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On virtual displacement and virtual work in Lagrangian dynamics

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Abstract

The confusion and ambiguity encountered by students in understanding virtual displacement and virtual work is discussed in this paper. A definition of virtual displacement is presented that allows one to express them explicitly for holonomic (velocity independent), non-holonomic (velocity dependent), scleronomous (time independent) and rheonomous (time dependent) constraints. It is observed that for holonomic, scleronomous constraints, the virtual displacements are the displacements allowed by the constraints. However, this is not so for a general class of constraints. For simple physical systems, it is shown that the work done by the constraint forces on virtual displacements is zero. This motivates Lagrange's extension of d'Alembert's principle to a system of particles in constrained motion. However, a similar zero work principle does not hold for the allowed displacements. It is also demonstrated that d'Alembert's principle of zero virtual work is necessary for the solvability of a constrained mechanical problem. We identify this special class of constraints, physically realized and solvable, as *the ideal constraints*. The concept of virtual displacement and the principle of zero virtual work by constraint forces are central to both Lagrange's method of undetermined multipliers and Lagrange's equations in generalized coordinates.

1. Introduction

Almost all graduate level courses in classical mechanics include a discussion of virtual displacement [1–11] and Lagrangian dynamics [1–12]. From the concept of zero work by constraint forces on virtual displacement, the Lagrange equations of motion are derived.

However, the definition presented in most accessible texts often seem vague and ambiguous to students. Even after studying the so-called definition, it is rather commonplace that a student fails to identify whether a supplied vector is suitable as a virtual displacement for a given constrained system. Though some of the more advanced and rigorous treatise [13, 14] present a more precise and satisfactory treatment, they are often not easily comprehensible to most students. In this paper, we attempt a simple, systematic and precise definition of virtual displacement, which clearly shows the connection between the constraints and the corresponding allowed and virtual displacements. This definition allows one to understand how far the virtual displacement is 'arbitrary' and how far it is 'restricted' by the constraint condition.

There are two common logical pathways of arriving at Lagrange's equation:

(i) Bernoulli's principle of virtual velocity [7] (1717), d'Alembert's principle of zero virtual work [7, 15] (1743), Lagrange's generalization of d'Alembert's principle to a constrained system of moving particles and Lagrange's equations of motion (1788) [7, 16, 17].
(ii) Hamilton's principle of least action [7, 18] (1834) and variational approach to Lagrange's equation.

The two methods are logically and mathematically independent and individually self-contained. The first method was historically proposed half a century earlier, and it presents the motivation of introducing the Lagrangian as a *new physical quantity*. The second method starts with the Lagrangian and the related action as quantities axiomatically describing the dynamics of the system. This method is applied without ambiguity in some texts [12, 13] and courses [19]. However, one also finds intermixing of the two approaches in the literature and popular texts, often leading to circular definition and related confusion. A rational treatment demands an independent presentation of the two methods and then a demonstration of their interconnection. In the present paper, we confine ourselves to the first method.

In this approach, due to Bernoulli, d'Alembert and Lagrange, one begins with a constrained system, defined by equations of constraints connecting positions, time and often velocities of the particles under consideration. The concept of virtual displacement is introduced in terms of the constraint equations. The external forces alone cannot maintain the constrained motion. This requires the introduction of forces of constraints. The imposition of the principle of zero virtual work by constraint forces gives us a 'special class of systems' that are solvable.

A proper definition of virtual displacement is necessary to make the said approach logically satisfactory. However, the various definitions found in popular texts are often incomplete and contradictory with one another. These ambiguities will be discussed in detail in the next section.

In the literature, e.g., Greenwood [2] equation (1.26) and Pars [14] equation (1.6.1), one encounters holonomic constraints of the form

$$\phi_j(x_1, x_2, \ldots, x_{3N}, t) = 0, \qquad j = 1, 2, \ldots, k. \tag{1}$$

The differential form of the above equations are satisfied by allowed infinitesimal displacements $\{\mathrm{d}x_i\}$ (Greenwood [2] equation (1.27) and Pars [14] equation (1.6.3)):

$$\sum_{i=1}^{3N} \frac{\partial \phi_j}{\partial x_i}\, \mathrm{d}x_i + \frac{\partial \phi_j}{\partial t}\, \mathrm{d}t = 0, \qquad j = 1, 2, \ldots, k. \tag{2}$$

For a system under above constraints the virtual displacements $\{\delta x_i\}$ satisfy the following equations (Greenwood [2] equation (1.28) and Pars [14] equation (1.6.5)):

$$\sum_{i=1}^{3N} \frac{\partial \phi_j}{\partial x_i} \delta x_i = 0, \qquad j = 1, 2, \ldots, k. \tag{3}$$

The differential equations satisfied by allowed and virtual displacements are different even for the non-holonomic case. Here, the equations satisfied by the allowed displacements $\{dx_i\}$ are (Goldstein [1] equation (2.20), Greenwood [2] equation (1.29) and Pars [14] equation (1.7.1))

$$\sum_{i=1}^{3N} a_{ji}\, dx_i + a_{jt}\, dt = 0, \qquad j = 1, 2, \ldots, m. \tag{4}$$

Whereas, the virtual displacements $\{\delta x_i\}$ satisfy (Goldstein [1] equation (2.21), Greenwood [2] equation (1.30) and Pars [14] equation (1.7.2))

$$\sum_{i=1}^{3N} a_{ji} \delta x_i = 0, \qquad j = 1, 2, \ldots, m. \tag{5}$$

Thus, there appear certain equations in the literature, namely equations (3) and (5), which are always satisfied by the *not so precisely defined* virtual displacements. It may be noted that these equations are connected to the constraints but are not simply the infinitesimal forms of the constraint equations, i.e., equations (2) and (4). This fact is well documented in the literature [1, 2, 14]. However, the nature of the difference between these sets of equations, i.e., equations (3) and (5) on one hand and equations (2) and (4) on the other hand, and their underlying connection are not explained in most discussions. One may consider equations (3) or (5) as independent defining equations for virtual displacement. But it remains unclear as to how the virtual displacements $\{\delta x_i\}$ defined by two different sets of equations for the holonomic and the non-holonomic cases, namely equations (3) and (5), correspond to the same concept of virtual displacement.

We try to give a physical connection between the definitions of allowed and virtual displacements for any given set of constraints. The proposed definition of virtual displacement (section 2.1) as the difference between two unequal allowed displacements (satisfying equation (2) or (4)) over the same time interval automatically ensures that virtual displacements satisfy equations (3) and (5) for holonomic and non-holonomic systems, respectively. We show that in a number of natural systems, e.g., a pendulum with fixed or moving support, a particle sliding along a stationary or moving frictionless inclined plane, the work done by the forces of constraint on virtual displacements is zero. We also demonstrate that this condition is necessary for the solvability of a constrained mechanical problem. Such systems form an important class of natural systems.

