_A + \boldsymbol{r}_A \times \boldsymbol{a}_A\big) \\ &= \sum_A \boldsymbol{r}_A \times \boldsymbol{F}_A \\ &= \sum_{A,B,A\neq B} \boldsymbol{r}_A \times \boldsymbol{F}_{AB}, \end{aligned}$$

where we have again involved Eq. (1.4.2). Let us examine the double sum for the special case of three particles. We have

$$\begin{aligned} \sum_{A,B,A\neq B} \boldsymbol{r}_A \times \boldsymbol{F}_{AB} &= \sum_A \big(\boldsymbol{r}_A \times \boldsymbol{F}_{A1} + \boldsymbol{r}_A \times \boldsymbol{F}_{A2} + \boldsymbol{r}_A \times \boldsymbol{F}_{A3}\big) \\ &= \big(\boldsymbol{r}_2 \times \boldsymbol{F}_{21} + \boldsymbol{r}_3 \times \boldsymbol{F}_{31}\big) + \big(\boldsymbol{r}_1 \times \boldsymbol{F}_{12} + \boldsymbol{r}_3 \times \boldsymbol{F}_{32}\big) \\ &\quad + \big(\boldsymbol{r}_1 \times \boldsymbol{F}_{13} + \boldsymbol{r}_2 \times \boldsymbol{F}_{23}\big) \\ &= \big(\boldsymbol{r}_1 - \boldsymbol{r}_2\big) \times \boldsymbol{F}_{12} + \big(\boldsymbol{r}_1 - \boldsymbol{r}_3\big) \times \boldsymbol{F}_{13} + \big(\boldsymbol{r}_2 - \boldsymbol{r}_3\big) \times \boldsymbol{F}_{23}, \end{aligned}$$

where we have used Eq. (1.4.3). The vector $\boldsymbol{r}_1 - \boldsymbol{r}_2$ is directed from body 2 to body 1. In most circumstances the force $\boldsymbol{F}_{12}$ also is directed from body 2 to body 1 (or in the opposite direction). Under these conditions the vector product $(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{F}_{12}$ is zero, and this is true for all other pairs of bodies. The double sum is therefore zero. These considerations generalize to an arbitrary number of bodies, and we conclude that

$$\dot{\boldsymbol{L}} = \boldsymbol{0} \tag{1.4.11}$$

whenever the force $\boldsymbol{F}_{AB}$ points in the direction of the relative separation $\boldsymbol{r}_A - \boldsymbol{r}_B$. Under these conditions we have conservation of the system's total angular

momentum.

Exercise 1.13. Calculate $d\boldsymbol{P}/dt$ and $d\boldsymbol{L}/dt$ when there are also external forces acting on the particles.

Let us express the position vector of body A as in Eq. (1.4.7),

$$\boldsymbol{r}_A = \boldsymbol{R} + \boldsymbol{r}'_A, \tag{1.4.12}$$

where $\boldsymbol{r}'_A$ is its position relative to the centre of mass. We write, similarly,

$$\boldsymbol{v}_A = \dot{\boldsymbol{R}} + \boldsymbol{v}'_A. \tag{1.4.13}$$

We make these substitutions into Eq. (1.4.10), and get

$$\begin{aligned}
\boldsymbol{L} &= \sum_A m_A(\boldsymbol{R} + \boldsymbol{r}'_A) \times (\dot{\boldsymbol{R}} + \boldsymbol{v}'_A) \\
&= \sum_A m_A(\boldsymbol{R} \times \dot{\boldsymbol{R}} + \boldsymbol{R} \times \boldsymbol{v}'_A + \boldsymbol{r}'_A \times \dot{\boldsymbol{R}} + \boldsymbol{r}'_A \times \boldsymbol{v}'_A) \\
&= (\boldsymbol{R} \times \dot{\boldsymbol{R}}) \sum_A m_A + \boldsymbol{R} \times \sum_A m_A \boldsymbol{v}'_A - \dot{\boldsymbol{R}} \times \sum_A m_A \boldsymbol{r}'_A + \sum_A m_A \boldsymbol{r}'_A \times \boldsymbol{v}'_A.
\end{aligned}$$

This mess simplifies. For the first term on the right-hand side we have $\sum_A m_A = M$, the total mass of the system. In the second term we recognize that $\sum_A m_A \boldsymbol{v}'_A$ is the system's total momentum as measured in the centre-of-mass frame; this is zero. The third term vanishes also, and we finally have

$$\boldsymbol{L} = M\boldsymbol{R} \times \dot{\boldsymbol{R}} + \sum_A m_A \boldsymbol{r}'_A \times \boldsymbol{v}'_A. \tag{1.4.14}$$

In this expression, the first term represents the angular momentum of the centre of mass, while the second term is the total angular momentum of the system of particles *relative* to the centre of mass. When the origin of the inertial frame is placed at the centre of mass, we have $\boldsymbol{R} = \boldsymbol{0}$ and the first term disappears. In general, we see that $\boldsymbol{L}$ depends on the choice of origin.

1.4.4 Conservation of energy

The presentation here parallels closely our discussion of Sec. 1.3.4 on energy conservation for a single particle. The notation of this section, however, will be slightly more cumbersome, because we now have to keep track of many particles.

We begin by calculating the total work done on all the particles as they move from a configuration labeled 1 to another configuration labeled 2. (This means that in the interval of time over which we follow the particles, each moves from a point 1 to a point 2 on its trajectory.) This is

$$W_{12} = \sum_A \int_1^2 \boldsymbol{F}_A \cdot d\boldsymbol{r}_A = \sum_A \int_1^2 \boldsymbol{F}_A \cdot \boldsymbol{v}_A \, dt, \tag{1.4.15}$$

where $d\boldsymbol{r}_A = \boldsymbol{v}_A \, dt$ is the displacement vector on the trajectory of particle A. Substituting the equations of motion (1.4.1) gives

$$W_{12} = \sum_A \int_1^2 m_A \frac{d\boldsymbol{v}_A}{dt} \cdot \boldsymbol{v}_A \, dt.$$

But since $\boldsymbol{v}_A \cdot d\boldsymbol{v}_A/dt = \frac{1}{2}dv_A^2/dt$, where $v_A^2 = \boldsymbol{v}_A \cdot \boldsymbol{v}_A$, this becomes

$$W_{12} = \sum_A \int_1^2 \frac{d}{dt}\left(\frac{1}{2}m_A v_A^2\right) dt = \sum_A \big[T_A(2) - T_A(1)\big]$$

where $T_A = \frac{1}{2}m_A v_A^2$ is the kinetic energy of particle A. Introducing the *total kinetic energy* of the system

$$T = \sum_A T_A = \sum_A \frac{1}{2}m_A v_A^2, \tag{1.4.16}$$

we have obtained the statement of the *work-energy theorem*,

$$W_{12} = T(2) - T(1). \tag{1.4.17}$$

In words, this states that the total work done on all the particles is equal to the difference in total kinetic energy between the configurations 2 and 1.

Exercise 1.14. Express the total kinetic energy of the system in terms of the centre-of-mass quantities $\boldsymbol{R}$, $\dot{\boldsymbol{R}}$ and the relative quantities $\boldsymbol{r}'_A$, $\boldsymbol{v}'_A$. You should find an expression analogous to Eq. (1.4.14).

To proceed further we shall assume that the mutual force $\boldsymbol{F}_{AB}$ can be derived from a potential $V_{AB} = V_{BA}$ that depends only on the distance r_{AB} between the bodies A and B. We shall therefore have

$$V_{AB} = V_{AB}(r_{AB}), \qquad r_{AB} \equiv |\boldsymbol{r}_{AB}|, \qquad \boldsymbol{r}_{AB} \equiv \boldsymbol{r}_A - \boldsymbol{r}_B. \tag{1.4.18}$$

The force acting on A exerted by B is given by

$$\boldsymbol{F}_{AB} = -\boldsymbol{\nabla}_A V_{AB}, \tag{1.4.19}$$

where $\boldsymbol{\nabla}_A = (\partial/\partial x_A, \partial/\partial y_A, \partial/\partial z_A)$ is the gradient operator with respect to the coordinates $\boldsymbol{r}_A = (x_A, y_A, z_A)$ of body A. Similarly, the force acting on B exerted by A is

$$\boldsymbol{F}_{BA} = -\boldsymbol{\nabla}_B V_{AB}, \tag{1.4.20}$$

where $\boldsymbol{\nabla}_B$ us the gradient operator with respect to the coordinates $\boldsymbol{r}_B = (x_B, y_B, z_B)$ of body B. (To be fully symmetrical we might have written $\boldsymbol{F}_{BA} = -\boldsymbol{\nabla}_B V_{BA}$, but this produces the same result because V_{BA} is by definition equal to V_{AB}.)

Let us verify that $\boldsymbol{F}_{BA} = -\boldsymbol{F}_{AB}$ and that the forces are directed along the vector $\boldsymbol{r}_A - \boldsymbol{r}_B$, that is, in the direction of the relative separation between the two bodies. Let us examine, say, the x component of $\boldsymbol{F}_{AB}$. According to Eq. (1.4.19) we have

$$F_{AB,x} = -\frac{\partial}{\partial x_A} V_{AB}.$$

Because V_{AB} depends on x_A only through its dependence on the distance r_{AB}, we apply the chain rule to evaluate the partial derivative:

$$F_{AB,x} = -\frac{dV_{AB}}{dr_{AB}}\frac{\partial r_{AB}}{\partial x_A} = -V'_{AB}\frac{\partial r_{AB}}{\partial x_A},$$

where the prime indicates differentiation with respect to r_{AB}. To calculate the partial derivative of r_{AB} with respect to x_A we start with the definition

$$r_{AB}^2 = \big(x_A - x_B\big)^2 + \big(y_A - y_B\big)^2 + \big(z_A - z_B\big)^2.$$

Differentiating both sides gives

$$2r_{AB}\frac{\partial r_{AB}}{\partial x_A} = 2\big(x_A - x_B\big),$$

and finally,

$$\frac{\partial r_{AB}}{\partial x_A} = \frac{x_A - x_B}{r_{AB}}.$$

Returning to our main calculation we find that the x component of the force is

$$F_{AB,x} = -\frac{x_A - x_B}{r_{AB}}V'_{AB},$$

and very similar calculations would reveal also the y and z components. The complete vectorial expression is

$$\boldsymbol{F}_{AB} = -\frac{\boldsymbol{r}_{AB}}{r_{AB}}V'_{AB}, \qquad V'_{AB} = \frac{dV_{AB}}{dr_{AB}}. \tag{1.4.21}$$

This shows that F_{AB} is indeed directed along $\boldsymbol{r}_{AB} = \boldsymbol{r}_A - \boldsymbol{r}_B$.

We now calculate $\boldsymbol{F}_{BA}$. Looking also at its x component we get from Eq. (1.4.20) that

$$F_{BA,x} = -\frac{dV_{AB}}{dr_{AB}}\frac{\partial r_{AB}}{\partial x_B} = -V'_{AB}\frac{\partial r_{AB}}{\partial x_B}.$$

Repeating the same steps as before we find that

$$\frac{\partial r_{AB}}{\partial x_B} = -\frac{x_A - x_B}{r_{AB}},$$

which *differs by a sign* from the preceding expression for $\partial r_{AB}/\partial x_A$. We finally obtain

$$F_{BA,x} = \frac{x_A - x_B}{r_{AB}}V'_{AB}$$

and the vectorial generalization

$$\boldsymbol{F}_{BA} = \frac{\boldsymbol{r}_{AB}}{r_{AB}}V'_{AB}. \tag{1.4.22}$$

This also is directed along $\boldsymbol{r}_{AB} = \boldsymbol{r}_A - \boldsymbol{r}_B$. Comparing Eqs. (1.4.21) and (1.4.22) shows that, as required, $\boldsymbol{F}_{BA} = -\boldsymbol{F}_{AB}$.

The calculations presented above are important and they occur frequently. To go through them with some efficiency it is useful to memorize the rule $\boldsymbol{\nabla}_B V_{AB} = -\boldsymbol{\nabla}_A V_{AB}$, which is valid whenever V_{AB} depends on $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ only through its dependence on $r_{AB} = |\boldsymbol{r}_A - \boldsymbol{r}_B|$.

Having made our assumptions regarding the mutual forces $\boldsymbol{F}_{AB}$, we now return to the work integral of Eq. (1.4.15). Substituting Eq. (1.4.2) gives

$$W_{12} = \sum_{A,B,A\neq B}\int_1^2 \boldsymbol{F}_{AB}\cdot d\boldsymbol{r}_A.$$

To examine this we again specialize to the case of three particles. We have

$$\begin{aligned} W_{12} &= \int_1^2 \big(\boldsymbol{F}_{21}\cdot d\boldsymbol{r}_2 + \boldsymbol{F}_{31}\cdot d\boldsymbol{r}_3 + \boldsymbol{F}_{12}\cdot d\boldsymbol{r}_1 + \boldsymbol{F}_{32}\cdot d\boldsymbol{r}_3 + \boldsymbol{F}_{13}\cdot d\boldsymbol{r}_1 + \boldsymbol{F}_{23}\cdot d\boldsymbol{r}_2\big) \\ &= \int_1^2 \Big[\boldsymbol{F}_{12}\cdot\big(d\boldsymbol{r}_1 - d\boldsymbol{r}_2\big) + \boldsymbol{F}_{13}\cdot\big(d\boldsymbol{r}_1 - d\boldsymbol{r}_3\big) + \boldsymbol{F}_{23}\cdot\big(d\boldsymbol{r}_2 - d\boldsymbol{r}_3\big)\Big]. \end{aligned}$$

But $d\boldsymbol{r}_1 - d\boldsymbol{r}_2 = d(\boldsymbol{r}_1 - \boldsymbol{r}_2) = d\boldsymbol{r}_{12}$, so this can be expressed as

$$W_{12} = \int_1^2 \big(\boldsymbol{F}_{12} \cdot d\boldsymbol{r}_{12} + \boldsymbol{F}_{13} \cdot d\boldsymbol{r}_{13} + \boldsymbol{F}_{23} \cdot d\boldsymbol{r}_{23}\big).$$

At this stage of the derivation we incorporate the fact that the mutual forces are derived from a potential. As we have seen, $\boldsymbol{F}_{12} = -\boldsymbol{\nabla}_1 V_{12}$, where $\boldsymbol{\nabla}_1 = (\partial/\partial x_1, \partial/\partial y_1, \partial/\partial z_1)$. But since V_{12} depends on (x_1, y_1, z_1) only through its dependence on (x_{12}, y_{12}, z_{12}) (where, for example, $x_{12} = x_1 - x_2$), the force can also be expressed as $\boldsymbol{F}_{12} = -\boldsymbol{\nabla}_{12} V_{12}$, where $\boldsymbol{\nabla}_{12}$ is the gradient operator with respect to $\boldsymbol{r}_{12} = (x_{12}, y_{12}, z_{12})$,

$$\boldsymbol{\nabla}_{12} = \left(\frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial y_{12}}, \frac{\partial}{\partial z_{12}}\right).$$

This is possible because $\partial x_{12}/\partial x_1 = 1$, and so on.

So we now have

$$W_{12} = \int_1^2 \big(-\boldsymbol{\nabla}_{12} V_{12} \cdot d\boldsymbol{r}_{12} - \boldsymbol{\nabla}_{13} V_{13} \cdot d\boldsymbol{r}_{13} - \boldsymbol{\nabla}_{23} V_{23} \cdot d\boldsymbol{r}_{23}\big).$$

Each integral can be evaluated (refer back to Sec. 1.3.1), giving

$$\begin{aligned} W_{12} &= -\big[V_{12}(2) - V_{12}(1)\big] - \big[V_{13}(2) - V_{13}(1)\big] - \big[V_{23}(2) - V_{23}(1)\big] \\ &\equiv -\big[V_{12} + V_{13} + V_{23}\big]_1^2. \end{aligned}$$

Since $V_{21} = V_{12}$ and so on, we may write this as

$$W_{12} = -\frac{1}{2}\big[V_{12} + V_{13} + V_{21} + V_{23} + V_{31} + V_{32}\big]_1^2,$$

where we now sum over all possible pairs of indices, provided that each index is not repeated. Generalizing to an arbitrary number of particles, this is

$$W_{12} = -\left[\frac{1}{2}\sum_{A,B,A\neq B} V_{AB}\right]_1^2.$$

We define the *total potential energy* of the system to be

$$V = \frac{1}{2}\sum_{A,B,A\neq B} V_{AB}. \tag{1.4.23}$$

With this notation our previous result is $W_{12} = -[V(2) - V(1)]$, and Eq. (1.4.17) becomes $-V(2) + V(1) = T(2) - T(1)$ or $T(1) + V(1) = T(2) + V(2)$.

We have finally established that the *total mechanical energy* of the system,

$$E = T + V = \sum_A \frac{1}{2} m_A v_A^2 + \frac{1}{2}\sum_{A,B,A\neq B} V_{AB}(r_{AB}), \tag{1.4.24}$$

stays unchanged as the particles move from configuration 1 to configuration 2. We recall that the mutual potentials V_{AB} are assumed to depend on $r_{AB} = |\boldsymbol{r}_A - \boldsymbol{r}_B|$ only; the mutual forces are then given by Eqs. (1.4.21) and (1.4.22). This is the statement of energy conservation for a system of particles.

Exercise 1.15. Starting from the definition of Eq. (1.4.24), prove directly that $dE/dt = 0$.

1.5 Kepler's problem

To give concreteness to the formal developments of the preceding section we examine, in this section, the specific situation of two bodies subjected to their mutual gravitational forces. This could be the Earth-Moon system, or the Sun-Jupiter system, or again a binary system of two main-sequence stars. Our goal is to determine the motion of the two bodies, that is, to find a solution to *Kepler's problem*.

1.5.1 Gravitational force

The force acting on body 1 due to the gravity of body 2 has a magnitude Gm_1m_2/r_{12}^2, where G is Newton's gravitational constant, m_1 the mass of body 1, m_2 the mass of body 2, and r_{12} is the distance between the two bodies. The force is directed along the vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$, which points from body 1 to body 2. Introducing the notation

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \qquad r = |\boldsymbol{r}_1 - \boldsymbol{r}_2| \equiv r_{12}, \tag{1.5.1}$$

we write

$$\boldsymbol{F}_{12} = -Gm_1m_2\frac{\boldsymbol{r}}{r^3}. \tag{1.5.2}$$

The force acting on body 2 due to the gravity of body 1 is

$$\boldsymbol{F}_{21} = Gm_1m_2\frac{\boldsymbol{r}}{r^3}, \tag{1.5.3}$$

and it is directed along $\boldsymbol{r}_1 - \boldsymbol{r}_2$, which points from body 2 to body 1.

These forces can be derived from a mutual potential

$$V_{12} = -\frac{Gm_1m_2}{r}. \tag{1.5.4}$$

This means that the force of Eq. (1.5.2) is given by

$$\boldsymbol{F}_{12} = -\boldsymbol{\nabla}_1 V_{12}, \tag{1.5.5}$$

where $\boldsymbol{\nabla}_1$ is the gradient operator with respect to the coordinates $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ of body 1. Similarly, the force of Eq. (1.5.3) can be expressed as

$$\boldsymbol{F}_{21} = -\boldsymbol{\nabla}_2 V_{12}, \tag{1.5.6}$$

where $\boldsymbol{\nabla}_2$ is the gradient operator with respect to the coordinates $\boldsymbol{r}_2 = (x_2, y_2, z_2)$ of body 2. To verify these statements, let us calculate, say, the z component of $\boldsymbol{F}_{21}$. We have

$$F_{21,z} = -\frac{\partial V_{12}}{\partial z_2} = -\frac{dV_{12}}{dr}\frac{\partial r}{\partial z_2}.$$

The first factor is

$$\frac{dV_{12}}{dr} = \frac{Gm_1m_2}{r^2},$$

and to calculate the second factor we start with

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

and differentiate both sides with respect to z_2. This gives

$$2r\frac{\partial r}{\partial z_2} = -2(z_1 - z_2)$$

or

$$\frac{\partial r}{\partial z_2} = -\frac{z_1 - z_2}{r}.$$

So finally,

$$F_{21,z} = Gm_1m_2\frac{z_1 - z_2}{r^3},$$

and this is clearly compatible with Eq. (1.5.3). Similar calculations would return all other components of $\boldsymbol{F}_{21}$ and all components of $\boldsymbol{F}_{12}$, and Eqs. (1.5.5) and (1.5.6) would be fully verified.

According to Eq. (1.4.23), the total potential energy of the two-body system is

$$V = \frac{1}{2}\sum_{A,B,A\neq B} V_{AB} = \frac{1}{2}(V_{12} + V_{21}),$$

or

$$V = V_{12}. \tag{1.5.7}$$

This result will allow us, in the following subsections, to omit the label "12" from the mutual potential; we shall write, simply, $V_{12} = V = -Gm_1m_2/r$.

1.5.2 Equations of motion

Newton's equations for the two bodies are $m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_{12} = -Gm_1m_2\boldsymbol{r}/r^3$ and $m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_{21} = Gm_1m_2\boldsymbol{r}/r^3$. Simplifying, we arrive at

$$\ddot{\boldsymbol{r}}_1 = -Gm_2\frac{\boldsymbol{r}}{r^3} \tag{1.5.8}$$

and

$$\ddot{\boldsymbol{r}}_2 = Gm_1\frac{\boldsymbol{r}}{r^3}, \tag{1.5.9}$$

where, we recall, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $r = |\boldsymbol{r}|$.

The position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be expressed in terms of $\boldsymbol{R}$, the position of the centre of mass, and $\boldsymbol{r}$, the relative position. We have, according to Eq. (1.4.4), $M\boldsymbol{R} = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2$, where $M = m_1 + m_2$ is the total mass. Simple algebra gives

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r} \tag{1.5.10}$$

and

$$\boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r}. \tag{1.5.11}$$

The motion of the centre of mass is determined by the equation $M\ddot{\boldsymbol{R}} = m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2 = -Gm_1m_2\boldsymbol{r}/r^3 + Gm_1m_2\boldsymbol{r}/r^3 = \boldsymbol{0}$. As we had discovered in Sec. 1.4.2, the centre of mass moves uniformly:

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)t. \tag{1.5.12}$$

The motion of the relative position, on the other hand, is determined by the equation $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = -Gm_2\boldsymbol{r}/r^3 - Gm_1\boldsymbol{r}/r^3$, or

$$\ddot{\boldsymbol{r}} = -GM\frac{\boldsymbol{r}}{r^3}, \qquad M = m_1 + m_2. \tag{1.5.13}$$

Exercise 1.16. Verify Eqs. (1.5.10) and (1.5.11).

The centre of mass defines the origin of an inertial frame, and the mathematical description of the two-body system is simplest in this reference frame. We shall therefore set $\boldsymbol{R} = \boldsymbol{0}$, which brings Eqs. (1.5.10) and (1.5.11) to the simpler form

$$\boldsymbol{r}_1 = \frac{m_2}{M}\boldsymbol{r}, \qquad \boldsymbol{r}_2 = -\frac{m_1}{M}\boldsymbol{r}. \tag{1.5.14}$$

The vector $\boldsymbol{r}(t)$ is determined by integrating Eq. (1.5.13). Once the solution is known we obtain immediately, from Eqs. (1.5.14), the vectors $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$, which describe the trajectories of the two individual bodies. In Eq. (1.5.13) we have the reduction of our original two-body problem to a simpler *effective one-body problem*. The effective body is fictitious; it moves with a position vector $\boldsymbol{r}(t)$ in the gravitational field of another fictitious mass $M = m_1 + m_2$ situated at the centre of mass of the original system.

1.5.3 Conservation of angular momentum

From Eq. (1.5.13) we can immediately derive the statement

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = -GM\frac{\boldsymbol{r} \times \boldsymbol{r}}{r^3} = \boldsymbol{0}.$$

But $d(\boldsymbol{r} \times \dot{\boldsymbol{r}})/dt = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = \boldsymbol{r} \times \ddot{\boldsymbol{r}}$, and it follows that $d(\boldsymbol{r} \times \dot{\boldsymbol{r}})/dt = \boldsymbol{0}$. The vector

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{1.5.15}$$

is therefore constant during the motion. This must be related to the system's total angular momentum which, according to the discussion of Sec. 1.4.3, is also a constant vector. The definition of Eq. (1.4.10) gives

$$\boldsymbol{L} = \sum_A m_A \boldsymbol{r}_A \times \dot{\boldsymbol{r}}_A = m_1 \boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2 \boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2.$$

Substitution of Eqs. (1.5.14) gives

$$\boldsymbol{L} = \frac{m_1 m_2^2}{M^2}\boldsymbol{r} \times \dot{\boldsymbol{r}} + \frac{m_2 m_1^2}{M^2}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \frac{m_1 m_2 (m_1 + m_2)}{M^2}\boldsymbol{r} \times \dot{\boldsymbol{r}}.$$

Simplification produces

$$\boldsymbol{L} = \frac{m_1 m_2}{M}\boldsymbol{h}, \tag{1.5.16}$$

and we find that, sure enough, the vector $\boldsymbol{h}$ is a rescaled version of the total angular-momentum vector. We shall call $\boldsymbol{h}$ the *reduced angular-momentum vector* of our two-body system.

The position vector $\boldsymbol{r}(t)$ must be at all times orthogonal to the constant vector $\boldsymbol{h}$, because $\boldsymbol{r} \cdot \boldsymbol{h} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = 0$. This simple fact has the far-reaching consequence that *the motion must always take place within a plane that is orthogonal to the fixed direction of the vector* $\boldsymbol{h}$. The planar nature of the motion is illustrated in Fig. 1.9.

Conservation of total angular momentum therefore implies planar motion. To describe this mathematically we orient the coordinate system so that the orbital plane is the x-y plane, and we direct the vector $\boldsymbol{h}$ along the z axis. We have

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}}, \tag{1.5.17}$$

$$\dot{\boldsymbol{r}}(t) = \dot{x}(t)\hat{\boldsymbol{x}} + \dot{y}(t)\hat{\boldsymbol{y}}, \tag{1.5.18}$$

and

$$\boldsymbol{h} = h\hat{\boldsymbol{z}}. \tag{1.5.19}$$

A simple calculation, based on Eqs. (1.5.15) and (1.5.17)–(1.5.19), reveals that

$$h = x\dot{y} - y\dot{x} = \text{constant}. \tag{1.5.20}$$

Exercise 1.17. Verify Eq. (1.5.20).

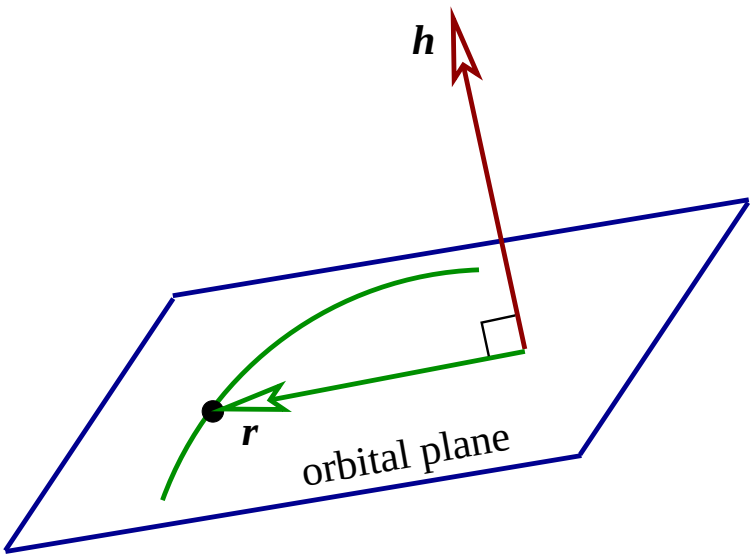


Figure 1.9: The position vector $\boldsymbol{r}(t)$ always lies in a plane orthogonal to the constant vector $\boldsymbol{h}$. This plane is called the *orbital plane*.

1.5.4 Polar coordinates

To proceed with our calculations it is convenient to involve the polar coordinates r and ϕ that were first introduced in Sec. 1.2. These, we recall, are defined by

$$x = r\cos\phi, \qquad y = r\sin\phi. \tag{1.5.21}$$

In terms of the new coordinates and the associated basis of unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$, we have

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}, \tag{1.5.22}$$

$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}, \tag{1.5.23}$$

$$\boldsymbol{a} = \big(\ddot{r} - r\dot{\phi}^2\big)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\big(r^2\dot{\phi}\big)\hat{\boldsymbol{\phi}} \tag{1.5.24}$$

for the position, velocity, and acceleration vectors, respectively.

If we now substitute Eq. (1.5.24) for $\boldsymbol{a} = \ddot{\boldsymbol{r}}$ into the equations of motion of Eq. (1.5.13), we obtain

$$\big(\ddot{r} - r\dot{\phi}^2\big)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\big(r^2\dot{\phi}\big)\hat{\boldsymbol{\phi}} = -\frac{GM}{r^3}\boldsymbol{r} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}.$$

Equating the radial components of both sides gives

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}, \tag{1.5.25}$$

while equating the angular components gives

$$\frac{d}{dt}\big(r^2\dot{\phi}\big) = 0. \tag{1.5.26}$$

These are the equations of motion of the effective one-body problem, expressed in their simplest form in terms of polar coordinates. In the following subsections we will endeavour to find solutions to these equations.

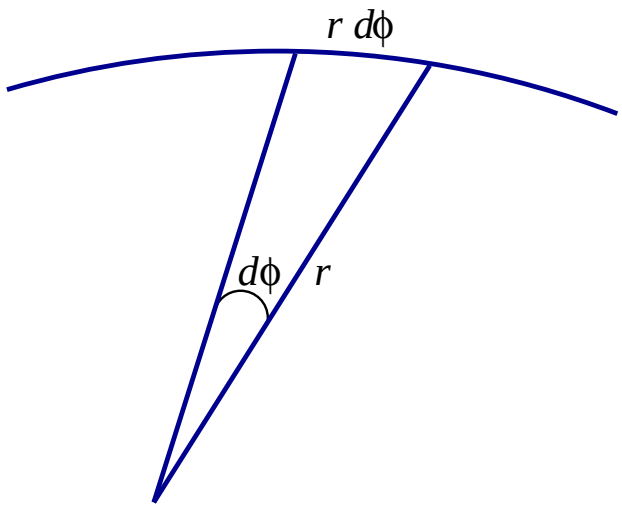


Figure 1.10: The position vector moves by an angle $d\phi$ during a time dt.

1.5.5 Kepler's second law

Kepler's second law states that *the position vector of a planet orbiting the Sun sweeps out equal areas in equal times*. The statement generalizes to any two massive bodies, and in this case the position vector refers specifically to the relative separation $\boldsymbol{r}(t)$ between the two bodies. This law comes as an immediate consequence of Eq. (1.5.26), which implies the conservation statement $r^2\dot{\phi} = \text{constant}$.

Let us first show that this constant is in fact h, the magnitude of the constant vector $\boldsymbol{h}$ defined by Eq. (1.5.15). According to Eq. (1.5.20), this is given by $h = x\dot{y} - y\dot{x}$. Making use of Eqs. (1.5.21), we write this as

$$h = (r\cos\phi)(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - (r\sin\phi)(\dot{r}\cos\phi - r\dot{\phi}\sin\phi).$$

Simplifying this we obtain

$$h = r^2\dot{\phi} = \text{constant}, \tag{1.5.27}$$

the expected result.

The fact that $r^2\dot{\phi}$ is conserved gives us our statement of the second law. Consider Fig. 1.10. During an interval dt of time the position vector moves by an angle $d\phi$ and sweeps out an area dA. To a good approximation the area is shaped as a triangle and we have $dA = \frac{1}{2}r(r\,d\phi) = \frac{1}{2}r^2\,d\phi$. The rate at which the position vector sweeps out this area is therefore

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{1}{2}h. \tag{1.5.28}$$

This is a constant, and we have the mathematical statement of Kepler's second law.

1.5.6 Conservation of energy

With the substitution $\dot{\phi} = h/r^2$ obtained from Eq. (1.5.27), Eq. (1.5.25) becomes

$$\ddot{r} + \frac{GM}{r^2} - \frac{h^2}{r^3} = 0. \tag{1.5.29}$$

This equation can immediately be integrated by multiplying all members by $\dot{r}$. (Recall that we used the same trick back in Sec. 1.3.7.) We have

$$\ddot{r}\dot{r} \;=\; \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right),$$

$$\begin{aligned}\frac{GM}{r^2}\dot{r} &= \frac{d}{dt}\left(-\frac{GM}{r}\right),\\ -\frac{h^2}{r^3}\dot{r} &= \frac{d}{dt}\left(\frac{h^2}{2r^2}\right),\end{aligned}$$

and the preceding equation becomes

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\right) = 0.$$

This implies $\frac{1}{2}\dot{r}^2 - GM/r + h^2/(2r^2) = \varepsilon$, where ε is the constant of integration.

We shall write this result in the form

$$\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon, \tag{1.5.30}$$

with

$$\nu(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}. \tag{1.5.31}$$

The first term on the left of Eq. (1.5.30) can be thought of as a *reduced kinetic energy* for the radial component of the motion. The second term is a *reduced effective potential* for this motion, and the constant ε is a *reduced total energy*. (Recall that we introduced this terminology back in Sec. 1.3.7; by "reduced" we mean a rescaled version of the usual quantities.)

The reduced energy ε is directly related to the system's true total energy E. Let us calculate it. The system's total kinetic energy is $T = \frac{1}{2}m_1|\dot{\boldsymbol{r}}_1|^2 + \frac{1}{2}m_2|\dot{\boldsymbol{r}}_2|^2$, and according to Eqs. (1.5.4) and (1.5.7), the potential energy is $V = -Gm_1m_2/r$. The total energy is therefore

$$E = \frac{1}{2}m_1|\dot{\boldsymbol{r}}_1|^2 + \frac{1}{2}m_2|\dot{\boldsymbol{r}}_2|^2 - \frac{Gm_1m_2}{r}.$$

After involving Eq. (1.5.14) and cleaning up the algebra, this becomes

$$E = \frac{m_1m_2}{M}\left(\frac{1}{2}|\dot{\boldsymbol{r}}|^2 - \frac{GM}{r}\right). \tag{1.5.32}$$

Now Eq. (1.5.23) states $\dot{\boldsymbol{r}} = \boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, so that $|\dot{\boldsymbol{r}}|^2 = \dot{r}^2 + r^2\dot{\phi}^2 = \dot{r}^2 + h^2/r^2$. So

$$E = \frac{m_1m_2}{M}\left(\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\right),$$

and comparing this with Eqs. (1.5.30) and (1.5.31) yields

$$E = \frac{m_1m_2}{M}\varepsilon. \tag{1.5.33}$$

As promised, ε is a rescaled version of the total energy E. Recall that back in Sec. 1.5.3 we had similarly obtained $\boldsymbol{L} = (m_1m_2/M)\boldsymbol{h}$.

Exercise 1.18. Verify Eq. (1.5.32) and check the algebra leading to Eq. (1.5.33).

1.5.7 Qualitative description of the orbital motion

The equations of motion have been reduced to the first-order form of Eqs. (1.5.27) and (1.5.30), with the effective potential $\nu(r)$ given by Eq. (1.5.31). The equation $\dot{\phi} = h/r^2$ informs us that ϕ increases monotonically with t: If h is positive $\dot{\phi}$ is

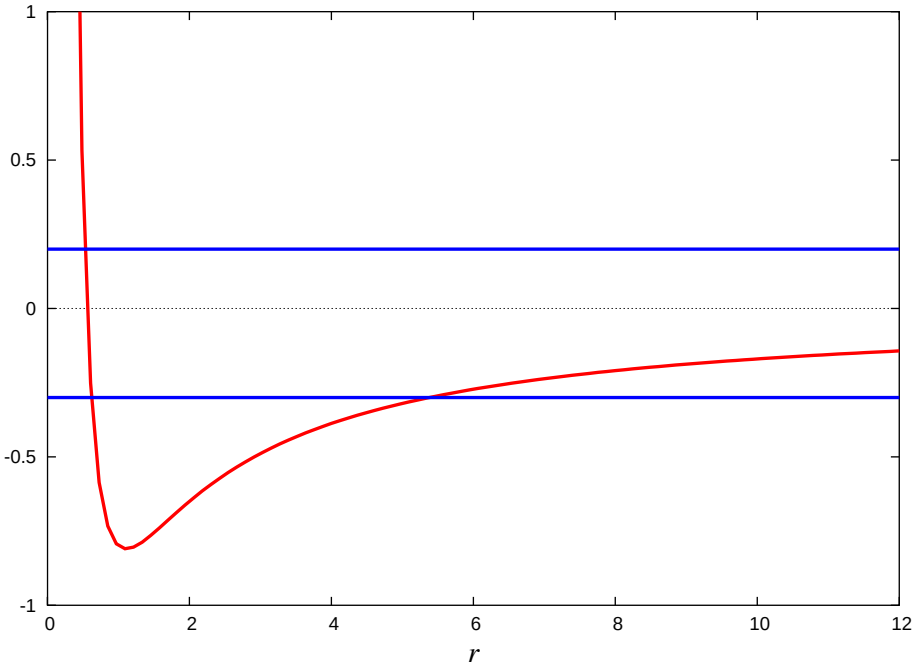


Figure 1.11: Energy diagram for the radial component of the motion. When $\varepsilon < 0$ the motion takes place between two turning points; this is elliptical motion. When $\varepsilon > 0$ the motion takes place on the right of a single turning point; this is hyperbolic motion. When $\varepsilon = 0$ the motion is parabolic.

always greater than zero and $\phi(t)$ is an increasing function; if h is negative $\dot{\phi}$ is always smaller than zero and $\phi(t)$ is then a decreasing function. (The case $h = 0$ will be dealt with separately later.) The equation

$$\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon$$

governs the radial component of the motion. As in Sec. 1.3.7 we will describe this qualitatively by constructing an *energy diagram*, a plot of the effective potential $\nu(r)$ together with the constant value of the reduced energy ε. The energy diagram is shown in Fig. 1.11. Recall the two main features of such diagrams: (i) The difference between ε and $\nu(r)$ represents $\frac{1}{2}\dot{r}^2$, the reduced kinetic energy, which must be positive; and (ii) points on the diagram for which $\nu(r) = \varepsilon$ represent *turning points* of the motion, at which $\dot{r}$ changes sign, either from positive to negative, or from negative to positive.

Because the effective potential can be negative, it is possible for ε to be either negative or positive. The nature of the motion depends sensitively on this sign.

When $\varepsilon < 0$ the motion takes place between two turning points at $r = r_{\min}$ and $r = r_{\max}$. The motion is bounded, and as we shall see, the orbit possesses an elliptical shape. When ε is equal to the minimum value of the effective potential, $\varepsilon = \nu_{\min} < 0$, motion can take place only at $r = r_0$, the radius at which the minimum occurs, which is defined by $\nu(r_0) = \nu_{\min}$. The orbit is then circular, because $\dot{r}$ is always zero and r can therefore never change with time.

When $\varepsilon > 0$ the motion takes place only to the right of a single turning point at $r = r_{\min}$. The motion proceeds from $r = \infty$ (where $\nu = 0$ and $\frac{1}{2}\dot{r}^2 = \varepsilon$) down to $r = r_{\min}$ (where $\dot{r}$ changes sign from negative to positive), and then back to $r = \infty$. The particle traces a hyperbola in the orbital plane, and the motion is said to be hyperbolic.

When $\varepsilon = 0$ the situation is qualitatively the same as before (for $\varepsilon > 0$). The only difference is that the particle now starts at $r = \infty$ with a zero radial velocity, because $\frac{1}{2}\dot{r}^2 = \varepsilon = 0$. The particle then traces a parabola in the orbital plane, and the motion is parabolic.

1.5.8 Circular orbits

Circular orbits are especially simple to describe. To have circular motion we need both $\dot{r}$ *and* $\ddot{r}$ to be zero, so that r always stays constant. The condition $\dot{r} = 0$ is not sufficient, because $\dot{r}$ might just happen to be in the process of changing sign at a turning point; we need a *permanent* turning point, which we get by also imposing $\ddot{r} = 0$. To get $\dot{r} = 0$ we need to impose

$$\varepsilon = \nu(r_0) = -\frac{GM}{r_0} + \frac{h^2}{2r_0^2},$$

where r_0 is the orbital radius. To get $\ddot{r} = 0$ we look back at Eq. (1.5.29) and impose

$$0 = \frac{GM}{r_0^2} - \frac{h^2}{r_0^3} = \nu'(r_0),$$

in which a prime indicates differentiation with respect to r. The second equation determines h in terms of r_0: we get $h^2 = GMr_0$, or

$$h = \sqrt{GMr_0}, \tag{1.5.34}$$

if we choose a positive sign for h. The first equation determines ε also in terms of r_0: we get $\varepsilon = -GM/r_0 + GM/(2r_0)$, or

$$\varepsilon = -\frac{GM}{2r_0}. \tag{1.5.35}$$

The angular velocity of a circular orbit is given by $\dot{\phi} = h/r_0^2 = \sqrt{GMr_0}/r_0^2$, or

$$\dot{\phi} = \sqrt{\frac{GM}{r_0^3}}. \tag{1.5.36}$$

The orbital period P is the time required for ϕ to advance by 2π. We have $\sqrt{GM/r_0^3} = 2\pi/P$, which gives

$$P = 2\pi\sqrt{\frac{r_0^3}{GM}}. \tag{1.5.37}$$

This equation states that $P^2 \propto r_0^3$, and we have the statement of *Kepler's third law* for circular orbits.

1.5.9 Shape of orbits

To go beyond the qualitative description of the orbit we must now fully integrate the equations of motion. We shall, to begin with, eliminate the time from the equations and focus on the geometrical appearance of the orbit; we shall, in other words, derive a differential equation for $r(\phi)$ and solve it. We will return later with a description of the motion in time.

We go back to Eq. (1.5.29),

$$\ddot{r} + \frac{GM}{r^2} - \frac{h^2}{r^3} = 0,$$

and to Eq. (1.5.27),

$$\dot{\phi} = \frac{h}{r^2}.$$

To eliminate t from these equations we write

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{h}{r^2}\frac{dr}{d\phi}.$$

We then have

$$\begin{aligned} \ddot{r} &= -\frac{2h}{r^3}\dot{r}\frac{dr}{d\phi} + \frac{h}{r^2}\frac{d^2r}{d\phi^2}\dot{\phi} \\ &= -\frac{2h^2}{r^5}\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^4}\frac{d^2r}{d\phi^2}. \end{aligned}$$

The equation that determines the shape of the orbit is therefore

$$\frac{h^2}{r^4}\frac{d^2r}{d\phi^2} - \frac{2h^2}{r^5}\left(\frac{dr}{d\phi}\right)^2 + \frac{GM}{r^2} - \frac{h^2}{r^3} = 0,$$

or

$$\frac{1}{r^2}\frac{d^2r}{d\phi^2} - \frac{2}{r^3}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{r} + \frac{GM}{h^2} = 0,$$

or

$$-\frac{1}{r^2}\frac{d^2r}{d\phi^2} + \frac{2}{r^3}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r} = \frac{GM}{h^2}. \tag{1.5.38}$$

This is a second-order, nonlinear differential equation for $r(\phi)$.

The standard trick that is used to solve Eq. (1.5.38) is to adopt $u = 1/r$ as the dependent variable. Then $r = 1/u$,

$$\frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi},$$

and

$$\frac{d^2r}{d\phi^2} = \frac{2}{u^3}\left(\frac{du}{d\phi}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\phi^2}.$$

With these transformations Eq. (1.5.38) becomes

$$-\frac{2}{u}\left(\frac{du}{d\phi}\right)^2 + \frac{d^2u}{d\phi^2} + \frac{2}{u}\left(\frac{du}{d\phi}\right)^2 + u = \frac{GM}{h^2},$$

or

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}, \qquad u = \frac{1}{r}. \tag{1.5.39}$$

The equation is still of second order, but it is now linear. If we write $v = u - GM/h^2$ it becomes even simpler:

$$\frac{d^2v}{d\phi^2} + v = 0.$$

The solution is $v = u_0\cos(\phi - \phi_0)$, where u_0 and ϕ_0 are the constants of integration. We therefore have

$$u = \frac{GM}{h^2} + u_0\cos(\phi - \phi_0) \equiv \frac{GM}{h^2}\big[1 + e\cos(\phi - \phi_0)\big] \equiv \frac{1}{p}\big[1 + e\cos(\phi - \phi_0)\big],$$

where we have put $u_0 = GMe/h^2$ and $p = h^2/(GM)$.

The final result for $r(\phi)$ is

$$r(\phi) = \frac{p}{1 + e\cos\phi}, \tag{1.5.40}$$

in which we have set $\phi_0 = 0$ to simplify the expression. This involves two constants: We have p, which plays the role of average radius and is known officially as the *semilatus rectum*, and we have e, which measures the range over which r varies and is known as the *eccentricity*. We have seen that p is related to the reduced angular momentum h by

$$h = \sqrt{GMp}. \tag{1.5.41}$$

The eccentricity, on the other hand, can be related to the reduced energy ε; as we shall calculate in a following paragraph,

$$\varepsilon = -\frac{GM}{2p}\big(1 - e^2\big). \tag{1.5.42}$$

This equation is valid for $e < 1$, which means that $\varepsilon < 0$, and it is valid also for $e \geq 1$, which means that $\varepsilon \geq 0$.

We have just observed that $\varepsilon < 0$ when $e < 1$. This is the case of bound motion, which takes place between two turning points at $r = r_{\min} = p/(1 + e)$ and $r = r_{\max} = p/(1 - e)$. As we see from Eq. (1.5.40), the motion proceeds from $r = r_{\min}$ (known as the orbit's *pericentre*) when $\phi = 0$, to $r = r_{\max}$ (known as the orbit's *apocentre*) when $\phi = \pi$, and then back to $r = r_{\min}$ when $\phi = 2\pi$. When $e < 1$ the equation $r = p/(1 + e\cos\phi)$ describes an *ellipse*. The maximum length of this ellipse is $r_{\min} + r_{\max} = p/(1 + e) + p/(1 - e) = 2p/(1 - e^2)$. Half of this is the ellipse's *semi-major axis*,

$$a = \frac{p}{1 - e^2}. \tag{1.5.43}$$

These statements give rise to *Kepler's first law*: A body moving under the gravitational influence of another body follows an elliptical orbit when the motion is bounded. When $e = 0$ we have that Eq. (1.5.40) reduces to $r(\phi) = p$, and the ellipse has become a circle. In this case we have $p \equiv r_0$, Eq. (1.5.41) becomes identical to Eq. (1.5.34), and Eq. (1.5.42) becomes Eq. (1.5.35).

Exercise 1.19. Look up a reference book on elementary geometry and review the properties of ellipses. Answer the following questions: (1) Is Eq. (1.5.40) really the equation of an ellipse? (2) Where are the two foci of the ellipse? (3) What is the semi-minor axis b of the ellipse? A good way to answer some of these questions is to show that Eq. (1.5.40) is equivalent to the usual description of an ellipse via the equation $X^2/a^2 + Y^2/b^2 = 1$. But be careful! In this equation X is not equal to $r\cos\phi$ and Y is perhaps not equal to $r\sin\phi$, because the two coordinate systems do not share the same origin.

We have also observed that $\varepsilon > 0$ when $e > 1$. This is the case of unbound motion, which takes place to the right of a single turning point at $r = r_{\min} = p/(1+e)$. In this case the equation $r = p/(1+e\cos\phi)$ describes a *hyperbola*, and we have hyperbolic motion. In the special case $e = 1$ we have $\varepsilon = 0$, and the equation $r = p/(1+\cos\phi)$ describes a *parabola*; in this special case we have parabolic motion. You may have learned that an ellipse, a hyperbola, and a parabola are all special cases of a general family of curves called *conic sections*. The conic sections are all described by the parametric equation $r(\phi) = p/(1 + e\cos\phi)$.

Let us now return to the derivation of Eq. (1.5.42). We go back to Eqs. (1.5.30) and (1.5.31),

$$\varepsilon = \frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2},$$

and we compute each member of the right-hand side. To evaluate $\dot{r}$ we start with Eq. (1.5.40) and get

$$\dot{r} = \frac{ep\sin\phi}{(1+e\cos\phi)^2}\dot{\phi} = \frac{e}{p}r^2\dot{\phi}\sin\phi = \frac{e}{p}h\sin\phi.$$

This gives us

$$\varepsilon = \frac{1}{2}\frac{e^2}{p^2}h^2\sin^2\phi - \frac{GM}{p}(1+e\cos\phi) + \frac{h^2}{2p^2}(1+e\cos\phi)^2,$$

and replacing h^2 by GMp in this equation yields

$$\varepsilon = \frac{GM}{2p}\Big[e^2\sin^2\phi - 2(1+e\cos\phi) + (1+e\cos\phi)^2\Big].$$

After simplification the expression within the square brackets becomes $e^2\sin^2\phi - 1 + e^2\cos^2\phi = -1 + e^2$, and we arrive at

$$\varepsilon = -\frac{GM}{2p}(1-e^2),$$

the same statement as in Eq. (1.5.42).

1.5.10 Motion in time

Now that $r(\phi)$ is known we must relate ϕ to the time t in order to have a complete description of the motion. The relevant equations are $\dot{\phi} = h/r^2$, $h = \sqrt{GMp}$, and $r = p/(1+e\cos\phi)$. Putting this all together, we obtain

$$\frac{d\phi}{dt} = \sqrt{\frac{GM}{p^3}}(1+e\cos\phi)^2. \tag{1.5.44}$$

This is the differential equation that must be solved in order to obtain $\phi(t)$. Unless e is very small, in which case approximate analytical results can be obtained, this equation must be integrated numerically. Results of numerical integrations are displayed in Fig. 1.12.

The integral form of Eq. (1.5.44) is

$$t = \sqrt{\frac{p^3}{GM}}\int\frac{d\phi}{(1+e\cos\phi)^2} + \text{constant}. \tag{1.5.45}$$

This indefinite integral cannot be evaluated in closed form, but it provides a nice way of calculating the orbital period P of bound orbits ($e < 1$). Because this is equal to the time required for ϕ to advance by 2π, or twice the time required for ϕ to advance by π, we have

$$P = 2\sqrt{\frac{p^3}{GM}}\int_0^\pi\frac{d\phi}{(1+e\cos\phi)^2}.$$

This definite integral can be evaluated, and the result is $\pi/(1-e^2)^{3/2}$. We therefore have

$$P = 2\pi\sqrt{\frac{[p/(1-e^2)]^3}{GM}}.$$

We obtain a cleaner form of this result by involving Eq. (1.5.43). In terms of the semi-major axis $a = p/(1-e^2)$, the orbital period is

$$P = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{1.5.46}$$

We have that $P^2 \propto a^3$, and this is the general statement of *Kepler's third law*.

