Vamos a discutir la posibilidad de asignar valores en ∂U a funciones en $W^{1,p}(U)$, permitiendo definir condiciones de borde.

Def: Dado $U \subseteq \mathbb{R}^n$
∂U es C^1 si $\forall x \in \partial U$
$\exists r > 0$ y $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ y C^1
tal que:

$$U \cap B(x,r) = \{x \in B(x,r) : x_n > \gamma(x_1, \dots, x_{n-1})\}$$

Tomaremos siempre $1 \leq p < \infty$

Teorema ①: Teorema de trazas

Sea U acotado y ∂U C^1, entonces existe $T : W^{1,p}(U) \longrightarrow L^p(\partial U)$ operador lineal acotado tq:

① $Tu = u|_{\partial U}$ $\quad \forall u \in W^{1,p}(U) \cap C(\bar{U})$

② $\|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}$

Donde C depende sólo de p y U

Dem:

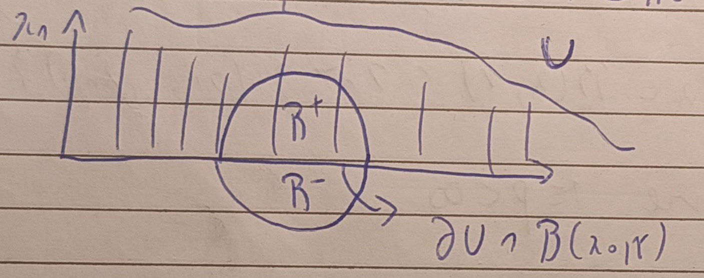
Caso ① Asumimos $u \in C^1(\bar{U})$

Sea $x_0 \in \partial U$ y asumimos ∂U plano cerca de x_0 y que esté en el plano $\{x_n = 0\}$. Es decir:

$\exists\, r > 0$ tal que

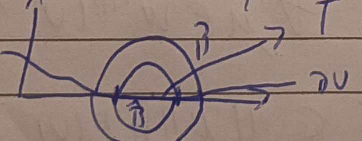
$B^+ = B(x_0, r) \cap \{x_n \geq 0\} \subseteq \bar{U}$

$B^- = B(x_0, r) \cap \{x_n \leq 0\} \subseteq \mathbb{R}^n \setminus U$



Ahora tomamos $\hat{B} = B(x_0, r/2)$ y $\theta \in C_c^\infty(B)$ tal que $\theta \geq 0$ en B y $\theta = 1$ en $\hat{B}$ (Suavizar $\chi_{\hat{B}}$)

Sea $T = \hat{B} \cap \partial U$ la parte de ∂U adentro de $\hat{B}$



Sea $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$

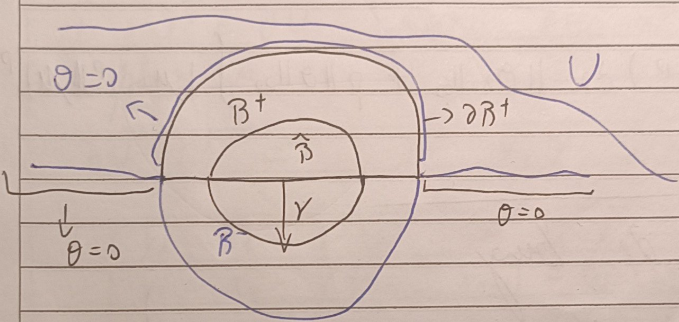
Entonces:

$$\int_T |u|^p \, dx' \leq \int_{\{x_n = 0\}} \theta |u|^p \, dx' = -\int_{B^+} (\theta |u|^p)_{x_n} \, dx$$

La última igualdad viene de:

Gauss - Green:

$u \in C^1(U) \Rightarrow \int_U u_{x_i}\, dx = \int_{\partial U} u\, \nu^i\, dS$



Entonces:

$$\int_{B^+} (\theta |u|^p)_{x_n}\, dx = \int_{\partial B^+} (\theta |u|^p)\cdot \nu_n\, dx'$$

~~θ es θ en todo ∂B^+ menos una parte~~
~~de $\{x_n = 0\} \cap B$~~

$\theta \in C_c^\infty(B) \Rightarrow \theta = 0$ en $\partial B^+ \cap \{x_n > 0\}$

$\theta = 0$ $(\{x_n = 0\} \cap B^+)^c$

$$\Rightarrow \int_{\partial B^+} (\theta |u|^p)\cdot \nu_n\, dx' = -\int_{\{x_n = 0\}} (\theta |u|^p)\, dx$$