1.1. Ambiguity in virtual displacement

In the literature certain statements appear in reference to virtual displacement which seem confusing and mutually inconsistent, particularly to a student. In the following we present few such statements found in common texts:

(1) It is claimed that (i) a virtual displacement $\delta\mathbf{r}$ is consistent with the forces and constraints imposed on the system at a given instant t [1]; (ii) a virtual displacement is an arbitrary, instantaneous, infinitesimal change of position of the system compatible with the conditions of constraint [7]; (iii) virtual displacements are, by definition, arbitrary displacements of the components of the system, satisfying the constraint [5]; (iv) virtual displacement does not violate the constraints [10]; (v) we define a virtual displacement as one which does not violate the kinematic relations [11]; (vi) the virtual displacements obey the constraint on the motion [4]. These statements imply that the virtual displacements satisfy the constraint conditions, i.e., the constraint equations. However, this is true only for holonomic, sclerenomous constraints. We shall show that for non-holonomic

constraints or rheonomous constraints, e.g., a pendulum with moving support, this definition violates the zero virtual work principle.

(2) It is also stated that (i) virtual displacements do not necessarily conform to the constraints [2]; (ii) the virtual displacements δq have nothing to do with actual motion. They are introduced, so to speak, as test quantities, whose function is to make the system reveal something about its internal connections and about the forces acting on it [7]; (iii) the word 'virtual' is used to signify that the displacements are arbitrary, in the sense that they need not correspond to any actual motion executed by the system [5]; (iv) it is not necessary that it (virtual displacement) represents any actual motion of the system [9]; (v) it is not intended to say that such a displacement (virtual) occurs during the motion of the particle considered or even that it could occur [3]; (vi) virtual displacement is any arbitrary infinitesimal displacement not necessarily along the constrained path [6]. From the above, we understand that the virtual displacements do not necessarily satisfy the constraint equations and they need not be the ones actually realized. We shall see that these statements are consistent with physical situations, but they cannot serve as a satisfactory definition of virtual displacement. Statements like 'not necessarily conform to the constraints' or 'not necessarily along the constrained path' only tell us what virtual displacement is not, they do not tell us what it really is. The reader should note that there is a conflict between the statements quoted under items 1 and 2.

Thus, it is not clear from the above whether the virtual displacements satisfy the constraints, i.e., the constraint equations, or not.

(3) It is also stated that (i) virtual displacement is to be distinguished from an actual displacement of the system occurring in a time interval $\mathrm{d}t$ [1]; (ii) it is an arbitrary, instantaneous, change of position of the system [7]; (iii) virtual displacement $\delta\mathbf{r}$ takes place without any passage of time [10]; (iv) virtual displacement has no connection with the time—in contrast to a displacement which occurs during actual motion and which represents a portion of the actual path [3]; (v) one of the requirements on acceptable virtual displacement is that the time is held fixed [4]. We even notice equations like '$\delta x_i = \mathrm{d}x_i$ for $\mathrm{d}t = 0$' [10]. The above statements are puzzling to a student. If position is a continuous function of time, a change in position during zero time has to be zero. In other words, this definition implies that the virtual displacement cannot possibly be an infinitesimal (or differential) of any continuous function of time. In words of Arthur Haas: *since its* (virtual displacement) *components are thus not functions of the time, we are not able to regard them as differentials, as we do for the components of the element of the actual path* [3]. It will be shown later (section 2) that virtual displacement can be looked upon as a differential. It is indeed a differential change in position or an infinitesimal displacement, consistent with virtual velocity $\tilde{\mathbf{v}}_k(t)$, taken over a time interval $\mathrm{d}t$ (see equation (13)).

(4) Virtual displacement is variously described as: *arbitrary*, *virtual* and *imaginary* [1, 7, 5, 9, 6]. These adjectives make the definition more mysterious to a student.

Together with the above ambiguities, students are often unsure whether it is sufficient to discuss virtual displacement as an abstract concept or it is important to have a quantitative definition. Some students appreciate that the virtual displacement as a vector should not be ambiguous. The principle of zero virtual work is required to derive Lagrange's equations. For a particle under constraint this means that the virtual displacement is always orthogonal to the force of constraint.

At this stage, a student gets further puzzled. Should he take the forces of constraint as supplied and the principle of zero virtual work as a definition of virtual displacement?

In that case, the principle reduces merely to a definition of a new concept, namely virtual displacement. Or should the virtual displacement be defined from the constraint conditions independently? The principle of zero virtual work may then be used to obtain the forces of constraint. These forces of constraint ensure that the constraint condition is maintained throughout the motion. Hence, it is natural to expect that they should be connected to and perhaps derivable from the constraint conditions.

2. Virtual displacement and forces of constraint

2.1. Constraints and virtual displacement

Let us consider a system of constraints that are expressible as equations involving positions and time. They represent some geometric restrictions (holonomic) either independent of time (sclerenomous) or explicitly dependent on it (rheonomous). Hence, for a system of N particles moving in three dimensions, a system of (s) holonomic, rheonomous constraints are represented by functions of $\{\mathbf{r}_k\}$ and (t),

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = 0, \qquad i = 1, 2, \ldots, s. \tag{6}$$

The system may also be subjected to non-holonomic constraints which are represented by equations connecting velocities $\{\mathbf{v}_k\}$, positions $\{\mathbf{r}_k\}$ and time (t):

$$\sum_{k=1}^{N} \mathbf{A}_{ik} \cdot \mathbf{v}_k + A_{it} = 0, \qquad i = 1, 2, \ldots, m, \tag{7}$$

where $\{\mathbf{A}_{ik}\}$ and $\{A_{it}\}$ are functions of positions $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ and time (t). The equations for non-holonomic constraints impose restrictions on possible or allowed velocity vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$, for given positions $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ and time (t). The holonomic constraints given by equation (6) are equivalent to the following equations imposing further restrictions on the possible or allowed velocities:

$$\sum_{k=1}^{N} \left(\frac{\partial f_i}{\partial \mathbf{r}_k}\right) \cdot \mathbf{v}_k + \frac{\partial f_i}{\partial t} = 0, \qquad i = 1, 2, \ldots, s. \tag{8}$$

For a system of N particles under (s) holonomic and (m) non-holonomic constraints, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ satisfying equations (7) and (8) are called allowed velocities. It is worth noting at this stage that there are many, in fact infinitely many, allowed velocities, since we have imposed only $(s + m)$ number of scalar constraints, equations (7) and (8), on $(3N)$ scalar components of the allowed velocity vectors.