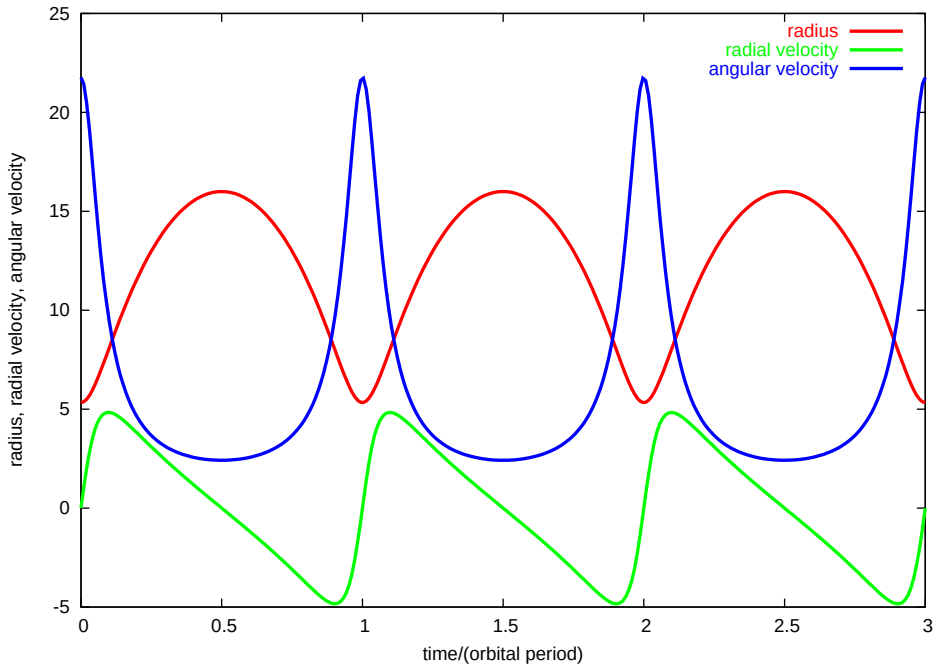


Figure 1.12: Numerical integration of the equations of motion for an orbit with eccentricity $e = 0.5$. The blue curve shows the angular velocity $\dot{\phi}$ as a function of time, the green curve shows the radial velocity $\dot{r}$ as a function of time, and the red curve shows the radial position r as a function of time. The time variable is scaled by the orbital period P, and three complete orbital cycles are displayed. Notice that the motion starts at the pericentre with maximum angular velocity and zero radial velocity.

1.5.11 Summary

The motion of two bodies subjected to their mutual gravity is described by the relative position vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. When the origin of the coordinate system is at the centre of mass we have

$$\boldsymbol{r}_1 = \frac{m_2}{M}\boldsymbol{r}, \qquad \boldsymbol{r}_2 = -\frac{m_1}{M}\boldsymbol{r}, \qquad M = m_1 + m_2.$$

The vector $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is constant, and $\boldsymbol{h}$ is related to the system's total angular momentum by $\boldsymbol{L} = (m_1 m_2/M)\boldsymbol{h}$. The fact that $\boldsymbol{h}$ is constant implies that the motion takes place in a fixed plane. Using polar coordinates, the motion is described by the functions $r(t)$ and $\phi(t)$. These are determined by the first-order differential equations

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2} = \varepsilon, \qquad \dot{\phi} = \frac{h}{r^2}.$$

The constant ε is related to the system's total energy by $E = (m_1 m_2/M)\varepsilon$. The shape of the orbit is described by

$$r(\phi) = \frac{p}{1 + e\cos\phi}.$$

The orbital elements (p, e) are related to (h, ε) by

$$h = \sqrt{GMp}, \qquad \varepsilon = -\frac{GM}{p}\big(1 - e^2\big).$$

The motion in time is determined by numerically integrating

$$\dot{\phi} = \sqrt{\frac{GM}{p^3}}(1 + e\cos\phi)^2.$$

When $e < 1$ the motion is elliptical, and the ellipse's semi-major axis is

$$a = \frac{p}{1 - e^2}.$$

The orbital period is then

$$P = 2\pi\sqrt{\frac{a^3}{GM}}.$$

1.6 Appendix: Numerical integration of differential equations

Some of the results presented in this Chapter were obtained by numerical integration. Some of our future results also will be obtained using numerical techniques. In this Appendix we explain the fundamental ideas behind these numerical methods. These ideas are implemented in various available packages, for example, within Maple, or within subroutines found in the book *Numerical Recipes*.

To begin, we examine a first-order differential equation of the form

$$\frac{dy}{dx} = f(y), \tag{1.6.1}$$

where x is the independent variable, y the dependent variable, and f an arbitrary function of y. A concrete example is

$$\frac{d\phi}{dt} = \sqrt{\frac{GM}{p^3}}(1 + e\cos\phi)^2,$$

which we encountered in Sec. 1.5; here $x \equiv t$, $y \equiv \phi$, and f stands for what appears on the right-hand side of the preceding equation.

We seek to determine $y(x)$ in the interval $x_{\text{initial}} < x < x_{\text{final}}$, starting from the known value y_{initial} at $x = x_{\text{initial}}$. The essential idea is to break down the continuum between x_{initial} and x_{final} into a finite number of discrete points separated by a small interval Δ. The computational grid is then

$$x_n = x_{\text{initial}} + n\Delta, \qquad n = 0, 1, 2, \cdots, N, \tag{1.6.2}$$

where N is the total number of points; we have $\Delta = (x_{\text{final}} - x_{\text{initial}})/N$. Correspondingly, we have the sampled values $y_n = y(x_n)$ of the dependent variable, which we wish to determine. We shall do so by turning the differential equation $dy/dx = f(y)$ into a *finite-difference equation.*

Consider the first step of moving from $x_0 = x_{\text{initial}}$ to x_1. We know $y_0 = y_{\text{initial}}$, and we wish to determine y_1. Because Δ is small it is safe to assume that the function $f(y)$ changes by very little in the interval between y_0 and y_1. We may approximate it by its Taylor expansion about $y = y_0$:

$$\begin{aligned} f(y) &= f(y_0) + f'(y_0)(y - y_0) + \cdots \\ &= f(y_0)\big[1 + f^{-1}f'(y_0)(y - y_0) + \cdots\big]. \end{aligned}$$

The differential equation gives

$$dx = \frac{dy}{f(y)} = \frac{1}{f(y_0)}\big[1 - f^{-1}f'(y_0)(y - y_0) + \cdots\big]\, dy,$$

where we have used the identity $(1 + \epsilon)^\alpha = 1 + \alpha\epsilon + O(\epsilon^2)$, which holds for any small quantity ϵ and any power α — this identity also can be established by Taylor expansion. Integrating the preceding equation gives

$$x_1 - x_0 = \frac{1}{f(y_0)}\left[(y_1 - y_0) - \frac{1}{2}f^{-1}f'(y_0)(y_1 - y_0)^2 + \cdots\right],$$

or

$$f(y_0)\Delta = (y_1 - y_0) - \frac{1}{2}f^{-1}f'(y_0)(y_1 - y_0)^2 + \cdots.$$

This equation can be solved formally for $y_1 - y_0$:

$$\begin{aligned} y_1 - y_0 &= f(y_0)\Delta + \frac{1}{2}f^{-1}f'(y_0)(y_1 - y_0)^2 + \cdots \\ &= f(y_0)\Delta + \frac{1}{2}f^{-1}f'(y_0)\big[f(y_0)\Delta + \cdots\big]^2 + \cdots \\ &= f(y_0)\Delta + \frac{1}{2}ff'(y_0)\Delta^2 + \cdots. \end{aligned}$$

We write this result as

$$y_1 = y_0 + f(y_0)\Delta + \frac{1}{2}ff'(y_0)\Delta^2 + O(\Delta^3), \tag{1.6.3}$$

indicating that the error of this approximation for y_1 is of order Δ^3 and therefore quite small.

A cruder approximation for y_1 is

$$y_1 = y_0 + f(y_0)\Delta + O(\Delta^2),$$

and this approximation is at the core of *Euler's method* to solve the differential equation: From the known value y_0 compute $f(y_0)$ and multiply by Δ; add the

result to y_0 to get y_1, and repeat the procedure to get y_2, y_3, and so on. Euler's method is very simple and economical, but because its error term is of order Δ^2, it is not very accurate. With a little cleverness, however, it is possible to improve the accuracy of the method so that its error term becomes of order $\Delta^3 \ll \Delta^2$. One way of achieving this would be to use Eq. (1.6.3) instead of its cruder version. The price to pay would be the need to evaluate $f'(y)$, the derivative of the function with respect to y. This may not be practical in some circumstances, and there is an alternative method.

Consider evaluating the function f not at $y = y_0$, but at the midpoint between y_0 and $y_1 \simeq y_0 + f(y_0)\Delta$:

$$f(y_0) \to f(y_0 + \tfrac{1}{2} f_0 \Delta),$$

where we use the notation $f_0 = f(y_0)$. By Taylor expansion we have

$$\begin{aligned} f(y_0 + \tfrac{1}{2} f_0 \Delta) &= f(y_0) + f'(y_0)(\tfrac{1}{2} f_0 \Delta) + O(\Delta^2) \\ &= f(y_0) + \frac{1}{2} f f'(y_0)\Delta + O(\Delta^2), \end{aligned}$$

and this shows that Eq. (1.6.3) is equivalent to

$$y_1 = y_0 + f(y_0 + \tfrac{1}{2} f_0 \Delta)\Delta + O(\Delta^3). \tag{1.6.4}$$

This approximation for y_1 has an error term of order Δ^3, and it is obtained simply by evaluating the function f at the midpoint; the value of its derivative is not needed.

By being increasingly clever it is possible to decrease further the size of the error term. The *fourth-order Runge-Kutta method* consists of the following recipe. Suppose that the differential equation has been integrated up to $x = x_n$, and that we wish to proceed to the next grid point, at $x = x_{n+1}$. We have therefore obtained y_n and we wish to calculate y_{n+1}. First we compute the auxiliary quantities

$$\begin{aligned} k_1 &= f(y_n)\Delta, \\ k_2 &= f(y_n + \tfrac{1}{2} k_1)\Delta, \\ k_3 &= f(y_n + \tfrac{1}{2} k_2)\Delta, \\ k_4 &= f(y_n + k_3)\Delta, \end{aligned}$$

and next we approximate y_{n+1} by

$$y_{n+1} = y_n + \frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4 + O(\Delta^5). \tag{1.6.5}$$

As indicated, the judicious choice of coefficients in front of k_1, k_2, k_3, and k_4 ensures that the error term is now of order Δ^5, and therefore very small. The Runge-Kutta method is easy to implement, it is accurate, and it is robust; it works well for most functions $f(y)$. The method can also be generalized to handle functions $f(x, y)$ that depend on both variables.

The method also generalizes to a set of differential equations

$$\frac{dy[i]}{dx} = f[i]\big(y[1], y[2], \cdots\big), \qquad i = 1, 2, \cdots \tag{1.6.6}$$

for a set of dependent variables $y[i]$. In this case the auxiliary quantities k_1, k_2, k_3, and k_4 acquire an index $[i]$; for example we now have

$$k_1[i] = f[i]\big(y_n[1], y_n[2], \cdots\big)\Delta$$

and

$$k_2[i] = f[i]\big(y_n[1] + \tfrac{1}{2}k_1[1], y_n[2] + \tfrac{1}{2}k_1[2], \cdots\big)\Delta.$$

This generalization is useful, because it allows us to use the method to integrate second-order differential equations. Consider, for example, the pendulum equation of Eq. (1.3.24),

$$\ddot{\theta} + \omega^2 \sin\theta = 0.$$

This can be recast as a system of two first-order differential equations. To do this we define $y[1] = \theta$ and $y[2] = \dot{\theta}$. When then have the system

$$\begin{aligned} \frac{dy[1]}{dt} &= y[2], \\ \frac{dy[2]}{dt} &= -\omega^2 \sin(y[1]). \end{aligned}$$

In this instance we find that $f[1](y[1], y[2]) = y[2]$ and $f[2](y[1], y[2]) = -\omega^2 \sin(y[1])$. This system of equations can be integrated straightforwardly, and the result for $y[1](t)$ is the numerical approximation to $\theta(t)$, the solution to the original second-order equation.

1.7 Problems

1. Let
$$\boldsymbol{A} = (3x^2 - 6yz)\hat{\boldsymbol{x}} + (2y + 3xz)\hat{\boldsymbol{y}} + (1 - 4xyz^2)\hat{\boldsymbol{z}}.$$
Calculate $\int_C \boldsymbol{A} \cdot d\boldsymbol{s}$ along the following paths that link the point $(0, 0, 0)$ to the point $(1, 1, 1)$:
   - **(a)** The curve described by $x = u$, $y = u^2$, and $z = u^3$, in which the parameter u is restricted to the interval $0 < u < 1$.
   - **(b)** The straight line that joins these points.

2. Let
$$\boldsymbol{A} = (2xy + z^3)\hat{\boldsymbol{x}} + (x^2 + 2y)\hat{\boldsymbol{y}} + (3xz^2 - 2)\hat{\boldsymbol{z}}.$$
Find the function f such that $\boldsymbol{A} = \boldsymbol{\nabla} f$. Then evaluate $\int_C \boldsymbol{A} \cdot d\boldsymbol{s}$ along any path C that links the point $(1, -1, 1)$ to the point $(2, 1, 2)$.

3. Evaluate $\oint_C \boldsymbol{r} \cdot d\boldsymbol{s}$ along *all* closed loops C, where $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ is the position vector.

4. A projectile is launched with initial speed v_0 at an angle α with the horizontal. Calculate:
   - **(a)** the position vector as a function of time;
   - **(b)** the time required to reach the highest point;
   - **(c)** the maximum height reached by the projectile;
   - **(d)** the time of flight back to the Earth's surface;
   - **(e)** the range of the projectile;
   - **(f)** the angle α which maximizes the range.

5. Suppose that in the preceding problem, the projectile is also subjected to a frictional force equal to $-k\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity vector and k a positive constant. Find:
   - **(a)** the velocity vector as a function of time;

(b) the position vector as a function of time;

(c) the terminal velocity of the projectile.

For ease of notation set $k = m/\tau$.

6. A particle of mass m is traveling in the x direction. At time $t = 0$ it is located at $x = 0$ and has a speed v_0. The particle is subjected to a frictional force which opposes the motion; its magnitude is equal to βv^2, where $v = v(t)$ is the particle's speed at time t and β is a positive constant.

   (a) What is the speed of the particle as a function of time?

   (b) What is the position of the particle as a function of time?

7. A particle of mass m rests on top of a sphere of radius R. The particle is then displaced slightly so that it starts to move down the sphere. (It is assumed that the particle slides down without rolling and without friction.) As it moves down the sphere, the particle makes an angle θ with the vertical direction. At some point the particle loses contact with the surface of the sphere, and it proceeds to fall freely. We are interested in the motion of the particle from the initial moment where it is at rest to the final moment where it leaves the sphere.

   (a) Derive an equation of motion for $\theta(t)$, and find an expression for N, the magnitude of the normal force.

   (b) At which angle θ does the particle leave the surface of the sphere?

   (c) What is the speed of the particle when it leaves the surface of the sphere?

   [Hint: This problem is involved. You may find it useful to resolve the force and acceleration vectors into a basis that consists of $\hat{\boldsymbol{r}}$, a unit vector that points in the direction normal to the sphere, and $\hat{\boldsymbol{\theta}}$, a unit vector that points in the direction of increasing θ.]

8. The planar pendulum of Sec. 1.2.7 is now subjected to a frictional force $\boldsymbol{F}_{\text{friction}} = -(m/\tau)\boldsymbol{v}$, where τ is a positive constant. Derive the new equation of motion for the swing angle θ.

9. The equation of motion of the preceding problem reduces to

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0$$

   when the oscillations have a very small amplitude; here γ is a positive constant that is related to τ in the preceding problem. Find the general solution to this equation. Assume that $\omega^2 > \gamma^2$, so that the oscillations are underdamped. [Be sure that your final expression for $\theta(t)$ is a real (not complex) function.]

10. A mass m is allowed to move along the x axis, either in the positive or in the negative direction. It is subjected to a constant force $+F$ when $x < 0$ and to a constant force $-F$ when $x > 0$ (here F is positive).

    (a) Describe the motion qualitatively with the help of an energy diagram.

    (b) Calculate the period of the motion; express your result in terms of m, F, and the amplitude A of the motion.

11. A cylindrical cork is partially immersed in a liquid of (mass) density ρ. The cork's axis is oriented with the vertical direction, and the cork floats in the liquid. The cylinder's cross-sectional area is A, and its mass is m. The cork is gently pushed down into the liquid and then released; it starts to

oscillate. Neglecting any damping effect, calculate the angular frequency ω of the oscillations.

[Hint: The restoring force is provided by the buoyancy of the liquid. According to the Archimedes principle, the buoyant force is equal to the weight of the liquid displaced by the cork.]

12. For Kepler's problem, prove that the square of the velocity vector $\boldsymbol{v} = \dot{\boldsymbol{r}}$ can be expressed as

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

where $a = p/(1-e^2)$ is the ellipse's semi-major axis. What is v at pericentre? What is v at apocentre? (Express your results in units of the average speed $\bar{v} = \sqrt{GM/a}$.)

13. We have seen that the description of Keplerian motion in time can be obtained by integrating Eq. (1.4.52). An alternative method is based on the representation

$$r = a(1 - e\cos\psi),$$

in which r is expressed in terms of the *eccentric anomaly* ψ; this is an angular parameter that ranges from 0 to 2π as each body completes one full orbit. We wish to find ψ as a function of time.

(a) Starting from the statement of energy conservation, $\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon$, in which you are to substitute $h^2 = GMa(1-e^2)$ and $\varepsilon = -GM/(2a)$, derive an expression for $\dot{\psi}$. Make sure that this expression is simplified to the full extent possible.

(b) Integrate the equation for $\dot{\psi}$ that was obtained in part (a). Show that the solution is

$$\psi - e\sin\psi = \sqrt{\frac{GM}{a^3}}(t - t_0),$$

where t_0 is the time at which $\psi = 0$. This is *Kepler's equation*, and it can be numerically inverted to yield $\psi(t)$. This method is the most convenient to find the behaviour of r as a function of time.

14. We examine a special case of Kepler's problem. We set $h = 0$, so that $\dot{\phi} = 0$. We have purely radial motion, and the equation for $r(t)$ reduces to

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \varepsilon,$$

where ε is the reduced total energy.

(a) Construct an energy diagram for this situation. Describe the motion qualitatively when $\varepsilon > 0$, when $\varepsilon = 0$, and when $\varepsilon < 0$.

In the rest of the problem we consider the subcase $\varepsilon < 0$ in some detail.

(b) Relate ε to $r_{\max}$, the maximum value of r at which a turning point occurs.

(c) Imagine that the motion proceeds from $r = r_{\max}$ to $r = 0$. We represent this mathematically in terms of an auxiliary variable, the angle η. We write

$$r(\eta) = \frac{1}{2}r_{\max}(1 + \cos\eta),$$

and we let η vary from $\eta = 0$ to $\eta = \pi$. Calculate $\dot{\eta} = d\eta/dt$ and solve this for $t(\eta)$. The motion is now completely determined. Provide a plot of r as a function of t.

(d) What is the total time required for the bodies to go from $r = r_{\max}$ to $r = 0$?

15. Two particles move about each other in circular orbits under the influence of their gravitational attraction; their orbital period is τ. Their motion is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other. Calculate the time required for them to collide; express your answer in terms of τ.

    [Hint: You will need the result of part (d) of the preceding problem.]

16. The *escape velocity* of a particle on Earth is the minimum velocity required on the Earth's surface for the particle to escape the Earth's gravitational field. Neglecting air resistance within the atmosphere, calculate this in terms of the Earth's mass M and its radius R. Evaluate this numerically and show that it is close to 11 km/s.

17. Two bodies of mass m_1 and m_2 are subjected to their mutual forces, so that the force acting on body 1 due to body 2 is $\boldsymbol{F}_{12} = -f(r)\hat{\boldsymbol{r}}$, while the force acting on body 2 due to body 1 is $\boldsymbol{F}_{21} = +f(r)\hat{\boldsymbol{r}}$. Here f is an arbitrary function of $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, the distance between the bodies, and the forces are directed along $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$; such forces are called *central forces*.

    (a) Derive an equation of motion for $\boldsymbol{R}$, the position of the centre of mass.

    (b) Derive an equation of motion for $\boldsymbol{r}$, the position of body 1 relative to body 2.

    (c) Prove that $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant vector; conclude that the motion takes place in a fixed plane.

    (d) Introduce the polar coordinates (r, ϕ) and prove that $|\boldsymbol{h}| = h = r^2\dot{\phi}$; conclude that Kepler's law — the law of areas — is valid for all central forces, and not just for gravity.

    (e) Show that the equation of motion for r reduces to

    $$\ddot{r} + \frac{f}{\mu} - \frac{h^2}{r^3} = 0,$$

    where $\mu = m_1 m_2/(m_1 + m_2)$ is known as the *reduced mass* of the two-body system.

    (f) Prove that the shape of the orbit is determined by

    $$\frac{d^2u}{d\phi^2} + u = \frac{f}{\mu h^2 u^2},$$

    where $u = 1/r$.

18. A central force $f = k/r^n$, where k is a constant and n an integer, is known to produce an orbit described by $r = ae^{-\phi}$, where a is a constant.

    (a) Plot this orbit in the x-y plane.

    (b) Determine the integer n.

19. A central force $f = k/r^n$, where k is a constant and n an integer, is known to produce an orbit described by $r = a\sqrt{\cos 2\phi}$, where a is a constant.

    (a) Plot this orbit in the x-y plane.

(b) Determine the integer n.

20. A two-body system moves under the influence of a central force given by

$$f = \frac{a}{r^2} + \frac{b}{r^3},$$

where a and b are constants.

(a) Show that the shape of the orbit is described by

$$r = \frac{p}{1 + e\cos(k\phi)},$$

where p, e, and k are constants. Express p and k in terms of a, b, h^2, and μ. (Assume that $b < \mu h^2$.)

(b) Plot the orbit in the x-y plane. Set $p = 1$, $e = 0.6$, $k = 0.99$, and let ϕ range from 0 to 16π. What is happening to the major axis of the ellipse?

1.8 Additional problems

1. An inclined plane makes an angle α with the horizontal. A projectile is launched from point A at the bottom of the inclined plane. Its initial speed is v_0, and its initial velocity vector makes an angle β with the horizontal. The projectile eventually hits the inclined plane at point B. Air resistance is negligible.

(a) Calculate the range R of the projectile, the distance between points A and B. Show that it it can be expressed in the form

$$R = R_0 \sin(\beta - \alpha)\cos\beta$$

and find an expression for R_0.

(b) Find the angle $\beta_{\max}$ which maximizes the range.

2. A particle traveling in the positive x direction is subjected to a force $F = kx^3$. The particle started from an initial position $x_0 < 0$. Draw an energy diagram for this situation and provide a qualitative description of the possible motions.

3. Two bodies of masses m_1 and m_2 are subjected to a mutual attractive force $\boldsymbol{F}_{12} = -km_1m_2\boldsymbol{r}$, where k is a constant and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the relative position vector.

(a) Show that the equation of motion for $r(t)$ can be put in the form of an energy equation,

$$\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon,$$

and find an expression for $\nu(r)$, the effective potential. Draw an energy diagram for this system and give a qualitative description of the possible motions.

(b) Prove that

$$r(\phi) = \frac{r_0}{\sqrt{2 - e - e\cos(2\phi)}}$$

describes the shape of the orbit, and solve for r_0 in terms of the constants e, $M = m_1 + m_2$, h, and k.

4. The *parabolic coordinates* u and v are sometimes useful to describe the motion of a particle in a two-dimensional plane. These are related to the Cartesian coordinates x and y by

$$x = uv, \qquad y = \frac{1}{2}(u^2 - v^2).$$

(a) Sketch the shapes of the curves $u =$ constant in the x-y plane.

(b) Sketch the shapes of the curves $v =$ constant in the x-y plane.

(c) Find the unit vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ associated with this coordinate system.

Chapter 2
Lagrangian mechanics

2.1 Introduction: From Newton to Lagrange

The methods of Newtonian mechanics, based on the vectorial equation $\boldsymbol{F} = m\boldsymbol{a}$, are very powerful and they can be applied to all mechanical systems. But they lack in *efficiency* when Cartesian coordinates (x, y, z) do not give the simplest description of a mechanical system. An example is the problem of the pendulum (Sec. 1.3.7), which is best analyzed in terms of the swing angle θ; we have seen that to derive the equation of motion for $\theta(t)$ requires somewhat laborious calculations, and the reason is precisely that θ is not a Cartesian coordinate. Another example is Kepler's problem (Sec. 1.5), which is best analyzed in terms of the polar coordinates (r, ϕ); again we saw (back in Sec. 1.5.4) that to derive equations of motion for $r(t)$ and $\phi(t)$ required some long calculations.

To increase the efficiency of the theoretical methods of mechanics, a number of scientists in the centuries following Newton endeavoured to recast the Newtonian laws into a more flexible formulation. The most famous players include Leonhard Euler (1707–1783), Joseph Lagrange (1736–1813), William Rowan Hamilton (1805–1865), and Carl Gustav Jacobi (1804–1851). Their new techniques proved extremely useful, and they allowed them and others to solve increasingly challenging problems, most notably in the context of celestial mechanics. These new powerful techniques are the topic of this chapter on Lagrangian mechanics, and the following chapter on Hamiltonian mechanics.

It is important to point out that the Lagrangian and Hamiltonian formulations of the laws of mechanics are largely restricted to forces that can be derived from a potential. For other problems, such as a particle subjected to air resistance, the new techniques cannot be applied in a very straightforward way, and it is usually best to go back to the old Newtonian methods. In this chapter and the next, we shall consider only forces that can be derived from a potential.

The entire content of *Lagrangian mechanics* is summarized in the following simple recipe:

1. Select *generalized coordinates* q_a to describe the degrees of freedom of a mechanical system. These coordinates are completely arbitrary. They need not be the original Cartesian coordinates associated with an inertial frame. Indeed, there is no need for the coordinates to even be attached to an inertial frame. The index $a = 1, 2, \cdots$ labels each one of the generalized coordinates; there is one coordinate for each degree of freedom.

2. In terms of the generalized coordinates, calculate the system's total kinetic energy T and total potential energy V. Then form what is known as the *Lagrangian function* of the system, which is denoted $L(q_a, \dot{q}_a)$; this depends on the generalized coordinates q_a and the *generalized velocities* $\dot{q}_a = dq_a/dt$.

The Lagrangian is defined by

$$L = T - V;$$

it is the *difference* between the kinetic and potential energies.

3. Substitute the Lagrangian into the *Euler-Lagrange (EL) equations*,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0.$$

This returns an equation of motion for each generalized coordinate $q_a(t)$. There is one EL equation for each generalized coordinate.

4. The rest of the recipe is concerned with solving the equations of motion. The methods for doing this are varied, and they depend on the particular situation, just as they do in the Newtonian formulation.

Let us first verify that the recipe is compatible with Newton's laws. Consider a particle moving in three-dimensional space and subjected to a potential $V(x, y, z)$. As indicated, we use Cartesian coordinates to describe the motion of the particle. In this case, therefore, the generalized coordinates are chosen as $q_1 = x$, $q_2 = y$, and $q_3 = z$. The particle's kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, and the Lagrangian function is

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z).$$

To substitute this into the EL equation for $q_1 = x$, say, we must first evaluate $\partial L/\partial \dot{x}$. This is the derivative of L with respect to $\dot{x}$, treating all other variables (including x) as constant parameters. This is given by

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

We next differentiate this with respect to t, and get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}.$$

Finally, we differentiate L with respect to x, treating all other variables (including $\dot{x}$) as constant parameters; this gives

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}.$$

Substituting these results into the EL equation for x, we arrive at

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0.$$

Repeating these calculations for y and z would eventually return the full vectorial equation

$$m\boldsymbol{a} + \boldsymbol{\nabla} V = \boldsymbol{0},$$

or $m\boldsymbol{a} = \boldsymbol{F}$ if we recall that the force is derived from the potential, so that $\boldsymbol{F} = -\boldsymbol{\nabla} V$. This exercise reveals that indeed, the Lagrangian recipe is compatible with the Newtonian law.

The true power of the recipe, however, is revealed when the generalized coordinates are not Cartesian. Let us see what the recipe produces in the case of the pendulum. Recall from Sec. 1.3.7 that the pendulum's single degree of freedom is best represented by the swing angle θ; this will be our generalized coordinate

for this problem, and we write $q \equiv \theta$. (We do not need a label a in this case, as there is only one generalized coordinate.) The relation between θ and the original Cartesian coordinates is $x = \ell \sin\theta$ and $z = \ell\cos\theta$, with ℓ denoting the length of the rod. The pendulum's kinetic energy is $T = \frac{1}{2}(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$. Its potential energy is $V = -mgz = -mg\ell\cos\theta = -m\ell^2\omega^2\cos\theta$, where we have reintroduced the quantity $\omega^2 \equiv g/\ell$. The pendulum's Lagrangian function is

$$L(\theta,\dot{\theta}) = m\ell^2\left(\frac{1}{2}\dot{\theta}^2 + \omega^2\cos\theta\right).$$

To substitute this into the EL equation we must first evaluate $\partial L/\partial\dot{\theta}$, the partial derivative of L with respect to $\dot{\theta}$. This is

$$\frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta}.$$

Next we differentiate this with respect to time, and obtain

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m\ell^2\ddot{\theta}.$$

Finally we calculate the partial derivative of L with respect to θ, which yields

$$\frac{\partial L}{\partial\theta} = -m\ell^2\omega^2\sin\theta.$$

Substituting these results into the EL equation produces

$$m\ell^2\big(\ddot{\theta} + \omega^2\sin\theta\big) = 0,$$

the same pendulum equation as in Eq. (1.3.24). Comparing the computations carried out here to those required in Sec. 1.3.7, the greater efficiency of the Lagrangian recipe should come out loud and clear.

It is possible to derive the Lagrangian recipe from Newton's law, $\boldsymbol{F} = m\boldsymbol{a}$. The derivation is fairly laborious, and it involves performing a transformation from the original Cartesian system (x, y, z) to the generalized coordinates q_a. It is possible, however, and more interesting, to derive the recipe from a new physical principle. Instead of postulating the validity of $\boldsymbol{F} = m\boldsymbol{a}$ as the starting point of Newtonian mechanics, we shall instead adopt the *principle of least action* as the starting point of Lagrangian mechanics. As we shall see in the next two sections, the Euler-Lagrange equations can be derived as a direct consequence of the principle of least action, and as we have already seen, these are fully compatible with Newton's law. What we have, therefore, is the Newtonian postulate arising as a consequence of the new principle. More importantly, we have the more flexible framework of the EL equations arising as a consequence of the principle of least action.

As we shall see below, the principle of least action states that of all the possible paths $q_a(t)$ that a mechanical system *could take* to go from configuration 1 to configuration 2, the paths that are *actually taken* are the ones which minimize the system's *action functional*, defined by

$$S[q_a(t)] = \int_{t_1}^{t_2} L(q_a, \dot{q}_a)\, dt.$$

This beautiful statement is mathematically equivalent to the full set of EL equations, which give rise to the equations of motion that determine the actual paths of the system. This formulation of the laws of mechanics, in terms of a least-action principle, is economical and conceptually compelling. It is also extremely powerful: Virtually all fundamental laws of physics (including field theories) can be formulated in terms of such an action principle.

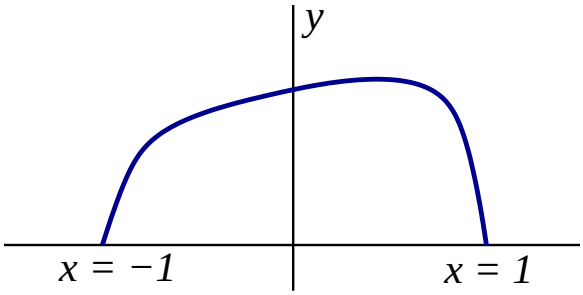


Figure 2.1: A curve in the x-y plane that links the points $(-1, 0)$ and $(1, 0)$.

2.2 Calculus of variations

In this section we introduce the mathematical tools — the calculus of variations — that are required in the derivation of the Euler-Lagrange (EL) equations from the principle of least action. We will look at this issue from a purely mathematical point of view, and return to the physics in the next section.

2.2.1 Curve of maximum area

Let us examine the following mathematical problem. We consider the infinite number of curves in the x-y plane that link the point $(x = -1, y = 0)$ to the point $(x = +1, y = 0)$; see Fig. 2.1. Of all these curves we select those that have a total arc length (the total distance traveled along the curve) equal to π. Of all the curves that are left we wish to find the one which *maximizes* the area under the curve. (Notice that the mathematical problem involves maximization of an area, while the physical problem involves minimization of an action. The mathematical techniques to be developed below work for both cases, maximization and minimization, and they do not care about the identity of the quantity to be *extremized*.)

We describe the family of curves introduced in the previous paragraph by parametric relations $x(s)$ and $y(s)$, in which the parameter s is the curve's arc length, calculated from the starting point $(-1, 0)$. Because all the curves within the family have a total arc length of π, the parameter s ranges from 0 to π as each curve runs from $(-1, 0)$ to $(+1, 0)$. We have $ds^2 = dx^2 + dy^2$, and this relation implies that the functions $x(s)$ and $y(s)$ are not independent of each other. The area under the curve is obtained by integration, $A = \int y\, dx$, which we write as

$$A = \int_0^\pi y(s) \frac{dx}{ds}\, ds.$$

We can replace the factor dx/ds by $\sqrt{1 - y'^2}$, where $y' = dy/ds$. This gives us, finally,

$$A = \int_0^\pi y\sqrt{1 - y'^2}\, ds. \tag{2.2.1}$$

We wish to find the function $y(s)$ that produces the largest possible value for A. Once this function is identified, $x(s)$ can be obtained by integrating the equation

$$x' = \sqrt{1 - y'^2}. \tag{2.2.2}$$

The maximal curve is then fully determined.

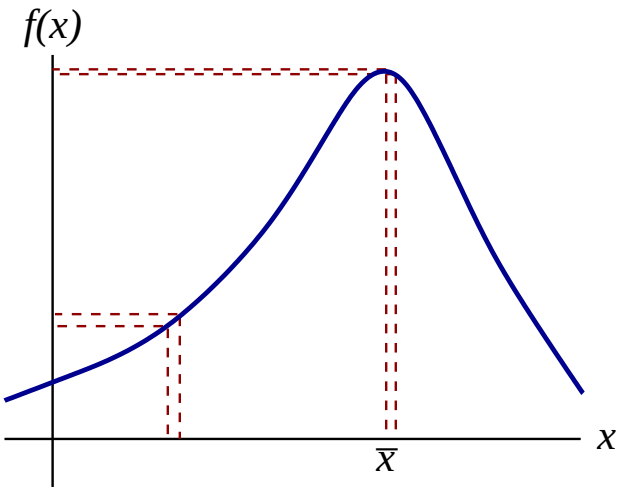


Figure 2.2: A function with a maximum point at $x = \bar{x}$. Because this is an extremum point, a displacement around $\bar{x}$ produces the smallest change in the function.

2.2.2 Extremum of a functional

To proceed it is helpful to broaden the scope of the preceding discussion and to examine the general structure of the mathematical problem. We are given a *functional* $A[y]$, a function A of a function $y(s)$, which we wish to maximize, or perhaps minimize, with respect to the choice of path $y(s)$. (In general we say that we wish to find the *extremum* of the functional, and we shall never need to distinguish between a maximum and a minimum.) The functional has the following structure:

$$A[y] = \int_{s_0}^{s_1} G(y, y')\, ds; \tag{2.2.3}$$

it is given by an integral over a parameter s of a function G which depends on the path $y(s)$ and its derivative $y'(s) = dy/ds$. The integral can be evaluated for any choice of trial function $y_{\text{trial}}(s)$, and the result is a number A_{trial}. We are looking for the function $\bar{y}(s)$ that produces the largest (or smallest) number. In mathematical terms, we are looking for the extremum of the functional $A[y]$.

The mathematical task of extremizing a function $f(x)$ with respect to its argument x — the argument being a number — is a simple one: We simply calculate the derivative of the function and set the result equal to zero; the solutions to $df/dx = 0$ are all extremum points (minima and maxima) of the function. To extremize a *functional* $A[y]$ with respect to a functional argument $y(s)$ is a much more delicate task. How does one do this?

Let us examine more closely the straightforward task of finding an extremum of a function $f(x)$. We imagine, for concreteness, that the function has a single maximum at $x = \bar{x}$; this is represented in Fig. 2.2. We have, of course, $f'(\bar{x}) = 0$, with a prime indicating differentiation with respect to x.

An important property of $\bar{x}$ is that it is the point from which the function $f(x)$ *changes the least* when x is displaced from $\bar{x}$ to a neighbouring point $\bar{x} + \delta x$. That this is so can easily be seen from the figure, but it is just as easy to prove it mathematically. Let us calculate δf, the change induced in the function when its argument x is moved to a neighbouring point $x + \delta x$. By Taylor's theorem we have

$$\begin{aligned} \delta f &\equiv f(x + \delta x) - f(x) \\ &= f'(x)\delta x + \frac{1}{2} f''(x)(\delta x)^2 + \cdots . \end{aligned}$$

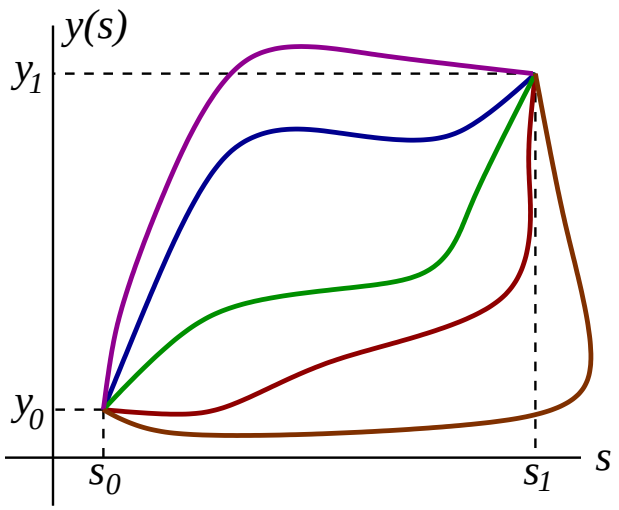


Figure 2.3: A family of paths which leave $y = y_0$ when $s = s_0$ and arrive at $y = y_1$ when $s = s_1$.

From this calculation we learn that in general, the change in the function is proportional to δx, as we might have expected. But when we let x become an extremum point $\bar{x}$, we get a different result. In this case we have $f'(\bar{x}) = 0$ and the preceding equation becomes

$$\delta f = \frac{1}{2} f''(\bar{x})(\delta x)^2 + \cdots .$$

Now the change in the function is proportional to $(\delta x)^2$, and this is much smaller than what we get in the general case. We have just found that the variation δf is smallest when it is taken at an extremum point. A useful way of characterizing an extremum point is therefore to say that it is a point from which a displacement δx produces a vanishing change δf, to linear order in δx. (The change is not actually zero, but it is of second order in δx, as we have shown.)

We shall use the same idea to find the extremum path of a functional. We will look for a path $\bar{y}(s)$ — analogous to the extremum point $\bar{x}$ — that has the property that a displacement away from this path produces no change in the functional $A[y]$, to linear order in the displacement $\delta y(s)$. In other words, if we evaluate the function on the extremum path $\bar{y}(s)$ and get the number $\bar{A}$, we will find that if we then evaluate the functional on the displaced path $y(s) = \bar{y}(s) + \delta y(s)$, we will still get the number $\bar{A}$, except for a correction of second order in the displacement; the change δA is zero to first order in $\delta y(s)$.

To flesh this out let us consider all paths $y(s)$ that leave the point $y = y_0$ when $s = s_0$ and arrive at the point $y = y_1$ when $s = s_1$; members of this family of curves are displayed in Fig. 1.3. Out of all these possible paths that link y_0 and y_1 we wish to find the one which extremizes the functional $A[y]$. Our strategy will be to assume the existence of an extremum path, which we denote $\bar{y}(s)$, and which we treat as a reference path. We shall examine what happens to $A[y]$ when we displace the path from $y(s) = \bar{y}(s)$ to $y(s) = \bar{y}(s) + \delta y(s)$. While we shall find that in general, this produces a change δA that is proportional to $\delta y(s)$, we will instead demand that δA vanish to first order in the displacement; as we shall see, this procedure will permit us to identify the extremum path $\bar{y}(s)$. To carry out this procedure properly it is important to ensure that all the considered paths begin and end at the same two end points. The reference path $\bar{y}(s)$ and the displaced paths $y(s) = \bar{y}(s) + \delta y(s)$ must all satisfy $y(s_0) = y_0$ and $y(s_1) = y_1$. This implies that the displacement $\delta y(s)$, which are completely arbitrary in the interval $s_0 < s < s_1$, must satisfy the

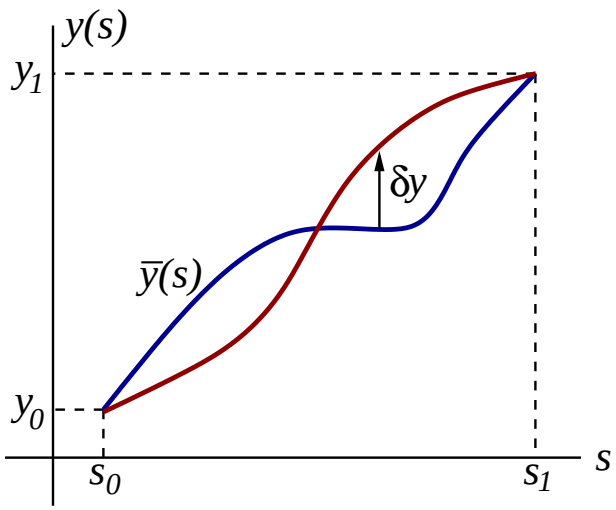


Figure 2.4: The reference path $\bar{y}(s)$ (in blue) and a displaced path $y(s) = \bar{y}(s) + \delta y(s)$ (in red). The displacement is arbitrary away from the two end points, but it must vanish at the end points.

boundary conditions

$$\delta y(s_0) = 0 = \delta y(s_1). \tag{2.2.4}$$

The situation is illustrated in Fig. 2.4.

We evaluate first the functional $A[y]$ on the reference path $\bar{y}(s)$; this is

$$\bar{A} = A[\bar{y}] = \int_{s_0}^{s_1} G(\bar{y}, \bar{y}')\, ds.$$

We next evaluate the functional on a displaced path $y(s) = \bar{y}(s) + \delta y(s)$; this is

$$A[\bar{y} + \delta y] = \int_{s_0}^{s_1} G(\bar{y} + \delta y, \bar{y}' + \delta y')\, ds,$$

where $\delta y' \equiv y' - \bar{y}' = d(y - \bar{y})/ds = d(\delta y)/ds$. The change in the functional is

$$\begin{aligned} \delta A &= A[\bar{y} + \delta y] - A[\bar{y}] \\ &= \int_{s_0}^{s_1} \Big[G(\bar{y} + \delta y, \bar{y}' + \delta y') - G(\bar{y}, \bar{y}') \Big]\, ds, \end{aligned}$$

and we wish to find conditions on $\bar{y}(s)$ that will allow us to set $\delta A = 0$, up to corrections of second order in δy.

The function G depends on two variables, $y(s)$ and $y'(s)$. By Taylor's theorem we have

$$G(\bar{y} + \delta y, \bar{y}' + \delta y') = G(\bar{y}, \bar{y}') + \left.\frac{\partial G}{\partial y}\right|_{y=\bar{y}, y'=\bar{y}'} \delta y + \left.\frac{\partial G}{\partial y'}\right|_{y=\bar{y}, y'=\bar{y}'} \delta y' + \cdots,$$

where we omit terms of higher order than first in the displacements $\delta y(s)$ and $\delta y'(s)$. The change in functional is therefore

$$\delta A = \int_{s_0}^{s_1} \left[\frac{\partial G}{\partial y}\, \delta y + \frac{\partial G}{\partial y'}\, \delta y' \right] ds,$$

where we again neglect higher-order terms, and where we discard the signs $|_{y=\bar{y}, y'=\bar{y}'}$ that instruct us to evaluate the partial derivatives on the reference path $\bar{y}(s)$; this operation will henceforth be understood.

Recalling that $\delta y' = d(\delta y)/ds$, we manipulate the second term within the integral:

$$\begin{aligned}\frac{\partial G}{\partial y'}\,\delta y'\,ds &= \frac{\partial G}{\partial y'}\,d(\delta y)\\ &= d\left(\frac{\partial G}{\partial y'}\,\delta y\right) - \delta y\,d\left(\frac{\partial G}{\partial y'}\right)\\ &= d\left(\frac{\partial G}{\partial y'}\,\delta y\right) - \delta y\,\frac{d}{ds}\frac{\partial G}{\partial y'}\,ds.\end{aligned}$$

This term can be integrated by parts, and we obtain

$$\delta A = \frac{\partial G}{\partial y'}\,\delta y\bigg|_{s_0}^{s_1} + \int_{s_0}^{s_1}\left[\frac{\partial G}{\partial y} - \frac{d}{ds}\frac{\partial G}{\partial y'}\right]\delta y(s)\,ds.$$

This result simplifies by virtue of Eq. (2.2.4): Because the displacement δy must vanish at the two end points, the boundary terms are necessarily zero. We end up with

$$\delta A = \int_{s_0}^{s_1}\left[\frac{\partial G}{\partial y} - \frac{d}{ds}\frac{\partial G}{\partial y'}\right]\delta y(s)\,ds. \tag{2.2.5}$$

The functional $A[\bar{y}]$ will be an extremum if δA vanishes for all displacements $\delta y(s)$ that satisfy the boundary conditions of Eq. (2.2.4). As we shall show presently, this will happen if and only if the quantity within square brackets vanishes. We therefore have the statement

$$\delta A = 0 \qquad \Rightarrow \qquad \frac{d}{ds}\frac{\partial G}{\partial y'} - \frac{\partial G}{\partial y} = 0. \tag{2.2.6}$$

This is the Euler-Lagrange (EL) equation associated with the function $G(y, y')$ which defines the functional $A[y]$. When fully worked out, the EL equation takes the form of a second-order differential equation for the function $y(s)$. Solving this equation gives the extremum path $\bar{y}(s)$.

To justify Eq. (2.2.6) we consider any integral of the form

$$\int_{s_0}^{s_1} E(s)n(s)\,ds,$$

which is known to vanish for *any choice* of function $n(s)$. [Here $E(s)$ plays the role of the quantity within square brackets in Eq. (2.2.5), and $n(s)$ plays the role of $\delta y(s)$.] What does this tell us about $E(s)$? To answer this let us design the arbitrary function $n(s)$ to suit our purposes. Let us imagine that it is everywhere positive and very sharply peaked near some value of s between s_0 and s_1, say $s = s^*$. Under these conditions the integral can be approximated by

$$E(s^*)\int_{s_0}^{s_1} n(s)\,ds,$$

and since the integral cannot be zero, we must conclude that $E(s^*) = 0$. Because the value of s^* is arbitrary, we can safely conclude that $E(s)$ must vanish everywhere in the interval $s_0 < s < s_1$. In this way we have shown that Eq. (2.2.5) leads to Eq. (2.2.6) whenever the displacement $\delta y(s)$ is arbitrary.

To sum up, we have shown in this subsection that an extremum path of the functional

$$A[y] = \int_{s_0}^{s_1} G(y, y')\,ds$$

is obtained by finding a solution $\bar{y}(s)$ to the EL equation

$$\frac{d}{ds}\frac{\partial G}{\partial y'} - \frac{\partial G}{\partial y} = 0.$$

This statement is true whether the extremum is a maximum or a minimum, and it is independent of the detailed nature of the function $G(y, y')$. Any function of the two variables $y(s)$ and $y'(s)$ can thus be substituted inside the functional, and our calculus of variations applies to a very wide range of situations.

2.2.3 Curve of maximum area (continued)

The function G that corresponds to our original problem is

$$G(y, y') = y\sqrt{1 - y'^2}. \tag{2.2.7}$$

Substitution of this function into the EL equation will produce a second-order differential equation for $y(s)$. Solving this will give us the curve that maximizes the area.

When we substitute Eq. (2.2.7) into Eq. (2.2.6) we must first calculate the derivative of G with respect to y', treating y as a constant parameter. This is

$$\frac{\partial G}{\partial y'} = -yy'\left[1 - y'^2\right]^{-1/2}.$$

We next differentiate this with respect to s. Because $\partial G/\partial y' \equiv G_{y'}$ depends on s through its dependence on both y and y', we must apply the chain rule. This gives

$$\begin{aligned}\frac{d}{ds}\frac{\partial G}{\partial y'} &= \frac{\partial G_{y'}}{\partial y}\frac{dy}{ds} + \frac{\partial G_{y'}}{\partial y'}\frac{dy'}{ds} \\ &= \frac{\partial G_{y'}}{\partial y}\,y' + \frac{\partial G_{y'}}{\partial y'}\,y''.\end{aligned}$$

We have

$$\frac{\partial G_{y'}}{\partial y} = -y'\left[1 - y'^2\right]^{-1/2}$$

and

$$\frac{\partial G_{y'}}{\partial y'} = -y\left[1 - y'^2\right]^{-3/2},$$

so that

$$\frac{d}{ds}\frac{\partial G}{\partial y'} = -y'^2\left[1 - y'^2\right]^{-1/2} - yy''\left[1 - y'^2\right]^{-3/2}.$$

The remaining quantity to calculate is

$$\frac{\partial G}{\partial y} = \left[1 - y'^2\right]^{1/2}.$$

After cleaning up the algebra we find that the EL equation is

$$yy'' - y'^2 + 1 = 0. \tag{2.2.8}$$

This is a nonlinear, second-order differential equation for the function $y(s)$.

Exercise 2.1. Make sure that you can reproduce the computations that lead to Eq. (2.2.8).