$\nu_n = -1$
en $\{x_n = 0\} \cap B^+$

$$\Longrightarrow \int_{\mathbb{R}^+} (\theta|u|^p)_{x_1}\, dx = -\int_{\{x_1=0\}} (\theta|u|^p)\, dx$$

$$\Longrightarrow \int_{T} |u|^p\, dx \le -\int_{\mathbb{R}^+} (\theta|u|^p)_{x_1}\, dx$$

$$= -\int_{\mathbb{R}^+} \left[|u|^p \theta_{x_1} + p|u|^{p-1}(\operatorname{sig} u)\, u_{x_1}\theta \right] dx$$

$$\le \int_{\mathbb{R}^+} |u|^p\, dx\, \|\theta_{x_1}\|_\infty + p\|\theta\|_\infty \int |u_{x_1}|\,|u|^{p-1}\, dx$$

Vale ahí

Desigualdad de Young

$1 < p, q < \infty \qquad 1/p + 1/q = 1$

$\Rightarrow ab \le a^p + C(\varepsilon)\, b^q \qquad a,b > 0 \quad \varepsilon > 0$

$C(\varepsilon)$ depende de ε y p

Usando la desigualdad con

$|u|^{p-1}$ y $|u_{x_1}| \qquad \varepsilon = 1 \qquad q = p/p-1$

Entonces $|u|^{p-1}|u_{x_1}| \le C\left[|u|^{q(p-1)} + |u_{x_1}|^p \right]$

$= C\left[|u|^p + |u_{x_1}|^p \right]$

$$\le C\int_{\mathbb{R}^+} |u|^p\, dx + C\int_{\mathbb{R}^+} |u|^p + C\int |u_{x_1}|^p$$

$C \geq \|\theta_{x_1}\|_\infty,\ \|\theta\|_\infty p,\ C(\varepsilon),\ 1$

$C' \geq 2C, 1$

$$\Rightarrow \int_{T} |u|^p \, dx \leq C' \left(\int_{B^+} |u|^p + |u_{x_n}|^p \right)$$

$$\leq C' \left(\int_{B^+} |u|^p + |Du|^p \, dx \right)$$

Para ②:

Si $x^0 \in \partial U$ pero ∂U no es totalmente plano en x_0

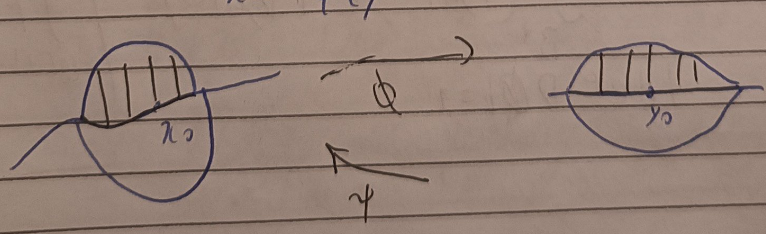
∂U C^1 $\Rightarrow$ Sea r, γ Si $x \in B(x_0, r)$

$$\begin{cases} y_i := x_i = \Phi^i(x) & i = 1, \ldots, n-1 \\ y_n := x_n - \gamma(x_1, \ldots, x_{n-1}) = \Phi^n(x) \end{cases}$$

$$\Phi(x) = y$$

$$\begin{cases} x_i := y_i = \psi^i(y) & i = 1, \ldots, n-1 \\ x_n := y_n + \gamma(x_1, \ldots, x_{n-1}) = \Phi^n(x) \end{cases}$$

$$x = \psi(y)$$



$\Phi = \psi^{-1}$ y $Det(D\Phi) = Det \, D\psi = 1$

Definiendo T como antes tomando $B = B(x_0, r/2)$
y aplicando para (1) a $\Phi(T)$
y $\mu \circ \psi$

$$\Rightarrow \int_{\Phi(T)|_{B(x_0,r/2)}} |\mu \circ \psi|^p \leq C\left[\int_{\Phi(B^+)|_{B(x_0,r/2)}} |\mu \circ \psi|^p + |D(\mu \circ \psi)|^p \right]$$

$$\frac{\partial \mu \circ \psi}{\partial y_m}(y) = \sum_{j=1}^{n} \frac{\partial \psi_j(y)}{\partial y_m} \left(\frac{\partial \mu}{\partial x_j}\right)(\psi(y))$$