At any given instant of time, the differences between any two such non-identical allowed sets of velocities, independently satisfying the constraint conditions, are called virtual velocities:

$$\tilde{\mathbf{v}}_k(t) = \mathbf{v}_k(t) - \mathbf{v}'_k(t) \qquad k = 1, 2, \ldots, N.$$

An infinitesimal displacement over time $(t, t + \mathrm{d}t)$, due to allowed velocities, will be called the *allowed infinitesimal displacement* or simply allowed displacement:

$$\mathrm{d}\mathbf{r}_k = \mathbf{v}_k(t)\,\mathrm{d}t, \qquad k = 1, 2, \ldots, N. \tag{9}$$

Allowed displacements $\{\mathrm{d}\mathbf{r}_k\}$ together with the differential of time $(\mathrm{d}t)$ satisfy the infinitesimal form of the constraint equations. From equations (8) and (7), we obtain

$$\sum_{k=1}^{N} \left(\frac{\partial f_i}{\partial \mathbf{r}_k}\right) \cdot \mathrm{d}\mathbf{r}_k + \frac{\partial f_i}{\partial t}\,\mathrm{d}t = 0, \qquad i = 1, 2, \ldots, s, \tag{10}$$

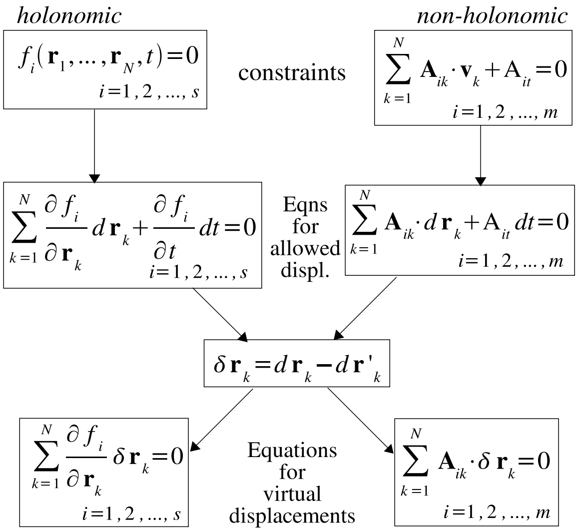

Figure 1. Virtual displacement defined as difference of allowed displacements.

$$\sum_{k=1}^{N} \mathbf{A}_{ik} \cdot \mathrm{d}\mathbf{r}_k + A_{it}\,\mathrm{d}t = 0, \qquad i = 1, 2, \ldots, m. \tag{11}$$

As there are many independent sets of allowed velocities, we have many allowed sets of infinitesimal displacements. We propose to define *virtual displacement* as the difference between any two such (unequal) allowed displacements taken over the same time interval $(t, t + \mathrm{d}t)$:

$$\delta\mathbf{r}_k = \mathrm{d}\mathbf{r}_k - \mathrm{d}\mathbf{r}'_k, \qquad k = 1, 2, \ldots, N. \tag{12}$$

Thus, virtual displacements are infinitesimal displacements over time interval $\mathrm{d}t$ due to virtual velocity $\widetilde{\mathbf{v}}_k(t)$:

$$\delta\mathbf{r}_k = \widetilde{\mathbf{v}}_k(t)\,\mathrm{d}t = (\mathbf{v}_k(t) - \mathbf{v}'_k(t))\,\mathrm{d}t, \qquad k = 1, 2, \ldots, N. \tag{13}$$

This definition is motivated by the possibility of (i) identifying a special class of '*ideal constraints*' (section 2.3) and (ii) verifying '*the principle of zero virtual work*' in common physical examples (section 3). It may be noted that, by this definition, virtual displacements $\{\delta\mathbf{r}_k\}$ are not instantaneous changes in position in zero time. They are rather smooth, differentiable objects.

The virtual displacements thus defined satisfy the homogeneous part of the constraint equations, i.e., equations (10) and (11) with $\partial f_i/\partial t = 0$ and $A_{it} = 0$. Hence,

$$\sum_{k=1}^{N} \frac{\partial f_i}{\partial \mathbf{r}_k} \cdot \delta\mathbf{r}_k = 0, \qquad i = 1, 2, \ldots, s, \tag{14}$$

$$\sum_{k=1}^{N} \mathbf{A}_{ik} \cdot \delta\mathbf{r}_k = 0, \qquad i = 1, 2, \ldots, m. \tag{15}$$

The logical connection between the equations of constraint, equations for allowed displacements and equations for virtual displacements is presented in figure 1.

The absence of $(\partial f_i/\partial t)$ and A_{it} in the above equations, equations (14) and (15), gives the precise meaning to the statement: '*virtual displacements are the allowed displacements in*

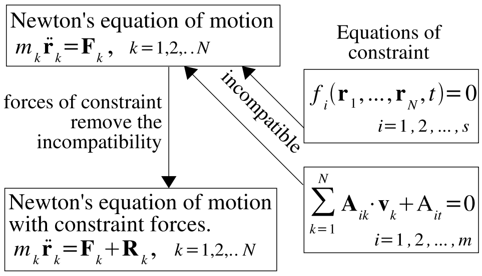

Figure 2. Existence of constraint forces.

the case of frozen constraints'. The constraints are frozen in time in the sense that we make the $(\partial f_i/\partial t)$ and A_{it} terms zero, though the $\partial f_i/\partial \mathbf{r}_k$ and $\mathbf{A}_{ik}$ terms still involve both position $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$ and time (t). In the case of stationary constraints, i.e., $f_i(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0$ and $\sum_k \mathbf{A}_{ik}(\mathbf{r}_1, \ldots, \mathbf{r}_N) \cdot \mathbf{v}_k = 0$, the virtual displacements are identical with allowed displacements as $(\partial f_i/\partial t)$ and A_{it} are identically zero.

2.2. Existence of forces of constraints

In the case of an unconstrained system of N particles described by position vectors $\{\mathbf{r}_k\}$ and velocity vectors $\{\mathbf{v}_k\}$, the motion is governed by Newton's law:

$$m_k \mathbf{a}_k = \mathbf{F}_k(\mathbf{r}_l, \mathbf{v}_l, t), \qquad k, l = 1, 2, \ldots, N \tag{16}$$

where m_k is the mass of the kth particle, $\mathbf{a}_k$ is its acceleration and $\mathbf{F}_k$ is the total external force acting on it. However, for a constrained system, the equations of constraint, namely equations (6) and (7), impose the following restrictions on the allowed accelerations:

$$\sum_{k=1}^{N} \frac{\partial f_i}{\partial \mathbf{r}_k} \cdot \mathbf{a}_k + \sum_{k=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f_i}{\partial \mathbf{r}_k}\right) \mathbf{v}_k + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f_i}{\partial t}\right) = 0, \qquad i = 1, 2, \ldots, s \tag{17}$$

$$\sum_{k=1}^{N} \mathbf{A}_{ik} \cdot \mathbf{a}_k + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_{ik} \cdot \mathbf{v}_k + \frac{\mathrm{d}}{\mathrm{d}t} A_{it} = 0, \qquad i = 1, 2, \ldots, m. \tag{18}$$

Given $\{\mathbf{r}_k\}$, $\{\mathbf{v}_k\}$ one is no longer free to choose all the accelerations $\{\mathbf{a}_k\}$ independently. Therefore, in general, the accelerations $\{\mathbf{a}_k\}$ allowed by equations (17) and (18) are incompatible with Newton's law, i.e., equation (16).