The general solution to Eq. (2.2.8) is

$$y = \frac{1}{c_1}\sin c_1(s + c_2),$$

where c_1 and c_2 are two constants. That this is indeed a solution can be verified by direct substitution; that this is the *general solution* can be seen from the fact that it depends on two arbitrary constants, the correct number for a second-order differential equation. These constants are determined by enforcing the boundary conditions $y(s=0)=0$ and $y(s=\pi)=0$, which follow from the requirement that the maximum curve must link the points $(-1,0)$ and $(+1,0)$. The first condition gives $(1/c_1)\sin(c_1c_2)=0$, which implies that $c_2=0$. The second condition gives $(1/c_1)\sin(c_1\pi)=0$, which implies that c_1 must be an integer, which we call n. We therefore have

$$y(s)=\frac{1}{n}\sin ns, \qquad y'(s)=\cos ns.$$

We may now look for $x(s)$, which is determined by Eq. (2.2.2),

$$x'=\sqrt{1-y'^2}=\sqrt{1-\cos^2 ns}=\sin ns.$$

This integrates to $x=x_0-(1/n)\cos ns$, where x_0 is another constant of integration. We must now impose the boundary conditions $x(s=0)=-1$ and $x(s=\pi)=+1$. The first condition gives $x_0-1/n=-1$, so that $x_0=-1+1/n$. The second condition gives $-1+(1-\cos n\pi)/n=1$, or $\cos n\pi=1-2n$, which implies that $n=1$. We therefore have $x=-\cos s$, and the constraint $n=1$ also implies $y=\sin s$.

Exercise 2.2. Verify that $y=(1/c_1)\sin c_1(s+c_2)$ is a solution to Eq. (2.2.8), and verify that the choices $c_1=1$ and $c_2=0$ are appropriate given the boundary conditions.

Our final result is this: The curve that maximizes the area A is described by the parametric relations

$$\bar{x}(s)=-\cos s, \qquad \bar{y}(s)=\sin s, \qquad 0<s<\pi. \tag{2.2.9}$$

This is a half-circle of unit radius that links the points $(-1,0)$ and $(+1,0)$. The maximum area is then given by

$$A_{\max}=\int_0^\pi \bar{y}(s)\frac{d\bar{x}}{ds}\,ds=\int_0^\pi \sin^2 s\,ds=\frac{\pi}{2}\simeq 1.5708.$$

To test whether this is really a maximum we evaluate A for a different choice of curve, one which consists of two straight segments. The first segment connects the points $(-1,0)$ and $(0,y_0)$, while the second segment connects the points $(0,y_0)$ and $(1,0)$. The length of each segment is $\ell=\sqrt{1+y_0^2}$. Because the total length of the curve must be equal to π, we must set $y_0=\sqrt{(\pi/2)^2-1}$. The area under this curve is the area of a triangle of base 2 and height y_0, so

$$A=\frac{1}{2}(2)(y_0)=\sqrt{(\pi/2)^2-1}\simeq 1.2114.$$

This area is indeed smaller than $A_{\max}$.

2.2.4 Path of minimum length

The calculus of variations, introduced in Sec. 2.2.2, can be employed to solve many different problems involving either the maximization or minimization of a functional. A simple example is the problem of finding the curve $y(x)$ that minimizes the distance between two fixed points in the x-y plane. We already know that the answer is a straight line, but it will be comforting to use the calculus to give a mathematical proof of this statement.

We shall take the two points to be $(0,0)$ and (x_1, y_1), respectively. We want to calculate the distance s measured along the curve $y(x)$, and we want to find the path $\bar{y}(x)$ that minimizes this distance. The increment of distance ds along the curve is easy enough to calculate; it is given by

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + y'^2}\, dx,$$

where we have set $y' = dy/dx$. The total distance along the curve is obtained by integration. We have

$$s = \int_0^{x_1} \sqrt{1 + y'^2}\, dx. \tag{2.2.10}$$

This is a functional of the path $y(x)$, and we wish to minimize this functional. So here s plays the role of $A[y]$, and x plays the role of the old parameter s. The function G is given by

$$G(y, y') = \sqrt{1 + y'^2}. \tag{2.2.11}$$

Notice that this depends only on y'; there is no explicit dependence on y.

The EL equation for this situation is

$$\frac{d}{dx}\frac{\partial G}{\partial y'} - \frac{\partial G}{\partial y} = 0.$$

Because G does not depend explicitly on y we have that $\partial G/\partial y = 0$. The EL equation implies

$$\frac{d}{dx}\frac{\partial G}{\partial y'} = 0,$$

and this states that the quantity $\partial G/\partial y'$ is in fact a constant, independent of x. We shall call this constant c. Calculating $\partial G/\partial y'$ gives $y'/\sqrt{1 + y'^2}$, and we have obtained the statement

$$\frac{y'}{\sqrt{1 + y'^2}} = c.$$

This equation can easily be solved for y', and we get

$$y' = \frac{c}{\sqrt{1 - c^2}} \equiv m,$$

where m is a new constant. Integration of this equation is straightforward, and we obtain

$$y(x) = mx + b,$$

where b is a final constant of integration. This is the equation of the straight line, the result we expected.

The constants m and b can be determined from the boundary conditions, $y(x = 0) = 0$ and $y(x = x_1) = y_1$. The first condition implies $b = 0$, while the second condition implies $m = y_1/x_1$. The final result is therefore that the path which minimizes the distance between $(0,0)$ and (x_1, y_1) is described by

$$\bar{y}(x) = \frac{y_1}{x_1}\, x. \tag{2.2.12}$$

That this is indeed a minimum, instead of a maximum, is obvious from the fact that the maximum distance between two points is always infinite.

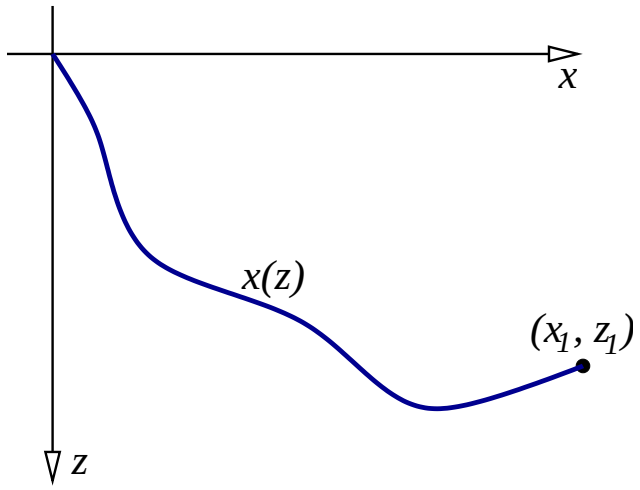


Figure 2.5: A particle falls on a slide whose shape is described by the function $x(z)$. The particle starts from rest at $(x = 0, z = 0)$ and it reaches the point (x_1, z_1) after a time t.

2.2.5 Brachistochrone

In this application of the calculus of variations we consider a particle released from rest on a slide of a specified shape. The particle is subjected to gravity, and it moves on the slide without friction. It eventually reaches the point (x_1, z_1) in a time t, as illustrated in Fig. 2.5. We wish to determine the shape of the slide that minimizes this time. This classic problem of mathematical physics is called the *brachistochrone*; it was first solved by Johann Bernoulli in 1696.

The shape of the slide is specified by the unknown function $x(z)$; this curve in the x-z plane is required to link the points $(0, 0)$ and (x_1, z_1). The increment of length on the curve is given by

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{1 + (dx/dz)^2}\, dz = \sqrt{1 + x'^2}\, dz,$$

where we now use x' to denote dx/dz. The speed of the particle on the slide is $v = ds/dt$, and the increment of time is given by $dt = ds/v$. The total time required by the particle to reach the point (x_1, z_1) is then

$$t = \int \frac{ds}{v} = \int_0^{z_1} \frac{\sqrt{1 + x'^2}}{v(z)}\, dz.$$

To calculate $v(z)$ we appeal to the conservation of mechanical energy. In this situation the particle moves under the action of gravity, and its total energy is $E = \frac{1}{2}mv^2 - mgz$. It is stated that the particle proceeds from rest ($v = 0$) at the upper point of the slide ($z = 0$), and we conclude from this that its total energy is zero. As a consequence we find that $\frac{1}{2}mv^2 = mgz$, or $v(z) = \sqrt{2gz}$. We therefore have

$$t = \frac{1}{\sqrt{2g}} \int_0^{z_1} \frac{\sqrt{1 + x'^2}}{\sqrt{z}}\, dz,$$

and the functional that we wish to minimize is

$$\sqrt{2g}\, t[x] = \int_0^{z_1} \frac{\sqrt{1 + x'^2}}{\sqrt{z}}\, dz. \tag{2.2.13}$$

Here the role of the parameter is played by z, and the function G is given by

$$G(x, x') = \frac{\sqrt{1 + x'^2}}{\sqrt{z}}. \tag{2.2.14}$$

Notice that this depends only on x'; there is no explicit dependence on x. Notice further that there is an explicit dependence on the parameter z.

The EL equation for this situation is

$$\frac{d}{dz}\frac{\partial G}{\partial x'} - \frac{\partial G}{\partial x} = 0.$$

We have $\partial G/\partial x = 0$, and we conclude immediately that

$$\frac{\partial G}{\partial x'} = \text{constant} \equiv \frac{1}{\sqrt{2a}}.$$

(It turns out to be convenient to make this choice of constant.) Calculation gives

$$\frac{\partial G}{\partial x'} = \frac{x'}{\sqrt{z}\sqrt{1+x'^2}},$$

and the EL equation reduces to

$$\frac{x'}{\sqrt{z}\sqrt{1+x'^2}} = \frac{1}{\sqrt{2a}}.$$

This can easily be solved for x', and we obtain

$$x' = \frac{z}{\sqrt{2az - z^2}}.$$

This equation, finally, can be integrated, and a formal solution to our problem is

$$x(z) = \int_0^z \frac{z\,dz}{\sqrt{2az - z^2}}. \tag{2.2.15}$$

It is this integral that determines the shape of the minimal slide.

Exercise 2.3. Make sure that you can reproduce the steps that lead to Eq. (2.2.15).

To evaluate the integral of Eq. (2.2.15) we change the variable of integration from z to θ using the transformation

$$z = a(1 - \cos\theta),$$

which implies $dz = a\sin\theta\,d\theta$. The angle θ runs from 0 when $z = 0$ to θ_1 when $z = z_1$. After a short calculation we find that $2az - z^2 = a^2\sin^2\theta$, and it follows that

$$x = a\int_0^\theta (1 - \cos\theta)\,d\theta = a(\theta - \sin\theta).$$

The shape of the slide is therefore described by the parametric equations

$$x(\theta) = a(\theta - \sin\theta), \qquad z(\theta) = a(1 - \cos\theta) \qquad 0 \le \theta \le \theta_1. \tag{2.2.16}$$

These describe a curve known as a *cycloid*. The constants a and θ_1 are determined by the condition that $x = x_1$ and $z = z_1$ when $\theta = \theta_1$. For example, if we choose $x_1 = 5$ and $z_1 = 1$, then we need $a \simeq 0.89483$ and $\theta_1 \simeq 4.5946$. This particular slide is shown in Fig. 2.6. The figure reveals that contrary to expectations, the slide does not always go down; it indeed turns around when $\theta = \pi \simeq 3.1416$.

Exercise 2.4. Make sure that you can reproduce the steps that lead to Eq. (2.2.16). Check that the constants $a \simeq 0.89483$ and $\theta_1 \simeq 4.5946$ do indeed produce $x_1 = 5$ and $z_1 = 1$. Can you devise a method to determine a and θ_1 given a choice for x_1 and z_1?

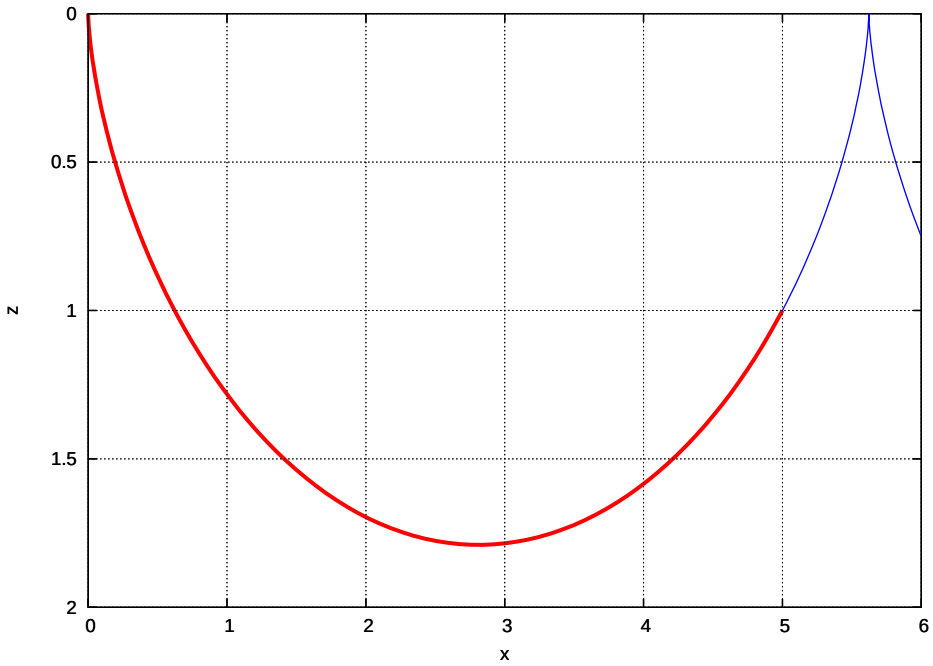


Figure 2.6: A cycloid that connects the points $(0, 0)$ and $(5, 1)$.

This feature of the minimal slide is surprising. Can we be sure that this slide truly minimizes the time? Would not a straight slide do a better job? To convince ourselves that we do have the minimal slide, let us compare the times required for the particle to go from $(0, 0)$ to $(5, 1)$ when it uses either the cycloid or a straight slide. We shall calculate $\sqrt{2g}t[x]$ for each case and compare the answers.

For the cycloid we have

$$\sqrt{2g}t_{\text{cycloid}} = \int_0^{z_1} \frac{\sqrt{1+x'^2}}{\sqrt{z}}\, dz.$$

With the change of variables introduced above we have $x' = (dx/d\theta)/(dz/d\theta) = (1-\cos\theta)/\sin\theta$, so that

$$\sqrt{1+x'^2} = \frac{\sqrt{\sin^2\theta + (1-\cos\theta)^2}}{\sin\theta} = \frac{\sqrt{2(1-\cos\theta)}}{\sin\theta}.$$

It follows that

$$\sqrt{2g}t_{\text{cycloid}} = \int_0^{\theta_1} \frac{\sqrt{2(1-\cos\theta)}}{\sin\theta} \frac{a\sin\theta\, d\theta}{\sqrt{a(1-\cos\theta)}} = \sqrt{2a}\int_0^{\theta_1} d\theta,$$

or

$$\sqrt{2g}t_{\text{cycloid}} = \sqrt{2a}\theta_1 \simeq 6.1466,$$

using the numerical values listed previously.

The shape of the straight slide is described by $x = 5z$, which implies that $x' = 5$. In this case we have

$$\sqrt{2g}t_{\text{straight}} = \int_0^1 \frac{\sqrt{26}}{\sqrt{z}}\, dz = 2\sqrt{26}z^{1/2}\Big|_0^1,$$

or

$$\sqrt{2g}t_{\text{straight}} = 2\sqrt{26} \simeq 10.198,$$

and this is a larger number.

We have found that, sure enough, $t_{\text{cycloid}} < t_{\text{straight}}$. The particle spends less time on the cycloid than on the straight slide, in spite of the fact that loses speed on the way up toward (x_1, z_1). The reason is that it picks up a lot of speed on the *way down*, and this more than makes up for the loss of speed on the way up. The straight slide just does not measure up.

2.2.6 Multiple paths

It is useful, and necessary, to generalize the calculus of variations to functionals A that depend not on one path only, but on a collection of paths. In this final subsection we consider the task of extremizing the multi-path functional

$$A[y_1, y_2, \cdots] = \int_{s_0}^{s_1} G(y_1, y_1'; y_2, y_2'; \cdots)\, ds \tag{2.2.17}$$

with respect to each individual path $y_a(s)$; the index $a = 1, 2, \cdots$ is used to label each path within the collection.

This generalization is straightforward. For each variable $y_a(s)$ within the collection we select a reference path $\bar{y}_a(s)$ and we calculate $A[\bar{y}_1, \bar{y}_2, \cdots]$. We then displace each path from $\bar{y}_a(s)$ to $\bar{y}_a(s) + \delta y_a(s)$ and calculate the new value $A[\bar{y}_1 + \delta y_1, \bar{y}_2 + \delta y_2, \cdots]$ for the functional. The extremum of A is found by demanding that the variation $\delta A = A[\bar{y}_1 + \delta y_1, \bar{y}_2 + \delta y_2, \cdots] - A[\bar{y}_1, \bar{y}_2, \cdots]$ vanish to first order in the displacements $\delta y_a(s)$. As before we impose that the reference and displaced paths all begin and end at the same end points, $y_a(s_0)$ and $y_a(s_1)$. We therefore impose that the variations δy_a all vanish at the end points, $\delta y_a(s_0) = 0 = \delta y_a(s_1)$.

The change in functional that occurs when we displace the paths from the reference paths $\bar{y}_a(s)$ is

$$\delta A = \int_{s_0}^{s_1} \Big[G(\bar{y}_1 + \delta y_1, \bar{y}_1' + \delta y_1'; \bar{y}_2 + \delta y_2, \bar{y}_2' + \delta y_2'; \cdots) - G(\bar{y}_1, \bar{y_1}'; \bar{y}_2, \bar{y}_2'; \cdots) \Big]\, ds.$$

By Taylor's theorem,

$$\begin{aligned} G(\bar{y}_1 + \delta y_1, \bar{y}_1' + \delta y_1'; \bar{y}_2 + \delta y_2, \bar{y}_2' + \delta y_2'; \cdots) &= G(\bar{y}_1, \bar{y_1}'; \bar{y}_2, \bar{y}_2'; \cdots) \\ &+ \left.\frac{\partial G}{\partial y_1}\right|_{y_1=\bar{y}_1, y_1'=\bar{y}_1'; y_2=\bar{y}_2, y_2'=\bar{y}_2'; \cdots} \delta y_1 + \left.\frac{\partial G}{\partial y_1'}\right|_{y_1=\bar{y}_1, y_1'=\bar{y}_1'; y_2=\bar{y}_2, y_2'=\bar{y}_2'; \cdots} \delta y_1' \\ &+ \left.\frac{\partial G}{\partial y_2}\right|_{y_1=\bar{y}_1, y_1'=\bar{y}_1'; y_2=\bar{y}_2, y_2'=\bar{y}_2'; \cdots} \delta y_2 + \left.\frac{\partial G}{\partial y_2'}\right|_{y_1=\bar{y}_1, y_1'=\bar{y}_1'; y_2=\bar{y}_2, y_2'=\bar{y}_2'; \cdots} \delta y_2' \\ &+ \cdots. \end{aligned}$$

Here G is differentiated with respect to each one of its variables, and the partial derivatives are evaluated on the reference paths; we discard all terms that are not linear in the displacements δy_a and $\delta y_a'$. We have, in a more compact notation,

$$\delta A = \int_{s_0}^{s_1} \sum_a \left(\frac{\partial G}{\partial y_a} \delta y_a + \frac{\partial G}{\partial y_a'} \delta y_a' \right) ds,$$

where we sum over all the variables and omit the warning that all partial derivatives must be evaluated on the reference paths, at $y_a = \bar{y}_a$ and $y_a' = \bar{y}_a'$.

We now write

$$\delta y_a' \equiv y_a' - \bar{y}_a' = \frac{d}{ds}\big(y_a - \bar{y}_a\big) = \frac{d}{ds}\delta y_a$$

and express the second term within the integral as

$$\frac{\partial G}{\partial y_a'} d(\delta y_a) = d\left(\frac{\partial G}{\partial y_a'} \delta y_a \right) - \delta y_a d\left(\frac{\partial G}{\partial y_a'} \right).$$

Integrating this term by parts gives

$$\delta A = \sum_a \frac{\partial G}{\partial y'_a}\delta y_a\bigg|_{s_0}^{s_1} + \sum_a \int_{s_0}^{s_1}\left(\frac{\partial G}{\partial y_a} - \frac{d}{ds}\frac{\partial G}{\partial y'_a}\right)\delta y_a\, ds.$$

Because the displacements must vanish at the end points, the two boundary terms disappear. And because each displacement $\delta y_a(s)$ is independent of any other displacement, and the displacements are arbitrary in the interval $s_0 < s < s_1$, we conclude that

$$\delta A = 0 \qquad \Rightarrow \qquad \frac{d}{ds}\frac{\partial G}{\partial y'_a} - \frac{\partial G}{\partial y_a} = 0. \tag{2.2.18}$$

We have one EL equation for each path $y_a(s)$. This simple statement provides the desired generalization of the calculus of variations to multi-path functionals.

2.3 Hamilton's principle of least action

In Chapter 1 we saw that Newton's law, $\boldsymbol{F} = m\boldsymbol{a}$, can serve as the very foundation of all of mechanics; conservation of momentum, angular momentum, and energy could be derived as a consequence of this dynamical law. In this section we officially replace this old foundation by a new one, which is at once more practical, more powerful, and more easily generalizable to other areas of physics. This new foundation will be *Hamilton's principle of least action*; the dynamical law $\boldsymbol{F} = m\boldsymbol{a}$, and the statements of conservation, will all be derived as consequences of this new principle.

The principle of least action states that of all the paths $q_a(t)$ that a system of particles *could take* to go from an initial configuration $q_a(t_0)$ to a final configuration $q_a(t_1)$, the paths $\bar{q}_a(t)$ that the particles *actually take* are the ones that minimize the action functional

$$S[q_a] = \int_{t_0}^{t_1} L(q_a, \dot{q}_a)\, dt, \tag{2.3.1}$$

where

$$L = T - V \tag{2.3.2}$$

is the *Lagrangian function* of the mechanical system. The Lagrangian is the difference between T, the system's total kinetic energy, and V, the total potential energy. The Lagrangian can be expressed in any system of generalized coordinates q_a that conveniently describe the system's degrees of freedom. Because the Lagrangian is a scalar function (as opposed to a vectorial function), the choice of coordinates is immaterial to the formulation of Hamilton's principle. In particular, it is not necessary to adopt Cartesian coordinates attached to an inertial frame. (Of course, nothing prevents us from making this choice if it is convenient.)

To find the paths $\bar{q}_a(t)$ that minimize the action functional we follow the techniques developed in Sec. 2.2. Here $S[q_a]$ is a multi-path functional, and the paths $q_a(t)$ play the role of the functions $y_a(s)$; the Lagrangian plays the role of the function G, and the parameter is the time t. There is no need to repeat the calculations described in Sec. 2.2.6; the conclusion is

$$\delta S = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0. \tag{2.3.3}$$

These are the Euler-Lagrange (EL) equations for the mechanical system; there is one EL equation for each degree of freedom. The EL equations, when fully worked out, become a set of second-order differential equations for the paths $q_a(t)$. The solutions to these equations, which much be subjected to the boundary conditions at $t = t_0$ and $t = t_1$, are the paths $\bar{q}_a(t)$ that minimize the action functional.

We have already seen in Sec. 2.1 that when the generalized coordinates $q_a(t)$ of a particle are Cartesian, so that the Lagrangian takes the form $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$, then the EL equations become the vectorial equation $m\boldsymbol{a} + \boldsymbol{\nabla} V = \mathbf{0}$. Recalling that the force acting on the particle is $\boldsymbol{F} = -\boldsymbol{\nabla} V$, this is obviously $\boldsymbol{F} = m\boldsymbol{a}$, and we have *derived* Newton's fundamental law from a deeper principle, Hamilton's principle of least action. The beauty of this Lagrangian formulation of mechanics, however, is not so much that Newton's equation follows from a deeper principle. Its beauty is much more in the fact that *Hamilton's principle frees us from the need to always set up the equations in terms of Cartesian coordinates.* Any system of generalized coordinates $q_a(t)$ will do; they all lead to the EL equations of Eq. (2.3.3), and the choice is entirely one of convenience.

In the following sections we will explore the power of Hamilton's principle in a number of applications. We will take full advantage of the generalized nature of the coordinates $q_a(t)$, and the EL equations will allow us to derive the equations of motion very efficiently, with far less effort than would be required in the traditional Newtonian formulation.

2.4 Applications of Lagrangian mechanics

2.4.1 Equations of motion in cylindrical coordinates

As was just stated, the principal advantage of the Lagrangian formulation of mechanics is that it is based on a scalar function L which can be expressed in any coordinate system whatever. We shall begin our discussion with a derivation of the equations of motion in cylindrical coordinates; the case of spherical coordinates will considered next.

Suppose that a particle moves in the presence of a potential V that is most simply expressed in terms of cylindrical coordinates (ρ, ϕ, z). These are related to the usual Cartesian coordinates (x, y, z) by

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z. \tag{2.4.1}$$

To use cylindrical coordinates would be advantageous, for example, when the potential is axially symmetric, so that it depends only on ρ and z, or cylindrically symmetric, when it depends only on ρ.

From Eq. (2.4.1) we obtain the total differentials

$$\begin{aligned} dx &= (\cos\phi)\,d\rho - (\rho\sin\phi)\,d\phi, \\ dy &= (\sin\phi)\,d\rho + (\rho\cos\phi)\,d\phi, \\ dz &= dz. \end{aligned}$$

It follows that the squared distance between two neighbouring points is given by $ds^2 = dx^2 + dy^2 + dz^2$, or

$$ds^2 = d\rho^2 + \rho^2\,d\phi^2 + dz^2. \tag{2.4.2}$$

The squared velocity is then

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2,$$

and the particle's kinetic energy is $T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$. The Lagrangian is therefore

$$L(\rho, \dot{\rho}; \phi, \dot{\phi}; z, \dot{z}) = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z). \tag{2.4.3}$$

Exercise 2.5. Verify Eq. (2.4.2).

The equations of motion for the particle are obtained by substituting L into the EL equations for $q_a = (\rho, \phi, z)$. We begin with the equation for ρ. We have, from Eq. (2.4.3),

$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}.$$

This implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = m\ddot{\rho}.$$

We also have

$$\frac{\partial L}{\partial \rho} = m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho},$$

and the EL equation gives

$$m\ddot{\rho} - m\rho\dot{\phi}^2 + \frac{\partial V}{\partial \rho} = 0. \tag{2.4.4}$$

We continue with the equation for ϕ. We now have

$$\frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi},$$

which implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = m\frac{d}{dt}\Big(\rho^2\dot{\phi}\Big).$$

Notice that we choose to leave the total time derivative unevaluated; to evaluate it would require some care, because both $\dot{\phi}$ and ρ^2 depend on time in this expression. We also have

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi},$$

and the EL equation gives

$$m\frac{d}{dt}\Big(\rho^2\dot{\phi}\Big) + \frac{\partial V}{\partial \phi} = 0. \tag{2.4.5}$$

We conclude with the equation for z. Here the computations are quite easy. We have

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z},$$

so that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z},$$

and we also have

$$\frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}.$$

The EL equation for z is therefore

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0. \tag{2.4.6}$$

The equations of motion (2.4.4)–(2.4.6) could also be derived by resolving Newton's equation $\boldsymbol{F} = m\boldsymbol{a}$ in the vectorial basis $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}})$. The results would be identical, but the computations would be much more laborious.

Exercise 2.6. Challenge yourself: Derive Eqs. (2.4.4)–(2.4.6) the hard way, as described in the previous paragraph. Begin by computing the acceleration vector $\boldsymbol{a}$ in terms of the cylindrical coordinates (ρ, ϕ, z). Next, find the basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ using the method outlined in Sec. 1.2. Finally, resolve the equation $m\boldsymbol{a} + \boldsymbol{\nabla} V = \boldsymbol{0}$ in this basis, and use the chain rule to calculate $\partial V/\partial\rho$ and $\partial V/\partial\phi$ in terms of $\partial V/\partial x$ and $\partial V/\partial y$. The end result should resemble Eqs. (2.4.4)–(2.4.6). If you are not already, after all this you *will* be fully convinced of the superiority of the Lagrangian methods!

2.4.2 Equations of motion in spherical coordinates

Suppose now that a particle moves in the presence of a potential V that is most simply expressed in terms of spherical coordinates (r, θ, ϕ). Their relation with the usual Cartesian coordinates (x, y, z) is

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{2.4.7}$$

The use of spherical coordinates would be advantageous, for example, when the potential is axially symmetric, so that it depends only on r and θ, or spherically symmetric, when it depends only on r.

From Eq. (2.4.7) we obtain the total differentials

$$\begin{aligned} dx &= (\sin\theta\cos\phi)\,dr + (r\cos\theta\cos\phi)\,d\theta - (r\sin\theta\sin\phi)\,d\phi, \\ dy &= (\sin\theta\sin\phi)\,dr + (r\cos\theta\sin\phi)\,d\theta + (r\sin\theta\cos\phi)\,d\phi, \\ dz &= (\cos\theta)\,dr - (r\sin\theta)\,d\theta. \end{aligned}$$

It follows that the squared distance between two neighbouring points is given by

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{2.4.8}$$

The squared velocity is then $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$, and the Lagrangian is

$$L(r, \dot{r}; \theta, \dot{\theta}; \phi, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r, \theta, \phi). \tag{2.4.9}$$

Exercise 2.7. Verify Eq. (2.4.8).

The equations of motion for the particle are obtained by substituting L into the EL equations for $q_a = (r, \theta, \phi)$. We have, from Eq. (2.4.9),

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

so that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}.$$

We also have

$$\frac{\partial L}{\partial r} = mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - \frac{\partial V}{\partial r},$$

and the EL equation for r is

$$m\ddot{r} - mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{\partial V}{\partial r} = 0. \tag{2.4.10}$$

Moving on, we now have

$$\frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta,$$

which implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m\frac{d}{dt}\Big(r^2\dot\theta\Big).$$

We also have

$$\frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot\phi^2 - \frac{\partial V}{\partial\theta},$$

and the EL equation gives

$$m\frac{d}{dt}\Big(r^2\dot\theta\Big) - mr^2\sin\theta\cos\theta\,\dot\phi^2 + \frac{\partial V}{\partial\theta} = 0. \tag{2.4.11}$$

Finally, we have

$$\frac{\partial L}{\partial \dot\phi} = mr^2\sin^2\theta\,\dot\phi,$$

which implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = m\frac{d}{dt}\Big(r^2\sin^2\theta\,\dot\phi\Big).$$

We also have

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial\phi},$$

and the EL equation gives

$$m\frac{d}{dt}\Big(r^2\sin^2\theta\,\dot\phi\Big) + \frac{\partial V}{\partial\phi} = 0. \tag{2.4.12}$$

The equations of motion (2.4.10)–(2.4.12) could also be derived by resolving Newton's equation $\boldsymbol{F} = m\boldsymbol{a}$ in the vectorial basis $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$. The results would be identical, but as in the preceding subsection the computations would be much more laborious.

Exercise 2.8. Challenge yourself once again: Derive Eqs. (2.4.10)–(2.4.12) the hard way, as described in the previous paragraph. Or finally cry uncle and pledge allegiance to the Lagrangian way of life!

2.4.3 Motion on the surface of a cone

As our first real application of the Lagrangian formalism, we consider a particle that is constrained to move on the surface of a cone, subjected to gravity. As shown in Fig. 2.7, the cone has an opening angle of 2α, and it is placed vertically in the gravitational field. The particle is at a distance $r(t)$ from the cone's apex, and at an angle $\phi(t)$ relative to the x axis. Because the particle is confined to the cone's surface, its angle θ with respect to the z axis is a constant; it is in fact equal to α.

The motion of the particle is best described in terms of spherical coordinates (r, θ, ϕ), with θ restricted at all times to the value α. According to the results of Sec. 2.4.2, its kinetic energy is $T = \frac{1}{2}m(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2)$, and its potential energy is $V = mgz = mgr\cos\alpha$. The Lagrangian is therefore

$$L(r, \dot r; \phi, \dot\phi) = \frac{1}{2}m(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2) - mgr\cos\alpha. \tag{2.4.13}$$

The equations of motion for $r(t)$ and $\phi(t)$ are obtained by substituting this Lagrangian into the EL equations.

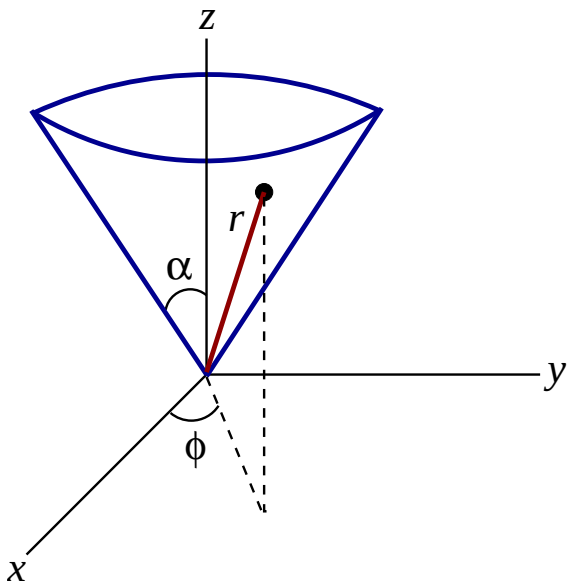


Figure 2.7: A particle of mass m moves on the surface of a cone of opening angle 2α. The motion is described by the coordinates $r(t)$ and $\phi(t)$.

We have

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

so that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}.$$

We also have

$$\frac{\partial L}{\partial r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha,$$

and the EL equation for r is

$$\ddot{r} - r\sin^2\alpha\,\dot{\phi}^2 + g\cos\alpha = 0. \tag{2.4.14}$$

Moving on, we observe that L is independent of ϕ, and the fact that $\partial L/\partial\phi = 0$ means that the EL equation for ϕ reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0.$$

This implies that the quantity $\partial L/\partial\dot{\phi}$ is a constant, which we shall call mh. Calculating the partial derivative gives $\partial L/\partial\dot{\phi} = mr^2\sin^2\alpha\,\dot{\phi}$, and we finally obtain the statement

$$r^2\sin^2\alpha\,\dot{\phi} = h = \text{constant}. \tag{2.4.15}$$

The quantity h is readily interpreted as the z component of the particle's reduced angular momentum vector, and it is a constant of the motion. Equation (2.4.15) shows that $\dot{\phi}$ is always of the same sign; the angular part of the motion is monotonic.

Substituting $\dot{\phi} = h/(r^2\sin^2\alpha)$ into Eq. (2.4.14) produces

$$\ddot{r} - \frac{h^2}{r^3\sin^2\alpha} + g\cos\alpha = 0.$$

This equation can be integrated by using the standard trick of multiplying each term by $\dot{r}$ (recall that we used this trick back in Sec. 1.5.6). We have

$$\ddot{r}\dot{r} - \frac{h^2\dot{r}}{r^3\sin^2\alpha} + g\dot{r}\cos\alpha = 0,$$

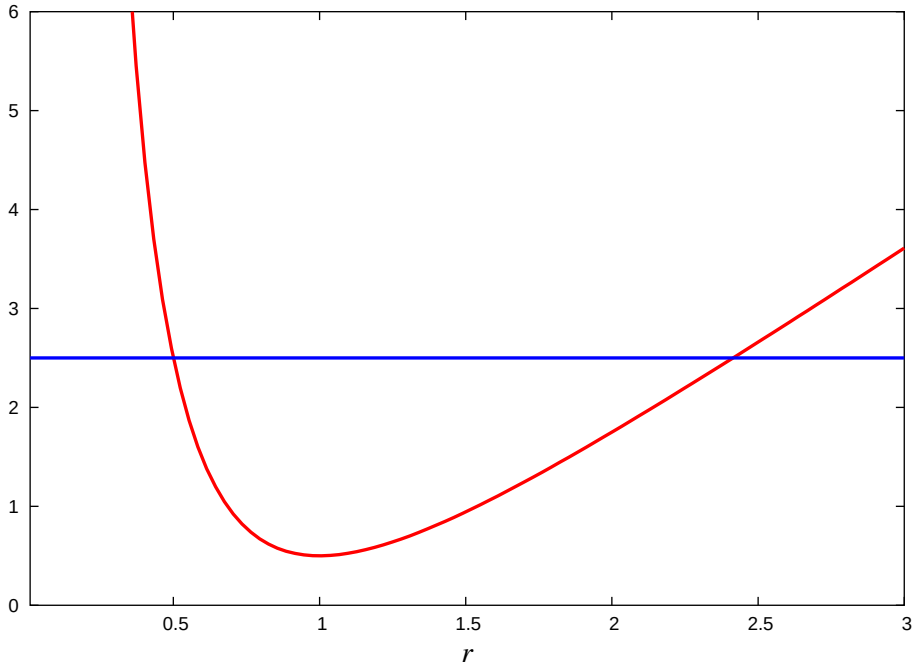


Figure 2.8: Energy diagram for a particle moving on a cone. The motion always takes place between two turning points at $r = r_\pm$.

or

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2\sin^2\alpha} + gr\cos\alpha\right) = 0.$$

This finally gives us the conservation statement

$$\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon = \text{constant}, \tag{2.4.16}$$

where ε is the particle's reduced total mechanical energy, and

$$\nu(r) = \frac{h^2}{2r^2\sin^2\alpha} + gr\cos\alpha \tag{2.4.17}$$

is an effective potential for the radial part of the motion. Equations (2.4.16) and (2.4.17) give rise to the energy diagram of Fig. 2.8. From this diagram we immediately conclude that the motion takes place between two turning points at $r = r_\pm$; these are determined by the condition $\nu(r_\pm) = \varepsilon$.

To obtain a full picture of the motion Eqs. (2.4.14) and (2.4.15) must be integrated numerically. Results of such a numerical integration are presented in Fig. 2.9. To carry out these integrations the equations are recast into the following set of first-order equations:

$$\dot{r} = v, \qquad \dot{v} = \frac{h^2}{r^3\sin\alpha} - g\cos\alpha, \qquad \dot{\phi} = \frac{h}{r^2\sin^2\alpha},$$

where we have introduced the auxiliary variable v. We start the integration at $r = r_-$, setting $v = 0$ (as we must) and $\phi = 0$. The constant h can be determined in terms of r_- and r_+ by using the relation $\nu(r_-) = \nu(r_+)$, which follows from Eq. (2.4.16). The result is

$$h^2 = 2g\sin^2\alpha\cos\alpha\frac{(r_+r_-)^2}{r_+ + r_-}.$$

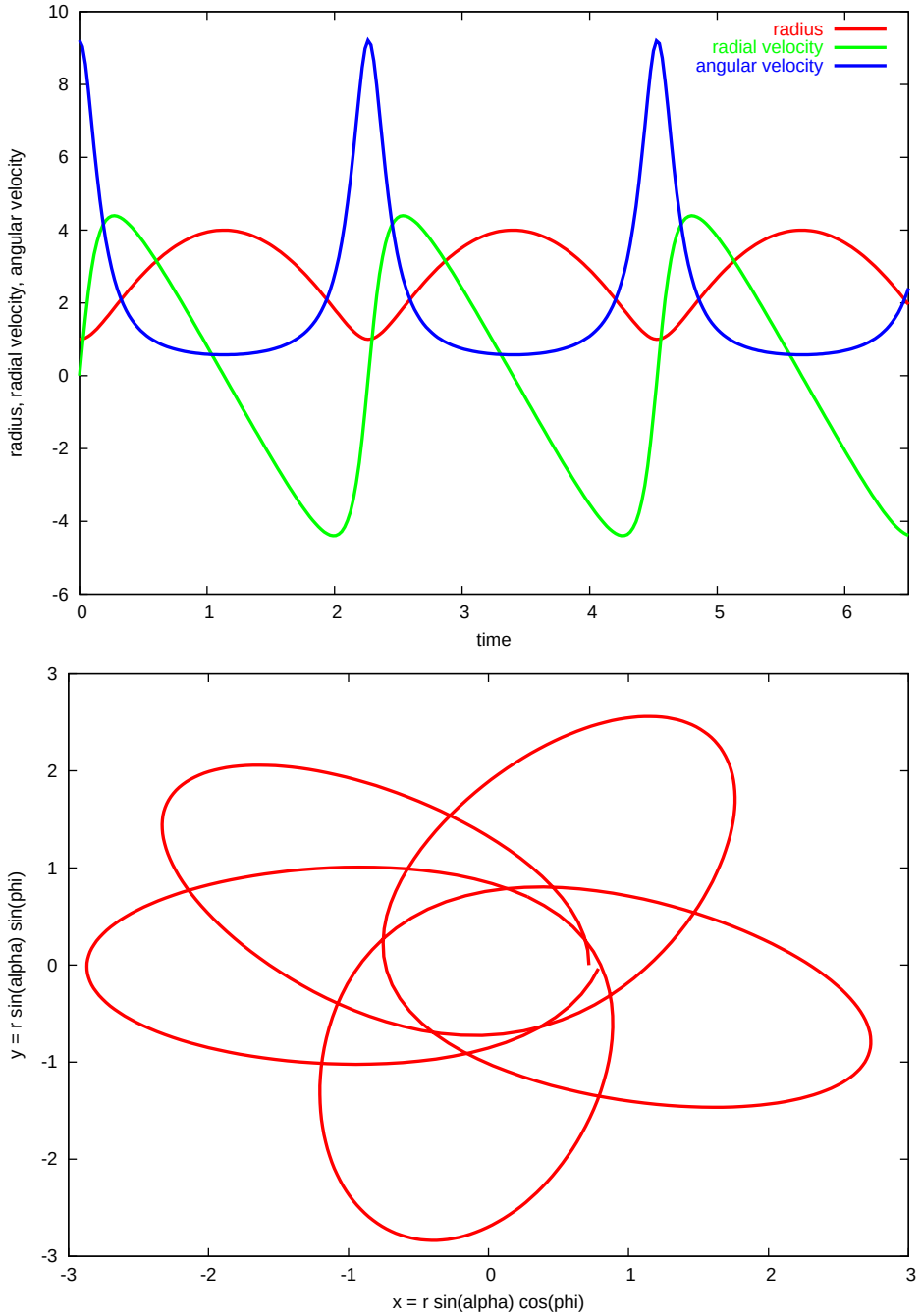


Figure 2.9: Numerical integration of the equations of motion for a particle moving on the surface of a cone. To produce these results we have chosen $\alpha = 0.8$, $r_- = 1$, and $r_+ = 4$. The upper panel shows the raw results for $r(t)$, $\dot{r}(t)$, and $\dot{\phi}(t)$. Notice that the radial velocity is zero whenever $r = r_\pm$ and that it oscillates between positive and negative values. Notice also that $\dot{\phi}$ is always positive; it is maximum whenever $r = r_-$ and minimum whenever $r = r_+$. The lower panel shows the projection of the particle's motion in the x-y plane. To obtain this we let $x(t) = r(t)\sin\alpha\cos\phi(t)$ and $y(t) = r(t)\sin\alpha\sin\phi(t)$. The motion proceeds counterclockwise and the figure is that of a regressing ellipse.

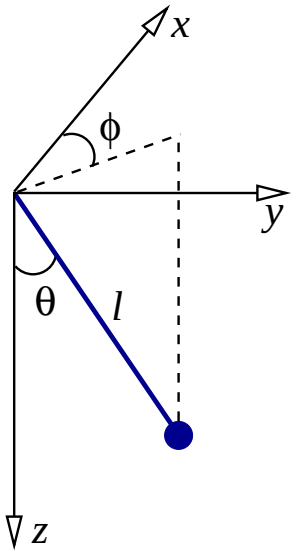


Figure 2.10: The motion of a spherical pendulum is described in terms of the angles $\theta(t)$ and $\phi(t)$.

Exercise 2.9. Verify the quoted relation between h^2 and $r_\pm$.

2.4.4 Spherical pendulum

We now examine the situation of a pendulum which is free to move in all directions about its pivot point. The pendulum has a mass m, a constant length ℓ, and its motion is described in terms of the two angles $\theta(t)$ and $\phi(t)$, as shown in Fig. 2.10. These coordinates are related to the standard Cartesian coordinates by

$$x = \ell \sin\theta \cos\phi, \qquad y = \ell \sin\theta \sin\phi, \qquad z = \ell \cos\theta.$$

As shown in the figure, the z axis is pointing down, in the direction of the gravitational acceleration $\boldsymbol{g}$. It is clear that we are once more dealing with spherical coordinates. This time, however, it is the radial coordinate r that is held fixed to the value ℓ. According to the results of Sec. 2.4.2 the pendulum's kinetic energy is $T = \frac{1}{2}m\ell^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$. Its potential energy is $V = -mgz = -mg\ell\cos\theta = -m\ell^2\omega^2\cos\theta$, where we have re-introduced the quantity

$$\omega = \sqrt{g/\ell}. \tag{2.4.18}$$

The pendulum's Lagrangian is

$$L(\theta,\dot\theta;\phi,\dot\phi) = \frac{1}{2}m\ell^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + m\ell^2\omega^2\cos\theta. \tag{2.4.19}$$

The equations of motion for $\theta(t)$ and $\phi(t)$ are obtained by substituting this Lagrangian into the EL equations.

We compute

$$\frac{\partial L}{\partial\dot\theta} = m\ell^2\dot\theta,$$

which implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m\ell^2\ddot\theta.$$

We also have

$$\frac{\partial L}{\partial \theta} = m\ell^2 \sin\theta\cos\theta\,\dot\phi^2 - m\ell^2\omega^2\sin\theta,$$

and the EL equation for θ is

$$\ddot\theta - \sin\theta\cos\theta\,\dot\phi^2 + \omega^2\sin\theta = 0. \tag{2.4.20}$$

Moving on, we observe that L is independent of ϕ, and the fact that $\partial L/\partial\phi = 0$ means that the EL equation for ϕ reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = 0.$$

This implies that the quantity $\partial L/\partial\dot\phi$ is a constant, which we shall call $m\ell^2 h$. Calculating the partial derivative gives $\partial L/\partial\dot\phi = m\ell^2\sin^2\theta\,\dot\phi$, and we finally obtain the statement

$$\sin^2\theta\,\dot\phi = h = \text{constant}. \tag{2.4.21}$$

The quantity h is once more interpreted as the z component of the pendulum's reduced angular momentum vector, and it is a constant of the motion. In the special case $h = 0$ the pendulum is prevented to move in the ϕ direction, and Eq. (2.4.20) for θ reduces to $\ddot\theta + \omega^2\sin\theta = 0$; this is the same equation that was first derived in Sec. 1.3.7, and then again in Sec. 2.1, and which describes the motion of a planar pendulum. In the general case ($h \neq 0$) we see that $\dot\phi$ is always of the same sign, so that $\phi(t)$ is a monotonic function of time; this means that the pendulum rotates in a consistent direction around the z axis.

With the substitution $\dot\phi = h/\sin^2\theta$ Eq. (2.4.20) becomes

$$\ddot\theta - \frac{h^2\cos\theta}{\sin^3\theta} + \omega^2\sin\theta = 0.$$

Multiplying each term by $\dot\theta$ allows us to integrate this equation. The result is the conservation statement

$$\frac{1}{2}\dot\theta^2 + \nu(\theta) = \varepsilon = \text{constant}, \tag{2.4.22}$$

where ε is the pendulum's reduced total mechanical energy, and

$$\nu(\theta) = \frac{h^2}{2\sin^2\theta} - \omega^2\cos\theta \tag{2.4.23}$$

is an effective potential for the motion in the θ direction. Equations (2.4.22) and (2.4.23) give rise to the energy diagram of Fig. 2.11. From this diagram we may immediately conclude that the motion takes place between two turning points at $\theta = \theta_\pm$; these are determined by the condition $\nu(\theta_\pm) = \varepsilon$.

Exercise 2.10. Verify that Eqs. (2.4.22) and (2.4.23) do indeed follow from the equations of motion.

In Fig. 2.12 we present the results of a numerical integration of the equations of motion, which we recast into the first-order form

$$\dot\theta = v, \qquad \dot v = \frac{h^2\cos\theta}{\sin^3\theta} - \omega^2\sin\theta, \qquad \dot\phi = \frac{h}{\sin^2\theta}.$$

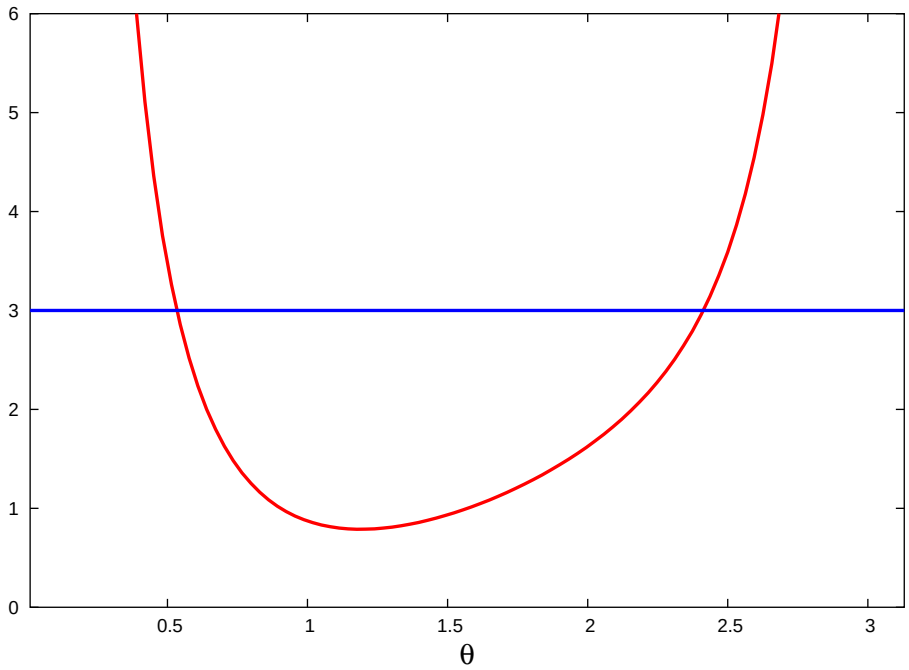


Figure 2.11: Energy diagram for the spherical pendulum. The motion always takes place between two turning points at $\theta = \theta_\pm$.

We start the integration at $\theta = \theta_-$, setting $v = 0$ and $\phi = 0$. The constant h can be determined in terms of θ_- and θ_+ by using the relation $\nu(\theta_-) = \nu(\theta_+)$, which follows from Eq. (2.4.22). The result, after some algebra, is

$$h^2 = 2\omega^2 \frac{(\sin\theta_+ \sin\theta_-)^2(\cos\theta_- - \cos\theta_+)}{(\sin\theta_+ - \sin\theta_-)(\sin\theta_+ + \sin\theta_-)}.$$

Exercise 2.11. Verify the quoted relation between h^2 and $\theta_\pm$.

2.4.5 Rotating pendulum

Another variation on the pendulum theme has the pivot point of a planar pendulum forced to rotate with a constant angular velocity Ω on a circle of radius a. This situation is shown in Fig. 2.13. Once more we describe the motion of the pendulum in terms of the swing angle $\theta(t)$, which is defined relative to the vertical direction; this we now associate with the y-direction.

