Ecuación del calor

Principio del máximo para el problema de Cauchy

Teorema.

Sea $u \in C^2_1(\mathbb{R}^N \times (0, T]) \cap C(\mathbb{R}^N \times [0, T])$ solución de

$$\begin{cases} u_t - \Delta u = 0 & \text{en } \mathbb{R}^N \times (0, T], \\ u(x, 0) = g(x) & \text{en } \mathbb{R}^N. \end{cases}$$

para la cual, existen constantes $A, \alpha > 0$ tal que

$$u(x, t) \leq A e^{\alpha |x|^2} \quad \text{en } \mathbb{R}^N \times [0, T].$$

Entonces

$$\sup\{u(x, t) \colon (x, t) \in \mathbb{R}^N \times [0, T]\} = \sup\{u(x, t) \colon x \in \mathbb{R}^N\}.$$

D/ 1º Paso: Suponemos que

$$4\alpha T < 1$$

Entonces p/ ε suficientemente chico

$$4\alpha(T+\varepsilon) < 1$$

Fijemos $y \in \mathbb{R}^N$, $\mu > 0$ y definimos

$$v(x,t) := u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{N/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$$

$$\Rightarrow \quad v_t - \Delta v = 0 \text{ en } \mathbb{R}^N \times (0,T]$$

Ahora tomemos $r>0$ y fijemos

$$U = B(y,r)$$

$$U_T = B(y,r)\times(0,T]$$

$\Rightarrow$ Por el principio del máximo

$$\max_{\overline{U_T}} v = \max_{\Gamma_T} v$$

2° Paso: Si $x\in\mathbb{R}^N$

$$v(x,0) = \underbrace{u(x,0)}_{g(x)} - \frac{\mu}{(T+\varepsilon)^{N/2}}\, e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \leq g(x) \qquad (1)$$

Si $|y-x| = r, \quad 0 \leq t \leq T$

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)} e^{\frac{r^2}{4(T+\varepsilon-t)}}$$

$$\underset{\text{Hip}}{\leq} A e^{\alpha|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{N/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}}$$

$$\leq A e^{\alpha(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{N/2}} e^{\frac{r^2}{4(T+\varepsilon)}}$$

Sabemos que

$4\alpha(T+\varepsilon) < 1 \Rightarrow \dfrac{1}{4(T+\varepsilon)} = \alpha + \gamma$ p/algun $\gamma > 0$

$$\Rightarrow v(x,t) \leq A e^{\alpha(|y|+r)^2 - \mu} \left(4(\alpha+\gamma)\right)^{\frac{N}{2}} e^{(\alpha+\gamma) r^2}$$

(2)

$\leq \sup_{\mathbb{R}^N} g$ p/ r suficientemente grande

Para r suficientemente grande por 1 y 2 tenemos que.

$$v(x,t) \le \sup_{\mathbb{R}^N} g \quad \text{en } \mathbb{R}^N \times [0,T]$$

$$\parallel$$

$$u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{N/2}}\, e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$$

$$\Big\downarrow \quad \mu \longrightarrow 0$$

$$u(x,t)$$

$$\Rightarrow \quad u(x,t) \le \sup_{\mathbb{R}^N} g \quad \text{en } \mathbb{R}^N \times [0,T]$$

3º Paso ¿Qué pasa si $4\alpha T < 1$?

$$T_1 = \frac{1}{8a}$$

$$T_n = \max\{T, 2T_{n-1}\} \quad n \geq 2$$

Aplico el resultado en

$$[0, T_1], \dots, [T_{n-1}, T_n], [T_n, T]$$

Ecuación del calor

Unicidad para el problema de Cauchy

Teorema.

Sean $g \in C(\mathbb{R}^N)$ y $f \in C(\mathbb{R}^N \times [0, T])$. Entonces existe a lo sumo uno solución $u \in C^2_1(\mathbb{R}^N \times (0, T]) \cap C(\mathbb{R}^N \times [0, T])$

$$\begin{cases} u_t - \Delta u = f & \text{en } \mathbb{R}^N \times (0, T], \\ u(x, 0) = g(x) & \text{en } \mathbb{R}^N. \end{cases}$$

que satisface

$$|u(x, t)| \leq A e^{\alpha |x|^2} \quad \text{en } \mathbb{R}^N \times [0, T]$$

para constantes $A, \alpha > 0$.