This implies that during the motion the constraint condition cannot be maintained by the external forces alone. Physically, some additional forces, e.g., normal reaction from the surface of constraint, tension in the pendulum string, come into play to ensure that the constraints are satisfied throughout the motion. Hence, one is compelled to introduce forces of constraints $\{\mathbf{R}_k\}$ and modify the equations of motion as

$$m_k \mathbf{a}_k = \mathbf{F}_k + \mathbf{R}_k, \qquad k = 1, 2, \ldots, N. \tag{19}$$

Figure 2 presents the connection between the equations of constraints and forces of constraints.

Now the problem is to determine the motion of N particles, namely their positions $\{\mathbf{r}_k(t)\}$, velocities $\{\mathbf{v}_k(t)\}$ and the forces of constraints $\{\mathbf{R}_k\}$, for a given set of external forces $\{\mathbf{F}_k\}$, constraint equations, equations (6) and (7), and initial conditions $\{\mathbf{r}_k(0), \mathbf{v}_k(0)\}$. It is important that the initial conditions are also compatible with the constraints. There are a total of $(6N)$

scalar unknowns, namely the components of $\mathbf{r}_k(t)$ and $\mathbf{R}_k$, connected by $(3N)$ scalar equations of motion, equation (19), and $(s+m)$ equations of constraints, equations (6) and (7). For $(6N > 3N + s + m)$, we have an under-determined system. Hence, to solve this problem we need $(3N - s - m)$ additional scalar relations.

2.3. Solvability and ideal constraints

In simple problems with stationary constraints, e.g., the motion of a particle on a smooth stationary surface, we observe that the allowed displacements are tangential to the surface. The virtual displacement being the difference between two such allowed displacements is also a vector tangential to it. For a frictionless surface, the force of constraint, the so-called 'normal reaction', is perpendicular to the surface. Hence, the work done by the constraint force is zero, on allowed as well as virtual displacement:

$$\sum_{k=1}^{N} \mathbf{R}_k \cdot \mathrm{d}\mathbf{r}_k = 0, \qquad \sum_{k=1}^{N} \mathbf{R}_k \cdot \delta\mathbf{r}_k = 0.$$

When the constraint surface is in motion, the allowed velocities and hence the allowed displacements are no longer tangent to the surface (see section 3). The virtual displacement however remains tangent to the constraint surface. As the surface is frictionless, it is natural to assume that the force of constraint is still normal to the instantaneous position of the surface. Hence, the work done by normal reaction on virtual displacement is zero. However, the work done by constraint force on allowed displacements is no longer zero:

$$\sum_{k=1}^{N} \mathbf{R}_k \cdot \mathrm{d}\mathbf{r}_k \neq 0, \qquad \sum_{k=1}^{N} \mathbf{R}_k \cdot \delta\mathbf{r}_k = 0. \tag{20}$$

In a number of physically interesting simple problems, such as the motion of a pendulum with fixed or moving support or the motion of a particle along a stationary and moving slope, we observe that the above interesting relation between the force of constraint and virtual displacement holds (see section 3). As the $3N$ scalar components of the virtual displacements $\{\delta\mathbf{r}_k\}$ are connected by $(s+m)$ equations, equations (14) and (15), only $n = (3N - s - m)$ of these scalar components are independent. If the $(s+m)$ dependent quantities are expressed in terms of remaining $(3N - s - m)$ independent objects we get

$$\sum_{j=1}^{n} \widetilde{R}_j \cdot \delta\widetilde{x}_j = 0, \tag{21}$$

where $\{\widetilde{x}_j\}$ are the independent components of $\{\mathbf{r}_k\}$. $\{\widetilde{R}_j\}$ are the coefficients of $\{\delta\widetilde{x}_j\}$ and are composed of different $\{\mathbf{R}_k\}$. Since the above components of virtual displacements $\{\delta\widetilde{x}_j\}$ are independent, one can equate each of their coefficients to zero $(\widetilde{R}_j = 0)$. This brings in exactly $(3N - s - m)$ new scalar conditions or equations that are needed to make the system solvable (see figure 3).

Thus we have found a special class of constraints, which is observed in nature (section 3) and which gives us a solvable mechanical system. We call this special class of constraints, where the forces of constraint do zero work on virtual displacement, i.e., $\sum_k \mathbf{R}_k \cdot \delta\mathbf{r}_k = 0$, the *ideal constraint*.

Our interpretation of the principle of zero virtual work, as a definition of an ideal class of constraints, agrees with Sommerfeld. In his exact words, '*a general* **postulate** *of mechanics: in any mechanical systems the virtual work of the reactions equals zero. Far be it from us to want to give a general proof of this postulate, rather we regard it practically as a* **definition of a mechanical system**' [7].

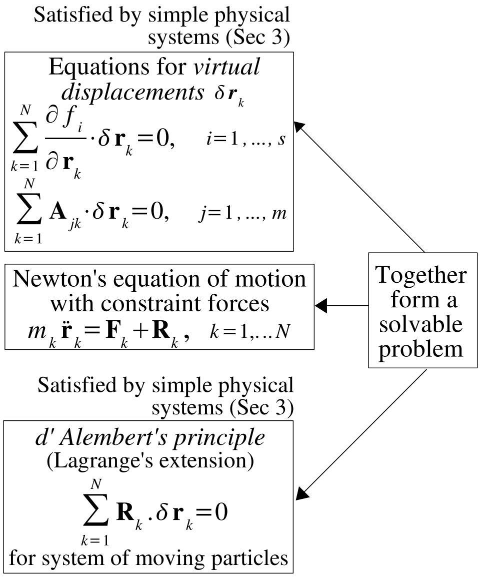

Figure 3. Solvability under ideal constraints.

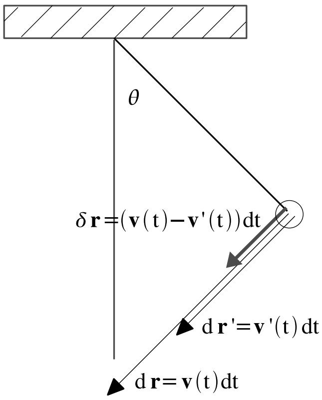

Figure 4. Allowed and virtual displacements of a pendulum with stationary support.

3. Examples of virtual displacements

3.1. Simple pendulum with stationary support

The motion of a pendulum is confined to a plane and its bob moves keeping a fixed distance from the point of suspension (see figure 4). The equation of constraint therefore is

$$f(x, y, t) \doteq x^2 + y^2 - r_0^2 = 0,$$

where r_0 is the length of the pendulum. Whence

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial t} = 0.$$

The constraint equation for allowed velocities, equation (8), becomes

$$x \cdot v_x + y \cdot v_y = 0.$$

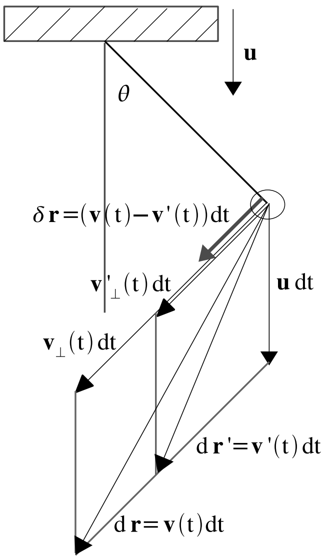

Figure 5. Allowed and virtual displacements of a pendulum with moving support.