The Cartesian coordinates of the pendulum, relative to the pivot point, are

$$x_{\text{relative}} = \ell \sin\theta, \qquad y_{\text{relative}} = -\ell \cos\theta.$$

The Cartesian coordinates of the pivot point are

$$x_{\text{pivot}} = a \cos\Omega t, \qquad y_{\text{pivot}} = a \sin\Omega t.$$

The Cartesian coordinates of the pendulum, relative to the origin of the coordinate system, are therefore

$$x = a\cos\Omega t + \ell \sin\theta, \qquad y = a \sin\Omega t - \ell\cos\theta. \tag{2.4.24}$$

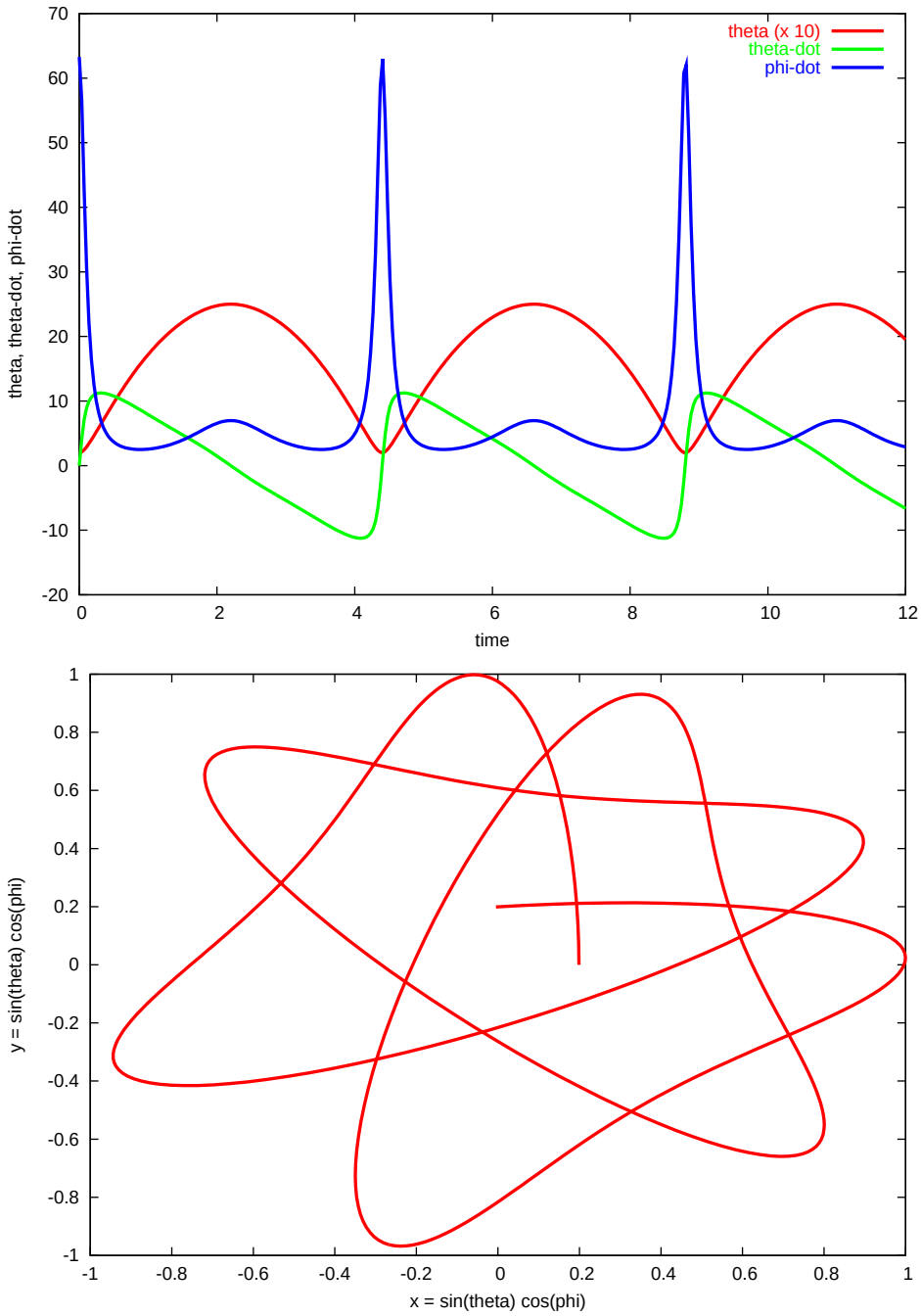


Figure 2.12: Numerical integration of the equations of motion for a spherical pendulum. To produce these results we have chosen $\theta_- = 0.2$, $\theta_+ = 2.5$, and set $\omega = 2\pi$. The upper panel shows the raw results for $\theta(t)$, $\dot{\theta}(t)$, and $\dot{\phi}(t)$. Notice that $\dot{\theta}$ is zero whenever $\theta = \theta_\pm$ and that it oscillates between positive and negative values. Notice also that $\dot{\phi}$ is always positive; it reaches a local maximum whenever $\theta = \theta_\pm$. The lower panel shows the projection of the pendulum's motion in the x-y plane. To obtain this we let $x(t) = \sin\theta(t)\cos\phi(t)$ and $y(t) = \sin\theta(t)\sin\phi(t)$. The motion proceeds counterclockwise.

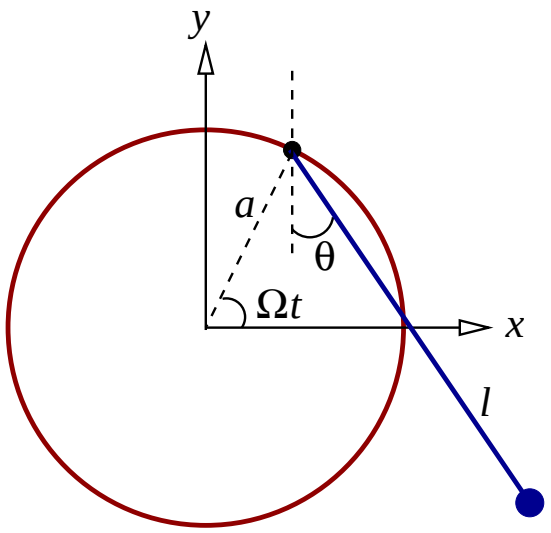


Figure 2.13: The motion of a rotating pendulum is described in terms of the swing angle $\theta(t)$. The pivot point rotates with a constant angular velocity Ω on a circle of radius a.

The components of the velocity vector are

$$\dot{x} = -a\Omega \sin \Omega t + \ell\dot{\theta}\cos\theta, \qquad \dot{y} = a\Omega\cos\Omega t + \ell\dot{\theta}\sin\theta.$$

The squared velocity is then calculated as

$$v^2 = \dot{x}^2 + \dot{y}^2 = (a\Omega)^2 + 2a\ell\Omega\dot{\theta}\sin(\theta - \Omega t) + \ell^2\dot{\theta}^2,$$

and the kinetic energy of the pendulum is $T = \frac{1}{2}mv^2$. Its potential energy is $V = mgy = mg(a\sin\Omega t - \ell\cos\theta)$.

Exercise 2.12. Verify the preceding result for v^2.

The Lagrangian of the rotating pendulum is, finally,

$$L(\theta,\dot{\theta};t) = \frac{1}{2}m\Big[(a\Omega)^2 + 2a\ell\Omega\dot{\theta}\sin(\theta - \Omega t) + \ell^2\dot{\theta}^2\Big] - m\ell\omega^2(a\sin\Omega t - \ell\cos\theta), \quad (2.4.25)$$

where we have once more introduced $\omega^2 = g/\ell$. A new feature of this Lagrangian is that it depends explicitly on time; this comes about because the pendulum is not left alone to its own devices, but is instead acted upon and forced to follow a rotational motion. In this circumstance we cannot expect the energy of the pendulum to be conserved: There will be at all times a transfer of energy between the pendulum and the external agent that is responsible for the rotational motion. Globally the total energy is conserved, but the energy of the pendulum is not individually conserved.

To obtain the equation for motion we must first calculate

$$\frac{\partial L}{\partial\dot{\theta}} = ma\ell\Omega\sin(\theta - \Omega t) + m\ell^2\dot{\theta}.$$

Differentiating this with respect to time gives

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = ma\ell\Omega\cos(\theta - \Omega t)(\dot{\theta} - \Omega) + m\ell^2\ddot{\theta}.$$

We next compute

$$\frac{\partial L}{\partial \theta} = ma\ell\Omega\dot{\theta}\cos(\theta - \Omega t) - m\ell^2\omega^2 \sin\theta,$$

and substituting all this into the EL equation produces

$$\ddot{\theta} + \omega^2 \sin\theta - (a/\ell)\Omega^2 \cos(\theta - \Omega t) = 0. \tag{2.4.26}$$

This is the equation of motion of a rotating pendulum.

Equation (2.4.26) cannot be integrated with the help of the $\dot{\theta}$ trick; this is prevented by the fact that the equation depends explicitly on time through the term in $\cos(\theta - \Omega t)$. As a consequence, the motion cannot be analyzed with the help of an energy diagram; this can be understood from the very fact that the total mechanical energy of the rotating pendulum is not conserved. The only tool that remains at our disposal to analyze the motion is numerical integration, and Fig. 2.14 displays the results.

The graphs reveal that when the pendulum is driven at a frequency Ω that is close to its natural frequency ω, the response is more violent: the amplitude of the oscillations is then much larger. This is the phenomenon of *resonance*. This phenomenon can be illustrated in the context of a simpler model, one which can be solved exactly. We consider a simple harmonic oscillator which is driven by an oscillating external force. The equations of motion for this simplified model is

$$\ddot{\theta} + \omega^2\theta = A\cos\Omega t. \tag{2.4.27}$$

When $\Omega \neq \omega$ a solution to this equation is

$$\theta(t) = \frac{A}{\omega^2 - \Omega^2}\cos\Omega t \qquad (\Omega \neq \omega). \tag{2.4.28}$$

In this situation the pendulum oscillates at the driving frequency Ω, and the oscillations have a constant amplitude. Notice, however, that the amplitude grows as Ω approaches the natural frequency ω. The solution of Eq. (2.4.28) is not valid when $\Omega = \omega$. In this case we have instead

$$\theta(t) = \frac{At}{2\omega}\sin\omega t \qquad (\Omega = \omega). \tag{2.4.29}$$

In this case the oscillations keep growing in amplitude; the simple harmonic oscillator is in resonance with the driving force.

Exercise 2.13. Verify that Eqs. (2.4.28) and (2.4.29) are solutions to Eq. (2.4.27). A more challenging question: What is the *general solution* to Eq. (2.4.27) when $\Omega \neq \omega$ and when $\Omega = \omega$? The general solution should be parameterized in terms of the initial angle $\theta(0)$ and the initial angular velocity $\dot{\theta}(0)$. What choices of initial conditions give rise to Eqs. (2.4.28) and (2.4.29)?

When the rotating pendulum is driven at resonance we observe a growth in the amplitude of oscillations, but this growth is bounded; it saturates and the amplitude then starts to decrease. This saturation is produced by nonlinear effects: When the amplitude grows the natural period of the oscillations changes (as we learned back in Sec. 1.3.7) and the pendulum is no longer driven at its natural frequency. As resonance stops the amplitude starts to decrease and the pendulum's natural frequency returns to its original value. At this stage the conditions are once more suitable for a resonant growth of the amplitude, and the cycle repeats.

For certain choices of parameters the driving force can have a dramatic influence on the pendulum. This is illustrated in Fig. 2.15, for which the driving frequency was set to $\Omega = 0.9\omega$. Here we see the driving force causing the pendulum to go beyond $\theta = \pi$, completing one or two revolutions before returning to a short oscillation cycle.

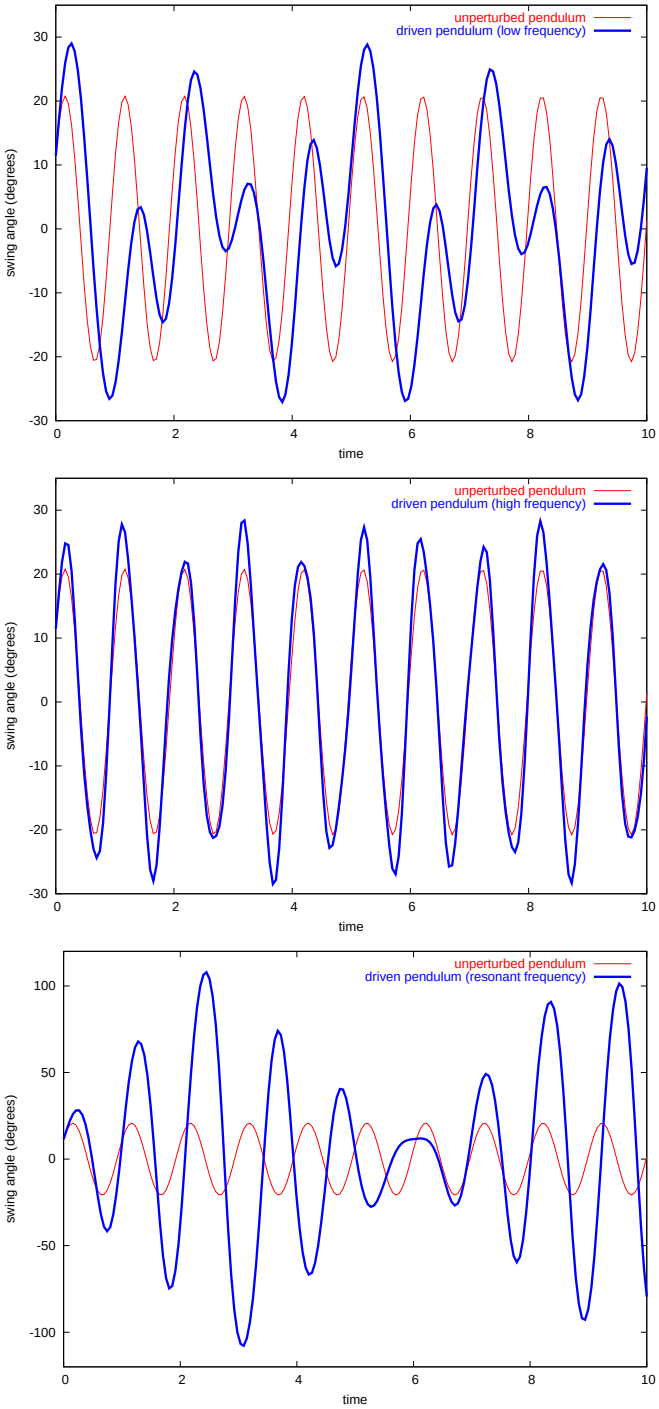


Figure 2.14: The motion of a rotating pendulum. Each graph shows the swing angle $\theta(t)$ of the driven pendulum in blue, and the swing angle of a free pendulum in red. In the first graph the pendulum is driven at a low frequency set at $\Omega/\omega = 0.4$. In the second graph the pendulum is driven at a high frequency set at $\Omega/\omega = 2.4$. In the third graph the pendulum is driven at resonant frequency, so that $\Omega/\omega = 1.0$; notice the large amplitude of oscillations in this case. In all cases we have set $(a/\ell)\Omega^2 = 0.2$, and the initial conditions are $\theta(0) = 0.2$ and $\dot{\theta}(0) = 0.3$.

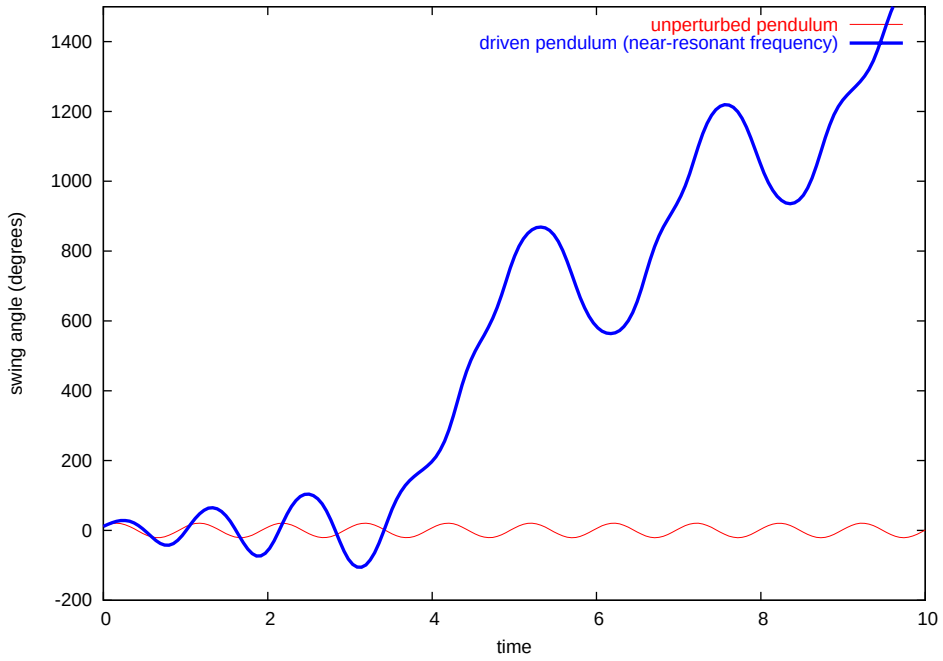


Figure 2.15: The motion of a rotating pendulum, with $\Omega/\omega = 0.9$, $(a/\ell)\Omega^2 = 0.2$, $\theta(0) = 0.2$, and $\dot{\theta}(0) = 0.3$.

2.4.6 Rolling disk

As our next application we consider a disk of mass m and radius R that rolls without slipping on an inclined plane of total length ℓ; the plane's inclination relative to the horizontal is α. As shown in Fig. 2.16, the distance from the top position on the plane to the disk's centre of mass — its geometric centre — is denoted s, and θ is the angle of a selected point on the disk's rim relative to an axis perpendicular to the inclined plane.

There is both a translational motion of the centre of mass and a rotational motion of the disk in this problem. The disk's kinetic energy is

$$T = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2,$$

where $I = \frac{1}{2}mR^2$ is the disk's moment of inertia. The coordinates s and θ, however, are not independent; they are related by the no-slip condition, which implies $s = R\theta$. So we have $\dot{s} = R\dot{\theta}$ and the kinetic energy becomes

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{4}mR^2\dot{\theta}^2 = \frac{3}{4}mR^2\dot{\theta}^2.$$

The disk's potential energy is $V = mgz = mg(l - s)\sin\alpha = mg(\ell - R\theta)\sin\alpha$.

The Lagrangian is therefore

$$L(\theta, \dot{\theta}) = \frac{3}{4}mR^2\dot{\theta}^2 - mg(\ell - R\theta)\sin\alpha. \tag{2.4.30}$$

To obtain the disk's equation of motion we substitute this into the EL equation. We first compute

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}mR^2\dot{\theta},$$

which implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}mR^2\ddot{\theta}.$$

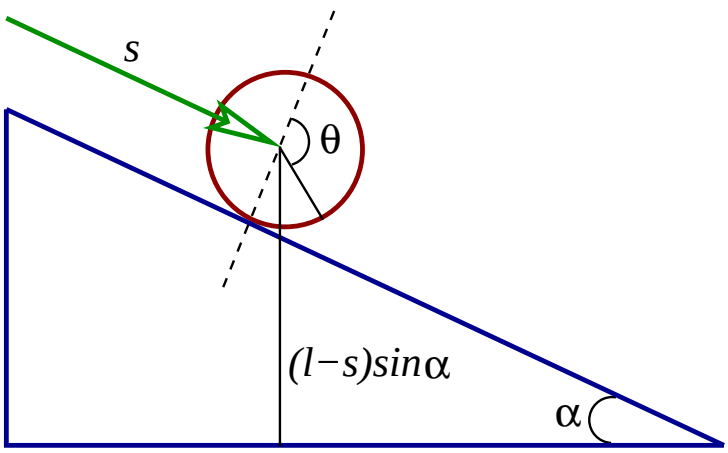


Figure 2.16: A disk rolling without slipping on an inclined plane. The plane has a length ℓ and its inclination angle is α. The distance from the disk's centre to the top of the plane is s; the height of the disk's centre is $(\ell - s)\sin\alpha$.

We also compute

$$\frac{\partial L}{\partial \theta} = mgR\sin\alpha.$$

The equation of motion is then

$$\ddot{\theta} = \frac{2g\sin\alpha}{3R}, \tag{2.4.31}$$

and we find that the disk is under constant angular acceleration.

If we assume that the disk started with zero angular velocity, then Eq. (2.4.31) integrates to

$$\theta(t) = \frac{g\sin\alpha}{3R}t^2. \tag{2.4.32}$$

The time $t_{\rm bottom}$ required for the disk to reach the bottom of the inclined plane is determined by the condition $\theta(t_{\rm bottom}) = \ell/R$. Solving this gives

$$t_{\rm bottom} = \sqrt{\frac{3\ell}{g\sin\alpha}}. \tag{2.4.33}$$

Notice that $t_{\rm bottom}$ is independent of R, the disk's radius.

2.4.7 Kepler's problem revisited

As a final application of the Lagrangian formalism we will rederive the main equations of Kepler's problem. As we shall see, the Lagrangian methods give a much more efficient way of obtaining these equations.

As in Sec. 1.5.2 we express the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of the two massive bodies in terms of the relative separation vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and the position $\boldsymbol{R}$ of the centre of mass, which is determined by $M\boldsymbol{R} = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2$, where $M = m_1 + m_2$ is the total mass. We have $\boldsymbol{r}_1 = \boldsymbol{R} + (m_2/M)\boldsymbol{r}$, $\boldsymbol{r}_2 = \boldsymbol{R} - (m_1/M)\boldsymbol{r}$, and after some algebra we find that the system's kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{\boldsymbol{r}}_1\cdot\dot{\boldsymbol{r}}_1 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2\cdot\dot{\boldsymbol{r}}_2 \\ &= \frac{1}{2}M\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}} + \frac{1}{2}\frac{m_1m_2}{M}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}. \end{aligned}$$

The system's potential energy, on the other hand, was calculated in Sec. 1.5.1, and it is given by

$$V = -\frac{Gm_1m_2}{r},$$

where $r = |\boldsymbol{r}| = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the distance between the two bodies. The system's Lagrangian is

$$L(\boldsymbol{R}, \dot{\boldsymbol{R}}; \boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{1}{2}M\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}} + \frac{1}{2}\mu\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{G\mu M}{r}, \tag{2.4.34}$$

where

$$\mu = \frac{m_1m_2}{M}, \qquad M = m_1 + m_2 \tag{2.4.35}$$

is known as the *reduced mass* of the two-body system.

Exercise 2.14. Go through the algebra that leads to our previous expression for the kinetic energy in terms of $\dot{\boldsymbol{R}}$ and $\dot{\boldsymbol{r}}$.

Notice that the Lagrangian of Eq. (2.4.34) separates into two independent pieces. The first piece depends on $\boldsymbol{R}$ only, and is independent of $\boldsymbol{r}$; this is the Lagrangian of the centre of mass,

$$L_{\rm CM}(\boldsymbol{R}, \dot{\boldsymbol{R}}) = \frac{1}{2}M\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}}.$$

The second piece depends on $\boldsymbol{r}$ only, and is independent of $\boldsymbol{R}$; this is the Lagrangian of the relative separation between the two bodies,

$$L_{\rm rel}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{1}{2}\mu\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{G\mu M}{r}.$$

Notice now that $L_{\rm CM}$ contains only a kinetic-energy term. The absence of a potential-energy term implies that the motion of the centre of mass is free. As a quick calculation will verify, the EL equations for $\boldsymbol{R}$ take the form $\ddot{\boldsymbol{R}} = \boldsymbol{0}$, for which the solution is $\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)t$. As we have seen in Sec. 1.5.2, the centre of mass moves freely, and it can be placed at the origin of an inertial frame. The relative Lagrangian, on the other hand, contains both a kinetic-energy term and a potential-energy term. It describes the motion of a fictitious particle of mass μ in the gravitational field of a central mass M, also fictitious. As we have witnessed before in Sec. 1.5.2, our original two-body problem has simplified into an effective one-body problem.

To proceed we may switch from the Cartesian coordinates $\boldsymbol{r} = (x, y, z)$ to any system of generalized coordinates q_a. Recalling from Sec. 1.5.3 that the motion takes place in the x-y plane (a fact that could be re-derived on the basis of Lagrangian mechanics), we adopt the polar coordinates (r, ϕ), related to x and y by $x = r\cos\phi$ and $y = r\sin\phi$. We have $\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$, $\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$, and it follows that

$$\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + (r\dot{\phi})^2.$$

The Lagrangian therefore becomes

$$L_{\rm rel}(r, \dot{r}; \phi, \dot{\phi}) = \frac{1}{2}\mu\Big[\dot{r}^2 + (r\dot{\phi})^2\Big] + \frac{G\mu M}{r}. \tag{2.4.36}$$

Notice that this Lagrangian is actually independent of ϕ, a feature that was encountered also in previous examples.

To obtain the equation of motion for r we compute

$$\frac{\partial L_{\rm rel}}{\partial \dot{r}} = \mu\dot{r} \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial L_{\rm rel}}{\partial \dot{r}} = \mu\ddot{r}$$

and

$$\frac{\partial L_{\rm rel}}{\partial r} = \mu r\dot{\phi}^2 - \frac{G\mu M}{r^2}.$$

This gives

$$\ddot{r} - r\dot{\phi}^2 + \frac{GM}{r^2} = 0. \tag{2.4.37}$$

This is the same statement as Eq. (1.5.25). To obtain the equation of motion for ϕ we observe that since $L_{\rm rel}$ is independent of ϕ, we must have that $\partial L_{\rm rel}/\partial\dot{\phi}$ is a constant of the motion. Calling this constant μh and calculating the partial derivative, we get

$$r^2\dot{\phi} = h = \text{constant}. \tag{2.4.38}$$

This is the same statement as Eq. (1.5.27), and h is identified as the reduced angular momentum of the two-body system.

The equations of motion (2.4.37) and (2.4.38) can be analyzed with the same mathematical techniques as those employed in Sec. 1.5. It should be clear that compared with the Newtonian methods of Chapter 1, the Lagrangian methods provide a much simpler way of obtaining these equations.

2.5 Generalized momenta and conservation statements

2.5.1 Conservation of generalized momentum

In the applications of Lagrangian mechanics presented in Sec. 2.4 it occurred a number of times that the Lagrangian was independent of one of the generalized coordinates (mostly it was the ϕ coordinate), and we saw that this fact always translated into the existence of a constant of the motion (which we usually called h). A specific example is the case of a particle moving on the surface of a cone (Sec. 2.4.3), for which the Lagrangian is indeed independent of ϕ and for which the constant of the motion was $h = r^2\sin^2\alpha\,\dot{\phi}$. A similar situation occurred for the spherical pendulum (Sec. 2.4.4) and for Kepler's problem (Sec. 2.4.7).

It is easy to generalize this discussion and to derive the very useful fact that whenever the Lagrangian does not depend explicitly on one (or more) of the generalized coordinates q_a, there exists a corresponding constant of the motion. To establish this statement we shall first introduce the notion of a *generalized momentum*.

Consider a Lagrangian $L(q_a, \dot{q}_a)$ that depends on a number of generalized coordinates q_a and a number of generalized velocities $\dot{q}_a$. The quantities

$$p_a = \frac{\partial L}{\partial \dot{q}_a} \tag{2.5.1}$$

feature prominently in the EL equations, which can be written in the form

$$\dot{p}_a = \frac{\partial L}{\partial q_a}. \tag{2.5.2}$$

The quantities p_a are the generalized momenta of the mechanical system. There is one generalized momentum p_a for each generalized coordinate q_a.

The generalized momenta can represent either a component of the linear-momentum vector or a component of the angular-momentum vector. Generally speaking, whenever q_a represents a linear variable the corresponding p_a will be a linear momentum; and whenever q_a represents an angular variable its corresponding p_a will be

an angular momentum. Consider, for example, the Lagrangian of a free particle in cylindrical coordinates (Sec. 2.4.1). This is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2).$$

The generalized momenta are

$$\begin{aligned} p_\rho &= \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \end{aligned}$$

In the case of p_ρ and p_z we clearly have quantities that represent components of a linear-momentum vector. But the case of p_ϕ is different. Here we have $p_\phi = m(\rho)(\rho\dot{\phi})$, and this clearly represents the component of an angular-momentum vector.

Exercise 2.15. Show that $p_\rho = \boldsymbol{p}\cdot\hat{\boldsymbol{\rho}}$ and $p_z = \boldsymbol{p}\cdot\hat{\boldsymbol{z}}$, where $\boldsymbol{p}$ is the particle's momentum vector. Show, on the other hand, that $p_\phi = \boldsymbol{L}\cdot\hat{\boldsymbol{z}}$, where $\boldsymbol{L}$ is the particle's angular-momentum vector.

Suppose now that a Lagrangian $L(q_1, \dot{q}_1; q_2, \dot{q}_2, \cdots)$ happens not to depend explicitly on one of its generalized coordinates, say q_*. Then

$$\frac{\partial L}{\partial q_*} = 0$$

and it follows from the EL equation for q_* that

$$\frac{dp_*}{dt} = 0,$$

where $p_* = \partial L/\partial\dot{q}_*$ is the generalized momentum associated with the coordinate q_*. This equation states that p_* is a constant of the motion, and we have established the following theorem:

> Whenever the Lagrangian of a mechanical system does not depend explicitly on a generalized coordinate q_*, the corresponding generalized momentum $p_* = \partial L/\partial\dot{q}_*$ is a constant of the motion.

A coordinate q_* that does not appear in L is sometimes called a *cyclic coordinate*. A Lagrangian may contain any number of cyclic coordinates.

As an example consider the following Lagrangian, again in cylindrical coordinates,

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho).$$

Here it is assumed that the potential energy V depends only on ρ; the mechanical system is cylindrically symmetric. This implies that ϕ and z are cyclic coordinates, and that $p_\phi = m\rho^2\dot{\phi}$ and $p_z = m\dot{z}$ are constants of the motion.

This theorem on cyclic coordinates and conserved quantities is extremely important and very useful. To find all the constants of the motion is usually a key step during the integration of the equations of motion, and the theorem provides a very efficient algorithm to identify at least some of them.

2.5.2 Conservation of energy

Conservation of total mechanical energy E is also an important aspect of the motion of a mechanical system and a key to solving the equations of motion. In this subsection we show that energy is conserved whenever the Lagrangian does not depend explicitly on time t.

To begin the discussion let us consider a Lagrangian $L(q_a, \dot{q}_a, t)$ that depends on a number of generalized coordinates q_a, a number of generalized velocities $\dot{q}_a$, and let us consider the possibility that it depends also explicitly on time. (An example is the Lagrangian of a rotating pendulum, which was written down in Sec. 2.4.5.) Applying the chain rule, we find that the total time derivative of the Lagrangian is given by

$$\frac{dL}{dt} = \sum_a \frac{\partial L}{\partial q_a}\,\dot{q}_a + \sum_a \frac{\partial L}{\partial \dot{q}_a}\,\ddot{q}_a + \frac{\partial L}{\partial t}.$$

The first term accounts for the time dependence contained in each $q_a(t)$, the second term for the time dependence contained in each $\dot{q}_a(t)$, and the third term accounts for the explicit dependence of the Lagrangian on t.

We have defined the generalized momenta p_a by

$$p_a = \frac{\partial L}{\partial \dot{q}_a}$$

and the EL equations can be expressed in the form

$$\dot{p}_a = \frac{\partial L}{\partial q_a}.$$

We make these substitutions in the previous equation, and obtain

$$\frac{dL}{dt} = \sum_a \Big(\dot{p}_a \dot{q}_a + p_a \ddot{q}_a\Big) + \frac{\partial L}{\partial t},$$

or

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_a p_a \dot{q}_a\right) + \frac{\partial L}{\partial t},$$

which is equivalent to the previous form by virtue of the chain rule.

We have obtained the equation

$$\frac{d}{dt}\left(\sum_a p_a \dot{q}_a - L\right) = -\frac{\partial L}{\partial t}, \tag{2.5.3}$$

and a statement of conservation follows immediately:

> Whenever L does not depend explicitly on time, so that $\partial L/\partial t = 0$, we have that
>
> $$h(q_a, \dot{q}_a) \equiv \sum_a p_a \dot{q}_a - L \tag{2.5.4}$$
>
> is a constant of the motion, $dh/dt = 0$.

Surely the function $h(q_a, \dot{q}_a)$ must have something to do with the system's total mechanical energy. Let us first figure out the relationship in the context of a simple example. We go back to the Lagrangian of a particle expressed in cylindrical coordinates,

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z),$$

but this time we place no constraints on the potential energy. The generalized momenta are $p_\rho = m\dot{\rho}$, $p_\phi = m\rho^2\dot{\phi}$, and $p_z = m\dot{z}$. We then have

$$\begin{aligned} h &= p_\rho\dot{\rho} + p_\phi\dot{\phi} + p_z\dot{z} - L \\ &= m\dot{\rho}^2 + m\rho^2\dot{\phi}^2 + m\dot{z}^2 - \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + V(\rho,\phi,z) \\ &= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + V(\rho,\phi,z). \end{aligned}$$

This is indeed the total mechanical energy, the sum of kinetic and potential energies.

To verify that $h(q_a, \dot{q}_a)$ is always equal to the total mechanical energy we use the fact that the kinetic energy is usually a quadratic function of the generalized velocities,

$$T = \frac{1}{2}\sum_{a,b} A_{ab}\dot{q}_a\dot{q}_b.$$

The coefficients A_{ab} may in general depend on the coordinates q_a, and without loss of generality we may assume that $A_{ba} = A_{ab}$. The Lagrangian is then

$$L = \frac{1}{2}\sum_{a,b} A_{ab}\dot{q}_a\dot{q}_b - V(q_a).$$

The generalized momentum p_a is obtained by differentiating L with respect to $\dot{q}_a$. To see what this amounts to let us consider a special case in which the mechanical system possesses three degrees of freedom. In this case we have, explicitly,

$$L = \frac{1}{2}A_{11}\dot{q}_1^2 + A_{12}\dot{q}_1\dot{q}_2 + A_{13}\dot{q}_1\dot{q}_3 + \frac{1}{2}A_{22}\dot{q}_2^2 + A_{23}\dot{q}_2\dot{q}_3 + \frac{1}{2}A_{33}\dot{q}_3^2 - V(q_1,q_2,q_3).$$

It follows that

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial\dot{q}_1} = A_{11}\dot{q}_1 + A_{12}\dot{q}_2 + A_{13}\dot{q}_3, \\ p_2 &= \frac{\partial L}{\partial\dot{q}_2} = A_{12}\dot{q}_1 + A_{22}\dot{q}_2 + A_{23}\dot{q}_3, \\ p_3 &= \frac{\partial L}{\partial\dot{q}_3} = A_{13}\dot{q}_1 + A_{23}\dot{q}_2 + A_{33}\dot{q}_3 \end{aligned}$$

are the generalized momenta. These relations are summarized by

$$p_a = \sum_b A_{ab}\dot{q}_b,$$

and the same expression is always obtained, regardless of the number of degrees of freedom. The function h is then

$$\begin{aligned} h &= \sum_a p_a\dot{q}_a - L \\ &= \sum_a\Big(\sum_b A_{ab}\dot{q}_b\Big)\dot{q}_a - \frac{1}{2}\sum_{a,b} A_{ab}\dot{q}_a\dot{q}_b + V(q_a) \\ &= \frac{1}{2}\sum_{a,b} A_{ab}\dot{q}_a\dot{q}_b + V(q_a), \end{aligned}$$

and we conclude that

$$h(q_a,\dot{q}_a) = T(q_a,\dot{q}_a) + V(q_a) = \text{total mechanical energy.} \tag{2.5.5}$$

In all generality, therefore, the function h is the system's total energy, and this is conserved whenever L does not depend explicitly on time.

2.5.3 Invariance of the EL equations under a change of Lagrangian

Suppose that we are given a Lagrangian $L(q_a, \dot{q}_a, t)$ and that we decide to define a second Lagrangian $L'(q_a, \dot{q}_a, t)$ by adding to the first Lagrangian a term of the form df/dt, where $f(q_a, t)$ is any function of the generalized coordinates q_a and of time t. What we have then is the operation

$$L \to L' = L + \frac{d}{dt} f(q_a, t). \tag{2.5.6}$$

Notice that f is quite arbitrary, but that it is not allowed to depend on the generalized velocities $\dot{q}_a$.

We assert that the equations of motion derived from L and L' will be identical. The Lagrangians L and L' are therefore *equivalent*, in the sense that they produce the same set of EL equations. In practice this formal property of Lagrangians can be useful: A complicated Lagrangian L' can be turned into a simpler Lagrangian L by removing a superfluous total time derivative. We will use this method of simplification in a later section.

Exercise 2.16. Read through Sec. 2.4 again and figure out where a Lagrangian could have been simplified using this method.

To prove our assertion we show that the change in Lagrangian,

$$\Delta L = \frac{df}{dt} = \sum_b \frac{\partial f}{\partial q_b} \dot{q}_b + \frac{\partial f}{\partial t},$$

produces no change in the equations of motion. The EL equations derived from L' are

$$0 = \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_a} - \frac{\partial L'}{\partial q_a}.$$

Writing $L' = L + \Delta L$, this becomes

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} + \frac{d}{dt} \frac{\partial \Delta L}{\partial \dot{q}_a} - \frac{\partial \Delta L}{\partial q_a}.$$

These will be identical to the EL equations derived from L if and only if

$$\frac{d}{dt} \frac{\partial \Delta L}{\partial \dot{q}_a} - \frac{\partial \Delta L}{\partial q_a} = 0.$$

Let us verify that this equation is always satisfied.

Because f does not depend on $\dot{q}_a$, we have that

$$\begin{aligned} \frac{\partial \Delta L}{\partial \dot{q}_a} &= \frac{\partial}{\partial \dot{q}_a} \left(\sum_b \frac{\partial f}{\partial q_b} \dot{q}_b + \frac{\partial f}{\partial t} \right) \\ &= \sum_b \frac{\partial f}{\partial q_b} \frac{\partial \dot{q}_b}{\partial \dot{q}_a} \\ &= \frac{\partial f}{\partial q_a}, \end{aligned}$$

because $\partial \dot{q}_b / \partial \dot{q}_a$ is 1 when $b = a$ and 0 otherwise. For example, $\dot{q}_1$ depends only on $\dot{q}_1$ and on no other variable, so that $\partial \dot{q}_1 / \partial \dot{q}_1 = 1$ while $\partial \dot{q}_1 / \partial \dot{q}_2 = \partial \dot{q}_1 / \partial \dot{q}_3 = \cdots = 0$. From this it follows that

$$\begin{aligned} \frac{d}{dt} \frac{\partial \Delta L}{\partial \dot{q}_a} &= \frac{d}{dt} \frac{\partial f}{\partial q_a} \\ &= \sum_b \frac{\partial^2 f}{\partial q_b \partial q_a} \dot{q}_b + \frac{\partial^2 f}{\partial t \partial q_a}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial \Delta L}{\partial q_a} &= \frac{\partial}{\partial q_a}\Big(\sum_b \frac{\partial f}{\partial q_b}\dot{q}_b + \frac{\partial f}{\partial t}\Big) \\ &= \sum_b \frac{\partial^2 f}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 f}{\partial q_a \partial t},\end{aligned}$$

and this is equal to the previous result, because the order in which one evaluates second partial derivatives does not matter. We conclude that

$$\frac{d}{dt}\frac{\partial \Delta L}{\partial \dot{q}_a} = \frac{\partial \Delta L}{\partial q_a}$$

and that the change of Lagrangian has no effect on the EL equations. The equations of motion that derive from $L' = L + df/dt$ are indeed identical to the equations that derive from L.

There is a more elegant way to prove this result. In this alternative derivation we appeal directly to Hamilton's principle. The action $S' = \int_{t_0}^{t_1} L'\,dt$ associated with L', and the action $S = \int_{t_0}^{t_1} L\,dt$ associated with L, are related by

$$\begin{aligned}S' &= S + \int_{t_0}^{t_1} \Delta L\,dt \\ &= S + \int_{t_0}^{t_1} \frac{df}{dt}\,dt \\ &= S + f\big(q_a(t_1), t_1\big) - f\big(q_a(t_0), t_0\big).\end{aligned}$$

The equations of motion are obtained from S' or S by varying the paths $q_a(t)$ and demanding that the variation of the action be zero to first order in the variations $\delta q_a(t)$. The variations, you may recall, must be subjected to the boundary conditions $\delta q_a(t_0) = \delta q_a(t_1) = 0$; the paths must all begin at the same $q_a(t_0)$ and end at the same $q_a(t_1)$. But under these conditions we find that the values $f(q_a(t_0), t_0)$ and $f(q_a(t_1), t_1)$ can never change under a variation of the paths, and we must conclude that

$$\delta S' = \delta S.$$

An extremum of S will also be an extremum of S', and the equations of motion derived from L and L' are guaranteed to be the same.

While the operation $L \to L' = L + df/dt$ does not affect the equations of motion, it may nevertheless change the expressions for the generalized momenta p_a and the total energy h. The new momenta p'_a are given by

$$p'_a = \frac{\partial L'}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{q}_a} + \frac{\partial \Delta L}{\partial \dot{q}_a},$$

or

$$p'_a = p_a + \frac{\partial f}{\partial q_a}, \tag{2.5.7}$$

according to our previous computations. The new energy function h' is given by $h' = \sum_a p'_a \dot{q}_a - L'$, so

$$\begin{aligned}h' &= \sum_a \Big(p_a + \frac{\partial f}{\partial q_a}\Big)\dot{q}_a - L - \Delta L \\ &= h + \sum_a \frac{\partial f}{\partial q_a}\dot{q}_a - \sum_b \frac{\partial f}{\partial q_b}\dot{q}_b - \frac{\partial f}{\partial t}.\end{aligned}$$

The sums cancel each other out, and we are left with

$$h' = h - \frac{\partial f}{\partial t}. \tag{2.5.8}$$

We find that the expression for the energy is affected only when f depends explicitly on time.

2.6 Charged particle in an electromagnetic field

The Lagrangian formulation of mechanics is well suited to mechanical systems for which the forces can all be derived from a potential-energy function $V(q_a)$; these forces will depend on the positions q_a, but that they might also depend on the velocities $\dot{q}_a$ is normally out of the question. There is, however, an important mechanical system for which the forces do depend on velocity: a charged particle moving in the presence of an electromagnetic field. In this case the particle is subjected to the Lorentz force, and the equations of motion are

$$m\boldsymbol{a} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{2.6.1}$$

Can this equation be derived on the basis of a Lagrangian?

The answer is in the affirmative. An interesting property of this Lagrangian is that it depends on the scalar potential Φ and vector potential $\boldsymbol{A}$ instead of depending on the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Recall that the fields can be expressed in terms of the potentials as

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{2.6.2}$$

The potentials are usually introduced to simplify the structure of Maxwell's equations. The definition of $\boldsymbol{E}$ implies

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \boldsymbol{A} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi);$$

since the curl of a gradient is always zero, this gives

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

one of the four Maxwell equations. Similarly, the definition of $\boldsymbol{B}$ implies

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A});$$

since the divergence of a curl is always zero, this gives

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

another one of the Maxwell equations. The remaining two equations can then be recast into equations that Φ and $\boldsymbol{A}$ must satisfy.

It is convenient to express the fields in terms of the potentials, but it is important to understand that the potentials do not have direct physical meaning. Indeed, it is even possible to change the potentials by a certain transformation and leave the fields unaffected. This transformation is given by

$$\Phi \to \Phi' = \Phi - \frac{\partial f}{\partial t}, \qquad \boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla} f, \tag{2.6.3}$$

where $f(\boldsymbol{r}, t)$ is an arbitrary function of position and time. Such a transformation of the potentials is known as a *gauge transformation*, and its defining property is

that the transformation leaves the fields invariant. Different sets of potentials that are related by a gauge transformation describe the same fields and therefore the same physical situation.

Exercise 2.17. Show that the transformation of Eq. (2.6.3) leaves the fields unaffected. That is, show that the transformation produces $\boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{E}$ and $\boldsymbol{B} \to \boldsymbol{B}' = \boldsymbol{B}$.

The Lagrangian for a particle of charge q in an electromagnetic field is

$$L = \frac{1}{2}mv^2 - q\Phi + q\boldsymbol{A}\cdot\boldsymbol{v}, \tag{2.6.4}$$

where $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$. As stated previously, this depends on the potentials Φ and $\boldsymbol{A}$ instead of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Another interesting property of the Lagrangian is that the potential-energy term $V = q\Phi - q\boldsymbol{A}\cdot\boldsymbol{v}$ depends on the velocity vector $\boldsymbol{v}$ as well as the position $\boldsymbol{r}$. The dependence on position, of course, comes from the potentials, which may also depend explicitly on time.

Let us verify that the Lagrangian of Eq. (2.6.4) does indeed give rise, via the EL equations, to the Lorentz-force equation of Eq. (2.6.1). It will suffice to verify the x component of the equation, which we write as

$$m\ddot{x} = qE_x + q(\boldsymbol{v}\times\boldsymbol{B})_x = qE_x + q(\dot{y}B_z - \dot{z}B_y).$$

Similar computations would allow us to verify also the y and z components, but we will not present these here.

We begin by presenting the Lagrangian in a more explicit form, as

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\Phi + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z).$$

We have

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x,$$

and this implies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} + \frac{\partial A_x}{\partial t}\right).$$

In this step we took into account the fact that A_x depends on time through its dependence on the coordinates $x(t)$, $y(t)$, and $z(t)$, and also through its own explicit dependence on t; the total time derivative had to be evaluated by using the chain rule. Finally, we have

$$\frac{\partial L}{\partial x} = -q\frac{\partial \Phi}{\partial x} + q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right).$$

The EL equations are

$$0 = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial \Phi}{\partial x}\right) + q\dot{x}\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x}\right) - q\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right),$$

or

$$m\ddot{x} = q\left(-\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x}\right) + q\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - q\dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right).$$

In the first set of brackets we recognize E_x, in the second B_z, and in the third B_y. We therefore have

$$m\ddot{x} = qE_x + q(\dot{y}B_z - \dot{z}B_y) = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_x,$$

and we have recovered the x component of the Lorentz-force equation, as required.

Exercise 2.18. Make sure that you can also recover the y and z components of Eq. (2.6.1).

In the course of this computation we came across the result

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x.$$

The left-hand side, we recall, is what was defined in Sec. 2.5.1 as the generalized momentum p_x associated with the coordinate x. Generalizing, we find that

$$\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A} \tag{2.6.5}$$

is the generalized momentum vector of a charged particle. It contains a direct contribution $m\boldsymbol{v}$ from the particle and an additional contribution $q\boldsymbol{A}$ from the electromagnetic field.

The energy function h of a charged particle is given by the general expression of Eq. (2.5.4), $h = \sum_a p_a \dot{q}_a - L$. We have

$$\begin{aligned} h &= \boldsymbol{p}\cdot\boldsymbol{v} - L \\ &= (m\boldsymbol{v} + q\boldsymbol{A})\cdot\boldsymbol{v} - \frac{1}{2}mv^2 + q\Phi - q\boldsymbol{A}\cdot\boldsymbol{v}, \end{aligned}$$

or

$$h = \frac{1}{2}mv^2 + q\Phi. \tag{2.6.6}$$

It is interesting to see that the terms containing $\boldsymbol{A}$ have canceled each other out; the energy function includes only the scalar potential Φ.

We might ask how L, $\boldsymbol{p}$, and h change under a gauge transformation. This is easily worked out. If we change the potentials from $(\Phi, \boldsymbol{A})$ to $(\Phi', \boldsymbol{A}')$ using Eq. (2.6.3) we find that the Lagrangian becomes

$$\begin{aligned} L' &= \frac{1}{2}mv^2 - q\Phi' + q\boldsymbol{A}'\cdot\boldsymbol{v} \\ &= \frac{1}{2}mv^2 - q\left(\Phi - \frac{\partial f}{\partial t}\right) + q(\boldsymbol{A} + \boldsymbol{\nabla} f)\cdot\boldsymbol{v} \\ &= L + q\left(\frac{\partial f}{\partial t} + \boldsymbol{\nabla} f\cdot\boldsymbol{v}\right). \end{aligned}$$

In other words,

$$L' = L + q\frac{df}{dt}. \tag{2.6.7}$$

Because the two Lagrangians differ by the total time derivative of a function $qf(\boldsymbol{r}, t)$, the equations of motion derived from L' and L will be identical (refer back to Sec. 2.5.3). And because the equations of motion involve the gauge-invariant fields $\boldsymbol{E}$ and $\boldsymbol{B}$, this conclusion should not come as a surprise.

Under a gauge transformation the generalized momentum vector becomes

$$\begin{aligned} \boldsymbol{p}' &= m\boldsymbol{v} + q\boldsymbol{A}' \\ &= m\boldsymbol{v} + q(\boldsymbol{A} + \boldsymbol{\nabla} f), \end{aligned}$$

so that

$$\boldsymbol{p}' = \boldsymbol{p} + q\boldsymbol{\nabla} f; \tag{2.6.8}$$

the momenta are not gauge invariant. The energy function, on the other hand, becomes

$$\begin{aligned} h' &= \frac{1}{2}mv^2 + q\Phi' \\ &= \frac{1}{2}mv^2 + q\left(\Phi - \frac{\partial f}{\partial t}\right), \end{aligned}$$

so that

$$h' = h - q\frac{\partial f}{\partial t}; \tag{2.6.9}$$

the energy function also is not invariant under a gauge transformation.

2.7 Motion in a rotating reference frame

It was mentioned previously that in Lagrangian mechanics, the generalized coordinates q_a are entirely arbitrary, and that in particular they do not have to be attached to an inertial frame. (For example, noninertial coordinates were employed in Sec. 2.4.5.) A consequence of this fact is that the Lagrangian methods can greatly facilitate the description of a mechanical system viewed in a reference frame that is not inertial. In this section we examine the motion of particles as viewed in rotating frames. We shall first consider the simple case of a particle moving on a turntable, and we shall next consider the more interesting case of a reference frame attached to a rotating Earth.

2.7.1 Motion on a turntable

We consider performing mechanical experiments on particles that move on, or above, a turntable that is rotating with a uniform angular velocity Ω. Our instruments are attached to the turntable, and we wish to analyze the motion of the particles as measured in the rotating frame. This frame, of course, is not an inertial frame, but the methods of Lagrangian mechanics can nevertheless be applied.