D/ suponer que existen dos soluciones u y v y aplicar el resultado anterior a $w = \pm(u - v)$

Tychonov (1935)

$$\varphi : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\varphi(z) = \begin{cases} e^{-1/z^2} & \text{p/ } z \neq 0 \\ 0 & \text{p/ } z = 0 \end{cases}$$

Definimos

$$u(x,t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{d^n}{dt^n} \varphi(t) \frac{x^{2n}}{(2n)!} & \text{p/ } t > 0 \\ 0 & \text{p/ } t = 0 \end{cases}$$

$$u_t = \sum_{n=0}^{\infty} \frac{d^{n+1}}{dt^{n+1}} \varphi(t) \frac{x^{2n}}{(2n)!}$$

$$u_{xx} = \sum_{n=0}^{\infty} \frac{d^{n}}{dt^{n}} \varphi(t) \frac{(2n)(2n-1)\, x^{2n-2}}{(2n)!}$$

$n = 2$

$(2n-2)!$

$$2j = 2n-2 \longrightarrow n = j+1$$

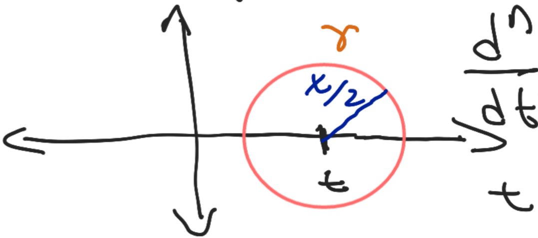
$$= \sum_{j=1}^{\infty} \frac{d^{j+1}}{dt^{j+1}} \varphi(t) \frac{x^{2j}}{(2j)!} = u_t$$

$\Rightarrow u_t - u_{xx} = 0$

$$\lim_{t \to 0} u(x,t) = \sum_{n=0}^{\infty} \frac{d^n}{dt^n} \varphi(t) \Big|_{t=0} \frac{x^{2n}}{n!} = 0$$

¿Porque vale todo?

$z \longrightarrow \varphi(z)$ es holomorfa en $\mathbb{C} - \{0\}$



$$\frac{d^n}{dt^n} \varphi(t) = \frac{n!}{2\pi i} \int_{\gamma} \frac{\varphi(z)}{(z-t)^{n+1}} \, dz$$

para todo $n \in \mathbb{N}$

$$\Rightarrow \left|\frac{d^n}{dt^n}\varphi(t)\right| \le \frac{n!}{2\pi}\int_\gamma \frac{e^{-\mathrm{Re}(z^{-2})}}{\underbrace{|z-t|^{n+1}}_{t/2}}|dz|$$

$$= \frac{n!}{2\pi}\left(\frac{2}{t}\right)^{n+1}\int_\gamma e^{-\mathrm{Re}(z^{-2})}\underbrace{|dz|}_{t/2}$$

$$\gamma = \left\{ z\in\mathbb{C} : z = t + \frac{t}{2}e^{i\theta} \,\middle|\, 0 < \theta \le 2\pi \right.$$

$$z^2 = t^2\left(1+\frac{1}{2}e^{i\theta}\right)^2 \Rightarrow \frac{1}{z^2} = \frac{1}{t^2}\,\frac{1+e^{-i\theta}+\frac{1}{4}e^{-2i\theta}}{\left|\left[1+\frac{1}{2}e^{i\theta}\right]\right|^2}$$

$$\mathrm{Re}\left(\frac{1}{t^2}\right) \geq \frac{1}{t^2}\frac{1}{|1+\frac{1}{2}e^{i\theta}|^4} \geq \frac{1}{(2t)^2}$$

$$\Longrightarrow \left|\frac{d^n}{dt^n}\varphi(t)\right| \leq n!\left(\frac{2}{t}\right)^n e^{-1/4t^2} \quad n \in \mathbb{N}$$

Fijemos $a > 0$. $\forall\, x \in B(0,a)$ se tiene que

$$\left|\frac{d^n}{dt^n}\varphi(t)\,\frac{x^{2n}}{(2n)!}\right| \leq e^{-1/4t^2}\left(\frac{2}{t}\right)^n n!\,\frac{a^{2n}}{(2n)!}$$

Por la desigualdad de Stirling

$$\frac{2^n n!}{(2n)!} \le \frac{1}{n!}$$

tenemos que

$$\left|\frac{d^n}{dt^n}\varphi(t)\,\frac{x^{2n}}{(2n)!}\right| \le \underbrace{e^{-1/4t}\left(\frac{1}{t}\right)^n \frac{a^{2n}}{n!}}_{\text{Mayorante}}$$

$$e^{-1/4t} \sum_{n=0}^{\infty} \frac{\left(\frac{a^2}{t}\right)^n}{n!} = e^{-\frac{1}{4t^2}} e^{a^2/t}$$

Luego la convergencia es uniforme a todas las series