Hence, the allowed velocity (v_x, v_y) is orthogonal to the instantaneous position (x, y) of the bob relative to the stationary support. The same may also be verified taking a plane polar coordinate.

The allowed velocities and the allowed displacements are perpendicular to the line of suspension. The virtual velocities and the virtual displacements, being the difference between two unequal allowed velocities and displacements, respectively, are also perpendicular to the line of suspension:

$$\mathrm{d}\mathbf{r} = \mathbf{v}(t)\,\mathrm{d}t, \qquad \mathrm{d}\mathbf{r}' = \mathbf{v}'(t)\,\mathrm{d}t, \qquad \delta\mathbf{r} = (\mathbf{v}(t) - \mathbf{v}'(t))\,\mathrm{d}t.$$

Although the virtual displacement is not uniquely specified by the constraint, it is restricted to being in a plane perpendicular to the instantaneous line of suspension. Hence, it is not 'completely arbitrary'.

The ideal string of the pendulum provides a tension (**T**) along its length, but no shear. The work done by this tension on both allowed and virtual displacements is zero:

$$\mathbf{T}\cdot\mathrm{d}\mathbf{r} = 0, \qquad \mathbf{T}\cdot\delta\mathbf{r} = 0.$$

3.2. Simple pendulum with moving support

Let us first consider the case when the support is moving vertically with a velocity u. The motion of the pendulum is still confined to a plane. The bob moves keeping a fixed distance from the moving point of suspension (figure 5). The equation of constraint is

$$f(x, y, t) \doteq x^2 + (y - ut)^2 - r_0^2 = 0,$$

where u is the velocity of the point of suspension along a vertical direction.

Whence,

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2(y - ut), \qquad \frac{\partial f}{\partial t} = -2u(y - ut).$$

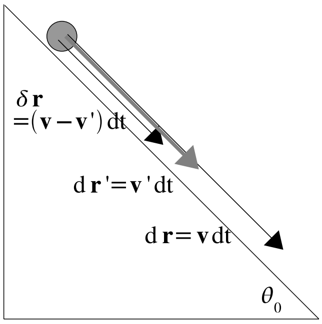

Figure 6. Allowed and virtual displacements of a particle sliding along a stationary slope.

Hence, the constraint equation gives

$$x \cdot v_x + (y - ut) \cdot v_y - u(y - ut) = 0,$$

or

$$x \cdot v_x + (y - ut) \cdot (v_y - u) = 0.$$

The allowed velocities (v_x, v_y) and the allowed displacements are not orthogonal to the instantaneous position of the bob relative to the instantaneous point of suspension $(x, y - ut)$. It is easy to verify from the above equation that the allowed velocity (v_x, v_y) is equal to the sum of a velocity vector $(v_x, v_y - u)$ perpendicular to the position of the bob relative to the point of suspension $(x, y - ut)$ and the velocity of the support $(0, u)$. If we denote $\mathbf{v}(t) = (v_x, v_y)$, $\mathbf{v}_\perp(t) = (v_x, v_y - u)$ and $\mathbf{u} = (0, u)$, then

$$\mathbf{v}(t) = \mathbf{v}_\perp(t) + \mathbf{u}.$$

The allowed displacements are vectors collinear to the allowed velocities. A virtual displacement being the difference between two allowed displacements is a vector collinear to the difference between allowed velocities. Hence it is orthogonal to the instantaneous line of suspension:

$$\mathrm{d}\mathbf{r} = \mathbf{v}(t)\,\mathrm{d}t = \mathbf{v}_\perp(t)\,\mathrm{d}t + \mathbf{u}\,\mathrm{d}t,$$
$$\mathrm{d}\mathbf{r}' = \mathbf{v}'(t)\,\mathrm{d}t = \mathbf{v}'_\perp(t)\,\mathrm{d}t + \mathbf{u}\,\mathrm{d}t,$$
$$\delta\mathbf{r} = (\mathbf{v}(t) - \mathbf{v}'(t))\,\mathrm{d}t = (\mathbf{v}_\perp(t) - \mathbf{v}'_\perp(t))\,\mathrm{d}t.$$

Hence none of these allowed or virtual vectors are ‘arbitrary’.

At any given instant, an ideal string provides a tension along its length with no shear. Hence the constraint force, namely tension $\mathbf{T}$, does zero work on virtual displacement:

$$\mathbf{T} \cdot \mathrm{d}\mathbf{r} = \mathbf{T} \cdot \mathbf{u}\,\mathrm{d}t \neq 0, \qquad \mathbf{T} \cdot \delta\mathbf{r} = \mathbf{T} \cdot \mathbf{v}_\perp\,\mathrm{d}t = 0$$

For the support moving in a horizontal or any arbitrary direction, one can show that the allowed displacement is not normal to the instantaneous line of suspension. But the virtual displacement, as defined in this paper, always remains perpendicular to the instantaneous line of support.

3.3. Motion along a stationary inclined plane

Let us consider a particle sliding along a stationary inclined plane as shown in figure 6. The constraint here is more conveniently expressed in polar coordinates. The constraint equation is

$$f(r, \theta) \doteq \theta - \theta_0 = 0$$

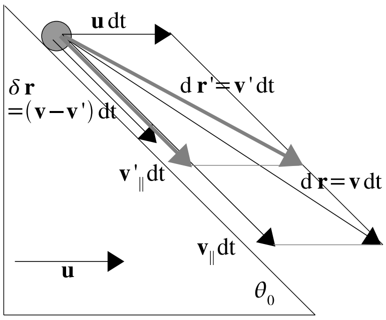

Figure 7. Allowed and virtual displacements of a particle sliding along a moving slope.

where θ_0 is the angle of the slope. Hence, the constraint equation for the allowed velocities, equation (8), gives

$$\begin{aligned}\left(\frac{\partial f}{\partial \mathbf{r}}\right)\cdot\mathbf{v}+\frac{\partial f}{\partial t} &= \left(\frac{\partial f}{\partial r}\right)v_r+\left(\frac{\partial f}{\partial \theta}\right)v_\theta+\frac{\partial f}{\partial t}\\ &= 0\cdot\dot{r}+1\cdot(r\dot{\theta})+0=0.\end{aligned}$$

Hence $\dot{\theta}=0$, implying that the allowed velocities are along the constant θ plane. The allowed velocity, and allowed and virtual displacements are

$$\begin{aligned}&\mathbf{v}=\dot{r}\widehat{\mathbf{r}}, \qquad \mathrm{d}\mathbf{r}=\dot{r}\widehat{\mathbf{r}}\,\mathrm{d}t,\\ &\mathbf{v}'=\dot{r}'\widehat{\mathbf{r}}, \qquad \mathrm{d}\mathbf{r}'=\dot{r}'\widehat{\mathbf{r}}\,\mathrm{d}t,\\ &\delta\mathbf{r}=(\mathbf{v}-\mathbf{v}')\,\mathrm{d}t=(\dot{r}-\dot{r}')\widehat{\mathbf{r}}\,\mathrm{d}t,\end{aligned}$$

where $\widehat{\mathbf{r}}$ is a unit vector along the slope.