We denote by S' the original inertial frame, and we let (x', y', z') be its associated system of Cartesian coordinates; the primes indicate that we will not, ultimately, describe the motion of our particles in this coordinate system. We denote by S the rotating frame of the turntable, and its associated system of Cartesian coordinates is (x, y, z). The turntable is placed in the x'-y' plane, and it is rotating around the z' axis, which coincides with the z axis of the rotating frame. As shown in Fig. 2.17, the angle between the x and x' axes is Ωt; this is also the angle between the y and y' axes.

To work out the relationship between the coordinate systems we use, as a tool, the spherical coordinates (r, θ, ϕ) and (r', θ', ϕ') assigned to an arbitrary point P. Because the frames S and S' share the same origin, we have in fact that $r' = r$. And because they share also the same z axis, we also have $\theta' = \theta$. The angles ϕ' and ϕ differ, however, and Fig. 2.17 makes it clear that they are related by $\phi' = \phi + \Omega t$. We have

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

and

$$x' = r\sin\theta\cos\phi', \qquad y' = r\sin\theta\sin\phi', \qquad z' = r\cos\theta.$$

The relationship is obtained by substituting $\phi' = \phi + \Omega t$ into the previous expressions. It is a bit more efficient to first construct the complex combinations

$$x + iy = r\sin\theta(\cos\phi + i\sin\phi) = r\sin\theta\, e^{i\phi}$$

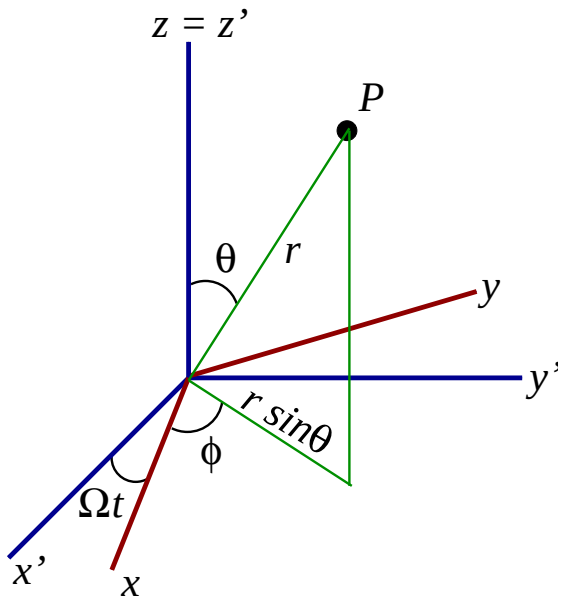


Figure 2.17: The rotating frame S of the turntable, as viewed in the inertial frame S'. A point P is referred to the inertial frame by its Cartesian coordinates (x', y', z') or its spherical coordinates $(r, \theta, \phi' = \phi + \Omega t)$. It is referred to the rotating frame by its Cartesian coordinates (x, y, z) or its spherical coordinates (r, θ, ϕ).

and

$$x' + iy' = r\sin\theta(\cos\phi' + i\sin\phi') = r\sin\theta\, e^{i\phi'}.$$

Then we have

$$x' + iy' = r\sin\theta\, e^{i(\phi+\Omega t)} = e^{i\Omega t} r\sin\theta\, e^{i\phi},$$

or

$$x' + iy' = e^{i\Omega t}(x + iy). \tag{2.7.1}$$

When fully expanded, this is

$$x' = x\cos\Omega t - y\sin\Omega t, \qquad y' = y\cos\Omega t + x\sin\Omega t, \qquad z' = z. \tag{2.7.2}$$

Exercise 2.19. Verify that Eq. (2.7.2) follows from Eq. (2.7.1). Then work out the inverse transformation, $(x', y', z') \to (x, y, z)$.

A particle moving in the rotating frame S with a position vector $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ moves in the inertial frame S' with a position vector $\boldsymbol{r'}(t) = [x'(t), y'(t), z'(t)]$; these are related by the transformation of Eq. (2.7.2). The components of the velocity vectors are then related by

$$\begin{aligned}
\dot{x}' &= \dot{x}\cos\Omega t - \dot{y}\sin\Omega t - \Omega(x\sin\Omega t + y\cos\Omega t),\\
\dot{y}' &= \dot{y}\cos\Omega t + \dot{x}\sin\Omega t - \Omega(y\sin\Omega t - x\cos\Omega t),\\
\dot{z}' &= \dot{z}.
\end{aligned}$$

After a fairly laborious calculation, we find that the squared velocity, as measured in the inertial frame, is

$$\begin{aligned}
v'^2 &= \dot{x}'^2 + \dot{y}'^2 + \dot{y}'^2\\
&= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2\Omega(y\dot{x} - x\dot{y}) + \Omega^2(x^2 + y^2).
\end{aligned} \tag{2.7.3}$$

The particle's kinetic energy is then $T = \frac{1}{2}mv'^2$. It contains a contribution from $\dot{\boldsymbol{r}}$, the particle's velocity vector as measured in the rotating frame, and contributions from the rotational motion of the frame (the terms that involve Ω).

Exercise 2.20. Verify Eq. (2.7.3). You will save yourself some work if you use the trick of forming complex combinations.

The particle's potential energy can be expressed in terms of the inertial coordinates (x', y', z'), but after the transformation of Eq. (2.7.2) it becomes a function of the rotating coordinates (x, y, z). Denoting this function $V(x, y, z)$, we find that the particle's Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\Omega(y\dot{x} - x\dot{y}) + \frac{1}{2}m\Omega^2(x^2 + y^2) - V(x, y, z). \tag{2.7.4}$$

The equations of motion for the particle are then obtained by substituting this into the EL equations for x, y, and z.

The computations that lead to the equations of motion will be left as an exercise for the reader. We find

$$m\ddot{x} = -\frac{\partial V}{\partial x} + 2m\Omega\dot{y} + m\Omega^2 x, \tag{2.7.5}$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} - 2m\Omega\dot{x} + m\Omega^2 y, \tag{2.7.6}$$

$$m\ddot{z} = -\frac{\partial V}{\partial z}. \tag{2.7.7}$$

These equations can be expressed in vectorial form if we introduce the angular-velocity vector $\boldsymbol{\Omega}$, defined by

$$\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}} = [0, 0, \Omega]. \tag{2.7.8}$$

The vectorial form is

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_{\text{applied}} + \boldsymbol{F}_{\text{Coriolis}} + \boldsymbol{F}_{\text{centrifugal}}, \tag{2.7.9}$$

where

$$\boldsymbol{F}_{\text{applied}} = -\boldsymbol{\nabla}V \tag{2.7.10}$$

is the *true applied force* on the particle, given by the gradient of the potential energy, while

$$\boldsymbol{F}_{\text{Coriolis}} = 2m\dot{\boldsymbol{r}} \times \boldsymbol{\Omega} = [2m\Omega\dot{y}, -2m\Omega\dot{x}, 0] \tag{2.7.11}$$

and

$$\boldsymbol{F}_{\text{centrifugal}} = m\boldsymbol{\Omega} \times (\boldsymbol{r} \times \boldsymbol{\Omega}) = [m\Omega^2 x, m\Omega^2 y, 0] \tag{2.7.12}$$

are *fictitious forces* that arise because the reference frame S is not an inertial frame (refer back to the discussion of Sec. 1.1). The *Coriolis force* is linear in the angular velocity Ω, and it depends on the particle's velocity vector $\dot{\boldsymbol{r}}$; its effect on the particle depends on its state of motion. The *centrifugal force* is quadratic in Ω, and it depends only on the position vector $\boldsymbol{r}$; this is always an outward force that points away from the centre of motion.

Exercise 2.21. Verify that Eqs. (2.7.5)–(2.7.7) follow from the Lagrangian of Eq. (2.7.4).

Exercise 2.22. Show that Eqs. (2.7.5)–(2.7.7) are equivalent to the vectorial equation

(2.7.9), together with the definitions of Eqs. (2.7.10)–(2.7.12).

2.7.2 Case study #1: Particle attached to a spring

As our first application of the rotating-frame formalism we examine a particle attached to a linear spring that is free to rotate around the z axis. The particle is thus subjected to the potential $V = \frac{1}{2}k(x^2 + y^2)$, which we write in the alternative form

$$V = \frac{1}{2}m\omega^2(x^2 + y^2), \tag{2.7.13}$$

in which $\omega^2 \equiv k/m$ is a stand-in for the spring constant k. For simplicity we assume that the particle is confined to the x-y plane and we set, accordingly, $z = \dot{z} = 0$ in all equations.

The equations of motion of Eqs. (2.7.5) and (2.7.6) become

$$\ddot{x} = 2\Omega\dot{y} + (\Omega^2 - \omega^2)x, \qquad \ddot{y} = -2\Omega\dot{x} + (\Omega^2 - \omega^2)y. \tag{2.7.14}$$

Let us first analyze these equations in the limit of no rotation, $\Omega = 0$. In this case they reduce to $\ddot{x} = -\omega^2 x$ and $\ddot{y} = -\omega^2 y$, the equations of simple harmonic motion. The general solution to the equations of motion is then

$$x_{\Omega=0}(t) = a\cos(\omega t + \alpha), \qquad y_{\Omega=0}(t) = b\cos(\omega t + \beta). \tag{2.7.15}$$

The four constants a, b, α, and β can be related to the four initial values $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$. Equations (2.7.5) are parametric equations for the motion of the particle in the x-y plane, and it is easy to show that the trajectory is elliptical.

To analyze the equations in the general case ($\Omega \neq 0$) we once more employ the clever trick of forming complex combinations. We introduce $\xi = x + iy$ and we combine the two equations (2.7.14) into a single equation for ξ:

$$\begin{aligned} \ddot{\xi} &= \ddot{x} + i\ddot{y} \\ &= 2\Omega(\dot{y} - i\dot{x}) + (\Omega^2 - \omega^2)(x + iy) \\ &= -2i\Omega(\dot{x} + i\dot{y}) + (\Omega^2 - \omega^2)(x + iy), \end{aligned}$$

or

$$\ddot{\xi} + 2i\Omega\dot{\xi} - (\Omega^2 - \omega^2)\xi = 0. \tag{2.7.16}$$

To find solutions to this equation we use a trial expression of the form $\xi = ce^{i\lambda t}$, where c and λ are complex constants. Substitution into Eq. (2.7.16) produces a quadratic equation for λ,

$$\lambda^2 + 2\Omega\lambda + (\Omega^2 - \omega^2) = 0,$$

which factorizes as

$$(\lambda + \Omega + \omega)(\lambda + \Omega - \omega) = 0.$$

The solutions, obviously, are $\lambda = -(\Omega \pm \omega)$, and the general solution for ξ is

$$\xi = c_1 e^{-i(\Omega+\omega)t} + c_2 e^{-i(\Omega-\omega)t},$$

or

$$\xi = e^{-i\Omega t}\big(c_1 e^{-i\omega t} + c_2 e^{i\omega t}\big),$$

where c_1 and c_2 are complex numbers.

To help us understand what we have just found, we observe that if we let Ω go to zero, our general solution for ξ becomes $c_1 e^{-i\omega t} + c_2 e^{i\omega t}$. This is the solution in

the limit of no rotation, and this must be equal to $x_{\Omega=0} + iy_{\Omega=0}$, which was given by Eq. (2.7.15) above. So we can write our general solution as

$$\xi(t) = e^{-i\Omega t}\big[x_{\Omega=0}(t) + iy_{\Omega=0}(t)\big]. \tag{2.7.17}$$

Written in full, this is

$$x(t) = x_{\Omega=0}(t)\cos\Omega t + y_{\Omega=0}(t)\sin\Omega t, \qquad y(t) = y_{\Omega=0}(t)\cos\Omega t - x_{\Omega=0}(t)\sin\Omega t. \tag{2.7.18}$$

What is the meaning of these results? The answer is simple. Comparison with the transformations of Eqs. (2.7.1) and (2.7.2) shows that the motion of the particle on the turntable is a rotated version of the motion that would take place in an inertial frame; the rotation angle is here $-\Omega t$ instead of $+\Omega t$. This is easy to understand: The motion $x_{\Omega=0}(t)$, $y_{\Omega=0}(t)$ is what the particle would do in an inertial frame; because, however, we are measuring this motion in a rotating frame, we see a rotated version of the inertial motion. This conclusion is confirmed by substituting our solution of Eqs. (2.7.17), (2.7.18) into Eqs. (2.7.1), (2.7.2); the result is

$$x'(t) = x_{\Omega=0}(t), \qquad y'(t) = y_{\Omega=0}(t),$$

as expected. The motion of the particle as measured in the rotating frame is shown in Fig. 2.18 for three selected values of Ω.

Exercise 2.23. Fill in the mathematical gaps that were left behind in the presentation of this subsection.

2.7.3 Motion on a rotating Earth. Kinematics

We wish to describe the motion of a mechanical system from the point of view of an observer attached to a point P on the surface of a rotating Earth. This will be done with the help of a Cartesian frame (x, y, z) whose origin will be at P, and which will rotate along with the Earth. We will construct this Cartesian coordinate system in two stages. In the first stage we will momentarily assume that the Earth does not, in fact, rotate around its polar axis; in the second stage we will incorporate the rotation.

We first place a Cartesian frame (x', y', z') at the centre of the nonrotating Earth. Neglecting the Earth's motion around the Sun, we consider this to be an inertial frame. Our end goal in this subsection is to relate (x, y, z), the local frame at P, to the inertial frame (x', y', z'). Our first step toward this goal is to introduce the spherical coordinates (r', θ', ϕ'), which are related to the original Cartesian coordinates by

$$x' = r'\sin\theta'\cos\phi', \qquad y' = r'\sin\theta'\sin\phi', \qquad z' = r'\cos\theta'.$$

As shown in Fig. 2.19, our point P on the Earth's surface is at a distance $r' = R$ from the centre, and its position on the sphere is determined by the colatitude θ' and the longitude ϕ'. (The latitude λ' is related to the colatitude by $\lambda' = \frac{\pi}{2} - \theta'$; thus the colatitude of the equator is 90° while its latitude is 0°.) Because the Earth is not yet rotating the longitude of P is a fixed angle; when we later incorporate the rotation into the picture we will put $\phi' = \Omega t$, with Ω denoting the Earth's angular velocity.

The spherical coordinates come with a set of basis vectors $(\hat{\boldsymbol{r}}', \hat{\boldsymbol{\theta}}', \hat{\boldsymbol{\phi}}')$. Following the discussion of Sec. 1.2, we derive that these vectors are related to the Cartesian

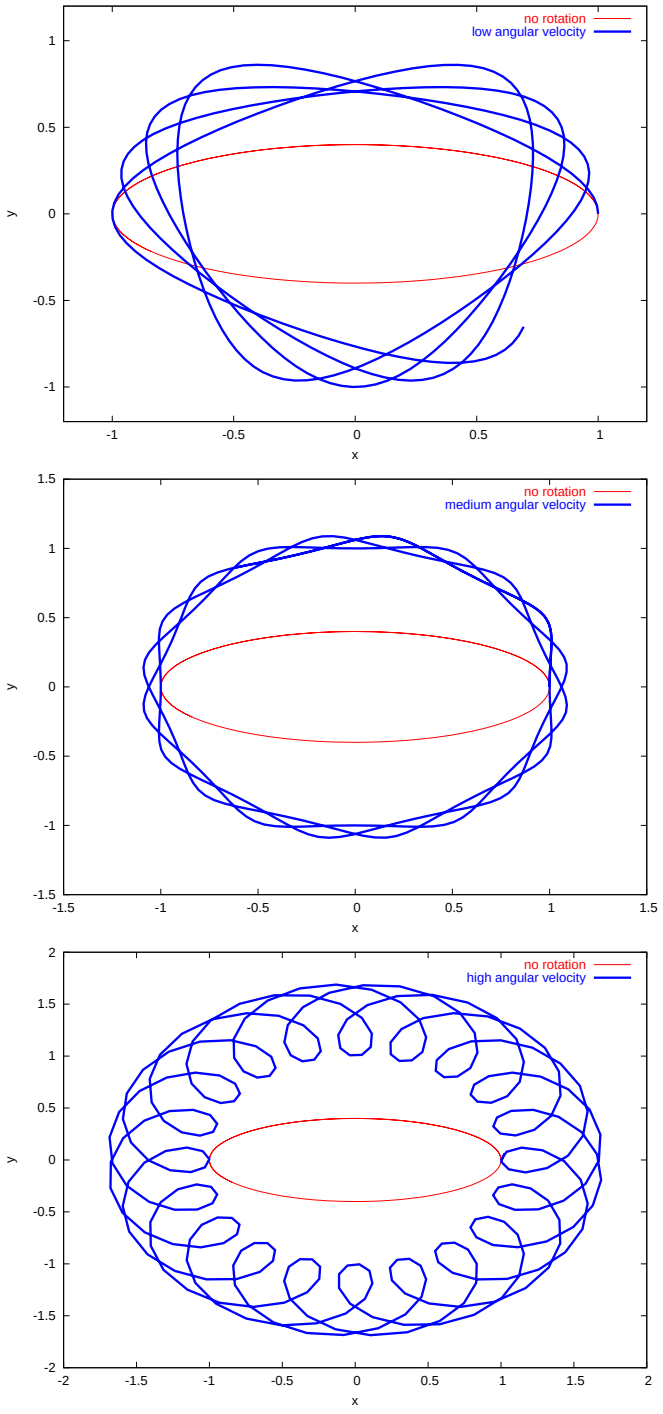


Figure 2.18: A particle attached to a linear spring is viewed from a rotating frame. The upper graph was generated with $\Omega/\omega = 0.3$, the middle graph with $\Omega/\omega = 0.7$, and the lower graph with $\Omega/\omega = 1.3$. In all cases the initial values were set to $x(0) = 1$, $\dot{x}(0) = 0$, $y(0) = 0$, and $\dot{y}(0) = 0.4$. The elliptical motion of the particle, which takes place when $\Omega = 0$ for the same initial conditions, is also shown for comparison.

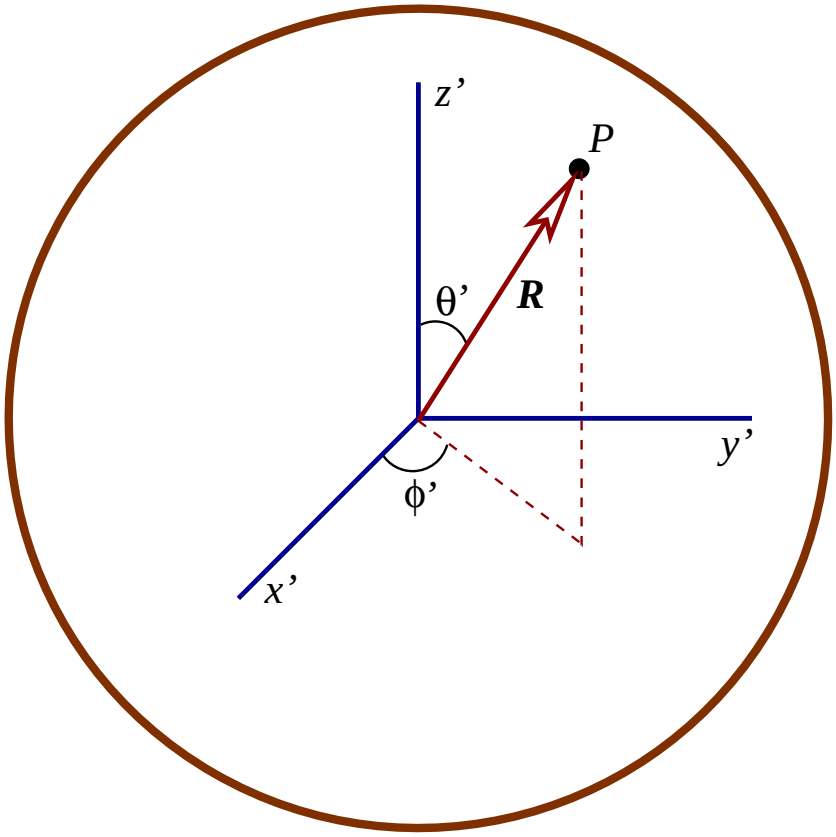


Figure 2.19: An inertial frame (x', y', z') attached to the centre of the Earth, and a point P on the surface, described by colatitude θ' and longitude ϕ'. The position vector of P relative to the inertial frame is $\boldsymbol{R}$.

basis $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}')$ by

$$\hat{\boldsymbol{r}}' = \frac{\partial \boldsymbol{r'}}{\partial r'} = \sin\theta' \cos\phi' \, \hat{\boldsymbol{x}}' + \sin\theta' \sin\phi' \hat{\boldsymbol{y}}' + \cos\theta' \, \hat{\boldsymbol{z}}', \tag{2.7.19}$$

$$\hat{\boldsymbol{\theta}}' = \frac{1}{r'} \frac{\partial \boldsymbol{r'}}{\partial \theta'} = \cos\theta' \cos\phi' \, \hat{\boldsymbol{x}}' + \cos\theta' \sin\phi' \hat{\boldsymbol{y}}' - \sin\theta' \, \hat{\boldsymbol{z}}', \tag{2.7.20}$$

$$\hat{\boldsymbol{\phi}}' = \frac{1}{r' \sin\theta'} \frac{\partial \boldsymbol{r'}}{\partial \phi'} = -\sin\phi' \, \hat{\boldsymbol{x}}' + \cos\phi' \hat{\boldsymbol{y}}'. \tag{2.7.21}$$

Here $\boldsymbol{r'}$ is the position vector expressed in terms of the spherical coordinates,

$$\boldsymbol{r'} = r' \sin\theta' \cos\phi' \, \hat{\boldsymbol{x}}' + r' \sin\theta' \sin\phi' \, \hat{\boldsymbol{y}}' + r' \cos\theta' \, \hat{\boldsymbol{z}}'. \tag{2.7.22}$$

The position vector of the point P on the surface is

$$\boldsymbol{R} = R \sin\theta' \cos\phi' \, \hat{\boldsymbol{x}}' + R \sin\theta' \sin\phi' \, \hat{\boldsymbol{y}}' + R \cos\theta' \, \hat{\boldsymbol{z}}'. \tag{2.7.23}$$

The spherical coordinates are useful to specify the position of the laboratory on the Earth's surface, but they are not so useful to describe the motion of mechanical bodies that would take place in this laboratory. For this purpose we introduce another Cartesian frame (x, y, z) whose origin will be at P. The orientation of this frame will be set by the directions of the basis vectors $(\hat{\boldsymbol{r}}', \hat{\boldsymbol{\theta}}', \hat{\boldsymbol{\phi}}')$. Thus, the z axis will point away from the surface, and will be aligned in the direction of $\hat{\boldsymbol{r}}'$; the x axis will point in the southern direction, and will be aligned in the direction of $\hat{\boldsymbol{\theta}}'$;

and the y axis will point in the eastern direction, and will be aligned in the direction of $\hat{\boldsymbol{\phi}}'$. We therefore set

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{\theta}}', \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\phi}}', \qquad \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}'. \tag{2.7.24}$$

The situation is illustrated in Fig. 2.20.

We denote by $\boldsymbol{r}$ the position vector of a particle located at a point Q near the surface of the Earth, relative to the surface point P to which our frame (x, y, z) is attached. As usual we will resolve this vector in the basis $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ and identify the components with the particle's coordinates. We have

$$\boldsymbol{r} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}},$$

and if we involve Eq. (2.7.24) we obtain

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}}' + y\,\hat{\boldsymbol{\phi}}' + z\,\hat{\boldsymbol{r}}'.$$

If we now substitute Eqs. (2.7.19)–(2.7.21) and rearrange what we get, we find

$$\begin{aligned} \boldsymbol{r} &= (x\cos\theta'\cos\phi' - y\sin\phi' + z\sin\theta'\cos\phi')\hat{\boldsymbol{x}}' \\ &\quad + (x\cos\theta'\sin\phi' + y\cos\phi' + z\sin\theta'\sin\phi')\hat{\boldsymbol{y}}' \\ &\quad + (-x\sin\theta' + z\cos\theta')\hat{\boldsymbol{z}}'. \end{aligned}$$

The position vector of the particle relative to the centre of the Earth is $\boldsymbol{R} + \boldsymbol{r}$. According to Eq. (2.7.23) and our previous result, this is

$$\begin{aligned} \boldsymbol{R} + \boldsymbol{r} &= \big[x\cos\theta'\cos\phi' - y\sin\phi' + (R+z)\sin\theta'\cos\phi'\big]\hat{\boldsymbol{x}}' \\ &\quad + \big[x\cos\theta'\sin\phi' + y\cos\phi' + (R+z)\sin\theta'\sin\phi'\big]\hat{\boldsymbol{y}}' \\ &\quad + \big[-x\sin\theta' + (R+z)\cos\theta'\big]\hat{\boldsymbol{z}}'. \end{aligned}$$

The components of this vector in the original Cartesian basis $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}')$ are the original Cartesian coordinates (x', y', z') of the particle at Q. We have obtained, therefore, the transformation

$$x' = x\cos\theta'\cos\phi' - y\sin\phi' + (R+z)\sin\theta'\cos\phi', \tag{2.7.25}$$

$$y' = x\cos\theta'\sin\phi' + y\cos\phi' + (R+z)\sin\theta'\sin\phi', \tag{2.7.26}$$

$$z' = -x\sin\theta' + (R+z)\cos\theta', \tag{2.7.27}$$

between the two systems of Cartesian coordinates.

Our considerations so far have relied on the fiction of a nonrotating Earth. To finally incorporate its rotation into the picture we set $\phi' = \Omega t$ into Eqs. (2.7.25)–(2.7.27), with Ω denoting the Earth's angular velocity. We also, at the same time, fix the colatitude of our laboratory to $\theta' = \alpha$. The transformation between the *local rotating frame* (x, y, z) and the original inertial frame (x', y', z') is finally given by

$$x' = x\cos\alpha\cos\Omega t - y\sin\Omega t + (R+z)\sin\alpha\cos\Omega t, \tag{2.7.28}$$

$$y' = x\cos\alpha\sin\Omega t + y\cos\Omega t + (R+z)\sin\alpha\sin\Omega t, \tag{2.7.29}$$

$$z' = -x\sin\alpha + (R+z)\cos\alpha. \tag{2.7.30}$$

We recall that R is the Earth's radius, that the x direction points due south, that the y direction points due east, and that the z direction points up, away from the surface.

Exercise 2.24. Fill in the mathematical gaps that were left behind in the presentation of this subsection.

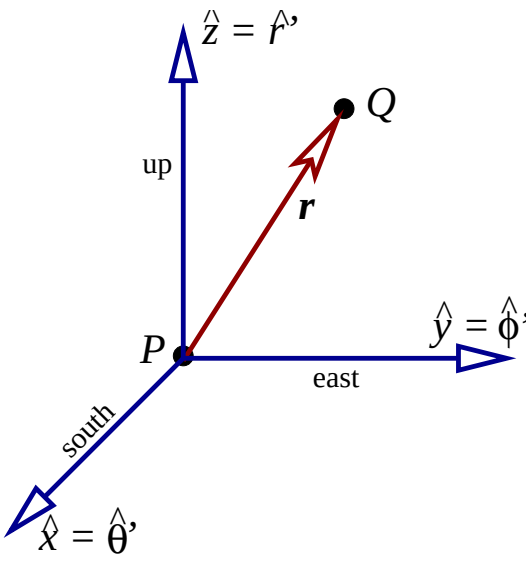


Figure 2.20: The local Cartesian frame (x, y, z) at P. The x direction points south, the y direction points east, and the z direction points up. The position vector of a point Q relative to P is denoted $\boldsymbol{r}$.

2.7.4 Motion on a rotating Earth. Dynamics

Having established the transformation of Eqs. (2.7.28)–(2.7.30) we may now turn to the task of describing the dynamics of a particle as viewed in the local rotating frame. The particle's coordinates (x, y, z) are changing with time and we let $(\dot{x}, \dot{y}, \dot{z})$ denote the components of the velocity vector in the local rotating frame. In the inertial frame, according to Eqs. (2.7.28)–(2.7.30), the components of the velocity vector are

$$\begin{aligned}
\dot{x}' &= \dot{x}\cos\alpha\cos\Omega t - \dot{y}\sin\Omega t + \dot{z}\sin\alpha\cos\Omega t \\
&\quad - \Omega\big[x\cos\alpha\sin\Omega t + y\cos\Omega t + (R+z)\sin\alpha\sin\Omega t\big], \\
\dot{y}' &= \dot{x}\cos\alpha\sin\Omega t + \dot{y}\cos\Omega t + \dot{z}\sin\alpha\sin\Omega t \\
&\quad + \Omega\big[x\cos\alpha\cos\Omega t - y\sin\Omega t + (R+z)\sin\alpha\cos\Omega t\big], \\
\dot{z}' &= -\dot{x}\sin\alpha + \dot{z}\cos\alpha.
\end{aligned}$$

A fairly laborious calculation then returns the squared velocity; we find

$$\begin{aligned}
v'^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\Omega\Big\{\big[(R+z)\dot{y} - y\dot{z}\big]\sin\alpha + \big[x\dot{y} - y\dot{x}\big]\cos\alpha\Big\} \\
&\quad + \Omega^2\Big\{\big[x\cos\alpha + (R+z)\sin\alpha\big]^2 + y^2\Big\}. \qquad (2.7.31)
\end{aligned}$$

The particle's kinetic energy is $T = \frac{1}{2}mv'^2$. Again (as in Sec. 2.7.1) we see that the kinetic energy has a contribution from the particle's motion within the local rotating frame, and contributions from the motion of the frame; these depend on Ω and the laboratory's colatitude α.

Exercise 2.25. Verify Eq. (2.7.31).

The particle's potential energy comes from two different sources, which we choose to distinguish in this subsection. The first is the Earth's gravity, and this contribution to the potential energy is $V_{\text{gravity}} = mgz$, as usual. The second contribution

comes from all the other forces acting on the particle; we write this as $V_{\text{other}} = U$. The total potential energy is then

$$V = mgz + U(x, y, z). \tag{2.7.32}$$

The particle's Lagrangian is, finally,

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m\Omega\Big\{\big[(R+z)\dot{y} - y\dot{z}\big]\sin\alpha + \big[x\dot{y} - y\dot{x}\big]\cos\alpha\Big\} \\ &\quad + \frac{1}{2}m\Omega^2\Big\{\big[x\cos\alpha + (R+z)\sin\alpha\big]^2 + y^2\Big\} - mgz - U(x, y, z). \end{aligned} \tag{2.7.33}$$

The equations of motion in the local rotating frame are obtained by substituting this into the EL equations.

Omitting the detail of the calculations, the equations of motion are

$$m\ddot{x} = -\frac{\partial U}{\partial x} + 2m\Omega\dot{y}\cos\alpha + m\Omega^2\big[x\cos\alpha + (R+z)\sin\alpha\big]\cos\alpha, \tag{2.7.34}$$

$$m\ddot{y} = -\frac{\partial U}{\partial y} - 2m\Omega(\dot{x}\cos\alpha + \dot{z}\sin\alpha) + m\Omega^2 y, \tag{2.7.35}$$

$$m\ddot{z} = -mg - \frac{\partial U}{\partial z} + 2m\Omega\dot{y}\sin\alpha + m\Omega^2\big[x\cos\alpha + (R+z)\sin\alpha\big]\sin\alpha. \tag{2.7.36}$$

These equations can be expressed in vectorial form if we introduce the angular-momentum vector $\boldsymbol{\Omega}$. This is given by $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}'$, and in the local rotating frame we have the components $\Omega_x = \boldsymbol{\Omega}\cdot\hat{\boldsymbol{x}} = \Omega\hat{\boldsymbol{z}}'\cdot\hat{\boldsymbol{\theta}}' = -\Omega\sin\alpha$, $\Omega_y = \boldsymbol{\Omega}\cdot\hat{\boldsymbol{y}} = \Omega\hat{\boldsymbol{z}}'\cdot\hat{\boldsymbol{\phi}}' = 0$, and $\Omega_z = \boldsymbol{\Omega}\cdot\hat{\boldsymbol{z}} = \Omega\hat{\boldsymbol{z}}'\cdot\hat{\boldsymbol{r}}' = \Omega\cos\alpha$. The vector is therefore given by

$$\boldsymbol{\Omega} = -\Omega\sin\alpha\,\hat{\boldsymbol{x}} + \Omega\cos\alpha\,\hat{\boldsymbol{z}} = [-\Omega\sin\alpha, 0, \Omega\cos\alpha] \tag{2.7.37}$$

in the local rotating frame. We also re-express the vector $\boldsymbol{R}$ of Eq. (2.7.23) as

$$\boldsymbol{R} = R\hat{\boldsymbol{z}} = [0, 0, R]; \tag{2.7.38}$$

this gives the position of the laboratory relative to the Earth's centre.

The vectorial equation is

$$m\ddot{\boldsymbol{r}} = m\boldsymbol{g} + \boldsymbol{F}_{\text{applied}} + \boldsymbol{F}_{\text{Coriolis}} + \boldsymbol{F}_{\text{centrifugal}}, \tag{2.7.39}$$

where $\boldsymbol{g} = -g\hat{\boldsymbol{z}} = [0, 0, -g]$ is the acceleration of gravity, $\boldsymbol{F}_{\text{applied}} = -\boldsymbol{\nabla}U$ is the net force coming from all other interactions,

$$\boldsymbol{F}_{\text{Coriolis}} = 2m\dot{\boldsymbol{r}}\times\boldsymbol{\Omega} \tag{2.7.40}$$

is the Coriolis force, and

$$\boldsymbol{F}_{\text{centrifugal}} = m\boldsymbol{\Omega}\times\big[(\boldsymbol{R}+\boldsymbol{r})\times\boldsymbol{\Omega}\big] \tag{2.7.41}$$

is the centrifugal force. In Eq. (2.7.39), $m\boldsymbol{g}$ and $\boldsymbol{F}_{\text{applied}}$ are genuine forces acting on the particle, while $\boldsymbol{F}_{\text{Coriolis}}$ and $\boldsymbol{F}_{\text{centrifugal}}$ are fictitious forces that arise from the rotational motion of the frame.

Exercise 2.26. Verify that Eqs. (2.7.34)–(2.7.36) follow from the Lagrangian of Eq. (2.7.33).

Exercise 2.27. Show that Eqs. (2.7.34)–(2.7.36) are equivalent to the vectorial equation (2.7.39), together with the definitions of Eqs. (2.7.37), (2.7.38), (2.7.40), and (2.7.41).

2.7.5 Case study #2: Particle released from rest

To gain some insight into the effects of Earth's rotation we shall determine what happens to a particle that is released from rest at a great height h in the Earth's gravitational field. Because the Earth's rotation is quite slow, it will be sufficient to work consistently to first order in the angular velocity Ω. We will therefore neglect all terms of order Ω^2 in the equations of motion; this means that we will keep the Coriolis term, but discard the centrifugal term.

At this level of accuracy the equations of motion (2.7.34)–(2.7.36) reduce to

$$\ddot{x} = 2\Omega\dot{y}\cos\alpha, \qquad \ddot{y} = -2\Omega(\dot{x}\cos\alpha + \dot{z}\sin\alpha), \qquad \ddot{z} = -g + 2\Omega\dot{y}\sin\alpha. \quad (2.7.42)$$

We impose the initial conditions $x(0) = y(0) = 0$, $z(0) = h$, as well as $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$.

We shall solve Eqs. (2.7.42) by the method of successive approximations. We first express the particle's coordinates $x(t)$, $y(t)$, and $z(t)$ as formal expansions in powers of Ω. Thus,

$$x(t) = x_0(t)+\Omega x_1(t)+\cdots, \quad y(t) = y_0(t)+\Omega y_1(t)+\cdots, \quad z(t) = z_0(t)+\Omega z_1(t)+\cdots.$$

In terms of these new quantities the initial conditions become $x_0(0) = y_0(0) = 0$, $z_0(0) = h$, $x_1(0) = y_1(0) = z_1(0) = 0$, as well as $\dot{x}_0(0) = \dot{y}_0(0) = \dot{z}_0(0) = \dot{x}_1(0) = \dot{y}_1(0) = \dot{z}_1(0) = 0$. Substituting the expansions into Eq. (2.7.42) yields

$$\begin{aligned} \ddot{x}_0 + \Omega\ddot{x}_1 + \cdots &= 2\Omega(\dot{y}_0 + \cdots)\cos\alpha, \\ \ddot{y}_0 + \Omega\ddot{y}_1 + \cdots &= -2\Omega(\dot{x}_0\cos\alpha + \dot{z}_0\sin\alpha + \cdots), \\ \ddot{z}_0 + \Omega\ddot{z}_1 + \cdots &= -g + 2\Omega(\dot{y}_0 + \cdots)\sin\alpha. \end{aligned}$$

Equating powers of Ω produces the set of equations

$$\ddot{x}_0 = 0, \qquad \ddot{y}_0 = 0, \qquad \ddot{z}_0 = -g,$$

$$\ddot{x}_1 = 2\dot{y}_0\cos\alpha, \qquad \ddot{y}_1 = -2\dot{x}_0\cos\alpha - 2\dot{z}_0\sin\alpha, \qquad \ddot{z}_1 = 2\dot{y}_0\sin\alpha.$$

The zeroth-order equations are easy to solve. In view of the initial conditions, the solutions are

$$x_0(t) = 0, \qquad y_0(t) = 0, \qquad z_0(t) = h - \frac{1}{2}gt^2.$$

With $\dot{x}_0 = \dot{y}_0 = 0$ and $\dot{z}_0 = -gt$ the first-order equations become

$$\ddot{x}_1 = 0, \qquad \ddot{y}_1 = 2gt\sin\alpha, \qquad \ddot{z}_1 = 0.$$

These equations also are easy to solve. Taking once more the initial conditions into account, we find that the solutions are

$$x_1(t) = 0, \qquad y_1(t) = \frac{1}{3}gt^3\sin\alpha, \qquad z_1(t) = 0.$$

The complete solution to the equations of motion is therefore

$$x(t) = 0 + O(\Omega^2), \quad (2.7.43)$$

$$y(t) = \frac{1}{3}(g\Omega\sin\alpha)t^3 + O(\Omega^2), \quad (2.7.44)$$

$$z(t) = h - \frac{1}{2}gt^2 + O(\Omega^2). \quad (2.7.45)$$

Observe that the factor $g\Omega\sin\alpha$ is positive for any colatitude α, except at the North and South poles where it is zero. The fact that y increases during the motion means

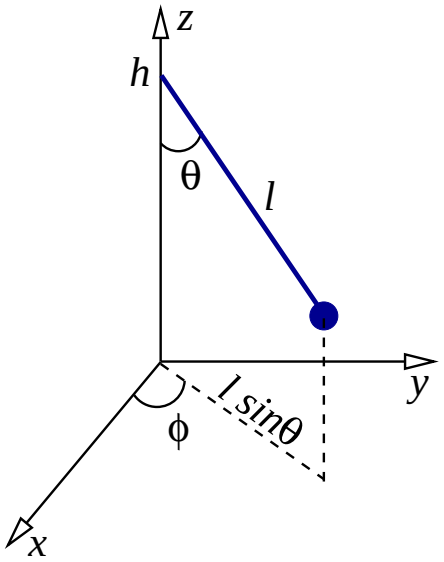


Figure 2.21: Geometry of Foucault's pendulum. The motion of the pendulum is described by the swing angle θ and the rotation angle ϕ.

that the particle, which would just fall straight down in the absence of rotation, is in fact drifting away in the eastward direction.

The time required for the particle to hit the ground is $t = (2h/g)^{1/2}$, and by this time to total eastward displacement is

$$y_{\text{total}} = \frac{1}{3}(g\Omega \sin\alpha)\left(\frac{2h}{g}\right)^{3/2}. \tag{2.7.46}$$

If the object is released from a height of 100 m at Guelph's colatitude (approximately 47°), this amounts to approximately 1.6 cm. The Coriolis force produces a rather small effect.

Exercise 2.28. Compute this number.

2.7.6 Case study #3: Foucault's pendulum

A dramatic demonstration of Earth's rotation came from Foucault's celebrated pendulum, which was first displayed in front of an audience at the Observatoire de Paris in 1851. The idea is that while the Earth rotates the pendulum keeps oscillating in a fixed plane as viewed from an inertial frame; as seen from the Earth's rotating frame, however, it is the pendulum that appears to be rotating. More precisely stated, as viewed in the local rotating frame the pendulum is swinging in a plane which rotates at a steady rate Ω_{plane}; this is directly related to Ω, the rate at which the Earth itself is rotating.

In this last application we will examine the motion of a pendulum in the local rotating frame. We aim to calculate Ω_{plane}, in an approximation in which we neglect the centrifugal effects (which are proportional to Ω^2) but retain the Coriolis effects (which are proportional to Ω), and in an approximation in which the amplitude of the pendulum's oscillations is assumed to be small.

In the spirit of Lagrangian mechanics we will use the generalized coordinates θ and ϕ to describe the motion of the pendulum, as illustrated in Fig. 2.21. (Notice that θ, as defined here, is not the standard spherical coordinate.) The relation

between these generalized coordinates and the original Cartesian system (x, y, z) is given by

$$x = \ell \sin\theta \cos\phi, \qquad y = \ell \sin\theta \sin\phi, \qquad z = h - \ell\cos\theta. \tag{2.7.47}$$

The pendulum has a constant length ℓ, and its pivot point is attached at a height h above the Earth's surface. As usual we introduce the quantity

$$\omega^2 = g/\ell, \tag{2.7.48}$$

and we will use it instead of g.

The two degrees of freedom of the pendulum are represented by the angles θ and ϕ. The first angle, θ, is the usual swing angle of the pendulum. In the absence of rotation, the pendulum would swing in a fixed plane, and ϕ would stay constant; in this situation there would be a single degree of freedom. But as we shall see, the Earth's rotation will force ϕ to change steadily with time, and the plane of the pendulum will rotate in the x-y plane; in this situation there are two degrees of freedom, and ϕ is the rotation angle of the swing plane.

According to Eq. (2.7.47) the pendulum's velocity vector has the components

$$\begin{aligned} \dot{x} &= \ell\cos\theta\cos\phi\,\dot{\theta} - \ell\sin\theta\sin\phi\,\dot{\phi}, \\ \dot{y} &= \ell\cos\theta\sin\phi\,\dot{\theta} + \ell\sin\theta\cos\phi\,\dot{\phi}, \\ \dot{z} &= \ell\sin\theta\,\dot{\theta}. \end{aligned}$$

It follows that the squared velocity is

$$v^2 = \ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\phi}^2.$$

We make this substitution, along with $z = h - \ell\cos\theta$, into the Lagrangian of Eq. (2.7.33), and we allow ourselves (once more) to neglect the centrifugal terms that are proportional to Ω^2. This yields

$$\begin{aligned} L &= \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \\ &\quad + m\Omega\Big\{\big[(R+h-\ell\cos\theta)\dot{y} - y\dot{z}\big]\sin\alpha + \big[x\dot{y} - y\dot{x}\big]\cos\alpha\Big\} \\ &\quad - mg(h - \ell\cos\theta). \end{aligned}$$

In this Lagrangian we recognize a term $m\Omega(R+h)\dot{y}\sin\alpha$ and another term $-mgh$ that can both be discarded. We can omit the first term because it is the time derivative of the function $m\Omega(R+h)y\sin\alpha$, and as we have learned in Sec. 2.5.3, such a term will not contribute to the equations of motion. And we can omit the second term for the simple reason that it is a constant; it will also contribute nothing to the equations of motion.

The Lagrangian simplifies to

$$\begin{aligned} L &= \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \\ &\quad + m\Omega\Big\{-\big[\ell\cos\theta\,\dot{y} + y\dot{z}\big]\sin\alpha + \big[x\dot{y} - y\dot{x}\big]\cos\alpha\Big\} \\ &\quad + mg\ell\cos\theta. \end{aligned}$$

This becomes

$$\begin{aligned} L &= m\ell^2\Big\{\frac{1}{2}\big(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\big) - \Omega\sin\alpha\big(\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}\big) \\ &\quad + \Omega\cos\alpha\sin^2\theta\,\dot{\phi} + \omega^2\cos\theta\Big\}, \end{aligned} \tag{2.7.49}$$

after involving the transformation between the Cartesian coordinates (x, y, z) and our generalized coordinates (θ, ϕ). This is the pendulum's Lagrangian, up to terms of order Ω^2 that have been neglected.

Exercise 2.29. Verify Eq. (2.7.49).

We may simplify the Lagrangian further if we assume that the amplitude of the pendulum's oscillations is sufficiently small that we can use the approximations

$$\sin\theta \simeq \theta, \qquad \cos\theta \simeq 1 - \frac{1}{2}\theta^2.$$

Neglecting all terms of order θ^3 and higher, we obtain

$$\begin{aligned} L &= m\ell^2\Big\{\frac{1}{2}\big(\dot\theta^2 + \theta^2\,\dot\phi^2\big) - \Omega\sin\alpha\big(\sin\phi\,\dot\theta + \theta\cos\phi\,\dot\phi\big) \\ &\quad + \Omega\cos\alpha\,\theta^2\dot\phi + \omega^2\Big(1 - \frac{1}{2}\theta^2\Big)\Big\}. \end{aligned}$$

In this simplified Lagrangian we recognize a term proportional to

$$\sin\phi\,\dot\theta + \theta\cos\phi\,\dot\phi = \frac{d}{dt}\Big(\theta\sin\phi\Big);$$

we can discard this term from the Lagrangian because this is a total time derivative. We may also remove the constant term ω^2. After these simplifications, our final Lagrangian will be

$$L = m\ell^2\Big\{\frac{1}{2}\big(\dot\theta^2 + \theta^2\,\dot\phi^2\big) + \Omega\cos\alpha\,\theta^2\dot\phi - \frac{1}{2}\omega^2\theta^2\Big\}. \tag{2.7.50}$$

This is the simplified Lagrangian for Foucault's pendulum, and it is valid in the limit of small swing angles.

The equations of motion are obtained by substituting the Lagrangian of Eq. (2.7.50) into the EL equations. Omitting all details, we find that the equation for θ is

$$\ddot\theta + \omega^2\theta - \dot\phi(\dot\phi + 2\Omega\cos\alpha)\theta = 0. \tag{2.7.51}$$

And the fact that the Lagrangian does not depend explicitly on ϕ implies that the (rescaled) generalized momentum

$$p_\phi = \theta^2(\dot\phi + \Omega\cos\alpha) \tag{2.7.52}$$

is a constant of the motion.

Exercise 2.30. Verify that the EL equations produce Eqs. (2.7.51) and the statement that p_ϕ, as defined by Eq. (2.7.52), is constant.

Solving Eq. (2.7.52) for $\dot\phi$ gives $\dot\phi = -\Omega\cos\alpha + p_\phi/\theta^2$. We see that unless $p_\phi = 0$, $\dot\phi$ would blow up as $\theta \to 0$, that is, whenever the pendulum crosses the z axis. To eliminate this unphysical behaviour we set $p_\phi = 0$, so that

$$\dot\phi = -\Omega\cos\alpha. \tag{2.7.53}$$

This is our key result. In the absence of rotation we would find that $\dot\phi = 0$, and we would conclude that the pendulum swings in a fixed plane, as we had foreseen

at the beginning of this subsection. With the Earth's rotation, however, we find instead that $\dot{\phi} = -\Omega\cos\alpha$, and this means that the swing plane is rotating with a constant angular velocity given by

$$\Omega_{\text{plane}} = -\Omega\cos\alpha. \tag{2.7.54}$$

This is the Foucault effect.

When the pendulum is located in the northern hemisphere, we have that $\cos\alpha > 0$ and we find that $\Omega_{\text{plane}} < 0$, so that the swing plane rotates clockwise. When, on the other hand, we go to the southern hemisphere, we have that $\cos\alpha < 0$ and $\Omega_{\text{plane}} > 0$, so that the swing plane rotates counterclockwise. The Foucault effect is maximum at the poles, and it vanishes at the equator.

Exercise 2.31. Calculate $\Delta\phi$, the angular displacement of the swing plane after 1 hour, when the Foucault pendulum is located in Guelph (colatitude 47°).

Exercise 2.32. This is the laboratory component of the course. There is a Foucault pendulum in the foyer of the MacNaughton building, and you are asked to determine its value for $\Delta\phi$. Measure the angular position of the swing plane when you first arrive in the Department of Physics, and record the time. Repeat the measurement when you are about to leave. Divide the difference in angular positions by the time interval measured in hours, and obtain your experimental value for $\Delta\phi$. How close is it to the theoretical value obtained in the previous exercise?

To finish off our discussion of the Foucault pendulum we return to Eq. (2.7.51), in which we substitute Eq. (2.7.53). The result is

$$\ddot{\theta} + \omega^2\theta - (-\Omega\cos\alpha)(+\Omega\cos\alpha)\theta = 0,$$

or

$$\ddot{\theta} + (\omega^2 + \Omega^2\cos^2\alpha)\theta = 0.$$

This is the equation for simple harmonic motion, and it appears to indicate that the natural frequency of the pendulum is shifted from ω to $\sqrt{\omega^2 + \Omega^2\cos^2\alpha}$ by the Earth's rotation. This conclusion, however, is premature. In the course of our calculations we have consistently neglected all terms of order Ω^2, starting with the Lagrangian of Eq. (2.7.49). We must continue to do so, and the previous equation must be approximated by

$$\ddot{\theta} + \omega^2\theta = 0. \tag{2.7.55}$$

This is still the equation for simple harmonic motion, with the original natural frequency $\omega = \sqrt{g/\ell}$. The general solution to this equation is $\theta(t) = \theta_0\cos(\omega t + \delta)$, where θ_0 and δ are constants.

To sum up, we have found that to first order in Ω, the pendulum swings as a simple harmonic oscillator, but that it does so in a plane that rotates around the vertical direction with an angular velocity $\Omega_{\text{plane}} = -\Omega\cos\alpha$.

2.8 Problems

1. **(a)** Find the curve $y(x)$ that passes through the endpoints $(0,0)$ and $(1,1)$ and minimizes the functional

$$I[y] = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 - y^2\right] dx.$$

(b) What is the minimum value of the functional?

(c) Evaluate $I[y]$ for a straight line that passes through the same two endpoints. Is this smaller or larger than your answer in part (b)?

2. You are mounting an expedition to reach the other side of a volcano, and you wish to determine the path that will minimize the distance traveled. To perform this calculation you decide to use cylindrical coordinates (ρ, ϕ, z) and you model the volcano as the conical surface $z = 1 - \rho$. [The cylindrical coordinates are defined by $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.] You describe the path by the function $\rho(\phi)$ and let the angle ϕ range through the interval $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$. The starting point of the expedition is $(\rho = 1, \phi = -\frac{\pi}{2}, z = 0)$ and the end point is $(\rho = 1, \phi = +\frac{\pi}{2}, z = 0)$. You wish to find the path $\rho(\phi)$ that minimizes the total distance traveled from this side of the volcano to the other side.

(a) Prove that the functional that must be minimized is

$$s[\rho] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2\rho'^2 + \rho^2}\, d\phi,$$

where $\rho' = d\rho/d\phi$.

(b) Find the differential equation that the minimal path must satisfy.

(c) Show that

$$\rho(\phi) = \frac{\cos\left(\frac{\pi}{2\sqrt{2}}\right)}{\cos\left(\frac{\phi}{\sqrt{2}}\right)}$$

is a solution to this differential equation, and that it satisfies the boundary conditions; conclude that this must be the minimal path. Produce a plot of $z = 1 - \rho$ as a function of ϕ.

(d) Calculate the minimum distance $s_{\min}$. Compare this with the distance that would be traveled if the path were instead chosen to be $\rho(\phi) = 1$.

3. A bead of mass m slides on a frictionless wire that is shaped in the form of a cycloid. This is described by the parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 + \cos\theta),$$

where a is a constant and the parameter θ ranges through the interval $0 \le \theta \le 2\pi$. The bead is subjected to gravity, and it oscillates back and forth on the wire.

(a) Using θ as a generalized coordinate, calculate the bead's Lagrangian.

(b) Show that the equation of motion for the bead is

$$2(1 - \cos\theta)\ddot{\theta} + \sin\theta\,\dot{\theta}^2 - \frac{g}{a}\sin\theta = 0.$$

(c) Show that the transformation $u = \cos(\frac{1}{2}\theta)$ brings this equation to the much simpler form

$$\ddot{u} + \omega^2 u = 0,$$

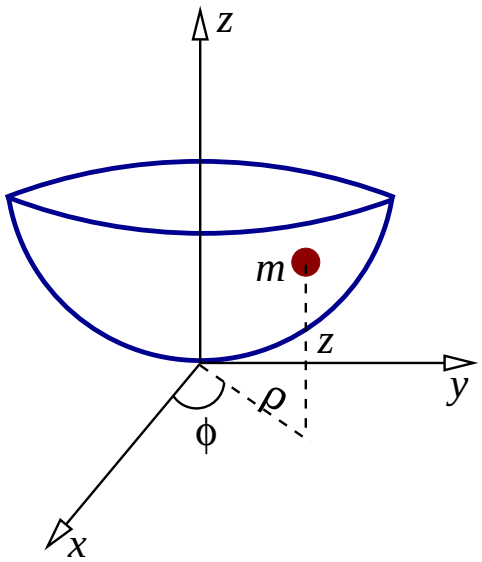
and find an expression for ω.

(d) What is the period of the bead's oscillations?

4. A particle of mass m moves on a paraboloid of revolution described by the equation

$$z = \frac{1}{a}(x^2 + y^2),$$

where a is a constant (see the figure). The particle is subjected to gravity, so that its potential energy is $V = mgz$. Using the cylindrical coordinates ρ and ϕ as generalized coordinates, find the Lagrangian of the particle. [The cylindrical coordinates are defined by $x = \rho\cos\phi$, $y = \rho\sin\phi$.]



5. A straight frictionless wire is attached at a height h to the z axis, and it makes an angle α relative to the z axis. The wire rotates around the z axis with a constant angular velocity Ω. A bead of mass m slides on the wire and is subjected to gravity; it is at a distance r from the point at which the wire is attached to the z axis (see the figure).

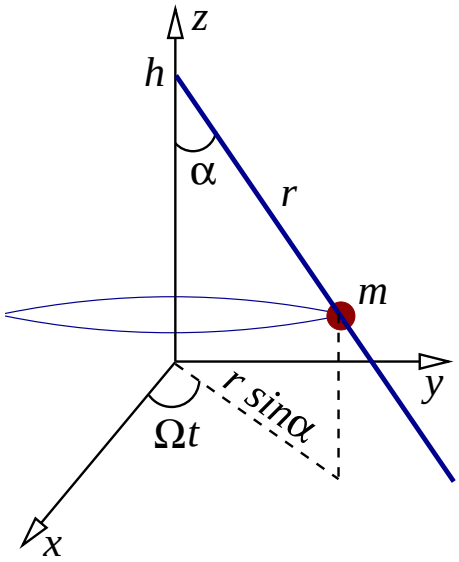
(a) Using r as a generalized coordinate, calculate the bead's Lagrangian.

(b) Obtain the equation of motion for the bead.

(c) Solve the equation of motion, assuming that the bead starts from rest at the point of attachment; this means that $r(t = 0) = 0$ and $\dot{r}(t = 0) = 0$. Show that your solution can be expressed in the form

$$r(t) = \frac{g\cos\alpha}{k^2}\Big[\cosh(kt) - 1\Big],$$

where $k = \Omega\sin\alpha$.



6. In Sec. 2.4.5 we examined the motion of a planar pendulum whose pivot point was forced to rotate with a constant angular velocity Ω. Here we consider instead a planar pendulum whose pivot point is forced to move *horizontally* with a constant acceleration a. This motion takes place in the x direction, and $x_{\text{pivot}} = \frac{1}{2}at^2$.

   (a) Using the swing angle θ as a generalized coordinate, find the pendulum's Lagrangian.

   (b) Derive the equation of motion for the pendulum.

   (c) Show that the pendulum can be in an equilibrium state in which $\theta(t) = \theta_{\text{eq}} = \text{constant}$. Show that the equilibrium position is determined by

   $$\tan\theta_{\text{eq}} = -\frac{a}{g}.$$

   (d) Suppose that the pendulum oscillates about its equilibrium position, so that $\theta = \theta_{\text{eq}} + \phi$, with ϕ denoting the angular deviation away from equilibrium. Assuming that ϕ is a small angle, show that its behaviour is governed by an equation of the form

   $$\ddot{\phi} + \omega^2\phi = 0.$$

   Find an expression for ω^2 in terms of g, a, and ℓ.

7. In this problem we examine the motion of the same pendulum as in the preceding problem, but we now let the pivot point move *vertically upward* with a constant acceleration a. Find the pendulum's Lagrangian and derive its equation of motion.

8. A particle of mass m is constrained to move on the surface of a cylinder. The cylinder is described in cylindrical coordinates by the equation $\rho = R$, where ρ is the distance from the z axis and R is the cylinder's radius. The particle is subjected to a force directed toward the origin of the coordinate system and proportional to the distance between the particle and the origin; this force is described by $\boldsymbol{F} = -k\boldsymbol{r}$, where k is a constant and $\boldsymbol{r}$ is the particle's position vector.

   (a) Using the cylindrical coordinates z and ϕ as generalized coordinates, find the particle's Lagrangian.

(b) Derive the particle's equations of motion and find their general solutions.

9. A particle of mass m and electric charge q moves in the presence of a vector potential

$$\boldsymbol{A} = \frac{1}{2} B_0 \big(-y\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}}\big),$$

where B_0 is a constant.

(a) What is the magnetic field $\boldsymbol{B}$?

(b) What is the particle's Lagrangian?

(c) What are the particle's equations of motion?

(d) What is the general solution to these equations? Describe how the particle moves in this magnetic field.

10. A particle of mass m and electric charge q moves in the presence of a vector potential

$$\boldsymbol{A} = \frac{E_0}{kc} \sin\big[k(z - ct)\big]\,\hat{\boldsymbol{x}},$$

where E_0 is a constant, c is the speed of light, and k is another constant.

(a) What are the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$? What kind of electromagnetic field does this vector potential represent?

(b) What is the particle's Lagrangian?

(c) What are the particle's equations of motion?

(d) Find the general solution to these equations in the nonrelativistic limit, in which $\dot{x}/c \ll 1$, $\dot{y}/c \ll 1$, and $\dot{z}/c \ll 1$.

11. A plumb bob at rest near Earth's surface, in a laboratory at colatitude α, is subjected to a force $\boldsymbol{F} = m\boldsymbol{g} + \boldsymbol{F}_{\text{centrifugal}}$ when viewed in a local rotating frame. We write this force as $\boldsymbol{F} = m\boldsymbol{g}_{\text{eff}}$ and define

$$\boldsymbol{g}_{\text{eff}} = \boldsymbol{g} + \boldsymbol{F}_{\text{centrifugal}}/m$$

as the effective gravitational field felt by the plumb bob. Assuming that $\Omega^2 R/g$ is a small number (which it is), calculate:

(a) The fractional difference $|g_{\text{eff}} - g|/g$ between the magnitudes of the effective and true gravitational fields; quote your result as a percentage.

(b) The angle that $\boldsymbol{g}_{\text{eff}}$ makes relative to the z direction; quote your result in degrees.

Assume that the laboratory is situated in Guelph, at a colatitude of 47°.

12. A projectile is launched from Earth's surface with an initial velocity $\boldsymbol{v}(t = 0) = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}} + v_3\hat{\boldsymbol{z}}$. The launch pad is situated at the origin of a local rotating frame at colatitude α, and the motion of the projectile is examined in this reference frame. Working consistently to first order in Ω (and therefore neglecting centrifugal effects), obtain the motion of the projectile for all times t. In other words, solve the equations of motion for the functions $x(t)$, $y(t)$, and $z(t)$, incorporating the initial conditions $x(0) = y(0) = z(0) = 0$, as well as $\dot{x}(0) = v_1$, $\dot{y}(0) = v_2$, and $\dot{z}(0) = v_3$.

13. A projectile is launched directly upward from Earth's surface (so that $v_1 = v_2 = 0$ but $v_3 \neq 0$). The launch pad is situated at the origin of a local rotating frame at colatitude α, and the motion of the projectile is examined in this reference frame. Working consistently to first order in Ω, calculate the projectile's position when it finally hits the ground. Express your result in terms of g, Ω, α, as well as h, the maximum height reached by the projectile. In which direction is the projectile displaced? [This problem is a special case of the preceding problem. You may find it useful to solve the general problem first.]

14. A cannonball is fired due east with an initial speed v_0, and with an angle θ with the horizontal. The cannon is placed at the origin of a local rotating frame at colatitude α, and the motion of the cannonball is examined in this reference frame. Working consistently to first order in Ω, calculate the cannonball's lateral displacement when it finally hits the ground. In which direction is it displaced? Does this direction depend on whether the cannon is in the northern or southern hemisphere? [The same remark as in the preceding problem applies.]

2.9 Additional problems

1. Two particles of mass m_1 and m_2 are attached to a string of constant length ℓ. The first particle moves on a frictionless table. The string goes through a hole in the middle of the table, and the second particle swings underneath the table. The first particle therefore moves in the x-y plane under the action of an attractive force directed toward the hole, and the second particle behaves as a planar pendulum with a variable distance to the pivot point.

   The first particle is at a distance r to the hole, and its position vector makes an angle χ relative to the x axis. The swing angle of the second particle (relative to the vertical) is denoted ψ.

   Using r, χ, and ψ as generalized coordinates, find the Lagrangian of this mechanical system.

2. In an experiment designed to measure the Coriolis effect, a particle of mass m is set to move on a large frictionless table. The table is placed in a laboratory at colatitude α, and the table is oriented along the x and y directions (with x pointing south and y pointing east). The Earth's angular velocity is Ω.

   At all times the particle moves on the table with $z = 0$. It begins its motion (when $t = 0$) at the origin $x = y = 0$ of the reference frame. Initially it is heading due south, with a velocity $\boldsymbol{v}(t = 0) = v_0\,\hat{\boldsymbol{x}}$ as measured in the local rotating frame. At later times the particle is observed to move laterally, as predicted by the Coriolis effect.

   Calculate the lateral displacement y as a function of the forward displacement x. You must perform the calculation consistently to first order in Ω, but you may neglect all terms of order Ω^2 (and higher powers).

   Express your result for y in terms of x, Ω, α, and v_0.

3. A particle of mass m moves on the interior surface of a hollow hemisphere of radius a. The particle's position on the hemisphere is determined by the usual angles θ and ϕ.

   (a) Show that the particle's Lagrangian is

   $$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mga\cos\theta.$$

(b) Derive the equations of motion for the particle. Show that the equation for θ can be expressed in the form

$$\frac{1}{2}\dot{\theta}^2 + \nu(\theta) = \varepsilon,$$

and find an expression for the effective potential $\nu(\theta)$. (The constant ε is proportional to the particle's total mechanical energy.)

(c) Provide a rough sketch of $\nu(\theta)$.

(d) Show that a possible solution to the equations of motion is $\theta(t) = \theta_0 =$ constant.

(e) Calculate the particle's speed v when it follows the path described in part (d); express your result in terms of a, g, and θ_0.

Chapter 3
Hamiltonian mechanics

3.1 From Lagrange to Hamilton

As we saw in Chapter 2, the Lagrangian formulation of the laws of mechanics offers increased flexibility and efficiency relative to the Newtonian methods, and it is based on an appealing principle of least action. In this chapter we add a layer of mathematical sophistication to this formulation of mechanics. The resulting *Hamiltonian formulation* of the laws of mechanics gives this area of theoretical physics an aura of perfection that has probably not been surpassed by any other area of theoretical physics.

The main goal of the Hamiltonian formulation is to displace the emphasis from the generalized velocities $\dot{q}_a$ to the generalized momenta p_a, and from the Lagrangian $L(q_a, \dot{q}_a, t)$ to a new function $H(q_a, p_a, t)$ called the *Hamiltonian function* of the mechanical system, which is numerically equal to the system's total mechanical energy. The motivation behind this shift of emphasis is clear: While the generalized velocities are rarely conserved quantities, the generalized momenta sometimes are, and while the Lagrangian is never conserved, the Hamiltonian usually is. The Hamiltonian formulation therefore involves all the dynamical quantities that have a chance of being constants of the motion, and this constitutes a useful and interesting refinement of the original Lagrangian methods.

3.1.1 Hamilton's canonical equations

To see how the reformulation is accomplished, let us go back to Eq. (2.5.4), which gives the definition of the function $h(q_a, \dot{q}_a, t)$, which is also numerically equal to the total mechanical energy of the system. This is

$$h(q_a, \dot{q}_a, t) = \sum_a p_a \dot{q}_a - L(q_a, \dot{q}_a, t), \tag{3.1.1}$$

where

$$p_a = \frac{\partial L}{\partial \dot{q}_a} \tag{3.1.2}$$

is the generalized momentum associated with the generalized coordinate q_a. Notice that here we allow L and h to depend explicitly on time. And notice that the energy function is denoted h, not H; we will explain this distinction later.

We construct the total differential of h:

$$dh = \sum_a (\dot{q}_a \, dp_a + p_a \, d\dot{q}_a) - dL.$$

To calculate dL we invoke the chain rule, and write

$$dL = \sum_a \left(\frac{\partial L}{\partial q_a} dq_a + \frac{\partial L}{\partial \dot{q}_a} d\dot{q}_a \right) + \frac{\partial L}{\partial t} dt.$$

Combining these results gives

$$dh = \sum_a \left[\dot{q}_a \, dp_a + \left(p_a - \frac{\partial L}{\partial \dot{q}_a} \right) d\dot{q}_a - \frac{\partial L}{\partial q_a} dq_a \right] - \frac{\partial L}{\partial t} dt.$$

From the definition of the generalized momentum we recognize that the coefficient of $d\dot{q}_a$ is zero. And since the Euler-Lagrange (EL) equations can be expressed in the form $\dot{p}_a = \partial L / \partial q_a$, what we have is

$$dh = \sum_a (\dot{q}_a \, dp_a - \dot{p}_a \, dq_a) - \frac{\partial L}{\partial t} dt. \tag{3.1.3}$$

Suppose now that h is given as a function of q_a, p_a, and t. Then it would follow as a matter of mathematical identity that the total differential of $h(q_a, p_a, t)$ is

$$dh = \sum_a \left(\frac{\partial h}{\partial q_a} dq_a + \frac{\partial h}{\partial p_a} dp_a \right) + \frac{\partial h}{\partial t} dt.$$

Comparing this with Eq. (3.1.3) reveals that we can make the identifications

$$\dot{q}_a = \frac{\partial h}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial h}{\partial q_a}, \qquad \frac{\partial L}{\partial t} = -\frac{\partial h}{\partial t}.$$

The first two equations are evolution equations for the dynamical variables $q_a(t)$ and $p_a(t)$. These are almost Hamilton's equations, except for one important subtlety.

The previous identifications can be made if and only if the function h is expressed in terms of q_a, p_a, and t. If it is so expressed, then we have learned that $\dot{q}_a$ is the partial derivative of h with respect to p_a keeping q_a constant, while $\dot{p}_a$ is (minus) the partial derivative of h with respect to q_a keeping p_a constant. Our function h, however, has not yet been expressed in terms of the new variables; it is still expressed in terms of the old variables q_a, $\dot{q}_a$, and t. Before we can write down Hamilton's equations we must solve for $\dot{q}_a$ in terms of q_a and p_a, and we must make the substitution in h. We must therefore evaluate

$$h\big(q_a, \dot{q}_a(q_a, p_a), t\big) \equiv H(q_a, p_a, t), \tag{3.1.4}$$

and *this* is what we shall call the Hamiltonian function of the mechanical system. The functions h and H are numerically equal, they both represent the total mechanical energy of the system, but only the Hamiltonian H is the required function of q_a, p_a, and t.

Having clarified this point and made the change of variables from $(q_a, \dot{q}_a)$ to (q_a, p_a), we can finally write down Hamilton's equations,

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}. \tag{3.1.5}$$

This system of equations is formally equivalent to the original set of EL equations. But instead of representing a set of n second-order differential equations for the coordinates $q_a(t)$ — there is one equation for each of the n degrees of freedom — Hamilton's equations represent a set of $2n$ first-order differential equations for the new dynamical variables $q_a(t)$ and $p_a(t)$. The generalized momenta are now put on an equal footing with the generalized coordinates.

The recipe to arrive at Hamilton's canonical equations goes as follows:

1. Begin with the Lagrangian $L(q_a, \dot{q}_a, t)$ of the mechanical system, expressed in terms of any set of generalized coordinates q_a and the corresponding generalized velocities $\dot{q}_a$.

2. Construct the generalized momenta $p_a = \partial L/\partial \dot{q}_a$, and solve for the generalized velocities to obtain $\dot{q}_a(q_a, p_a, t)$.

3. Construct the Hamiltonian function

$$H(q_a, p_a, t) = \sum_a p_a \dot{q}_a - L$$

and express the result entirely in terms of q_a, p_a, and t; at this stage the generalized velocities have completely disappeared from sight.

4. Formulate Hamilton's equations,

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}$$

and solve the equations of motion for $q_a(t)$ and $p_a(t)$; observe that the generalized velocities $\dot{q}_a$ have reappeared, but now as a consequence of the dynamical equations.

Concrete applications of this recipe will be given in the next section. For the time being we prefer to explore some of the formal consequences of Hamilton's formulation of the laws of mechanics.

3.1.2 Conservation statements

We begin with an examination of what Hamilton's equations have to say regarding the existence of constants of the motion.

It follows immediately from the dynamical equation

$$\dot{p}_a = -\frac{\partial H}{\partial q_a}$$

that if the Hamiltonian H happens not to depend explicitly on one of the generalized coordinates, say q_*, then $\partial H/\partial q_* = 0$ and $\dot{p}_* = 0$. This means that p_* will be a constant of the motion, and we have established the theorem:

> Whenever the Hamiltonian of a mechanical system does not depend explicitly on a generalized coordinate q_*, the corresponding generalized momentum p_* is a constant of the motion.

We made a similar statement back in Sec. 2.5.1, but in terms of the Lagrangian instead of the Hamiltonian.

Hamilton's equations allow us also to state another theorem, which is very similar: Whenever the Hamiltonian of a mechanical system does not depend explicitly on a generalized momentum p_*, the corresponding generalized coordinate q_* is a constant of the motion. This statement is a true consequence of Hamiltonian dynamics, but it is less useful in practice: If q_* were a constant of the motion it is likely that it would not have been selected as a coordinate in the first place!

What do Hamilton's equations have to say about conservation of energy? To answer this let us consider a general Hamiltonian of the form $H(q_a, p_a, t)$, which includes an explicit dependence on t. Its total time derivative is

$$\frac{dH}{dt} = \sum_a \left(\frac{\partial H}{\partial q_a}\dot{q}_a + \frac{\partial H}{\partial p_a}\dot{p}_a \right) + \frac{\partial H}{\partial t}.$$

By Hamilton's equations this becomes

$$\frac{dH}{dt} = \sum_a \left(-\dot{p}_a \dot{q}_a + \dot{q}_a \dot{p}_a \right) + \frac{\partial H}{\partial t},$$

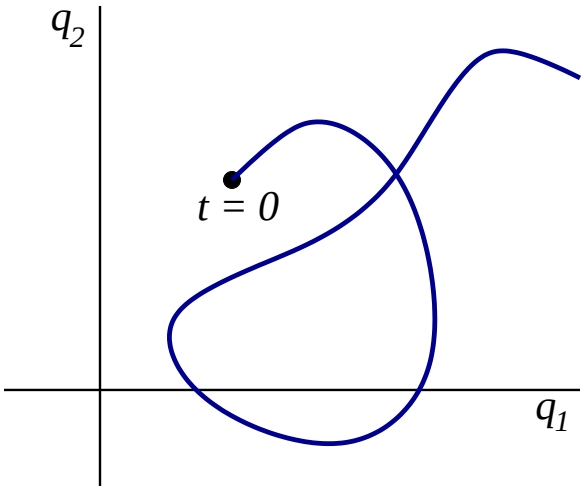


Figure 3.1: A trajectory in a two-dimensional configuration space. It is possible for the trajectory to intersect itself, because the system can go back to the same position after a given interval of time.

or

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{3.1.6}$$

This gives us the statement

> Whenever the Hamiltonian of a mechanical system does not depend explicitly on t, it is a constant of the motion: $dH/dt = 0$.

Recall that back in Sec. 2.5.2 we derived the relation $dh/dt = -\partial L/\partial t$. Here we have instead $dH/dt = \partial H/\partial t$. These statements are compatible by virtue of the fact that $\partial L/\partial t = -\partial H/\partial t$; you will recall that we came across this identification back in Sec. 3.1.1.

3.1.3 Phase space

Suppose that a mechanical system possesses n degrees of freedom represented by n generalized coordinates q_a (with $a = 1, 2, \cdots, n$ labeling each one of the n coordinates, as usual). The n-dimensional space spanned by the q_a's is called the *configuration space* of the mechanical system. The motion of the entire system can be represented by a trajectory in configuration space, and the generalized velocities $\dot{q}_a$ represent the tangent to this trajectory. This is illustrated in Fig. 3.1. The figure shows that a trajectory in configuration space can cross itself: The system could return later to a position q_a with a different velocity $\dot{q}_a$.

The Hamiltonian formulation of the laws of mechanics gives us an alternative way of representing the motion. Because the coordinates q_a and the momenta p_a are placed on an equal footing, it is natural to form a $2n$-dimensional space that will be spanned by the n coordinates and the n momenta. This new space is called the *phase space* of the mechanical system. While the phase space is twice as large as the configuration space, it allows a much simpler representation of the motion. The reason is that a point (q_a, p_a) in phase space represents the complete state of motion of a mechanical system at a given time; by identifying the point we obtain the complete information about the positions and momenta of all the particles within the system. (By contrast, in configuration space the complete state of motion would be represented by a point and the tangent to a trajectory that passes through this point.) As the coordinates and momenta change with time the mechanical system

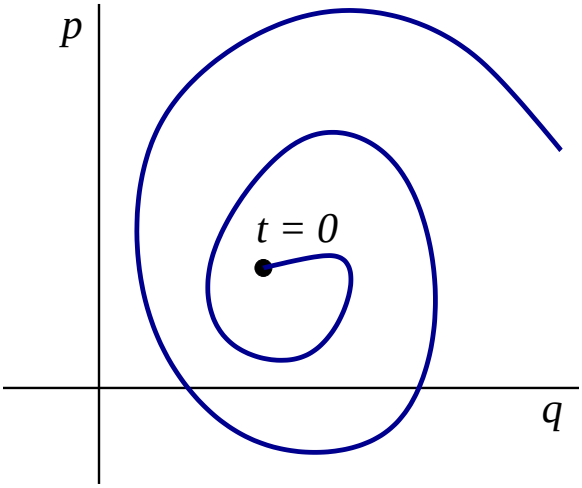


Figure 3.2: A trajectory in a two-dimensional phase space. So long as the Hamiltonian does not depend explicitly on time, it is impossible for the trajectory to intersect itself.

traces a trajectory in phase space; each point on this curve represents a new time and a new state of motion. The tangent to a phase-space trajectory gives the phase-space velocity field, $(\dot{q}_a, \dot{p}_a)$. This is illustrated in Fig. 3.2.

So long as the Hamiltonian does not depend explicitly on time, *a trajectory in phase space can never intersect itself*; as we have seen, this is quite unlike a trajectory in configuration space. This property of phase-space trajectories is another reason why motion in phase space is simpler than motion in configuration space. It follows from the fact that the Hamiltonian $H(q_a, p_a)$ is a single-valued function of its arguments: there is only one value of H at each point in phase space. To see the connection, observe that if H is single-valued, then its partial derivatives $\partial H/\partial q_a$ and $\partial H/\partial p_a$ will be single-valued also; and by Hamilton's equations this implies that the tangent $(\dot{q}_a, \dot{p}_a)$ to a phase-space trajectory is a single-valued vector field over phase space. If the tangent of a trajectory is unique at every phase-space point, the trajectory can never intersect itself.

Exercise 3.1. Explain why this conclusion does not apply when the Hamiltonian depends explicitly on time.

When the total energy of the mechanical system is conserved we find that the coordinates and momenta are constrained by the energy equation $H(q_a, p_a) = E =$ constant. In this case the motion will proceed on a fixed "surface" in phase space; this "surface", which is called an energy surface, has an intrinsic dimensionality of $2n - 1$. The existence of other constants of the motion would also restrict the motion to a "surface" of lower dimensionality. We will encounter specific examples of such "surfaces" in the next section.

3.1.4 Hamilton's equations from Hamilton's principle

Hamilton's equations can be derived directly from Hamilton's principle of least action, $\delta S = 0$. For this purpose the action functional must be expressed in terms of the Hamiltonian instead of the Lagrangian. Because $H = \sum_a p_a \dot{q}_a - L$, we write it as

$$S = \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1} \left(\sum_a p_a \dot{q}_a - H \right) dt,$$

or

$$S = \int_{t_0}^{t_1} \Big(\sum_a p_a \, dq_a - H \, dt \Big). \tag{3.1.7}$$

The action $S[q_a, p_a]$ must now be thought of as a functional of the variables $q_a(t)$ and $p_a(t)$; all these variables are considered to be independent of each other — they become connected only *after* the action has been extremized and the dynamical equations have been imposed. What we have, therefore, is a multi-path functional that depends on n paths $q_a(t)$ and n additional paths $p_a(t)$. Alternatively, we may think of $S[q_a, p_a]$ as a functional of a *single path* in a $2n$-dimensional phase space.

We intend to compute how the action of Eq. (3.1.7) changes when the paths are displaced relative to some reference paths $\bar{q}_a(t)$ and $\bar{p}_a(t)$. We will derive the equations of motion by demanding that $\delta S = 0$ to first order in the displacements $\delta q_a(t)$ and $\delta p_a(t)$. We will impose the boundary conditions $\delta q_a(t_0) = \delta q_a(t_1) = 0$: As in the usual form of the variational principle, all paths must begin and end at the same end points in configuration space, $q_a(t_0)$ and $q_a(t_1)$. We will not, however, impose any conditions on the variations $\delta p_a(t)$; these remain completely free, including at $t = t_0$ and $t = t_1$.

The variation of the action is given by

$$\begin{aligned} \delta S &= \int_{t_0}^{t_1} \Big[\sum_a \big(dq_a \, \delta p_a + p_a \, d\delta q_a \big) - \sum_a \Big(\frac{\partial H}{\partial q_a} \delta q_a + \frac{\partial H}{\partial p_a} \delta p_a \Big) \, dt \Big] \\ &= \int_{t_0}^{t_1} \sum_a \Big[p_a \, d\delta q_a + \Big(dq_a - \frac{\partial H}{\partial p_a} dt \Big) \delta p_a - \frac{\partial H}{\partial q_a} dt \, \delta q_a \Big]. \end{aligned}$$

To simplify this we write

$$p_a \, d\delta q_a = d\big(p_a \delta q_a \big) - \delta q_a \, dp_a,$$

and we integrate the first term. This gives

$$\delta S = \sum_a p_a \delta q_a \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_a \Big[\Big(dq_a - \frac{\partial H}{\partial p_a} dt \Big) \delta p_a - \Big(dp_a + \frac{\partial H}{\partial q_a} dt \Big) \delta q_a \Big],$$

or

$$\delta S = \int_{t_0}^{t_1} \sum_a \Big[\Big(\frac{dq_a}{dt} - \frac{\partial H}{\partial p_a} \Big) \delta p_a - \Big(\frac{dp_a}{dt} + \frac{\partial H}{\partial q_a} \Big) \delta q_a \Big] dt$$

by virtue of the boundary conditions on $\delta q_a(t)$. Because the variations δq_a and δp_a are arbitrary and independent of each other in the interval $t_0 < t < t_1$, we conclude that

$$\delta S = 0 \qquad \Rightarrow \qquad \dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}. \tag{3.1.8}$$

These are, once more, Hamilton's canonical equations.

3.2 Applications of Hamiltonian mechanics

3.2.1 Canonical equations in Cartesian coordinates

The Lagrangian of a particle moving in a potential $V(x, y, z)$ expressed in Cartesian coordinates is

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{x}^2) - V(x, y, z). \tag{3.2.1}$$

The momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and so on, and the Hamiltonian is $H = p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-L$. Expressing this entirely in terms of the coordinates and momenta, we obtain

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z). \tag{3.2.2}$$

At this state the velocities $\dot{x}$, $\dot{y}$, and $\dot{z}$ are no longer part of our description.

The canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

and so on, as well as

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

and so on. Summarizing these equations in vectorial form, we have

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{p}}{m}, \qquad \dot{\boldsymbol{p}} = -\boldsymbol{\nabla} V. \tag{3.2.3}$$

Notice that the first equation reproduces the relationship between $\boldsymbol{p}$ and $\dot{\boldsymbol{r}}$ that was worked out previously. This first-order system of differential equations is of course equivalent to the second-order system $m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla} V$, which is just Newton's old law; this is obtained by eliminating $\boldsymbol{p}$ from Eqs. (3.2.3).

3.2.2 Canonical equations in cylindrical coordinates

The Lagrangian of a particle moving in a potential $V(\rho, \phi, z)$ expressed in cylindrical coordinates was worked out in Sec. 2.4.1. According to Eq. (2.4.3), it is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z). \tag{3.2.4}$$

Recall that the cylindrical coordinates are related to the Cartesian coordinates by $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.

The momenta are

$$\begin{aligned} p_\rho &= \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \end{aligned}$$

The Hamiltonian is $H = p_\rho\dot{\rho} + p_\phi\dot{\phi} + p_z\dot{z} - L$. Expressing this entirely in terms of the coordinates and the momenta, we obtain

$$H = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + V(\rho, \phi, z). \tag{3.2.5}$$

At this stage the velocities $\dot{\rho}$, $\dot{\phi}$, and $\dot{z}$ are no longer part of our description.

Exercise 3.2. Go through the algebra that leads to Eq. (3.2.5).

The first set of canonical equations are

$$\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}, \tag{3.2.6}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}, \tag{3.2.7}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}. \tag{3.2.8}$$

Notice that these equations reproduce the relationships between the momenta and velocities that were worked out previously. The second set of canonical equations are

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = -\frac{\partial V}{\partial \rho} + \frac{p_\phi^2}{m\rho^3}, \tag{3.2.9}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}, \tag{3.2.10}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}. \tag{3.2.11}$$

If we eliminated the momenta from this system of first-order differential equations we would find that they are equivalent to the second-order equations listed in Eqs. (2.4.4)–(2.4.6).

Exercise 3.3. Verify this last statement.

3.2.3 Canonical equations in spherical coordinates

The Lagrangian of a particle moving in a potential $V(r, \theta, \phi)$ expressed in spherical coordinates was worked out in Sec. 2.4.2. According to Eq. (2.4.9), it is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - V(r, \theta, \phi). \tag{3.2.12}$$

Recall that the spherical coordinates are related to the Cartesian coordinates by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.

The momenta are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta\, \dot{\phi}. \end{aligned}$$

The Hamiltonian is $H = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$. Expressing this entirely in terms of the coordinates and the momenta, we obtain

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r, \theta, \phi). \tag{3.2.13}$$

At this stage the velocities $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ are no longer part of our description.

Exercise 3.4. Go through the algebra that leads to Eq. (3.2.13).

The first set of canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \tag{3.2.14}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \tag{3.2.15}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}. \tag{3.2.16}$$

Notice that these equations reproduce the relationships between the momenta and velocities that were worked out previously. The second set of canonical equations are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\partial V}{\partial r} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta}, \tag{3.2.17}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta} + \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}, \tag{3.2.18}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}. \tag{3.2.19}$$

If we eliminated the momenta from this system of first-order differential equations we would find that they are equivalent to the second-order equations listed in Eqs. (2.4.10)–(2.4.12).

Exercise 3.5. Verify this last statement.

3.2.4 Planar pendulum

For our first real application of the Hamiltonian framework we reintroduce the planar pendulum of Sec. 1.3.7. The Lagrangian of this mechanical system was first written down in Sec. 2.1; it is

$$L = m\ell^2\left(\frac{1}{2}\dot{\theta}^2 + \omega^2 \cos\theta\right). \tag{3.2.20}$$

Here m is the mass of the pendulum, ℓ is the length of its rigid rod, θ is the swing angle, and $\omega^2 = g/\ell$, where g is the acceleration of gravity. This mechanical system has a single degree of freedom that is represented by the generalized coordinate θ.

The generalized momentum associated with θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta},$$

and this equation can be inverted to give $\dot{\theta}$ is terms of p_θ. The pendulum's Hamiltonian is $H = p_\theta\dot{\theta} - L$, or

$$H = \frac{p_\theta^2}{2m\ell^2} - m\ell^2\omega^2 \cos\theta. \tag{3.2.21}$$

As usual, we find that at this stage the generalized velocity $\dot{\theta}$ is no longer part of our description.

Exercise 3.6. Verify Eq. (3.2.21).

The canonical equations for the Hamiltonian of Eq. (3.2.21) are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \tag{3.2.22}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -m\ell^2\omega^2 \sin\theta. \tag{3.2.23}$$

If we eliminate p_θ from this system of equations we eventually obtain the second-order equation $\ddot{\theta} + \omega^2 \sin\theta = 0$, which is the same as Eq. (1.3.24).

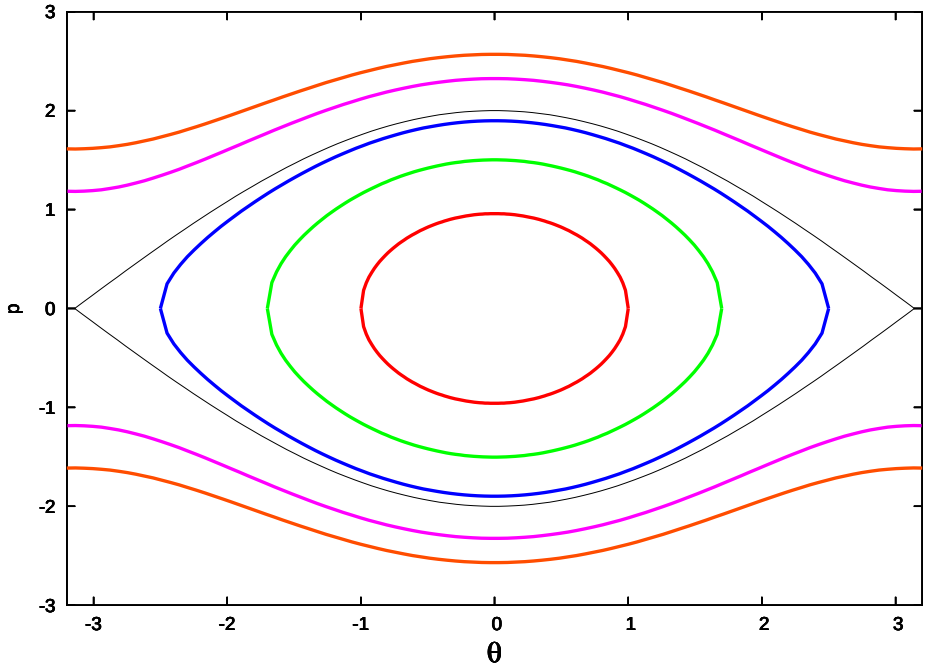


Figure 3.3: Phase trajectories of a planar pendulum. The innermost curves have a rescaled energy ε smaller than ω^2; they represent the bounded oscillations of a pendulum between the angles $\pm\theta_0$, where $\varepsilon = -\omega^2 \cos\theta_0$. The outermost curves have a rescaled energy larger than ω^2; they represent a pendulum undergoing complete revolutions instead of bounded oscillations. The thin black curve represents the marginal case $\varepsilon = \omega^2$.

The canonical equations must be integrated numerically if we wish to determine the functions $\theta(t)$ and $p_\theta(t)$. Because they are already presented as a system of first-order differential equations for the set (θ, p_θ) of dynamical variables, the numerical techniques introduced in Sec. 1.6 can be applied directly. This is one advantage of the Hamiltonian formulation of the laws of mechanics: the first-order form of the equations of motion means that they are directly amenable to numerical integration.

Because the pendulum's Hamiltonian does not depend explicitly on time, it is a constant of the motion. The dynamical variables of the mechanical system are therefore constrained by the equation

$$\frac{p_\theta^2}{2m\ell^2} - m\ell^2\omega^2 \cos\theta = E = \text{constant}, \tag{3.2.24}$$

where E is the pendulum's total mechanical energy. This equation describes a one-dimensional curve in the two-dimensional phase space of the mechanical system. This curve is the trajectory of the pendulum in phase space. A number of such phase trajectories are shown in Fig. 3.3.

To describe what is going on in Fig. 3.3 it is helpful to introduce the rescaled momentum $\mathsf{p} = p_\theta/(m\ell^2)$ and the rescaled energy $\varepsilon = E/(m\ell^2)$. In terms of these variables Eq. (3.2.24) becomes

$$\frac{1}{2}\mathsf{p}^2 - \omega^2 \cos\theta = \varepsilon,$$

and the phase trajectories are obtained by solving this for $\mathsf{p}(\theta)$. There are two solutions, one for which p is positive, and the other for which p is negative.

When $\varepsilon < \omega^2$ we find that the momentum vanishes whenever $\theta = \pm\theta_0$; the amplitude θ_0 of the motion is determined by $\varepsilon = -\omega^2 \cos\theta_0$. The motion is then

limited to the interval $-\theta_0 \le \theta \le \theta_0$, and we have the usual situation of a pendulum oscillating back and forth between the limits $\pm\theta_0$. The phase trajectories representing this bounded, oscillatory motion are closed curves that pass through $\mathsf{p} = 0$ whenever θ achieves its limiting values $\pm\theta_0$. The fact that these phase trajectories are closed reflects the fact that the motion of the pendulum is periodic.

When $\varepsilon > \omega^2$ we find that turning points can no longer occur: p never changes sign, and $\theta(t)$ increases (or decreases) monotonically. In this case the pendulum does not oscillate; instead it undergoes complete revolutions. The phase trajectories representing this unbounded motion are open curves in the two-dimensional phase space.

The phase trajectory that represents the motion of a pendulum with $\varepsilon = \omega^2$ separates the closed curves that represent oscillatory motion and the open curves that represent the complete revolutions. This curve is called a *separatrix*.

3.2.5 Spherical pendulum

We next turn to the spherical pendulum, a mechanical system with two degrees of freedom. Its Lagrangian was derived in Sec. 2.2.4; according to Eq. (2.4.19), it is

$$L = m\ell^2\left(\frac{1}{2}\dot\theta^2 + \frac{1}{2}\sin^2\theta\,\dot\phi^2 + \omega^2\cos\theta\right).$$

Here m is the mass of the pendulum, ℓ is the length of its rigid rod, θ and ϕ give the angular position of the pendulum (the angles are defined in Fig. 2.10), and $\omega^2 = g/\ell$. The factor $m\ell^2$ in L multiplies each term, and its purpose is simply to give an overall scale to the Lagrangian; the factor accomplishes nothing else, and it would just come along for the ride in our further developments. To save ourselves some trouble we will eliminate this factor by rescaling our quantities. Thus we will deal with the rescaled Lagrangian $\mathsf{L} = L/(m\ell^2)$, the rescaled momenta $\mathsf{p}_\theta = p_\theta/(m\ell^2)$ and $\mathsf{p}_\phi = p_\phi/(m\ell^2)$, and the rescaled Hamiltonian $\mathsf{H} = H/(m\ell^2)$.

The (rescaled) Lagrangian is

$$\mathsf{L} = \frac{1}{2}\dot\theta^2 + \frac{1}{2}\sin^2\theta\,\dot\phi^2 + \omega^2\cos\theta, \tag{3.2.25}$$

and the (rescaled) momenta are

$$\begin{aligned}\mathsf{p}_\theta &= \frac{\partial\mathsf{L}}{\partial\dot\theta} = \dot\theta\\ \mathsf{p}_\phi &= \frac{\partial\mathsf{L}}{\partial\dot\phi} = \sin^2\theta\,\dot\phi.\end{aligned}$$

The (rescaled) Hamiltonian is $\mathsf{H} = \mathsf{p}_\theta\dot\theta + \mathsf{p}_\phi\dot\phi - \mathsf{L}$, and this becomes

$$\mathsf{H} = \frac{1}{2}\mathsf{p}_\theta^2 + \frac{\mathsf{p}_\phi^2}{2\sin^2\theta} - \omega^2\cos\theta \tag{3.2.26}$$

after expressing the velocities in terms of the momenta.

Exercise 3.7. Verify Eq. (3.2.26).

The canonical equations are

$$\dot\theta = \frac{\partial\mathsf{H}}{\partial\mathsf{p}_\theta} = \mathsf{p}_\theta, \tag{3.2.27}$$

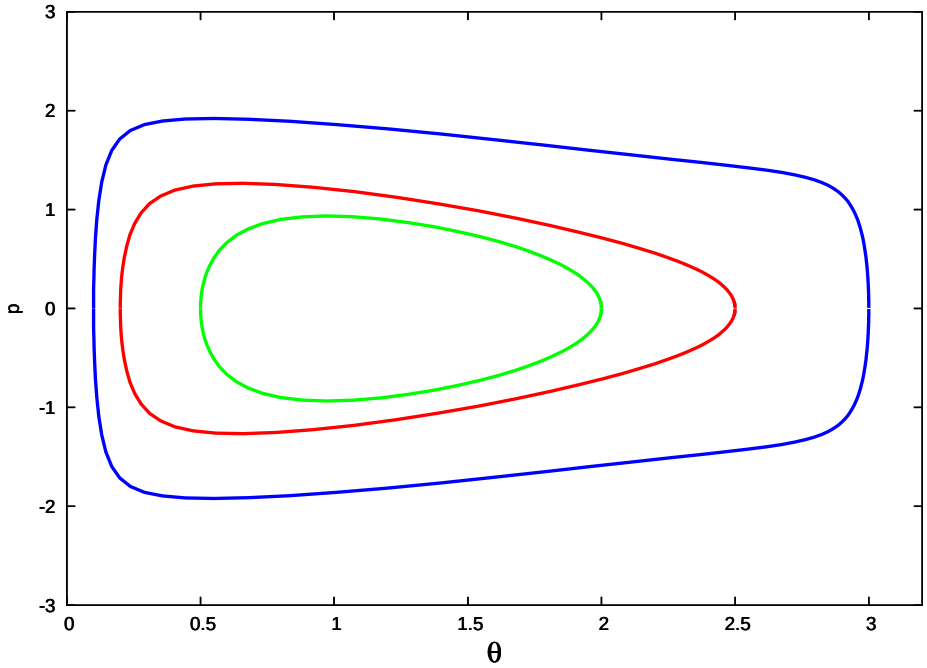


Figure 3.4: Phase trajectories of a spherical pendulum. The different curves have different values of ε but they share the same value of h. The fact that all the trajectories are closed indicates that the motion is always bounded and periodic.

$$\dot{\phi} = \frac{\partial \mathsf{H}}{\partial \mathsf{p}_\phi} = \frac{\mathsf{p}_\phi}{\sin^2\theta}, \tag{3.2.28}$$

$$\dot{\mathsf{p}}_\theta = -\frac{\partial \mathsf{H}}{\partial \theta} = \frac{\mathsf{p}_\phi^2 \cos\theta}{\sin^3\theta} - \omega^2 \sin\theta, \tag{3.2.29}$$

$$\dot{\mathsf{p}}_\phi = -\frac{\partial \mathsf{H}}{\partial \phi} = 0. \tag{3.2.30}$$

The last equation implies that p_ϕ is a constant of the motion; we shall set $\mathsf{p}_\phi = h =$ constant, as we have done in Eq. (2.4.21). Equations (3.2.27) and (3.2.29) must be integrated numerically to determine the functions $\theta(t)$ and $\mathsf{p}_\theta(t)$; when these are known Eq. (3.2.28) can be integrated for $\phi(t)$.

Exercise 3.8. Show that Eqs. (3.2.27) and (3.2.29) are equivalent to the second-order differential equation of Eq. (2.4.20).

The motion of the spherical pendulum in phase space can be described analytically. Because $\mathsf{p}_\phi \equiv h$ and $\mathsf{H} \equiv \varepsilon$ are constants of the motion, the phase trajectories are described by

$$\frac{1}{2}\mathsf{p}_\theta^2 + \frac{h^2}{2\sin^2\theta} - \omega^2 \cos\theta = \varepsilon. \tag{3.2.31}$$

This equation can be solved for p_θ, and the resulting curves are displayed in Fig. 3.4. Here the motion always takes place within the bounded interval $\theta_- < \theta < \theta_+$, where the limits $\theta_\pm$ are determined by the values of h and ε (the details are provided in Sec. 2.4.4). It should be noted that the phase space of the spherical pendulum is, strictly speaking, four-dimensional, because it is spanned by the coordinates θ, p_θ, ϕ, and p_ϕ. We have reduced this to an effective two-dimensional phase space by examining a "surface" $p_\phi =$ constant $= h$, and by discarding the ϕ direction.

3.2.6 Rotating pendulum

The rotating pendulum was first examined in Sec. 2.4.5. Here we have a planar pendulum whose pivot point is attached to a centrifuge: it rotates with an angular velocity Ω on a circle of radius a. The Lagrangian of this mechanical system is displayed in Eq. (2.4.25):

$$L = \frac{1}{2}m\Big[(a\Omega)^2 + 2a\ell\Omega\dot\theta\sin(\theta-\Omega t) + \ell^2\dot\theta^2\Big] - m\ell\omega^2(a\sin\Omega t - \ell\cos\theta).$$

Before we proceed we simplify this Lagrangian using the rules derived in Sec. 2.5.3: We discard the term $\frac{1}{2}m(a\Omega)^2$ because it is merely a constant, and we discard the term $-m\ell\omega^2 a\sin\Omega t$ because it is the time derivative of $(m\ell\omega^2 a/\Omega)\cos\Omega t$. The simplified Lagrangian is

$$L = m\ell^2\left[\frac{1}{2}\dot\theta^2 + \frac{a\Omega\dot\theta}{\ell}\sin(\theta-\Omega t) + \omega^2\cos\theta\right].$$

We simplify this further by rescaling away the common factor of $m\ell^2$. Our final (rescaled) Lagrangian is therefore

$$\mathsf{L} = \frac{1}{2}\dot\theta^2 + \frac{a\Omega\dot\theta}{\ell}\sin(\theta-\Omega t) + \omega^2\cos\theta. \tag{3.2.32}$$

It is noteworthy that this Lagrangian depends explicitly on time; this comes as consequence of the fact that the pendulum is driven at a frequency Ω.

The (rescaled) momentum associated with θ is

$$\mathsf{p} = \frac{\partial\mathsf{L}}{\partial\dot\theta} = \dot\theta + \frac{a\Omega}{\ell}\sin(\theta-\Omega t).$$

Notice that this is not simply equal to $\dot\theta$; here the momentum differs in an essential way from the generalized velocity. The (rescaled) Hamiltonian is $\mathsf{H} = \mathsf{p}\dot\theta - \mathsf{L}$. After expressing $\dot\theta$ in terms of p, this becomes

$$\mathsf{H} = \frac{1}{2}\left[\mathsf{p} - \frac{a\Omega}{\ell}\sin(\theta-\Omega t)\right]^2 - \omega^2\cos\theta. \tag{3.2.33}$$

Exercise 3.9. Verify Eq. (3.2.33).

The canonical equations are

$$\dot\theta = \frac{\partial\mathsf{H}}{\partial\mathsf{p}} = \mathsf{p} - \frac{a\Omega}{\ell}\sin(\theta-\Omega t), \tag{3.2.34}$$

$$\dot{\mathsf{p}} = -\frac{\partial\mathsf{H}}{\partial\theta} = \frac{a\Omega}{\ell}\cos(\theta-\Omega t)\left[\mathsf{p} - \frac{a\Omega}{\ell}\sin(\theta-\Omega t)\right] - \omega^2\sin\theta. \tag{3.2.35}$$

These equations must be integrated numerically, and the results can be displayed as curves in the two-dimensional phase space spanned by the generalized coordinate θ and its (rescaled) momentum p. This is done in Fig. 3.5 for selected values of Ω/ω.

Exercise 3.10. Show that Eqs. (3.2.34) and (3.2.35) are equivalent to Eq. (2.4.26).