As the inclined slope is frictionless, the constraint force $\mathbf{N}$ is normal to the surface. The work done by this force on allowed as well as virtual displacement is zero:

$$\mathbf{N}\cdot\mathrm{d}\mathbf{r}=0, \qquad \mathbf{N}\cdot\delta\mathbf{r}=0.$$

3.4. Motion along a moving inclined plane

For an inclined plane moving along the horizontal side (figure 7), the constraint is given by

$$\begin{aligned}&\frac{(x-ut)}{y}-\cot(\theta_0)=0,\\ &f(x,y)\doteq(x-ut)-\cot(\theta_0)y=0.\end{aligned}$$

Whence the constraint for allowed velocities, equation (8), becomes

$$(\dot{x}-u)-\cot(\theta_0)\dot{y}=0.$$

Hence, the allowed velocity $(\dot{x},\dot{y})$ is the sum of two vectors, one along the plane $(\dot{x}-u,\dot{y})$ and the other equal to the velocity of the plane itself $(u,0)$:

$$\mathbf{v}(t)=\mathbf{v}_\parallel(t)+\mathbf{u}.$$

The allowed displacements are vectors along the allowed velocities, however the virtual displacement is still a vector along the instantaneous position of the plane:

$$\begin{aligned}&\mathrm{d}\mathbf{r}=(\mathbf{v}_\parallel(t)+\mathbf{u})\,\mathrm{d}t, \qquad \mathrm{d}\mathbf{r}'=(\mathbf{v}'_\parallel(t)+\mathbf{u})\,\mathrm{d}t,\\ &\delta\mathbf{r}=(\mathbf{v}(t)-\mathbf{v}'(t))\,\mathrm{d}t=(\mathbf{v}_\parallel(t)-\mathbf{v}'_\parallel(t))\,\mathrm{d}t.\end{aligned}$$

For the moving frictionless slope, the constraint force provided by the surface is perpendicular to the plane. Hence, the work done by the constraint force on virtual displacement remains zero:

$$\mathbf{N} \cdot \mathrm{d}\mathbf{r} \neq 0, \qquad \mathbf{N} \cdot \delta\mathbf{r} = 0.$$

4. Lagrange's method of undetermined multipliers

A constrained system of particles obey the equations of motion given by

$$m_k \mathbf{a}_k = \mathbf{F}_k + \mathbf{R}_k, \qquad k = 1, 2, \ldots, N$$

where m_k is the mass of the kth particle, and $\mathbf{a}_k$ is its acceleration. $\mathbf{F}_k$ and $\mathbf{R}_k$ are the total external force and force of constraint on the kth particle. If the constraints are *ideal*, we can write

$$\sum_{k=1}^{N} \mathbf{R}_k \cdot \delta\mathbf{r}_k = 0, \tag{22}$$

whence we obtain

$$\sum_{k=1}^{N} (m_k \mathbf{a}_k - \mathbf{F}_k) \cdot \delta\mathbf{r}_k = 0. \tag{23}$$

If the components of $\{\delta\mathbf{r}_k\}$ were independent, we could recover Newton's law for an unconstrained system from this equation. However, for a constrained system $\{\delta\mathbf{r}_k\}$ are dependent through the constraint equations, equations (14) and (15), for holonomic and non-holonomic systems, respectively:

$$\sum_{k=1}^{N} \frac{\partial f_i}{\partial \mathbf{r}_k} \delta\mathbf{r}_k = 0, \qquad i = 1, 2, \ldots, s \tag{14}$$

$$\sum_{k=1}^{N} \mathbf{A}_{jk} \cdot \delta\mathbf{r}_k = 0, \qquad j = 1, 2, \ldots, m. \tag{15}$$

We multiply equation (14) successively by (s) scalar multipliers $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$, equation (15) successively by (m) scalar multipliers $\{\mu_1, \mu_2, \ldots, \mu_m\}$ and then subtract them from the zero virtual work equation, namely equation (22):

$$\sum_{k=1}^{N} \left(\mathbf{R}_k - \sum_{i=1}^{s} \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k} - \sum_{j=1}^{m} \mu_j \mathbf{A}_{jk} \right) \delta\mathbf{r}_k = 0. \tag{24}$$

These multipliers $\{\lambda_i\}$ and $\{\mu_j\}$ are called the Lagrange multipliers. Explicitly in terms of components

$$\sum_{k=1}^{N} \left[R_{k,x} - \sum_{i=1}^{s} \lambda_i \frac{\partial f_i}{\partial x_k} - \sum_{j=1}^{m} \mu_j (\mathbf{A}_{jk})_x \right] \delta x_k + \sum_{k=1}^{N} [Y]_k \delta y_k + \sum_{k=1}^{N} [Z]_k \delta z_k = 0, \tag{25}$$

where $[Y]_k$ and $[Z]_k$ denote the coefficients of δy_k and δz_k, respectively.

The constraint equations, equations (14) and (15), allow us to write the $(s+m)$ dependent virtual displacements in terms of the remaining $n = (3N - s - m)$ independent ones. We choose $(s + m)$ multipliers $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and $\{\mu_1, \mu_2, \ldots, \mu_m\}$, such that the coefficients of $(s + m)$ dependent components of virtual displacement vanish. The remaining virtual

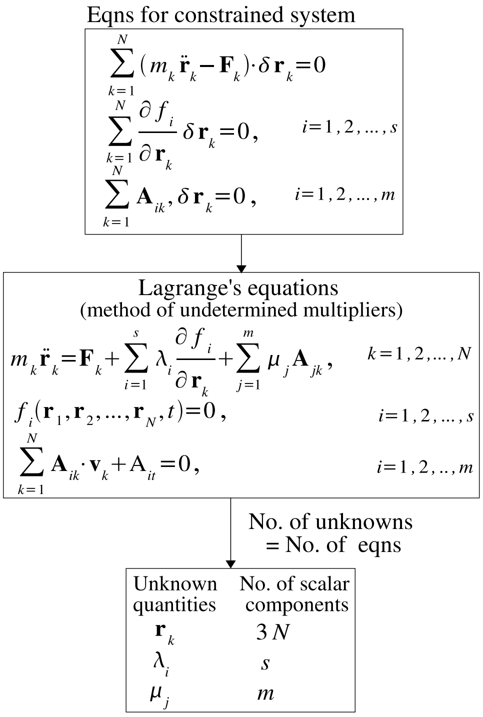

Figure 8. Lagrange's method of undetermined multipliers: solvability.