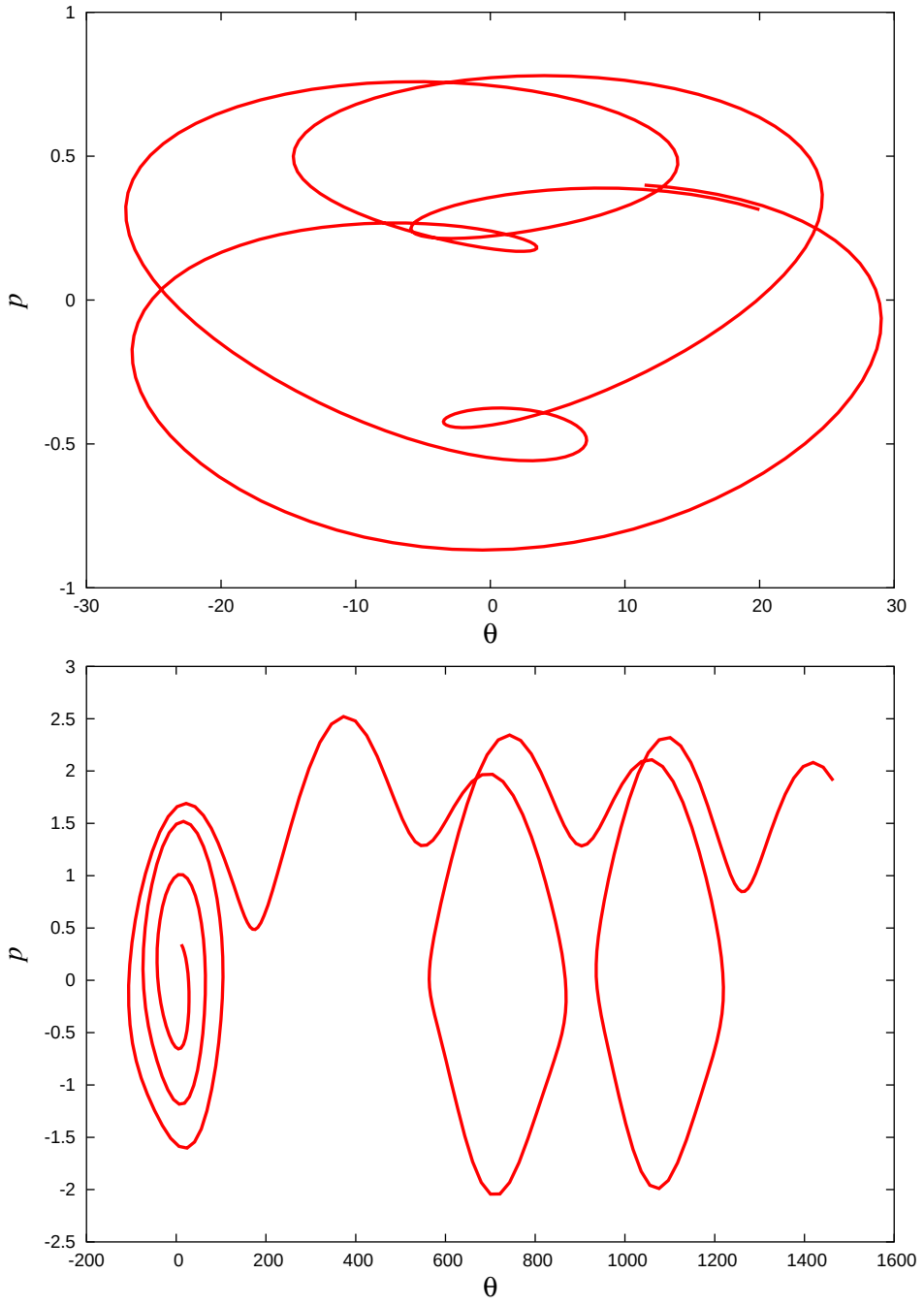


Figure 3.5: Phase trajectories of a rotating pendulum. The upper graph shows the motion in phase space of a pendulum driven at a frequency $\Omega = 0.4\omega$; the motion in configuration space can be seen in the upper graph of Fig. 2.14. The lower graph has $\Omega = 0.9\omega$ instead, and the motion in configuration space can be seen in Fig. 2.15. In both cases we set $(a/\ell)\Omega^2 = 0.2$ and use the same initial conditions as in Figs. 2.14 and 2.15. The motion in the upper graph is always confined to the interval $-30^\circ < \theta < 30^\circ$. The motion in the lower graph is not bounded: After oscillating a few times the pendulum is driven to go through a number of complete revolutions before going back to a brief oscillation cycle. In both cases the phase trajectories intersect themselves; this is possible because the Hamiltonian depends explicitly on time.

3.2.7 Rolling disk

The rolling disk was first examined in Sec. 2.4.6. Its Lagrangian was obtained in Eq. (2.4.30) and it is

$$L = \frac{3}{4}mR^2\dot{\theta}^2 - mg(\ell - R\theta)\sin\alpha.$$

Here m is the mass of the disk, R its radius, ℓ is the total length of the inclined plane, and α is the inclination angle; the disk's motion is represented by the angle θ, and these quantities are all illustrated in Fig. 2.16. We can simplify the Lagrangian by discarding the constant term $-mg\ell\sin\alpha$; we obtain

$$L = \frac{3}{4}mR^2\dot{\theta}^2 + mgR\sin\alpha\,\theta. \tag{3.2.36}$$

The momentum associated with θ is $p \equiv p_\theta = \partial L/\partial\dot{\theta} = \frac{3}{2}mR^2\dot{\theta}$, and the Hamiltonian is $H = p\dot{\theta} - L$, or

$$H = \frac{p^2}{3mR^2} - mgR\sin\alpha\,\theta \tag{3.2.37}$$

after expressing $\dot{\theta}$ in terms of p.

Exercise 3.11. Verify Eq. (3.2.37).

The canonical equations are

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{2p}{3mR^2}, \tag{3.2.38}$$

$$\dot{p} = -\frac{\partial H}{\partial\theta} = mgR\sin\alpha. \tag{3.2.39}$$

By eliminating p from this system of equations we obtain $\ddot{\theta} = \frac{2}{3}g\sin\alpha/R$, which is the same as Eq. (2.4.31). A particular solution to this second-order differential equation was displayed in Eq. (2.4.32). From this solution it is easy to calculate $p(t)$.

Exercise 3.12. Obtain the solution to the canonical equations which enforces the initial conditions $\theta(t=0) = 0$ and $p(t=0) = p_0$, where p_0 is an arbitrary constant. Plot the motion of the disk in phase space for selected values of p_0, and verify that your plots look similar to those featured in Fig. 3.6. Finally, show that the phase trajectories are described by the equation

$$p^2 - 3m^2gR^3\sin\alpha\,\theta = 3mR^2E,$$

where E is the disk's total mechanical energy; find the relationship between E and p_0.

3.2.8 Kepler's problem

Kepler's problem was first considered in Sec. 1.5. It was revisited in Sec. 2.4.7, where the Lagrangian of two bodies subjected to their mutual gravity was decomposed into a centre-of-mass Lagrangian that governs the overall motion of the centre of mass, and a relative Lagrangian that governs the relative separation $\boldsymbol{r}$ between the two bodies. The relative Lagrangian was expressed in polar coordinates in Eq. (2.4.36), which we copy here:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{G\mu M}{r}. \tag{3.2.40}$$

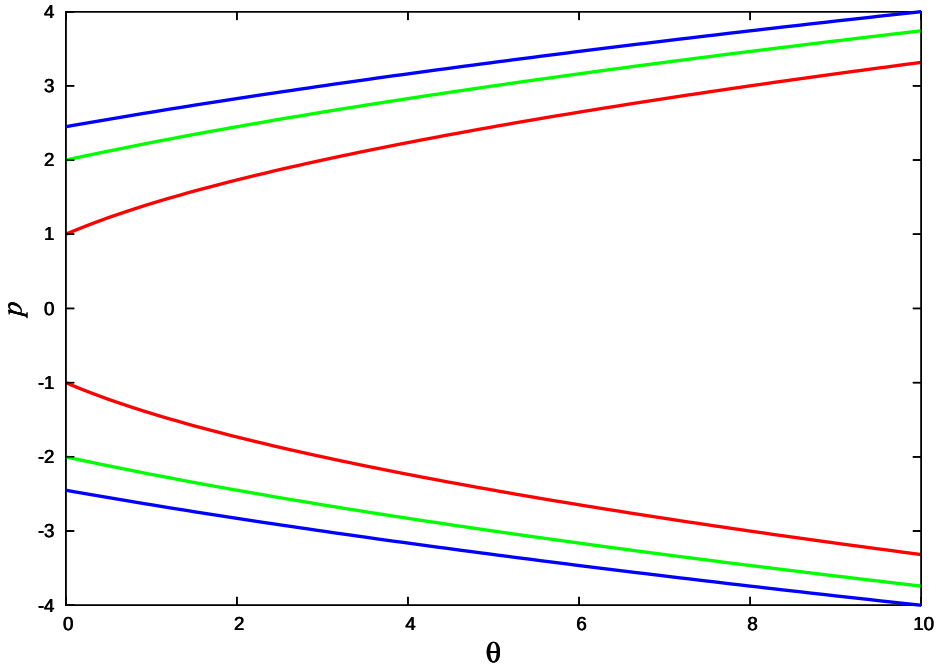


Figure 3.6: Phase trajectories of a rolling disk. The curves with p positive represent a disk rolling down the inclined plane. The curves with p negative represent a disk rolling up.

The quantity $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the gravitating system, and $M = m_1 + m_2$ is the total mass. The distance between the two bodies is r, and ϕ is the orbital angle. Our effective one-body system possesses two degrees of freedom.

The momenta associated with r and ϕ are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}, \end{aligned}$$

and the Hamiltonian is $H = p_r \dot{r} + p_\phi \dot{\phi} - L$, or

$$H = \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2} - \frac{G\mu M}{r} \tag{3.2.41}$$

after eliminating the velocities in favour of the momenta.

Exercise 3.13. Verify Eq. (3.2.41).

The canonical equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}, \tag{3.2.42}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\mu r^2}, \tag{3.2.43}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{\mu r^3} - \frac{G\mu M}{r^2}, \tag{3.2.44}$$

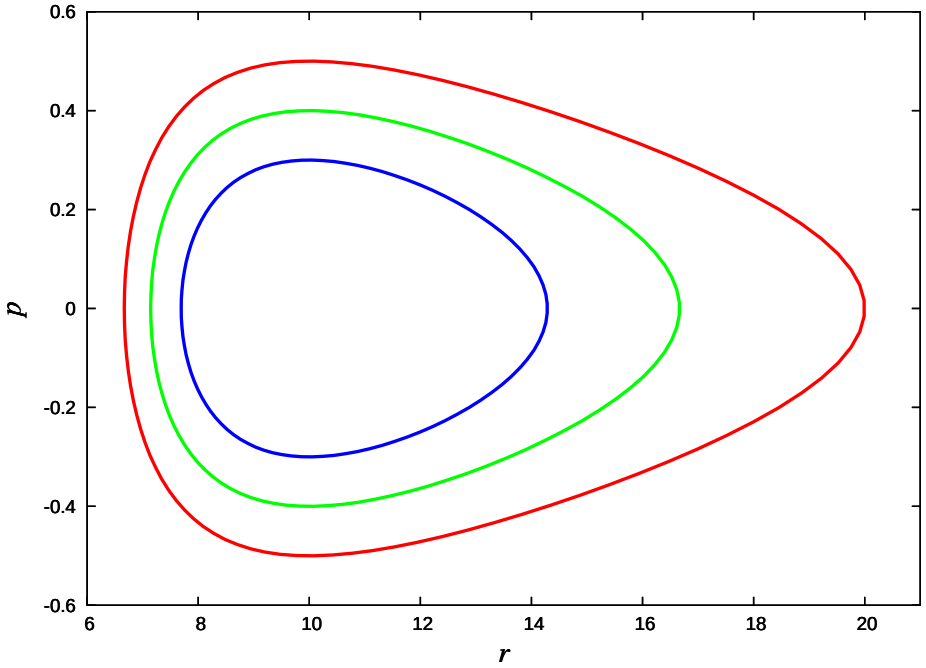


Figure 3.7: Phase trajectories of Kepler's problem, in a two-dimensional subspace of the complete four-dimensional phase space. The different curves have different values of ε but they share the same value of h. The fact that all the trajectories are closed indicates that the motion is bounded and periodic. The curves are most easily produced by using the results of Sec. 1.5.9: We use the parametric representation $r = p/(1 + e\cos\phi)$ and $\mathsf{p} = \dot{r} = e\sqrt{GM/p}\sin\phi$, where p is the semilatus rectum and e the eccentricity. In terms of these we have $h = \sqrt{GMp}$ and $\varepsilon = -GM(1-e^2)/(2p)$. The phase trajectories displayed here have eccentricities of 0.3, 0.4, and 0.5, respectively; they all share the same value of p.

$$\dot{p}_\phi \quad = \quad -\frac{\partial H}{\partial \phi} = 0. \tag{3.2.45}$$

The last equation implies that p_ϕ is a constant of the motion; we shall express this as $p_\phi/\mu = h =$ constant, as we have done in Eq. (2.4.38). Equations (3.2.42) and (3.2.44) can be shown to be equivalent to Eq. (2.4.37).

Exercise 3.14. Verify this last statement.

The solutions to the equations of motion were studied back in Sec. 1.5. The motion in phase space is described by the equation $H = E =$ constant, which expresses the fact that H also is a constant of the motion. Introducing the rescaled quantities $\mathsf{p} = p_r/\mu$ and $\varepsilon = E/\mu$, this equation states that

$$\frac{1}{2}\mathsf{p}^2 + \frac{h^2}{2r^2} - \frac{GM}{r} = \varepsilon.$$

This equation can be solved for p and the result is displayed in Fig. 3.7.

3.2.9 Charged particle in an electromagnetic field

For our last application we consider a particle of charge q moving in an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The fields can be expressed as $\boldsymbol{E} = -\partial\boldsymbol{A}/\partial t - \boldsymbol{\nabla}\Phi$

and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ in terms of potentials Φ and $\boldsymbol{A}$; and as we saw in Sec. 2.6, the Lagrangian of the particle is

$$L = \frac{1}{2}mv^2 - q\Phi + q\boldsymbol{A} \cdot \boldsymbol{v}, \tag{3.2.46}$$

where $v^2 = \boldsymbol{v} \cdot \boldsymbol{v} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. The momentum vector $\boldsymbol{p}$ associated with the position vector $\boldsymbol{r}$ was obtained back in Eq. (2.6.5); it is $\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}$. The Hamiltonian is $H = \boldsymbol{p} \cdot \boldsymbol{v} - L$, or

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A}) \cdot (\boldsymbol{p} - q\boldsymbol{A}) + q\Phi. \tag{3.2.47}$$

Exercise 3.15. Verify Eq. (3.2.47).

The canonical equations governing the evolution of x and p_x are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}(p_x - qA_x), \tag{3.2.48}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{q}{m}(\boldsymbol{p} - q\boldsymbol{A}) \cdot \frac{\partial \boldsymbol{A}}{\partial x} - q\frac{\partial \Phi}{\partial x}, \tag{3.2.49}$$

and similar equations can be obtained for the pairs (y, p_y) and (z, p_z). The second-order differential equation that is obtained by eliminating p_x from the system of Eqs. (3.2.48) and (3.2.49) is

$$m\ddot{x} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})_x,$$

and this is the x component of the Lorentz-force law. The other components are obtained by similar manipulations.

Exercise 3.16. Go through the algebra that leads to Eqs. (3.2.48) and (3.2.49), and then repeat these calculations for the other four canonical equations. Finally, show that these equations are indeed equivalent to the Lorentz-force law, $m\ddot{\boldsymbol{x}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. Be warned: this last calculation can be a bit tricky!

3.3 Liouville's theorem

3.3.1 Formulation of the theorem

We wish to examine the motion of a large number N of identical particles in phase space; each particle has its own position and momentum, but all are subjected to the same potential V. We may imagine that the N particles co-exist peacefully, without interacting with one another. Or we may imagine that the N particles are in fact mental copies of one and the same particle, on which we are carrying out N separate experiments. In all cases we shall refer to the N particles as an *ensemble of particles*, and we wish to follow the motion of this ensemble in phase space.

Supposing (for concreteness) that each particle possesses three degrees of freedom, which it would if it were to move in a three-dimensional space, we could form a $6N$-dimensional phase space of all positions and momenta of all the particles, and we could display the motion of the whole ensemble as a trajectory in this super phase space. We shall not follow this strategy, although it is a viable one. Instead, we will simultaneously represent the motion of all N particles in the six-dimensional phase space of an individual particle (which one we pick does not matter, because

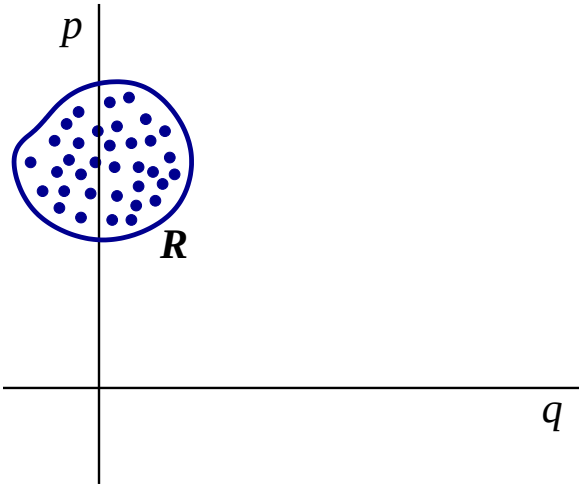


Figure 3.8: The initial state of motion of an ensemble of N identical particles is represented by N representative points in the phase space of an individual particle. These points are distributed within a bounded region $\mathcal{R}$ of the phase space.

the particles are all identical); this individual phase space is spanned by the three position variables and the three momentum variables. We will have, therefore, a collection of N separate trajectories in phase space.

We have seen that a point in phase space — the phase space of an individual particle — gives a complete representation of the state of motion of that particle at an instant of time. By identifying the point we automatically know the particle's position and momentum, and this is all there is to know about the state of motion of the particle at that instant of time; the equations of motion then tell us how the state of motion will change from this time to the next time. As was mentioned previously, the particle will trace a trajectory in phase space, and each point on this curve will represent a state of motion corresponding to a given instant of time.

Suppose that we wish to represent, at an initial moment $t = 0$, the state of motion of our ensemble of N particles, and that we wish to do so in the phase space of an individual particle. We will need to identify N points in the phase space, and each of these points will represent the state of motion of one of the particles. We will call them *representative points*. Because we give each particle its own set of initial conditions, the representative points will be spread out in phase space, and they will define a region $\mathcal{R}$ of phase space. We will assume that this region is bounded; this is illustrated in Fig. 3.8.

Each particle within the ensemble moves according to Hamilton's equations, and each representative point traces a trajectory in phase space. Because the initial conditions are different for each particle, each trajectory is different. In a time t the initial region $\mathcal{R}(0)$ of phase space will be mapped to a distinct region $\mathcal{R}(t)$; this mapping is illustrated in Fig. 3.9. The shape of $\mathcal{R}(t)$ will in general be very different from the shape of the initial region $\mathcal{R}(0)$. But according to *Liouville's theorem*:

> The "volume" of the region $\mathcal{R}(t)$ of phase space,
>
> $$V = \int_{\mathcal{R}(t)} dq_1 dq_2 \cdots dp_1 dp_2 \cdots ,$$
>
> is independent of the time t; the volume does not change as the region $\mathcal{R}(t)$ evolves in accordance with the motion of each representative point in phase space.

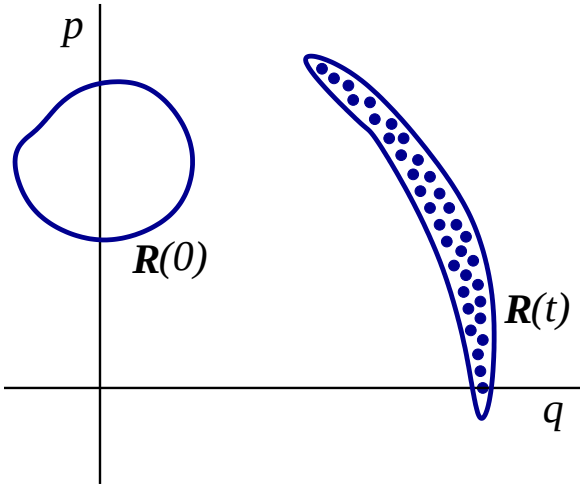


Figure 3.9: An initial region $\mathcal{R}(0)$ of phase space containing the N representative points is mapped by Hamilton's equations to a new region $\mathcal{R}(t)$. The shape of the new region can be very different from the shape of the initial region. But according to Liouville's theorem, their phase-space "volumes" are equal.

So Liouville's theorem states that while the Hamiltonian evolution of the ensemble will produce a deformation of the region $\mathcal{R}(t)$, the evolution will nevertheless preserve its "volume", as defined by the integration over $\mathcal{R}(t)$ of the "volume element" $dV = dq_1 dq_2 \cdots dp_1 dp_2 \cdots$ in phase space.

The proof of this important theorem will be presented below. Illustrations of the theorem are displayed in Fig. 3.10, which shows how an initial region $\mathcal{R}(0)$ of a two-dimensional phase space evolves over time. It is important to note that Liouville's theorem is formulated in phase space, and that its validity is therefore restricted to phase space. An attempt to formulate such a theorem in configuration space, or in a position-velocity space, would fail: Volumes in such spaces are not preserved under the time evolution of the ensemble.

3.3.2 Case study: Linear pendulum

The examples displayed in Fig. 3.10 involved a nonlinear pendulum, and the nonlinearities of the dynamics produced interesting distortions of the initial (rectangular) region $\mathcal{R}(0)$ of the system's phase space. These distortions, however, are difficult (probably impossible) to describe mathematically, and this means that the validity of Liouville's theorem would be difficult to check directly.

To help build confidence in these new ideas we will simplify the problem further and eliminate the nonlinear aspects of the dynamics. We will therefore examine the motion of an ensemble of *linear pendula*. Each pendulum possesses the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\theta^2, \tag{3.3.1}$$

which can be obtained from Eq. (3.2.21) by (i) invoking the approximation $\cos\theta = 1 - \frac{1}{2}\theta^2$, (ii) discarding an irrelevant constant term, and (iii) rescaling the variables according to $H/(m\ell^2) \to H$ and $p_\theta/(m\ell^2) \to p_\theta \equiv p$.

The canonical equations are

$$\dot{\theta} = p, \qquad \dot{p} = -\omega^2\theta, \tag{3.3.2}$$

and they are equivalent to the second-order differential equation $\ddot{\theta} + \omega^2\theta = 0$ that

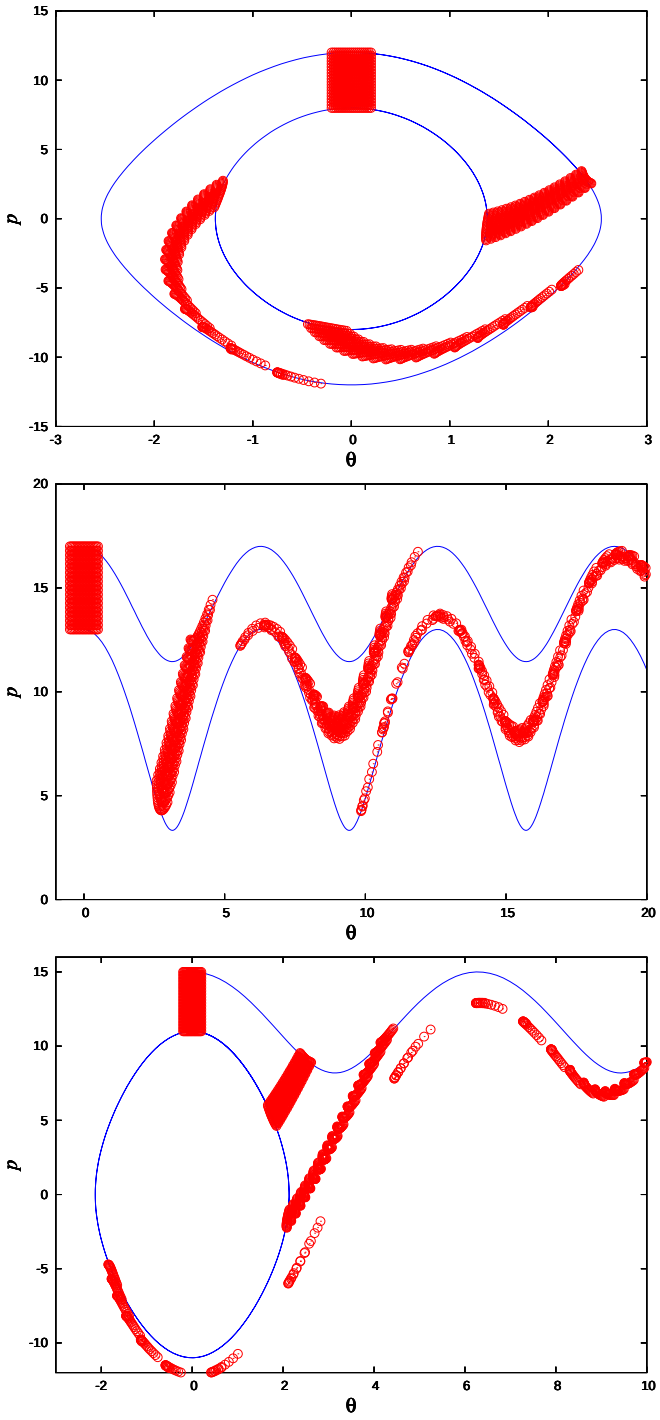


Figure 3.10: Evolution of a region $\mathcal{R}(t)$ in a two-dimensional phase space; the mechanical system is the planar pendulum of Sec. 3.2.4. For all three plots the initial region $\mathcal{R}(0)$ is the rectangular region near the top of the page, and $\mathcal{R}(t)$ is drawn at three successive times. Each region consists of 400 representative points, drawn as open circles, and the motion of each point in phase space is determined by numerical integration of Hamilton's equations. Two bounding phase trajectories are also shown to guide the eye. In the first (upper) graph the initial conditions are such that the motion each pendulum is bounded, limited to an interval $-\theta_0 < \theta < \theta_0$. In the second (middle) graph the initial conditions are such that the motion of each pendulum is not bounded; each goes through complete revolutions. In the third (lower) graph the initial conditions are such that about half the pendula undergo bounded motion, while the other half undergo unbounded motion. All three graphs reveal a significant distortion of the region $\mathcal{R}(t)$ as the motion of each pendulum proceeds; but the area of this region is the same at all times.

governs simple harmonic motion. The general solution to Eqs. (3.3.2) is

$$\theta(t) = \theta(0)\cos\omega t + \frac{p(0)}{\omega}\sin\omega t, \tag{3.3.3}$$

$$p(t) = p(0)\cos\omega t - \omega\theta(0)\sin\omega t, \tag{3.3.4}$$

where $\theta(0) \equiv \theta(t=0)$ is the initial position of a pendulum sampled from the ensemble, and $p(0) \equiv p(t=0)$ is its initial momentum. The energy of each pendulum is conserved, and the trajectory of each pendulum in phase space is an ellipse described by

$$\frac{1}{2}p^2 + \frac{1}{2}\omega^2\theta^2 = \varepsilon = \frac{1}{2}p^2(0) + \frac{1}{2}\omega^2\theta^2(0).$$

By varying the initial conditions among all the pendula within the ensemble, we trace ellipses of varying shapes and sizes.

Each pendulum within the ensemble has its own set of initial conditions $\theta(0)$ and $p(0)$. The spread of initial conditions in phase space defines the initial region $\mathcal{R}(0)$. Suppose that we choose initial conditions such that the N values for $\theta(0)$ are centered around $\theta(0) = 0$ and are within a deviation σ_θ away from zero, either in the negative or positive direction. Suppose also that the N values for $p(0)$ are centered around $p(0) = p_0$ and are within a deviation σ_p away from p_0. What we have, then, is a region $\mathcal{R}(0)$ in phase space that is centered at $(\theta, p) = (0, p_0)$ and has a typical extension of σ_θ in the position direction, and a typical extension of σ_p is the momentum direction. This region will evolve to $\mathcal{R}(t)$ in a time t, as each pendulum within the ensemble moves in phase space. We wish to describe this evolution, and in particular, we wish to show that the "volume" of $\mathcal{R}(t)$ is independent of time.

Concretely we choose the boundary of $\mathcal{R}(0)$ to be described by an ellipse of semiaxes σ_θ and σ_p, centered at $\theta(0) = 0$ and $p(0) = p_0$. (It is important to understand that this ellipse has nothing to do with the elliptical motion of each pendulum in phase space. We have two unrelated ellipses: one representing the motion of each pendulum in phase space, the other representing the distribution of initial conditions.) We describe this boundary by the parametric equations

$$\theta(0;\alpha) = -\sigma_\theta\cos\alpha, \qquad p(0;\alpha) = p_0 + \sigma_p\sin\alpha, \tag{3.3.5}$$

in which the parameter α ranges from 0 to 2π. All the representative points are initially located within this ellipse, and the region $\mathcal{R}(0)$ is therefore a solid ellipse; this is illustrated in Fig. 3.11. The phase-space "volume" of this region is, in this two-dimensional context, the surface area of the solid ellipse. This "volume" can be calculated as

$$\begin{aligned} V(0) &= \int_{\mathcal{R}(0)} d\theta(0)dp(0) \\ &= \int_{-\sigma_\theta}^{\sigma_\theta} p_+(0)\,d\theta(0) + \int_{\sigma_\theta}^{-\sigma_\theta} p_-(0)\,d\theta(0). \end{aligned}$$

The first integral is the area under the upper branch of the ellipse (the one for which $p \geq p_0$), and the second integral is (minus) the area under the lower branch (the one for which $p \leq p_0$). This can be expressed cleanly as

$$\begin{aligned} V(0) &= \int_0^{2\pi} p(0;\alpha)\frac{d\theta(0;\alpha)}{d\alpha}\,d\alpha \\ &= \int_0^{2\pi} \big[p_0 + \sigma_p\sin\alpha\big]\big[\sigma_\theta\sin\alpha\big]\,d\alpha, \end{aligned}$$

and integration gives

$$V(0) = \pi\sigma_\theta\sigma_p, \tag{3.3.6}$$

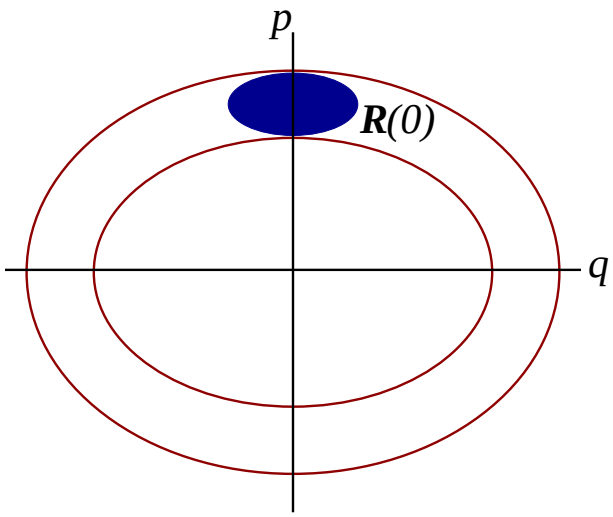


Figure 3.11: The initial region $\mathcal{R}(0)$ in phase space has an elliptical boundary. The ellipse is centered at $(\theta, p) = (0, p_0)$ and it has semiaxes σ_θ and σ_p. An angle α (not shown) parameterizes the position on the boundary.

the expected result for an ellipse of semiaxes σ_θ and σ_0. This is the phase-space "volume" of our initial region $\mathcal{R}(0)$. We now wish to determine how this region evolves in time, and how its volume changes.

Exercise 3.17. Verify Eq. (3.3.6).

As time moves forward each point $\theta(0;\alpha)$, $p(0;\alpha)$ on the boundary of the region $\mathcal{R}(0)$ is mapped to a corresponding point $\theta(t;\alpha)$, $p(t;\alpha)$ on the boundary of the new region $\mathcal{R}(t)$. The coordinates of the new point are given by Eqs. (3.3.3) and (3.3.4), which we write as

$$\theta(t;\alpha) = \theta(0;\alpha)\cos\omega t + \frac{p(0;\alpha)}{\omega}\sin\omega t, \tag{3.3.7}$$

$$p(t;\alpha) = p(0;\alpha)\cos\omega t - \omega\theta(0;\alpha)\sin\omega t. \tag{3.3.8}$$

The new regions $\mathcal{R}(t)$ are displayed in Fig. 3.12 for selected values of t. Their "volume" is given by

$$\begin{aligned} V(t) &= \int_{\mathcal{R}(t)} d\theta(t)dp(t) \\ &= \int_0^{2\pi} p(t;\alpha)\frac{d\theta(t;\alpha)}{d\alpha}\,d\alpha. \end{aligned}$$

After involving Eqs. (3.3.7), (3.3.8) and performing the integration, we arrive at

$$V(t) = \pi\sigma_\theta\sigma_p, \tag{3.3.9}$$

the same result as in Eq. (3.3.6). The volume of the phase-space region $\mathcal{R}(t)$ is indeed independent of time.

Exercise 3.18. Go through the calculational steps that lead to Eq. (3.3.9).

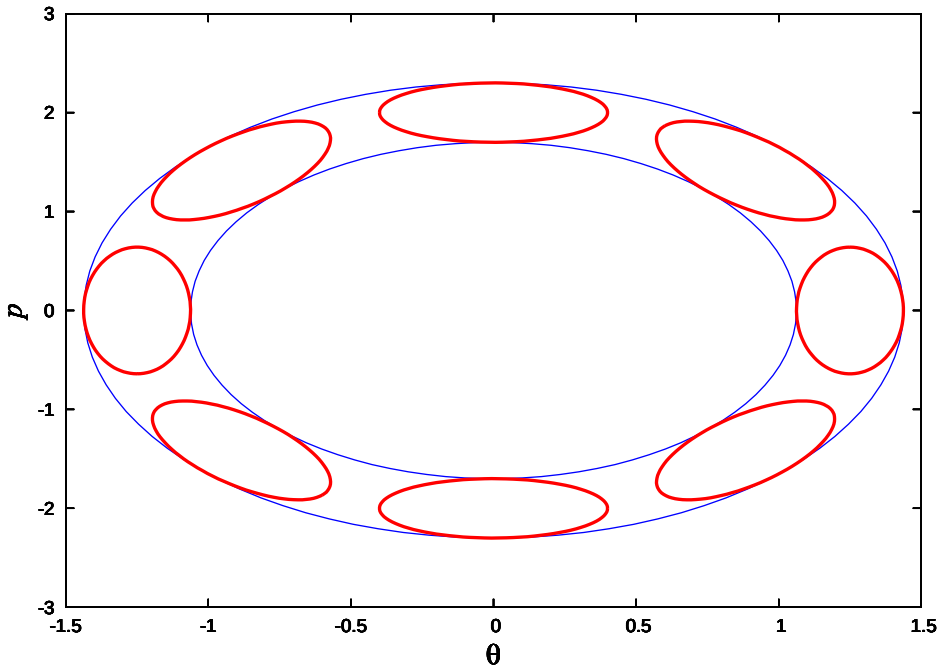


Figure 3.12: The initial region $\mathcal{R}(0)$ in phase space is mapped to a new region $\mathcal{R}(t)$ after a time t. These regions are shown for $\omega t = 0$ (uppermost ellipse), $\omega t = \pi/4$, $\omega t = \pi/2$ (rightmost ellipse), $\omega t = 3\pi/4$, and so on.

3.3.3 Proof of Liouville's theorem

There are, in fact, two versions of Liouville's theorem. The first version is concerned with a quantity ρ, the density of representative points in phase space, which we shall introduce below; it states that under a Hamiltonian evolution,

$$\frac{d\rho}{dt} = 0, \tag{3.3.10}$$

so that ρ is a constant of the motion. The second version is concerned with the volume $V(t)$ of a region $\mathcal{R}(t)$ of phase space that is defined by an ensemble of representative points; it states that $V(t)$ is constant under a Hamiltonian evolution. The second version of the theorem is a corollary of the first. We will prove the first version first, and then obtain the second version.

We have a region $\mathcal{R}(t)$ of phase space that contains a large number N of representative points; this region has a volume $V = \int_{\mathcal{R}(t)} dV$, where $dV = dq_1 dq_2 \cdots dp_1 dp_2 \cdots$ is the element of phase-space volume. We imagine that N is sufficiently large that we can introduce a notion of phase-space density ρ of representative points; this, by definition, is the number dN of representative points contained within a small region of phase space, divided by its volume dV. We have, therefore, $\rho = dN/dV$, and the density can vary from point to point in phase space: $\rho = \rho(q_a, p_a, t)$; we also allow the density to depend explicitly on time.

The phase-space density ρ plays essentially the same role here as the density of electric charge ρ_e plays in electromagnetism. If we introduce a velocity field $\boldsymbol{v} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{p}_1, \dot{p}_2, \cdots)$ in phase space, then the current density $\boldsymbol{j} = \rho\boldsymbol{v}$ will play essentially the same role here as the electric current density $\boldsymbol{j}_e$ plays in electromagnetism. It is known that in electromagnetism, the charge and current densities are related by an *equation of continuity*,

$$\frac{\partial \rho_e}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_e = 0.$$

We will show that a very similar equation of continuity applies to ρ and $\boldsymbol{j}$ in phase space. In electromagnetism the equation of continuity is a differential statement of charge conservation; in phase space it will be a differential statement of the fact that the number of representative points is conserved.

Exercise 3.19. Consult your favorite textbook on electromagnetism and review its derivation of the equation of continuity for electric charge.

Consider a region V of phase space which is bounded by the "surface" S. This region is completely arbitrary, but it is assumed to be fixed in time; unlike the region $\mathcal{R}(t)$ considered previously, this one does not move around in phase space. The representative points contained in $\mathcal{R}(t)$ do move, however, and in time they will move in and out of the region V. The number of representative points contained in V at any given time t is given by the phase-space integral $\int_V \rho\, dV$. The number of representative points that *move out* of V per unit time is then given by

$$-\frac{d}{dt}\int_V \rho\, dV.$$

If the total number of representative points is to be conserved, this number must be equal to the number of representative points that cross the bounding surface S, in the outward direction, per unit time. By definition of the current density $\boldsymbol{j}$, this is

$$\oint_S \boldsymbol{j}\cdot d\boldsymbol{a},$$

where $d\boldsymbol{a}$ is an element of "surface area" in phase space; this vector is directed along the outward normal to the surface, and its magnitude is equal to the area of an element of surface in phase space. Equating these two expressions gives

$$-\int_V \frac{\partial\rho}{\partial t}\, dV = \oint_S (\rho\boldsymbol{v})\cdot d\boldsymbol{a}.$$

We next use the phase-space version of Gauss's theorem to express the right-hand side as a volume integral,

$$\int_V \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\, dV,$$

where $\boldsymbol{\nabla} = (\partial/\partial q_1, \partial/\partial q_2, \cdots, \partial/\partial p_1, \partial/\partial p_2, \cdots)$ is the gradient operator in phase space. We now have

$$\int_V \left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\right] dV = 0,$$

and since this equation must be valid for all regions V of phase space, we conclude that

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0.$$

This is the phase-space version of the equation of continuity.

A more explicit form of the equation of continuity is

$$0 = \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial q_1}(\rho\dot{q}_1) + \frac{\partial}{\partial q_2}(\rho\dot{q}_2) + \cdots + \frac{\partial}{\partial p_1}(\rho\dot{p}_1) + \frac{\partial}{\partial p_2}(\rho\dot{p}_2) + \cdots,$$

or

$$0 = \frac{\partial\rho}{\partial t} + \sum_a \left[\frac{\partial}{\partial q_a}(\rho\dot{q}_a) + \frac{\partial}{\partial p_a}(\rho\dot{p}_a)\right].$$

The phase-space coordinates of the representative points change in accordance with Hamilton's equations. We may therefore substitute $\dot{q}_a = \partial H/\partial p_a$ and $\dot{p}_a = -\partial H/\partial q_a$ into the preceding equation. We obtain

$$\begin{aligned} 0 &= \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial}{\partial q_a}\left(\rho \frac{\partial H}{\partial p_a}\right) - \frac{\partial}{\partial p_a}\left(\rho \frac{\partial H}{\partial q_a}\right)\right] \\ &= \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial \rho}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial \rho}{\partial p_a}\frac{\partial H}{\partial q_a} + \rho\left(\frac{\partial^2 H}{\partial q_a \partial p_a} - \frac{\partial^2 H}{\partial p_a \partial q_a}\right)\right] \\ &= \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial \rho}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial \rho}{\partial p_a}\frac{\partial H}{\partial q_a}\right], \end{aligned}$$

or

$$0 = \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial \rho}{\partial q_a}\dot{q}_a + \frac{\partial \rho}{\partial p_a}\dot{p}_a\right]$$

after involving Hamilton's equations one more time.

We have obtained the first version of Liouville's theorem: If the phase-space density ρ is a function of q_a, p_a, and t, then by virtue of the chain rule its total time derivative is

$$\frac{d\rho}{dt} = \sum_a \left[\frac{\partial \rho}{\partial q_a}\dot{q}_a + \frac{\partial \rho}{\partial p_a}\dot{p}_a\right] + \frac{\partial \rho}{\partial t}, \tag{3.3.11}$$

and according to our previous results, this is zero. We have therefore established Eq. (3.3.10) on the basis of the equation of continuity in phase space.

To arrive at the second version of Liouville's theorem, consider the N representative points that are contained in the moving region $\mathcal{R}(t)$ of phase space. By definition of the phase-space density, we have

$$N = \int_{\mathcal{R}(t)} \rho \, dV,$$

and we know that this number is preserved as we follow the evolution of $\mathcal{R}(t)$ over time. We now also know that the density ρ is a constant of the motion. This means that if, for example, the density is initially chosen to be uniform over $\mathcal{R}(0)$, then it will stay uniform over $\mathcal{R}(t)$ throughout the Hamiltonian evolution. In this case we may bring ρ outside of the integral, and we obtain the statement that $N/\rho = \int_{\mathcal{R}(t)} dV \equiv V(t)$ is preserved during the evolution. This is the second version of Liouville's theorem.

3.3.4 Poisson brackets

The expression of Eq. (3.3.11) for the total time derivative of the phase-space density,

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial \rho}{\partial q_a}\dot{q}_a + \frac{\partial \rho}{\partial p_a}\dot{p}_a\right] \\ &= \frac{\partial \rho}{\partial t} + \sum_a \left[\frac{\partial \rho}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial \rho}{\partial p_a}\frac{\partial H}{\partial q_a}\right], \end{aligned}$$

is in fact a mathematical identity that holds for any function $\rho(q_a, p_a, t)$ defined in phase space. Because this expression is so general, it occurs often, and it has proved convenient to introduce a notation to recast it in a more compact form.

Let $f(q_a, p_a, t)$ and $g(q_a, p_a, t)$ be any two functions on phase space. Their *Poisson bracket* is defined by

$$[f, g] = \sum_a \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a}\right). \tag{3.3.12}$$

The Poisson bracket possesses the following properties: It is antisymmetric,

$$[g, f] = -[f, g]; \tag{3.3.13}$$

it is linear with respect to each of its arguments,

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g], \qquad [f, g_1 + g_2] = [f, g_1] + [f, g_2]; \tag{3.3.14}$$

it satisfies the product rule of differential calculus,

$$[f_1 f_2, g] = f_1[f_2, g] + [f_1, g]f_2, \qquad [f, g_1 g_2] = g_1[f, g_2] + [f, g_1]g_2; \tag{3.3.15}$$

and finally, it satisfies the *Jacobi identity*,

$$\big[f, [g, h]\big] + \big[h, [f, g]\big] + \big[g, [h, f]\big] = 0. \tag{3.3.16}$$

Exercise 3.20. Show that these are all true properties of the Poisson bracket. Be warned: To establish the Jacobi identity requires a lengthy calculation.

Particular applications of the Poisson bracket are

$$[f, q_a] = -\frac{\partial f}{\partial p_a}, \qquad [f, p_a] = \frac{\partial f}{\partial q_a}. \tag{3.3.17}$$

Special cases of these identities are

$$[q_a, q_b] = 0, \qquad [q_a, p_b] = \delta_{ab}, \qquad [p_a, p_b] = 0. \tag{3.3.18}$$

Exercise 3.21. Verify Eqs. (3.3.17) and (3.3.18).

In terms of the Poisson bracket, the total derivative with respect to time of a function $f(q_a, p_a, t)$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]. \tag{3.3.19}$$

If we apply this identity to the Hamiltonian H we obtain $dH/dt = \partial H/\partial t + [H, H] = \partial H/\partial t$, by virtue of the antisymmetric property of the Poisson bracket. If the Hamiltonian does not depend explicitly on time, we obtain the statement $dH/dt = 0$ and the conclusion that the Hamiltonian is a constant of the motion. This is a well-known result by now, but notice how quickly the result follows from the Poisson-bracket formalism.

Exercise 3.22. Verify Eq. (3.3.19). Then show that it leads to the expected answers for dq_a/dt and dp_a/dt.

3.4 Canonical transformation

3.4.1 Introduction

A theme that has been central to our development of Lagrangian and Hamiltonian mechanics is the arbitrariness of the generalized coordinates q_a that are adopted to describe the motion of a mechanical system. The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0$$

and the canonical equations

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}$$

are all *invariant* under a transformation of the generalized coordinates, from the set q_a of old coordinates to any set Q_a of new coordinates; these can by any functions $Q_a(q_1, q_2, \cdots)$ of the old coordinates.

In this section we show that the Hamiltonian formulation of the laws of mechanics admits a much wider class of possible transformations. In this context it is possible to change the phase-space coordinates from an old set (q_a, p_a) to a new set (Q_a, P_a), with

$$Q_a = Q_a(q_1, q_2, \cdots, p_1, p_2, \cdots), \qquad P_a = P_a(q_1, q_2, \cdots, p_1, p_2, \cdots). \tag{3.4.1}$$

Notice that the new generalized coordinates Q_a are now functions of the old coordinates *and* the old momenta; the new generalized momenta P_a also are functions of all the old phase-space coordinates. Under some conditions, which will be specified below, such a transformation will leave the canonical equations invariant: In the new system of phase-space coordinates there will exist a transformed Hamiltonian $H'(Q_a, P_a, t)$ such that

$$\dot{Q}_a = \frac{\partial H'}{\partial P_a}, \qquad \dot{P}_a = -\frac{\partial H'}{\partial Q_a}. \tag{3.4.2}$$

Under these conditions the transformation is known as a *canonical transformation*; transformations of the phase-space coordinates that are not canonical have no value, and they will not be considered.

Canonical transformations have the interesting and useful property that they leave the element of phase-space volume invariant. Thus,

$$dV = dq_1 dq_2 \cdots dp_1 dp_2 \cdots = dQ_1 dQ_2 \cdots dP_1 dP_2 \cdots . \tag{3.4.3}$$

In other words, the Jacobian of the transformation is equal to one. This gives us a means of checking whether a specified transformation is canonical or not: If the Jacobian of the transformation is not equal to one, the transformation is not canonical. This property of canonical transformations is rather deep, and it implies that the validity of Liouville's theorem is not restricted to a particular choice of phase-space coordinates; the volume of a region $\mathcal{R}(t)$ of phase space is invariant under a canonical transformation.

Because a canonical transformation produces new coordinates that are a mixture of old coordinates and old momenta, they can dramatically alter the physical meaning of the phase-space coordinates. Thus, a given Q_a may not necessarily represent a position variable, and a given P_a may not represent a momentum variable. A trivial example is the canonical transformation $Q_a = p_a$, $P_a = -q_a$, which clearly leaves the canonical equations invariant; here the new coordinates are the old momenta, the new momenta are the old coordinates, and the new phase-space coordinates do not retain their traditional physical meaning. Because the new "coordinates" Q_a and the new "momenta" P_a may not have straightforward physical interpretations after a canonical transformation, it is customary to refer to the new phase-space coordinates simply as *conjugate variables*.

3.4.2 Case study: Linear pendulum

Before we present the general theory of canonical transformations in the next subsection, we shall take the time to get acquainted with some of the fundamental ideas

by examining a specific example. We return once more to the linear pendulum of Sec. 3.3.2, with its Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \tag{3.4.4}$$

where we have identified the generalized coordinate q with the swing angle θ. The canonical equations for this mechanical system are

$$\dot{q} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q. \tag{3.4.5}$$

We intend to show that a canonical transformation can turn this rather simple mechanical system into a completely trivial one. Solving for the motion of the trivial system will allow us to find the solution to Eqs. (3.4.5) without having to solve these equations directly. This, in a nutshell, is the power and purpose of canonical transformations.

Let us consider the following transformation of the phase-space coordinates:

$$Q = \arctan\Big(\frac{\omega q}{p}\Big), \qquad P = \frac{1}{2\omega}\big(p^2 + \omega^2 q^2\big). \tag{3.4.6}$$

The new "momentum" P is proportional to the Hamiltonian; a curve $P =$ constant is therefore represented as an ellipse in the old phase space. A curve $Q =$ constant, on the other hand, is represented as a straight line that passes through the origin; this line has a slope $p/q = \omega/\tan Q$, and Q is an angle relative to the p axis. The inverse transformation is

$$q = \sqrt{\frac{2P}{\omega}}\sin Q, \qquad p = \sqrt{2\omega P}\cos Q. \tag{3.4.7}$$

It is easy to check that the transformation has a unit Jacobian. This is given by

$$\begin{aligned} J &= \begin{vmatrix} \partial q/\partial Q & \partial q/\partial P \\ \partial p/\partial Q & \partial p/\partial P \end{vmatrix} = \begin{vmatrix} \sqrt{2P/\omega}\cos Q & \sin Q/\sqrt{2\omega P} \\ -\sqrt{2\omega P}\sin Q & \sqrt{\omega/2P}\cos Q \end{vmatrix} \\ &= \cos^2 Q + \sin^2 Q = 1, \end{aligned}$$

and J is indeed equal to one. This gives us a successful partial check on whether the transformation is properly canonical.

Exercise 3.23. Check that Eq. (3.4.6) is the inverse transformation to Eq. (3.4.7). Then check all the partial derivatives that have been involved in the computation of the Jacobian.

The transformation of Eq. (3.4.6) will be canonical if and only if it preserves the form of the canonical equations. We shall now show that this is indeed the case. We will find that the evolution equations for Q and P are given by

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega, \qquad \dot{P} = -\frac{\partial H}{\partial Q} = 0, \tag{3.4.8}$$

with a Hamiltonian now expressed as

$$H = \omega P, \tag{3.4.9}$$

which follows by substituting Eq. (3.4.6) for P into Eq. (3.4.4) for H. In this particular instance of a canonical transformation, the new Hamiltonian H' is the same as the old Hamiltonian H.

We can verify the results of Eq. (3.4.8) by computing $\dot{Q}$ and $\dot{P}$ directly from their definitions in Eq. (3.4.6). We begin with the relation $\tan Q = \omega q/p$, which we differentiate with respect to t. We get

$$(1 + \tan^2 Q)\dot{Q} = \frac{\omega \dot{q}}{p} - \frac{\omega q \dot{p}}{p^2},$$

and if we now involve Eq. (3.4.5), this becomes

$$(1 + \tan^2 Q)\dot{Q} = \omega\left(1 + \frac{\omega^2 q^2}{p^2}\right) = \omega(1 + \tan^2 Q).$$

This gives, finally, $\dot{Q} = \omega$, as we had stated in Eq. (3.4.8). The right-hand side of this equation happens to be equal to $\partial H/\partial P$, and we have recovered one of the two canonical equations. The second equation follows much more easily. Because $P = H/\omega$ it is obvious that its equation of motion is $\dot{P} = 0$, as was stated in Eq. (3.4.8). The right-hand side of this equation happens to be equal to $\partial H/\partial Q$, and we have recovered our second canonical equation.