displacements being independent, their coefficients must vanish as well. Thus, it is possible to choose $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and $\{\mu_1, \mu_2, \ldots, \mu_m\}$ such that all coefficients $\{[X]_k, [Y]_k, [Z]_k\}$ of virtual displacements $\{\delta x_k, \delta y_k, \delta z_k\}$ in equation (25) vanish. Hence, we can express the forces of constraint in terms of the Lagrange multipliers:

$$\mathbf{R}_k = \sum_{i=1}^{s} \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k} + \sum_{j=1}^{m} \mu_j \mathbf{A}_{jk}, \qquad k = 1, 2, \ldots, N. \tag{26}$$

Thus, the problem reduces to finding a solution for the equations

$$m_k \mathbf{a}_k = \mathbf{F}_k + \sum_{i=1}^{s} \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k} + \sum_{j=1}^{m} \mu_j \mathbf{A}_{jk}, \qquad k = 1, 2, \ldots, N \tag{27}$$

together with the equations of constraint

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = 0, \qquad i = 1, 2, \ldots, s \tag{6}$$

and

$$\sum_{k=1}^{N} \mathbf{A}_{ik} \cdot \mathbf{v}_k + A_{it} = 0, \qquad i = 1, 2, \ldots, m. \tag{7}$$

Here, we have to solve $(3N+s+m)$ scalar equations involving $(3N+s+m)$ unknown scalar quantities, namely $\{x_k, y_k, z_k, \lambda_i, \mu_j\}$ (see figure 8). After solving this system of equations for $\{x_k, y_k, z_k, \lambda_i, \mu_j\}$, one can obtain the forces of constraint $\{\mathbf{R}_k\}$ using equation (26).

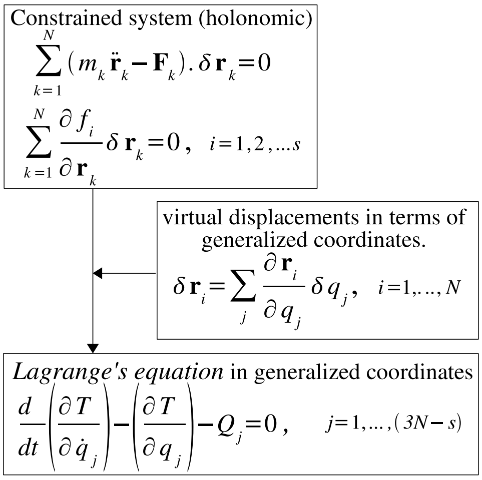

Figure 9. Lagrange's equations in generalized coordinates.

5. Lagrange's equations in generalized coordinates

For the sake of completeness, we discuss very briefly Lagrange's equations in generalized coordinates. A more complete discussion can be found in most texts [1–14]. Consider a system of N particles under (s) holonomic and (m) non-holonomic constraints. In certain suitable cases, one can express $(s+m)$ dependent coordinates in terms of the remaining $(3N-s-m)$ independent ones. It may be noted that such a complete reduction is not possible for general cases of non-holonomic- and time-dependent constraints [7, 14]. If we restrict our discussion to cases where this reduction is possible, one may express all the $3N$ scalar components of position $\{\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N\}$ in terms of $(3N-s-m)$ independent parameters $\{q_1,q_2,\ldots,q_n\}$ and time (t):

$$\mathbf{r}_k=\mathbf{r}_k(q_1,q_2,\ldots,q_n,t),\qquad k=1,2,\ldots,N. \tag{28}$$

The allowed and virtual displacements are given by

$$\begin{aligned} \mathrm{d}\mathbf{r}_k &= \sum_{j=1}^{n}\frac{\partial\mathbf{r}_k}{\partial q_j}\delta q_j+\frac{\partial\mathbf{r}_k}{\partial t}\,\mathrm{d}t, \qquad k=1,2,\ldots,N\\ \delta\mathbf{r}_k &= \sum_{j=1}^{n}\frac{\partial\mathbf{r}_k}{\partial q_j}\delta q_j, \qquad k=1,2,\ldots,N. \end{aligned} \tag{29}$$

From equation (23), we obtain

$$\sum_{k=1}^{N}m_k\frac{\mathrm{d}\dot{\mathbf{r}}_k}{\mathrm{d}t}\left(\sum_{j=1}^{n}\frac{\partial\mathbf{r}_k}{\partial q_j}\delta q_j\right)-\sum_{k=1}^{N}\mathbf{F}_k\left(\sum_{j=1}^{n}\frac{\partial\mathbf{r}_k}{\partial q_j}\delta q_j\right)=0. \tag{30}$$

Introduce the expression of kinetic energy

$$T=\frac{1}{2}\sum_{k=1}^{N}m_k\dot{\mathbf{r}}_k^2,$$

and that of the generalized force

$$Q_j=\sum_{k=1}^{N}\mathbf{F}_k\frac{\partial\mathbf{r}_k}{\partial q_j},\qquad j=1,2,\ldots,n. \tag{31}$$

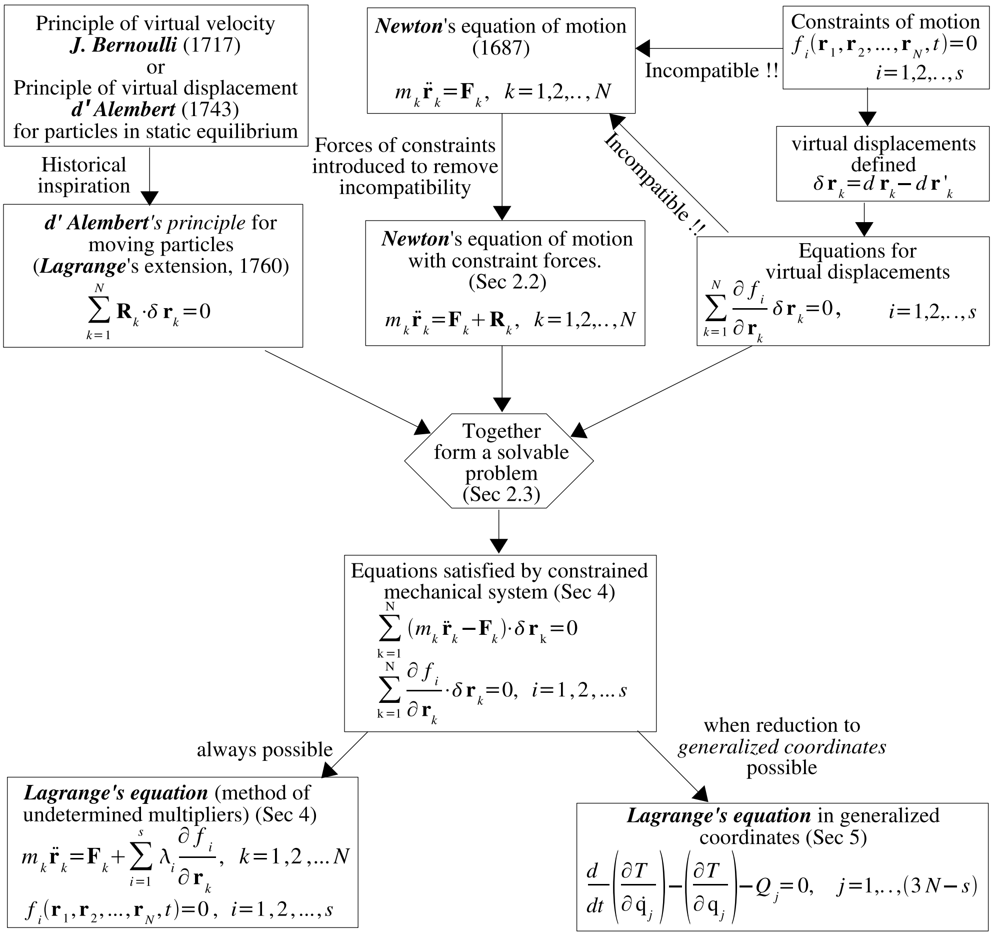

Figure 10. Logical connection between constraint equations, virtual displacements, forces of constraints, d'Alembert's principle and Lagrange's equations.

After some simple algebra one finds

$$\sum_{j=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0. \tag{32}$$

As $\{q_1, q_2, \ldots, q_n\}$ are independent coordinates, the coefficient of each δq_j must be zero separately. Hence (see figure 9),

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, 2, \ldots, n. \tag{33}$$

These are called the Lagrange equations in generalized coordinates. To proceed further one has to impose additional conditions on the nature of forces $\{\mathbf{F}_k\}$ or $\{Q_j\}$.

In problems where forces $\{\mathbf{F}_k\}$ are derivable from a scalar potential $\widetilde{V}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$,

$$\mathbf{F}_k = -\boldsymbol{\nabla}_k \widetilde{V}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N), \qquad k = 1, 2, \ldots, N. \tag{34}$$

One can obtain the generalized force as

$$Q_j = -\boldsymbol{\nabla}_k \widetilde{V} \cdot \left(\frac{\partial \mathbf{r}_k}{\partial q_j} \right) = -\frac{\partial V}{\partial q_j}, \qquad j = 1, 2, \ldots, n, \tag{35}$$

where V is the potential $\widetilde{V}$ expressed as a function of $\{q_1, q_2, \ldots, q_n\}$. In addition as the potential V is independent of the generalized velocities, we obtain from equation (33)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(T-V)}{\partial\dot{q}_j} - \frac{\partial(T-V)}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n. \tag{36}$$

At this stage one introduces the Lagrangian function, $L = T - V$. In terms of the Lagrangian, the equations of motion take up the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n. \tag{37}$$

6. Conclusion

In this paper, we make an attempt to present a quantitative definition of the virtual displacement. We show that for certain simple cases the work done by the forces of constraint on virtual displacement is zero. We also demonstrate that this zero virtual work principle gives us a solvable class of problems. Hence, we define this special class of constraint as *the ideal constraint*. We demonstrate in brief how one can solve a general mechanical problem by: (i) Lagrange's method of undetermined multipliers and (ii) Lagrange's equations in generalized coordinates.

In the usual presentations of Lagrange's equation based on virtual displacement and d'Alembert's principle, equations (3) and (5) are satisfied by the virtual displacements. One may consider these equations as the definition of virtual displacements. However, the situation is far from satisfactory, as separate defining equations are required for different classes of constraints. This ad hoc definition also fails to clarify the actual connection between the virtual displacements and the equations of constraints.

At this stage, one introduces d'Alembert's principle of zero virtual work. Bernoulli [7] (1717) and d'Alembert [7, 15] (1743) originally proposed this principle for a system in static equilibrium. The principle states that the forces of constraint do zero work on virtual displacement. For systems in static equilibrium, virtual displacement meant an imaginary displacement of the system that keeps its statical equilibrium unchanged. Lagrange generalized this principle to a constrained system of particles in motion. This principle is crucial in arriving at Lagrange's equation. However, most texts do not clearly address the questions: (i) why does one needs to extend d'Alembert's principle to particles in motion and (ii) why is the work done by constraint forces on virtual displacements, and not on allowed displacements, zero?

In the present paper, the allowed infinitesimal displacements are defined as ones that satisfy the infinitesimal form of the constraint equations. They are the displacements that could have been possible if only the constraints were present. Actual dynamics, under the given external forces, would choose one of these various sets of displacements as the actual displacement of the system. The definition of virtual displacement as the difference between two unequal allowed displacements over the same infinitesimal time interval $(t, t+\mathrm{d}t)$ gives a unified definition of virtual displacement. This definition of virtual displacement satisfies the appropriate equations found in the literature, for both holonomic and non-holonomic systems.

It is shown that Newton's equation of motion with external forces alone is inconsistent with equations of constraint. Hence, the forces of constraint are introduced. Now there are $3N$ equations of motion and $(s+m)$ equations of constraint involving $6N$ unknown scalars $\{\mathbf{r}_k(t), \mathbf{R}_k\}$. Without additional condition (d'Alembert principle), the problem is underspecified and unsolvable.

It is verified that for simple physical systems, the virtual displacements, as defined in this paper, satisfy the d'Alembert principle for particles in motion. The rheonomous examples discussed in section 3 show why the forces of constraint do zero work on virtual displacements and not on allowed displacements. The additional equations introduced by the d'Alembert principle make the problem solvable. These justify (i) the *peculiar* definition of virtual displacements and the equations they satisfy, (ii) Lagrange's extension of d'Alembert's principle to particles in motion, (iii) why the zero work principle is related to virtual displacement and not to allowed displacement. Once the system is solvable, two methods, originally proposed by Lagrange can be used. For Lagrange's method of undetermined multipliers, one solves $(3N + s)$ equations to obtain the motion of the system $\{\mathbf{r}_k(t), \ldots, \mathbf{r}_N(t)\}$ and Lagrange's multipliers $\{\lambda_i, \ldots, \lambda_s\}$. The forces of constraint $\{\mathbf{R}_k, \ldots, \mathbf{R}_N\}$ are expressed in terms of these multipliers. For Lagrange's equations in generalized coordinates, one solves $(3N - s)$ equations to obtain the time evolution of the generalized coordinates $\{q_j = q_j(t), j = 1, \ldots, 3N - s\}$. This gives the complete description of the motion $\{\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \ldots, q_n, t), k = 1, \ldots, N\}$, ignoring the calculation of the constraint forces. It may be noted that about a century later, Appell's equations [7, 14, 20] were introduced for efficiently solving non-holonomic systems.

It is interesting to note that both the above-mentioned methods require the principle of zero virtual work by constraint forces as a crucial starting point. In the case of Lagrange's method of undetermined multipliers, we start with the ideal constraint condition, equation (22). From there we obtain equations (23)–(27). Equation (26) expresses the constraint forces in terms of Lagrange's multipliers. For Lagrange's equations in generalized coordinates, we start with the ideal constraint, equation (22). We work our way through equations (23), (30), (32) and finally obtain Lagrange's equations in generalized coordinates, equations (33) and (37). The last figure, figure (10), gives the complete logical flow of this paper.

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