The main purpose of the canonical transformation of Eq. (3.4.6) is to bring the Hamiltonian to the simple form of Eq. (3.4.9). This Hamiltonian is proportional to the new momentum P, and it does not depend on the new coordinate Q. As a result, the equations of motion are exceptionally simple, and they can be solved easily: The new momentum is a constant of the motion and the new coordinate Q behaves in time according to $Q(t) = \omega t + \delta$, where δ is a constant of integration. The transformation has therefore turned the original problem into a very simple one. With the solution to the simple problem in hand, we may return to the original problem and express its solution as

$$q(t) = \sqrt{\frac{2P}{\omega}}\sin(\omega t + \delta), \qquad p(t) = \sqrt{2\omega P}\cos(\omega t + \delta),$$

by substituting our solution for $Q(t)$ into Eqs. (3.4.7). Our linear pendulum evidently undergoes simple harmonic motion. The frequency of the motion is ω, and its amplitude is $\sqrt{2P/\omega}$.

3.4.3 General theory of canonical transformations

When is a transformation of the phase-space coordinates,

$$Q_a = Q_a(q_b, p_b, t), \qquad P_a = P_a(q_b, p_b, t),$$

a canonical transformation? The fundamental criterion is that the transformation must preserve the form of Hamilton's canonical equations: The transformation must produce a new Hamiltonian H' such that

$$\dot{Q}_a = \frac{\partial H'}{\partial P_a}, \qquad \dot{P}_a = -\frac{\partial H'}{\partial Q_a}.$$

The question is: Under what conditions does this occur? We will provide a number of answers to this question, ranging from the formal to the practical.

Let us recall from Sec. 3.1.4 that Hamilton's equations for the original set (q_a, p_a) of phase-space coordinates can be derived on the basis of Hamilton's principle of least action. The principle can be expressed in the form

$$\delta \int_{t_0}^{t_1} \left(\sum_a p_a\, dq_a - H\, dt\right) = 0;$$

the variations $\delta q_a(t)$, $\delta p_a(t)$ are all independent, and they are subjected to the boundary conditions $\delta q_a(t_0) = \delta q_a(t_1) = 0$. If Hamilton's equations are to hold also for the new set (Q_a, P_a) of phase-space coordinates, they must also follow from Hamilton's principle. We must then have, simultaneously,

$$\delta \int_{t_0}^{t_1} \left(\sum_a P_a \, dQ_a - H' \, dt \right) = 0;$$

here it is the variations $\delta Q_a(t)$, $\delta P_a(t)$ that are taken to be independent and subjected to the boundary conditions $\delta Q_a(t_0) = \delta Q_a(t_1) = 0$. The two formulations of Hamilton's principle will be compatible with each other if and only if the integrands $\sum_a p_a \, dq_a - H \, dt$ and $\sum_a P_a \, dQ_a - H' \, dt$ differ by the total derivative dF_1 of a function $F_1(q_a, Q_a, t)$ of the old and new coordinates. For we would have, in this case, a difference of integrals given by

$$\int_{t_0}^{t_1} dF_1 = F_1\big(q_a(t_1), Q_a(t_1), t_1\big) - F_1\big(q_a(t_0), Q_a(t_0), t_0\big),$$

and $\delta \int_{t_0}^{t_1} dF_1 = 0$ would follow immediately by virtue of the boundary conditions on the variations δq_a and δQ_a.

The first answer to our question is therefore this: A transformation of the phase-space coordinates is a canonical transformation when there exists a function $F_1(q_a, Q_a, t)$ such that

$$\sum_a p_a \, dq_a - H \, dt = \sum_a P_a \, dQ_a - H' \, dt + dF_1. \tag{3.4.10}$$

The function $F_1(q_a, Q_a, t)$ is called the *generating function* of the canonical transformation. This is a formal answer to our question; we will provide more practical answers at a later stage.

The total derivative of F_1 can be expressed as

$$dF_1 = \sum_a \frac{\partial F_1}{\partial q_a} dq_a + \sum_a \frac{\partial F_1}{\partial Q_a} dQ_a + \frac{\partial F_1}{\partial t} dt.$$

On the other hand, Eq. (3.4.10) can be rewritten as

$$dF_1 = \sum_a p_a \, dq_a - \sum_a P_a \, dQ_a + (H' - H) \, dt.$$

Because both equations must be true, we obtain the identifications

$$p_a = \frac{\partial F_1}{\partial q_a}, \qquad P_a = -\frac{\partial F_1}{\partial Q_a}, \qquad H' = H + \frac{\partial F_1}{\partial t}. \tag{3.4.11}$$

The first two equations give us the old momenta p_a and the new momenta P_a in terms of the derivatives of the generating function. The last equation gives us the new Hamiltonian H'; if the generating function does not depend explicitly on time, the new Hamiltonian is the same as the old.

As a trivial application of the foregoing, let us consider the generating function $F_1 = \sum_b q_b Q_b$. The old momenta are $p_a = \partial F_1 / \partial q_a = Q_a$ and the new momenta are $P_a = -\partial F_1 / \partial Q_a = -q_a$. This generating function therefore produces the trivial transformation $Q_a = p_a$, $P_a = -q_a$ that was encountered previously. This is a canonical transformation because it is generated by the function F_1. Because this function does not depend explicitly on time, the transformation does not change the Hamiltonian: $H' = H$. And the transformation, evidently, preserves the form of the canonical equations.

A more interesting application involves the function $F_1 = \frac{1}{2}\omega q^2 \text{cotan} Q$, which generates the transformation of Eqs. (3.4.6) and (3.4.7). We have

$$p = \frac{\partial F_1}{\partial q} = \omega q \,\text{cotan} Q = \frac{\omega q}{\tan Q}$$

and

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2}\omega q^2(1 + \text{cotan}^2 Q) = \frac{1}{2}\omega q^2\left(1 + \frac{p^2}{\omega^2 q^2}\right) = \frac{1}{2\omega}(p^2 + \omega^2 q^2),$$

as was anticipated in Eqs. (3.4.6). Because F_1 does not depend explicitly on t, we have that $H' = H = \omega P$, as was stated in Eq. (3.4.9).

3.4.4 Alternative generating functions

It is possible to introduce new generating functions that depend on an alternative choice of variables. Consider, for example, the new function

$$F_2 = F_1 + \sum_a Q_a P_a.$$

Its total derivative is

$$\begin{aligned} dF_2 &= dF_1 + \sum_a P_a\, dQ_a + \sum_a Q_a\, dP_a \\ &= \sum_a p_a\, dq_a - \sum_a P_a\, dQ_a + (H' - H)\, dt + \sum_a P_a\, dQ_a + \sum_a Q_a\, dP_a \\ &= \sum_a p_a\, dq_a + \sum_a Q_a\, dP_a + (H' - H)\, dt; \end{aligned}$$

in the second line we substituted a previous expression for dF_1, and in the last line we canceled out the terms $\sum_a P_a\, dQ_a$. The fact that dF_2 involves the differentials dq_a, dP_a, and dt informs us that F_2 must be a function of q_a, P_a, and t. We have, therefore,

$$F_2 = F_1 + \sum_a Q_a P_a = F_2(q_a, P_a, t), \tag{3.4.12}$$

and this new generating function does indeed depend on a different set of variables. Our previous calculation allows us to make the identifications

$$p_a = \frac{\partial F_2}{\partial q_a}, \qquad Q_a = \frac{\partial F_2}{\partial P_a}, \qquad H' = H + \frac{\partial F_2}{\partial t}. \tag{3.4.13}$$

This freedom to introduce alternative generating functions adds flexibility to the framework of canonical transformations. We will make use of this in the next section.

Exercise 3.24. Consider the new generating function $F_3 = F_1 - \sum_a q_a p_a$. On which variables does F_3 depend? Find expressions for P_a, q_a, and H' in terms of partial derivatives of F_3.

Exercise 3.25. Consider now the new generating function $F_4 = F_1 + \sum_a Q_a P_a - \sum_a q_a p_a$. On which variables does F_4 depend? Find expressions for q_a, Q_a, and H' in terms of partial derivatives of F_4.

3.4.5 Direct conditions

It is rarely convenient to test whether a transformation is canonical by attempting to find its generating function. More direct tests are available, fortunately, and these do not require knowledge of the generating function. We shall describe these tests in this and the following subsection. For simplicity we assume that the transformation does not depend explicitly on time; this means that $H' = H$.

The transformation $Q_a = Q_a(q_b, p_b)$ implies that the time derivative of the new coordinates can be expressed as

$$\begin{aligned}\dot{Q}_a &= \sum_b \frac{\partial Q_a}{\partial q_b}\dot{q}_b + \sum_b \frac{\partial Q_a}{\partial p_b}\dot{p}_b \\ &= \sum_b \frac{\partial Q_a}{\partial q_b}\frac{\partial H}{\partial p_b} - \sum_b \frac{\partial Q_a}{\partial p_b}\frac{\partial H}{\partial q_b}.\end{aligned}$$

If the transformation is canonical, this will be equal to $\partial H/\partial P_a$. With H written as a function of the old phase-space coordinates, this is

$$\frac{\partial H}{\partial P_a} = \sum_b \frac{\partial H}{\partial q_b}\frac{\partial q_b}{\partial P_a} + \sum_b \frac{\partial H}{\partial p_b}\frac{\partial p_b}{\partial P_a}.$$

Hamilton's equations therefore imply

$$\begin{aligned}0 &= \dot{Q}_a - \frac{\partial H}{\partial P_a} \\ &= \sum_b \left(\frac{\partial Q_a}{\partial q_b} - \frac{\partial p_b}{\partial P_a}\right)\frac{\partial H}{\partial p_b} - \sum_b \left(\frac{\partial Q_a}{\partial p_b} + \frac{\partial q_b}{\partial P_a}\right)\frac{\partial H}{\partial q_b}.\end{aligned}$$

This equation will be satisfied if and only if

$$\frac{\partial}{\partial q_b}Q_a(q_b,p_b) = \frac{\partial}{\partial P_a}p_b(Q_a,P_a), \qquad \frac{\partial}{\partial p_b}Q_a(q_b,p_b) = -\frac{\partial}{\partial P_a}q_b(Q_a,P_a). \quad (3.4.14)$$

This first set of conditions must therefore be met if the transformation is to be a canonical transformation.

The second set of conditions is obtained by starting instead with the transformation $P_a = P_a(q_b, p_b)$. This time we have

$$\begin{aligned}\dot{P}_a &= \sum_b \frac{\partial P_a}{\partial q_b}\dot{q}_b + \sum_b \frac{\partial P_a}{\partial p_b}\dot{p}_b \\ &= \sum_b \frac{\partial P_a}{\partial q_b}\frac{\partial H}{\partial p_b} - \sum_b \frac{\partial P_a}{\partial p_b}\frac{\partial H}{\partial q_b}.\end{aligned}$$

If the transformation is canonical, this will be equal to $-\partial H/\partial Q_a$. With H written as a function of the old phase-space coordinates, this is

$$\frac{\partial H}{\partial Q_a} = \sum_b \frac{\partial H}{\partial q_b}\frac{\partial q_b}{\partial Q_a} + \sum_b \frac{\partial H}{\partial p_b}\frac{\partial p_b}{\partial Q_a}.$$

Hamilton's equations therefore imply

$$\begin{aligned}0 &= \dot{P}_a + \frac{\partial H}{\partial Q_a} \\ &= \sum_b \left(\frac{\partial P_a}{\partial q_b} + \frac{\partial p_b}{\partial Q_a}\right)\frac{\partial H}{\partial p_b} - \sum_b \left(\frac{\partial P_a}{\partial p_b} - \frac{\partial q_b}{\partial Q_a}\right)\frac{\partial H}{\partial q_b}.\end{aligned}$$

This equation will be satisfied if and only if

$$\frac{\partial}{\partial q_b}P_a(q_b,p_b) = -\frac{\partial}{\partial Q_a}p_b(Q_a,P_a), \qquad \frac{\partial}{\partial p_b}P_a(q_b,p_b) = \frac{\partial}{\partial Q_a}q_b(Q_a,P_a). \quad (3.4.15)$$

This is the second set of conditions that must be met if the transformation is to be a canonical transformation.

Equations (3.4.14) and (3.4.15) are called the *direct conditions* for a canonical transformation: all these conditions will be satisfied if the transformation $Q_a(q_a,p_a)$ and $P_a(q_a,p_a)$ is a canonical transformation. For a mechanical system with n degrees of freedom we have a total of $4n^2$ conditions. As we shall see, these are not all independent. In the next subsection we will identify a smaller, and more convenient, set of necessary and sufficient conditions.

3.4.6 Canonical invariants

As was stated in Sec. 3.4.1, a canonical transformation has the property of leaving the element of phase-space volume invariant:

$$dV = dq_1dq_2\cdots dp_1dp_2\cdots = dQ_1dQ_2\cdots dP_1dP_2\cdots. \quad (3.4.16)$$

A canonical transformation of the phase-space coordinates therefore has a unit Jacobian, $J = 1$. This statement can be shown to be a consequence of the direct conditions, Eqs. (3.4.14) and (3.4.15).

Another consequence of the direct conditions is the fact that canonical transformations leave all Poisson brackets invariant. Thus, if

$$[f,g]_{q,p} = \sum_a\left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a}\right)$$

is the Poisson bracket in the old phase-space coordinates, and if

$$[f,g]_{Q,P} = \sum_a\left(\frac{\partial f}{\partial Q_a}\frac{\partial g}{\partial P_a} - \frac{\partial f}{\partial P_a}\frac{\partial g}{\partial Q_a}\right)$$

is the Poisson bracket in the new coordinates, then

$$[f,g]_{q,p} = [f,g]_{Q,P} \quad (3.4.17)$$

if the transformation is canonical.

It is this statement which provides us with an efficient method to test whether a transformation $Q_a = Q_a(q_b,p_b,t)$, $P_a = P_a(q_b,p_b,t)$ is canonical: By virtue of the automatic relations (refer back to Sec. 3.3.4)

$$[Q_a,Q_b]_{Q,P} = 0, \qquad [Q_a,P_b]_{Q,P} = \delta_{ab}, \qquad [P_a,P_b]_{Q,P} = 0$$

and the invariance of the Poisson bracket, we must have that the relations

$$[Q_a,Q_b]_{q,p} = 0, \qquad [Q_a,P_b]_{q,p} = \delta_{ab}, \qquad [P_a,P_b]_{q,p} = 0 \quad (3.4.18)$$

hold if the transformation is canonical. Similarly, a transformation $q_a = q_a(Q_b,P_b,t)$, $p_a = p_a(Q_b,P_b,t)$ is canonical if the Poisson-bracket relations

$$[q_a,q_b]_{Q,P} = 0, \qquad [q_a,p_b]_{Q,P} = \delta_{ab}, \qquad [p_a,p_b]_{Q,P} = 0 \quad (3.4.19)$$

are satisfied. The conditions of Eqs. (3.4.18) or (3.4.19) can be shown to be sufficient and necessary. For a mechanical system with n degrees of freedom, we have a total

of $(2n-1)n$ conditions to satisfy; when $n=1$ there is only one relevant condition, $[Q,P]_{q,p}=1$ or $[q,p]_{Q,P}=1$.

Exercise 3.26. Verify that $[Q,P]_{q,p}=1$ in the case of the canonical transformation presented in Sec. 3.4.2.

We will not present a general proof of the statements that the phase-space volume element and the Poisson bracket are canonical invariants. We will, instead, present a proof that is restricted to a two-dimensional phase space. The restricted proof is easy to produce; the general proof would be much more difficult.

We consider a canonical transformation of the form $Q=Q(q,p)$, $P=P(q,p)$. The direct conditions for this transformation are

$$\frac{\partial Q}{\partial q}=\frac{\partial p}{\partial P},\qquad \frac{\partial Q}{\partial p}=-\frac{\partial q}{\partial P},\qquad \frac{\partial P}{\partial q}=-\frac{\partial p}{\partial Q},\qquad \frac{\partial P}{\partial p}=\frac{\partial q}{\partial Q}.$$

The volume elements are related by

$$dqdp=|J|\,dQdP,\qquad dQdP=|J|^{-1}\,dqdp,$$

in which J is the Jacobian of the transformation, and J^{-1} its inverse. The Jacobian is

$$J=\begin{vmatrix}\partial q/\partial Q & \partial q/\partial P\\ \partial p/\partial Q & \partial p/\partial P\end{vmatrix}=\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P}-\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q}.$$

Its inverse is

$$J^{-1}=\begin{vmatrix}\partial Q/\partial q & \partial Q/\partial p\\ \partial P/\partial q & \partial P/\partial p\end{vmatrix}=\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}.$$

By involving the direct conditions we may write this as

$$J^{-1}=\frac{\partial p}{\partial P}\frac{\partial q}{\partial Q}-\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q},$$

and this is equal to J. We therefore have $J^{-1}=J$, or $J^2=1$, and we conclude that $|J|=1$. This proves that the volume element is indeed preserved under a canonical transformation.

The Poisson bracket in the new phase-space coordinates is

$$[f,g]_{Q,P}=\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}-\frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}.$$

If we consider f and g to be functions of q and p, we may use the chain rule and express this as

$$\begin{aligned}
[f,g]_{Q,P} &= \left(\frac{\partial f}{\partial q}\frac{\partial q}{\partial Q}+\frac{\partial f}{\partial p}\frac{\partial p}{\partial Q}\right)\left(\frac{\partial g}{\partial q}\frac{\partial q}{\partial P}+\frac{\partial g}{\partial p}\frac{\partial p}{\partial P}\right)\\
&\quad-\left(\frac{\partial f}{\partial q}\frac{\partial q}{\partial P}+\frac{\partial f}{\partial p}\frac{\partial p}{\partial P}\right)\left(\frac{\partial g}{\partial q}\frac{\partial q}{\partial Q}+\frac{\partial g}{\partial p}\frac{\partial p}{\partial Q}\right)\\
&= \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}\left(\frac{\partial q}{\partial Q}\frac{\partial q}{\partial P}-\frac{\partial q}{\partial P}\frac{\partial q}{\partial Q}\right)+\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P}-\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q}\right)\\
&\quad+\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\left(\frac{\partial p}{\partial Q}\frac{\partial q}{\partial P}-\frac{\partial p}{\partial P}\frac{\partial q}{\partial Q}\right)+\frac{\partial f}{\partial p}\frac{\partial g}{\partial p}\left(\frac{\partial p}{\partial Q}\frac{\partial p}{\partial P}-\frac{\partial p}{\partial P}\frac{\partial p}{\partial Q}\right)\\
&= \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)\left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P}-\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q}\right)\\
&= J[f,g]_{q,p}.
\end{aligned}$$

We have learned that $|J| = 1$, and in practice we may always design the canonical transformation so that its Jacobian is in fact $J = +1$. This gives us, then, the statement that $[f, g]_{Q,P} = [f, g]_{q,p}$, and the Poisson bracket is indeed invariant under a canonical transformation.

3.5 Hamilton-Jacobi equation

3.5.1 Action as a function of the coordinates and time

The action functional of a mechanical system is

$$S = \int_{t_0}^{t_1} L\, dt,$$

where $L(q_a, \dot{q}_a, t)$ is the system's Lagrangian. We first encountered the action in the context of Hamilton's principle of least action, in which one compares the value of S for different trial paths $q_a^{\text{trial}}(t)$ and attempts to find the paths $\bar{q}_a(t)$ that minimize this value. In the course of these investigations, back in Secs. 2.2 and 2.3, we derived the result

$$\delta S = \sum_a \frac{\partial L}{\partial \dot{q}_a} \delta q_a \bigg|_{t_0}^{t_1} + \sum_a \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_a} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} \right) \delta q_a \, dt$$

for the variation of the action about reference paths $\bar{q}_a(t)$. We obtained the Euler-Lagrange equations by demanding that $\delta S = 0$ for variations $\delta q_a(t)$ that respect the boundary conditions $\delta q_a(t_0) = \delta q_a(t_1) = 0$.

We now intend to examine this result from a different perspective. Suppose that we compute S for *actual paths* $\bar{q}_a(t)$ that satisfy the Euler-Lagrange equations; we assume that our actual paths leave the positions q_a^{begin} at $t = t_0$ and arrive at the positions q_a^{end} at $t = t_1$. The result would be the number $\bar{S}$, and this number would depend on the choices made for q_a^{begin}, q_a^{end}, t_0, and t_1.

We now ask the question: Suppose that we next evaluate S on displaced paths $q_a(t) = \bar{q}_a(t) + \delta q_a(t)$ that all leave q_a^{begin} at $t = t_0$ but arrive at the different positions $q_a^{\text{end}} + \delta q_a^{\text{end}}$ at the time $t = t_1$; how will this value of S differ from $\bar{S}$? The answer is this: Because the reference paths all satisfy the Euler-Lagrange equations, and because the variations δq_a all vanish at $t = t_0$, the change in the action has to be

$$\delta S = \sum_a \frac{\partial L}{\partial \dot{q}_a} \bigg|_{t=t_1} \delta q_a^{\text{end}}.$$

Writing $\partial L / \partial \dot{q}_a = p_a$ and $\delta q_a^{\text{end}} = \delta q_a(t_1)$, this is

$$\delta S = \sum_a p_a(t_1)\, \delta q_a(t_1). \tag{3.5.1}$$

This result indicates that the action S is a function of the variables $q_a(t_1) \equiv q_a^{\text{end}}$, $S = S(q_a(t_1))$, and that

$$p_a(t_1) = \frac{\partial S}{\partial q_a(t_1)}. \tag{3.5.2}$$

It is understood that here, the partial derivative is evaluated while holding t_1 fixed.

Let us now consider a different variation of the action. This time we choose displaced paths $q_a(t) = \bar{q}_a(t) + \delta q_a(t)$ that still all leave q_a^{begin} at $t = t_0$, but that now arrive at the *same positions* q_a^{end} at a *different time* $t = t_1 + \delta t_1$; we wish to calculate by how much S differs from $\bar{S}$ under this change of paths.

To figure this out it is helpful to recall that the *total derivative* of the action with respect to t_1 is given by

$$\frac{dS}{dt_1} = L(t_1),$$

in which the Lagrangian function is evaluated at $t = t_1$. We already know that the action depends on t_1 through its dependence on $q_a^{\rm end} = q_a(t_1)$. We should also expect that the action contains an explicit dependence on t_1. Its total time derivative must therefore be expressed as

$$\frac{dS}{dt_1} = \frac{\partial S}{\partial t_1} + \sum_a \frac{\partial S}{\partial q_a(t_1)} \dot{q}_a(t_1).$$

In view of Eq. (3.5.2), this is

$$\frac{dS}{dt_1} = \frac{\partial S}{\partial t_1} + \sum_a p_a(t_1) \dot{q}_a(t_1).$$

From all this we obtain

$$\frac{\partial S}{\partial t_1} = L(t_1) - \sum_a p_a(t_1) \dot{q}_a(t_1).$$

The right-hand side is (minus) the Hamiltonian function evaluated at $t = t_1$, and our final result is

$$\frac{\partial S}{\partial t_1} = -H(t_1). \tag{3.5.3}$$

It is understood that here, the partial derivative is evaluated while holding the final positions $q_a(t_1)$ fixed. This gives us the answer to our question: The variation considered here has fixed final positions and a varying time; the change in the action $S(q_a(t_1), t_1)$ is $\delta S = (\partial S/\partial t_1)\, \delta t_1$, or

$$\delta S = -H(t_1)\, \delta t_1. \tag{3.5.4}$$

The complete variation of the action, if we allow all of $q_a^{\rm end}$ and t_1 to be varied, is given by the sum of the partial deviations computed above. The general statement is

$$dS = \sum_a p_a(t_1)\, dq_a(t_1) - H(t_1)\, dt_1.$$

Because this statement is true at any time t_1, we may express it as

$$dS = \sum_a p_a\, dq_a - H\, dt, \tag{3.5.5}$$

where the momenta and the Hamiltonian are now evaluated at the arbitrary time t. This relation informs us that when the action is evaluated on the actual paths $\bar{q}_a(t)$, it can be viewed as a function of the coordinates $q_a(t)$ and of time t:

$$S = S(q_a, t). \tag{3.5.6}$$

Its partial derivatives are then given by

$$\frac{\partial S}{\partial q_a} = p_a, \qquad \frac{\partial S}{\partial t} = -H. \tag{3.5.7}$$

As a concrete illustration of these notions, let us evaluate $S(q, t)$ in the case of the linear pendulum of Sec. 3.4.2. The pendulum's Lagrangian is

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2, \tag{3.5.8}$$

and the Euler-Lagrange equation for $q(t)$ is $\ddot{q}+\omega^2 q = 0$. The actual path is therefore given by

$$q(t) = q_0 \cos\omega t + \frac{\dot{q}_0}{\omega}\sin\omega t, \tag{3.5.9}$$

where $q_0 \equiv q(t=0)$ and $\dot{q}_0 \equiv \dot{q}(t=0)$ are the initial conditions. We substitute this inside Eq. (3.5.8) and obtain

$$L = \frac{1}{2}(\dot{q}_0^2 - \omega^2 q_0^2)\cos 2\omega t - \omega q_0 \dot{q}_0 \sin 2\omega t$$

after some simplification. Setting $t_0 = 0$ and $t_1 = t$, the action is $S = \int_0^t L\,dt$, and this evaluates to

$$S = \frac{1}{2\omega}(\dot{q}_0^2 - \omega^2 q_0^2)\sin\omega t\cos\omega t - q_0\dot{q}_0 \sin^2\omega t.$$

This does not yet have the expected form $S(q,t)$ with $q \equiv q(t)$. To put the action in this form we solve Eq. (3.5.9) for $\dot{q}_0$ and substitute this into our expression for S. After some simple algebra, we obtain our final answer,

$$S(q,t) = \frac{\omega}{2\sin\omega t}\Big[(q^2+q_0^2)\cos\omega t - 2q_0 q\Big], \tag{3.5.10}$$

in which q stands for $q(t)$, the changing position of the pendulum. It is easy to check that $\partial S/\partial q = \dot{q}_0\cos\omega t - \omega q_0 \sin\omega t = \dot{q} = p$ and $-\partial S/\partial t = \frac{1}{2}\dot{q}_0^2 + \frac{1}{2}\omega^2 q_0^2 = H$, in agreement with Eqs. (3.5.7).

Exercise 3.27. Go through all the algebra that leads to Eq. (3.5.10), starting from Eqs. (3.5.8) and (3.5.9). Then check that Eqs. (3.5.7) do indeed follow for this action.

3.5.2 Hamilton-Jacobi equation

We have seen that the partial derivative with respect to time of the action is related to the Hamiltonian by

$$H + \frac{\partial S}{\partial t} = 0.$$

The Hamiltonian is a function of the coordinates q_a and the momenta p_a, so that $H = H(q_1, q_2, \cdots; p_1, p_2, \cdots; t)$. But we have also seen that the momenta are related to the action by $p_a = \partial S/\partial q_a$. Putting this all together, we arrive at the equation

$$H\left(q_1, q_2, \cdots; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \cdots; t\right) + \frac{\partial S}{\partial t} = 0. \tag{3.5.11}$$

This is a partial differential equation for the function $S(q_1, q_2, \cdots; t)$, and this equation is known as the *Hamilton-Jacobi equation*. As we shall now explain, solving the Hamilton-Jacobi equation for S provides a round-about way of obtaining a complete solution to the original mechanical problem, which is to calculate how the coordinates q_a behave as a function of time. This technique is intricate, but it can be very powerful.

Suppose that we can find a solution to the Hamilton-Jacobi equation, and suppose that it has the general form

$$S = S(q_1, q_2, \cdots, q_n; \alpha_1, \alpha_2, \cdots, \alpha_n; t), \tag{3.5.12}$$

where n is the number of degrees of freedom, and where the α_a's are n independent constants of integration. Such a solution is called a *complete solution* to the

Hamilton-Jacobi equation, because it possesses a number of integration constants that corresponds to the number of independent variables q_a. We assume that a complete solution exists and can be obtained; we do not assume that this solution is unique — indeed it is not — nor that it is the most general solution to the Hamilton-Jacobi equation — which it is not.

To establish a connection between $S(q_a, \alpha_a, t)$ and the original mechanical problem we identify it with $F_2(q_a, P_a, t)$, the generating function of a canonical transformation. Here the new momenta P_a are identified with the constants α_a, and we will see in a moment that the dynamics generated by the new Hamiltonian H' is indeed such that $\dot{P}_a = 0$. The general theory of canonical transformations developed in Secs. 3.4.3 and 3.4.4 implies that the old momenta p_a are given by

$$p_a = \frac{\partial F_2}{\partial q_a} = \frac{\partial S}{\partial q_a},$$

and this statement is certainly compatible with our derivation of the Hamilton-Jacobi equation. The general theory also implies that the new coordinates Q_a are given by

$$Q_a = \frac{\partial F_2}{\partial P_a} = \frac{\partial S}{\partial \alpha_a}.$$

The evolution of the new phase-space variables is governed by the new Hamiltonian H', which is

$$H' = H + \frac{\partial F_2}{\partial t} = H + \frac{\partial S}{\partial t} = 0.$$

The new Hamiltonian vanishes by virtue of the Hamilton-Jacobi equation! There is no dynamics in the new variables, because $\dot{Q}_a = \partial H'/\partial P_a = 0$ and $\dot{P}_a = -\partial H'/\partial Q_a = 0$. We have already anticipated the fact that the new momenta $P_a \equiv \alpha_a$ are constants of the motion; we now have learned that the new coordinates $Q_a \equiv \beta_a$ are constants also.

The entire content of the Hamilton-Jacobi framework boils down to this: Once a complete solution $S(q_a, \alpha_a, t)$ to the Hamilton-Jacobi equation has been identified, the coordinates $q_a(t)$ of the mechanical system are obtained by unwrapping the equations

$$\beta_a = \frac{\partial}{\partial \alpha_a} S(q_b, \alpha_b, t), \tag{3.5.13}$$

where the n quantities β_a, like the n quantities α_a, are constants. Solving these equations will return equations of the from $q_a = q_a(\alpha_b, \beta_b, t)$, and the coordinates will be seen to depend on time as well as a number $2n$ of constants; this is as it should be, because we have n variables q_a and they each satisfy a second-order differential equation. The momenta can then be computed as

$$p_a = \frac{\partial}{\partial q_a} S(q_b, \alpha_b, t), \tag{3.5.14}$$

and these will also be of the form $p_a = p_a(\alpha_b, \beta_b, t)$. The motion in phase space is thus completely determined, and the constants α_a and β_a can be related to the initial conditions $q_a(t = 0)$ and $p_a(t = 0)$.

3.5.3 Case study: Linear pendulum

To see how this all works, let us return once more to the linear pendulum and its Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{3.5.15}$$

The Hamilton-Jacobi equation for this Hamiltonian is

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 + \frac{\partial S}{\partial t} = 0. \tag{3.5.16}$$

Because the mechanical system has a single degree of freedom, we wish to find a complete solution of the form $S(q, \alpha)$, with α playing the role of a constant of integration.

We can separate the variables by adopting

$$S = W(q) - \alpha t \tag{3.5.17}$$

as a form of solution; this already involves the constant α. Substituting Eq. (3.5.17) into Eq. (3.5.16) produces an ordinary differential equation for W:

$$\frac{1}{2}(W')^2 + \frac{1}{2}\omega^2 q^2 - \alpha = 0.$$

This can easily be solved for W',

$$W' = \sqrt{2\alpha}\sqrt{1 - \frac{\omega^2}{2\alpha}q^2},$$

and integration yields

$$W = \sqrt{2\alpha}\int \sqrt{1 - \frac{\omega^2}{2\alpha}q^2}\, dq.$$

To evaluate the integral we introduce a new variable of integration Φ, which is defined by

$$\sin\Phi = \frac{\omega}{\sqrt{2\alpha}}q. \tag{3.5.18}$$

Substituting this, along with $dq = (\sqrt{2\alpha}/\omega)\cos\Phi\, d\Phi$ into the integral for W produces

$$W = \frac{2\alpha}{\omega}\int \cos^2\Phi\, d\Phi.$$

The integral works out to be $\frac{1}{2}(\Phi + \sin\Phi\cos\Phi)$, and we arrive at

$$W = \frac{\alpha}{\omega}(\Phi + \sin\Phi\cos\Phi). \tag{3.5.19}$$

This could be expressed directly in terms of q by involving Eq. (3.5.18), but it is more convenient in practice to leave $W(q)$ in this implicit form.

Our final result for the action is

$$S(q, \alpha, t) = \frac{\alpha}{\omega}(\Phi + \sin\Phi\cos\Phi) - \alpha t, \qquad q = \frac{\sqrt{2\alpha}}{\omega}\sin\Phi. \tag{3.5.20}$$

Let us now use this information to determine the motion of the pendulum. According to Eq. (3.5.13) we must first calculate $\partial S/\partial\alpha$ and set the result equal to a new constant β. The dependence of S on α is both explicit — S is proportional to α — and implicit, because S depends also on Φ which itself depends on α. We therefore write

$$\begin{aligned}\frac{\partial S}{\partial\alpha} &= \frac{1}{\omega}(\Phi + \sin\Phi\cos\Phi) - t + \frac{\alpha}{\omega}(1 + \cos^2\Phi - \sin^2\Phi)\frac{\partial\Phi}{\partial\alpha} \\ &= \frac{1}{\omega}(\Phi + \sin\Phi\cos\Phi) + \frac{2\alpha}{\omega}\cos^2\Phi\frac{\partial\Phi}{\partial\alpha} - t,\end{aligned}$$

and we evaluate the remaining partial derivative using Eq. (3.5.18) as a starting point. Here we treat q as a constant and differentiate the two sides of the equation with respect to α; this gives

$$\cos\Phi\frac{\partial\Phi}{\partial\alpha} = -\frac{\omega q}{2\sqrt{2}\alpha^{3/2}} = -\frac{1}{2\alpha}\sin\Phi,$$

after reinvolving Eq. (3.5.18) in the last step. We have obtained

$$\begin{aligned}\frac{\partial S}{\partial\alpha} &= \frac{1}{\omega}(\Phi+\sin\Phi\cos\Phi) - \frac{1}{\omega}\cos^2\Phi\frac{\sin\Phi}{\cos\Phi} - t\\ &= \frac{1}{\omega}\Phi - t\\ &= \beta,\end{aligned}$$

or

$$\Phi = \omega(t+\beta). \tag{3.5.21}$$

The motion of the pendulum is finally determined by substituting Eq. (3.5.21) into Eq. (3.5.20); our final result is

$$q(t,\alpha,\beta) = \frac{\sqrt{2\alpha}}{\omega}\sin\omega(t+\beta). \tag{3.5.22}$$

This evidently describes simple harmonic motion of amplitude $\sqrt{2\alpha}/\omega$ at a frequency ω; this well-known result has been obtained in a very novel way, by solving the Hamilton-Jacobi equation. While the use of this fancy technique hardly seems justified for such a simple problem (the phrase cracking a nut with a sledgehammer comes to mind), the Hamilton-Jacobi framework has been shown to be very powerful in other, more complicated, situations.

We can easily relate the constants α and β to the initial conditions of the motion. Evaluating Eq. (3.5.22) at $t = 0$ gives

$$q_0 \equiv q(t=0) = \frac{\sqrt{2\alpha}}{\omega}\sin\omega\beta,$$

while differentiating Eq. (3.5.22) with respect to time and then evaluating at $t = 0$ gives

$$\dot{q}_0 \equiv \dot{q}(t=0) = \sqrt{2\alpha}\cos\omega\beta.$$

These relations can easily be solved for α and β. The constant β has a direct physical meaning: it determines the initial phase of the pendulum. The constant α also has a clear physical meaning: Solving for α yields

$$\alpha = \frac{1}{2}\dot{q}_0^2 + \frac{1}{2}\omega^2 q_0^2, \tag{3.5.23}$$

and this is the pendulum's total mechanical energy.

Exercise 3.28. Calculate the momentum p of the pendulum starting from Eq. (3.5.14); show that $p(t,\alpha,\beta) = \dot{q}(t,\alpha,\beta)$.

Exercise 3.29. You may have noticed that the action of Eq. (3.5.20) is very different from the action of Eq. (3.5.10), which we rewrite as

$$S(q,\alpha',t) = \frac{\omega}{2\sin\omega t}\Big[(q^2+\alpha'^2)\cos\omega t - 2\alpha' q\Big],$$

with $\alpha' \equiv q_0$. Despite the functional differences, these are two different representations of the same physical quantity, expressed in terms of two different constants, α and α'. Show that the action given here is also a solution to the Hamilton-Jacobi equation. Determine the motion of the pendulum by setting $\partial S/\partial\alpha'$ equal to a new constant β'; what is the physical meaning of β'?

3.6 Problems

1. A bead of mass m slides on a frictionless wire that is shaped in the form of a cycloid. This is described by the parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 + \cos\theta),$$

where a is a constant and the parameter θ ranges through the interval $0 \leq \theta \leq 2\pi$. The bead is subjected to gravity, and it oscillates back and forth on the wire. (See problem #3 from Chapter 2.)

(a) Using θ as a generalized coordinate, calculate the bead's Hamiltonian.

(b) Obtain Hamilton's canonical equations of motion for the bead.

2. A particle of mass m moves on a paraboloid of revolution described by the equation

$$z = \frac{1}{a}\left(x^2 + y^2\right),$$

where a is a constant. (See the figure for problem #4 from Chapter 2.) The particle is subjected to gravity, so that its potential energy is $V = mgz$. Using the cylindrical coordinates ρ and ϕ as generalized coordinates, find the Hamiltonian of the particle. [The cylindrical coordinates are defined by $x = \rho\cos\phi$, $y = \rho\sin\phi$.]

3. A straight frictionless wire is attached at a height h to the z axis, and it makes an angle α relative to the z axis. The wire rotates around the z axis with a constant angular velocity Ω. A bead of mass m slides on the wire and is subjected to gravity; it is at a distance r from the point at which the wire is attached to the z axis. (See the figure for problem #5 from Chapter 2.)

(a) Using r as a generalized coordinate, calculate the bead's Hamiltonian.

(b) Obtain Hamilton's canonical equations of motion for the bead.

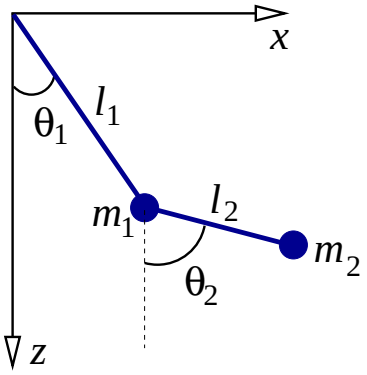
4. A particle of mass m is constrained to move on the surface of a cylinder. The cylinder is described in cylindrical coordinates by the equation $\rho = R$, where ρ is the distance from the z axis and R is the cylinder's radius. The particle is subjected to a force directed toward the origin of the coordinate system and proportional to the distance between the particle and the origin; this force is described by $\boldsymbol{F} = -k\boldsymbol{r}$, where k is a constant and $\boldsymbol{r}$ is the particle's position vector. (See problem #8 from Chapter 2.)

(a) Using the cylindrical coordinates z and ϕ as generalized coordinates, find the particle's Hamiltonian.

(b) Obtain Hamilton's canonical equations of motion for the particle. Show in particular that p_ϕ is a constant of the motion.

(c) Draw the particle's motion in the reduced phase space spanned by z and p_z.

5. A pendulum of mass m_2 and length ℓ_2 is attached to another pendulum of mass m_1 and length ℓ_1 (see diagram). The first pendulum is at an angle $\theta_1(t)$ relative to the vertical, while the second pendulum is at an angle $\theta_2(t)$. We wish to determine the motion of this *double pendulum*.



(a) Show that the Lagrangian of the double pendulum is given by

$$L = \frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2+m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+(m_1+m_2)g\ell_1\cos\theta_1+m_2g\ell_2\cos\theta_2.$$

(b) Calculate the generalized momenta p_1 and p_2 and express $\dot{\theta}_1$ and $\dot{\theta}_2$ in terms of them.

(c) Find the Hamiltonian of the double pendulum.

(d) Show that Hamilton's equations are

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\ell_2 p_1 - \ell_1 p_2\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2\big[m_1+m_2\sin^2(\theta_1-\theta_2)\big]},\\
\dot{\theta}_1 &= \frac{(m_1+m_2)\ell_1 p_2 - m_2\ell_2 p_1\cos(\theta_1-\theta_2)}{\ell_1\ell_2^2\big[m_1+m_2\sin^2(\theta_1-\theta_2)\big]},\\
\dot{p}_1 &= -A+B-(m_1+m_2)g\ell_1\sin\theta_1,\\
\dot{p}_2 &= A-B-m_2g\ell_2\sin\theta_2,
\end{aligned}$$

where

$$A = \frac{p_1p_2\sin(\theta_1-\theta_2)}{\ell_1\ell_2\big[m_1+m_2\sin^2(\theta_1-\theta_2)\big]}$$

and

$$B = \frac{m_2\ell_2^2p_1^2+(m_1+m_2)\ell_1^2p_2^2-2m_2\ell_1\ell_2p_1p_2\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2^2\big[m_1+m_2\sin^2(\theta_1-\theta_2)\big]^2}\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2).$$

Appendix A
Term project: Motion around a black hole

This term project is to be handed in on the last day of classes. While working on this project you are permitted to have discussions with your colleagues on any aspect of the project. However, and this is important, you are not allowed to directly collaborate when carrying out the tasks listed below. You must work through these by yourself, independently of anyone else. Cheating will not be tolerated.

In this term project you will examine the motion of a particle in the strong gravitational field of a (nonrotating) black hole. The equations of motion for the particle are derived from Einstein's general relativity, according to which *the particle must follow a geodesic* — the straightest possible path — *in the curved spacetime of the black hole*. The motion of the particle is represented in polar coordinates by its radial position $r(t)$ and its angular position $\phi(t)$. Because the gravitational field of the black hole is spherically symmetric, the motion takes place within a plane (just as in Newtonian theory).

A.1 Equations of motion

The general relativistic equations of motion are

$$\dot{\phi} = \frac{h}{r^2} \tag{A.1.1}$$

for the angular position, where h is a constant (related to the particle's angular momentum), and

$$\ddot{r} + \frac{GM}{r^2} - \frac{h^2}{r^3}\left(1 - \frac{3R}{2r}\right) = 0 \tag{A.1.2}$$

for the radial position. Here an overdot indicates differentiation with respect to t, and

$$R = \frac{2GM}{c^2} \tag{A.1.3}$$

is the *Schwarzschild radius* of the black hole. Notice that Eq. (A.1.1) is identical to its Newtonian analogue, and that Eq. (A.1.2) is very similar to it. In fact, the equations of motion reduce to their Newtonian expressions when $R/r \ll 1$, that is, when the particle is very far from the black hole so that the hole's gravitational field is very weak.

Task 1. Calculate R for a black hole whose mass is equal to the Sun's. Then calculate $R/R_\odot$, the ratio of the Sun's Schwarzschild radius R to its actual radius

$R_\odot$. This number characterizes the size of general relativistic effects in the solar system. These effects are small, but they have been measured.

Task 2. Integrate Eq. (A.1.2) and express your result in the usual form

$$\frac{1}{2}\dot{r}^2 + \nu(r) = \varepsilon, \tag{A.1.4}$$

in which $\nu(r)$ is the reduced effective potential and ε the reduced total mechanical energy (which is constant).

Task 3. Sketch the form of the effective potential $\nu(r)$ and give a complete qualitative description of the possible motions. Be careful: there is a wider range of possibilities than in Newtonian theory. When you plot the potential be sure to choose values of h that are both above and below the critical value

$$h_c = \sqrt{6GMR}. \tag{A.1.5}$$

Explain what happens to $\nu(r)$ when h decreases below h_c.

A.2 Circular orbits

The equations of motion admit solutions in which $r(t)$ stays constant, $r(t) = r_0 =$ constant; these solutions represent circular orbits around the black hole.

Task 4. For a circular orbit of radius r_0, calculate its angular momentum h, its energy ε, and its angular velocity $\dot{\phi}$. Show that your relativistic results reduce to the Newtonian expressions when R/r_0 is very small.

A.3 Eccentric orbits

Eccentric motion is possible when $\varepsilon < 0$; there are solutions to the equations of motion which describe a particle that moves between two turning points at $r = r_-$ and $r = r_+$, where $r_- < r_+$. It is convenient to introduce the same parameterization as in Newtonian theory, and to set

$$r_- = \frac{p}{1+e} = \text{pericentre}, \qquad r_+ = \frac{p}{1-e} = \text{apocentre}, \tag{A.3.1}$$

where p is an average radius and e an eccentricity. When $e = 0$ we have that $r_- = r_+ = p \equiv r_0$, and the orbit is circular.

The results of Task 3 above will have revealed that unlike in Newtonian theory, where there are only two turning points, there exists in the relativistic situation a third turning point at $r = r_<$. (For this to be true we need $\varepsilon < 0$, which was already assumed, and $h > h_c$, which is understood.) We have $r_< < r_-$, and for our purposes this third turning point plays a mathematical role, but has no physical

significance. The three turning points are located by setting $\dot{r} = 0$ in Eq. (A.1.4), which implies $\nu(r) - \varepsilon = 0$.

Task 5. Show that $\nu(r) - \varepsilon$ has the structure of a cubic polynomial in $1/r$. This can therefore be presented in the factorized form

$$\nu(r) - \varepsilon = k\left(\frac{1}{r} - \frac{1}{r_<}\right)\left(\frac{1}{r} - \frac{1}{r_-}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right), \tag{A.3.2}$$

in which k is a constant of proportionality. As required, $\nu(r) - \varepsilon$ vanishes whenever r becomes equal to either one of $r_<$, r_-, or r_+. Use this observation to:

1. Calculate k.
2. Express $r_<$ in terms of r_-, r_+, and R.
3. Express h^2 in terms of r_-, r_+, R, and GM.
4. Express ε in terms of r_-, r_+, R, and GM.

Finally, clean up these results by substituting Eq. (A.3.1). Show that

$$h^2 = \frac{GMp}{1 - \frac{1}{2}(3 + e^2)R/p} \tag{A.3.3}$$

and

$$\varepsilon = -\frac{GM}{2p}(1 - e^2)\frac{1 - 2R/p}{1 - \frac{1}{2}(3 + e^2)R/p}. \tag{A.3.4}$$

Notice that Eqs. (A.3.3) and (A.3.4) reduce to the Newtonian expressions when $R/p \ll 1$, as should be expected.

Task 6. Prove that the condition $r_< < r_-$ implies

$$p > (3 + e)R. \tag{A.3.5}$$

This means that the particle must be at a safe distance away from the black hole to be able to keep an eccentric orbit. If this condition is not met the particle will be forced to plunge into the hole.

A.4 Numerical integration of the equations of motion

A viable strategy to integrate the equations of motion would be to recast Eqs. (A.1.1) and (A.1.2) as a system of first-order equations, such as $\dot{r} = v$, $\dot{v} = -GM/r^2 + h^2(1 - 3R/2r)/r^3$, and $\dot{\phi} = h/r^2$. One would then select values for e and p, evaluate h from Eq. (A.3.3), and integrate the equations starting from the initial values $r(0) = r_- = p/(1 + e)$, $v(0) = 0$, and $\phi(0) = 0$. From the numerical information thus obtained one could then reconstruct the shape of each selected orbit.

We shall adopt instead an alternative, simpler strategy that will allow us to obtain the orbit more directly, at the price of eliminating t from the system of equations. We will thus obtain complete shape information, but give up on any

temporal information. To formulate this strategy we need to pursue the analytical work a bit more.

It is convenient to introduce an angular parameter χ and to mathematically represent the radial part of the motion as

$$r(\chi) = \frac{p}{1 + e\cos\chi}. \tag{A.4.1}$$

We see that as χ proceeds from 0 to π and then to 2π, the orbital radius proceeds from $r_- = p/(1+e)$ to $r_+ = p/(1-e)$ and then back to r_-; as χ ranges through the interval $0 < \chi < 2\pi$ the particle undergoes what we shall call *a complete radial orbit*.

Task 7. Combine Eqs. (A.1.4), (A.3.2), and (A.4.1) and derive an expression for $\dot{\chi}$. The right-hand side should involve GM, R, p, e, and $\cos\chi$ only. Try to simplify this expression as much as possible.

Task 8. Combine the result of Task 7 with Eqs. (A.1.1) and (A.3.3) and derive the equation

$$\frac{d\phi}{d\chi} = \frac{1}{\sqrt{1 - (3 + e\cos\chi)R/p}}. \tag{A.4.2}$$

This equation shows that χ becomes equal to ϕ in the nonrelativistic limit $R/p \ll 1$.

Equation (A.4.2) can be numerically integrated for $\phi(\chi)$. This, together with Eq. (A.4.1), give the exact shape of the relativistic orbit around the black hole.

Task 9. Using whatever method at your disposal, integrate Eq. (A.4.2) numerically and obtain the orbit of the particle. Do this for the following values of p and e:

orbit 1: $p/R = 5.5$, $e = 0.60377$;

orbit 2: $p/R = 4.2$, $e = 0.66155$;

orbit 3: $p/R = 3.7$, $e = 0.42387$;

orbit 4: $p/R = 3.7$, $e = 0.61976$;

orbit 5: your own selected values;

orbit 6: your own selected values (different from above).

In the numerical work it is a good idea to measure p (and r) in units of R; this eliminates R from all equations. For each case listed above, plot the shape of the orbit in the x-y plane. For each case let the parameter χ range through the interval $0 < \chi < 4\pi$, to make sure that the particle undergoes two complete radial orbits.

Task 10. Provide a summary of your numerical results by listing the following quantities in a table: p/R, e, r_-/R, r_+/R, and $\Delta\phi/(2\pi)$. The last quantity,

$$\frac{\Delta\phi}{2\pi} = \frac{\phi(\chi = 2\pi) - \phi(\chi = 0)}{2\pi}, \tag{A.4.3}$$

is the total change in ϕ during one complete radial orbit, divided by 2π; this is the number of revolutions that the particle completes during one radial orbit. In Newtonian theory this number would always be equal to unity, and the orbit would close on itself. In general relativity this number is generally larger than unity, and the orbit typically does no close. (For carefully selected sets of orbital parameters, the orbit may close after the particle completes a certain number of radial orbits.)