Como solo nos interesa en $\Phi(B(x_0, r/2))$
y como $\psi \in C^1(B(x_0, r))$
$\Rightarrow C = \sup \| \psi|_{\bar{B}(x_0,r/2)} \|$

$$\Rightarrow \left| \frac{\partial \mu \circ \psi}{\partial y_m} \Big| \Phi(\bar{B}(x_0, r/2)) \right|^p$$

Usando $p \geq 1$
$\left|\sum_{i=1}^{n} x_i\right|^p \leq n^{p-1} \sum_i |x_i|^p$

$$\leq \sum_{j=1}^{n} \left| \frac{\partial \mu}{\partial x_j}(\psi(y)) \right|^p C_m^*$$

$C^* = \max\{C_m^*\}$

$$\Rightarrow \int_{\Phi|_{B(x_0,r/2)}(T)} |\mu \circ \psi|^p \leq C^* \int_{B^+} \left(|\mu|^p + |D\mu|^p\right)$$

$\int_{\Phi(U)} f = \int_U f(\Phi)|\det D\Phi|$

$|\det D\Phi| = 1$

$$\parallel$$

$$\int_{T \cap B(x_0, r/2)} |\mu|^p$$

Paso (3)

Como ∂U compacto $\Rightarrow$ existen finitos $x_0^i \in \partial U$

y finitos abiertos de ∂U $T^i \subseteq \partial U$
tal que $\partial U = \bigcup_{i=1}^{N} T^i$

y $\|u\|_{L^p(T_i)} \leq C \|u\|_{W^{1,p}(U)}$ $i=1,\dots,N$

Definimos $Tu := u_{|\partial U}$

$\Rightarrow$ $\|u\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}$

Para una C que no depende de u

Paso (4) Sea $u \in W^{1,p}(U) \Rightarrow$ existe $u_n \in C^\infty(\bar{U})$
tal que $u_n \to u$ $W^{1,p}(U)$

$\Rightarrow \|Tu_n - Tu_l\| \leq C \|u_n - u_l\|$

$\Rightarrow \{Tu_n\}$ de Cauchy en $L^p(\partial U)$
$\Rightarrow \lim_{n\to\infty} Tu_n := Tu$

(Esta def no depende de $\{u_n\}$)
Sea $\{z_n\} \subseteq C^\infty(\bar{U})$ $z_n \to u$
$\Rightarrow$ $\|Tz_n - Tu_n\| \leq C \|u_n - z_n\|$
$\leq C\left[\|u_n - u\| + \|z_n - u\|\right] \to 0$

Finalmente si $\mu \in W^{1,p}(U) \cap C(\overline{U})$

las $\mu_n \in C^\infty(\overline{U})$ construidas en el teorema de aproximación global converge uniforme a μ en $\overline{U}$, por ende la convergencia unif se da en ∂U

$\Rightarrow T\mu = \mu_{|\partial U}$ // $T\mu := \lim_{n\to\infty} T\mu_n = \lim_{n\to\infty} \mu_{n|\partial U} = \mu_{|\partial U}$ $L^p(\partial U)$

Teorema ②: Funciones de traza 0

Sea U acotado, ∂U C^1 y $\mu \in W^{1,p}(U)$, entonces:

$\mu \in W_0^{1,p}(U) \iff T\mu = 0$

Dem: $(\Rightarrow)$ Si $\mu \in W_0^{1,p}(U) = \overline{C_c^\infty(U)}^{W^{1,p}}$

$\Rightarrow$ existe $\{\mu_n\} \subseteq C_c^\infty(U)$ tq $\mu_n \to \mu$ $W^{1,p}(U)$

las $\mu_n \in C_c^\infty(U)$ soporte compacto en U entonces podemos extender a $\overline{U}$ tal que sea 0 fuera de sop μ_n.

$$\hat{\mu}_n = \begin{cases} \mu_n & \text{sop}\,\mu_n \\ 0 & (\text{sop}\,\mu_n)^c \cap \overline{U} \end{cases}$$

$\Rightarrow \hat{\mu}_n \longrightarrow \mu$ $W^{1,p}(U)$

$T: W^{1,p}(U) \longrightarrow L^p(\partial U)$ acotado $\Rightarrow$

$T\hat{\mu}_n \longrightarrow T\mu$, $\hat{\mu}_n \in C_c^\infty(\overline{U}) \Rightarrow$

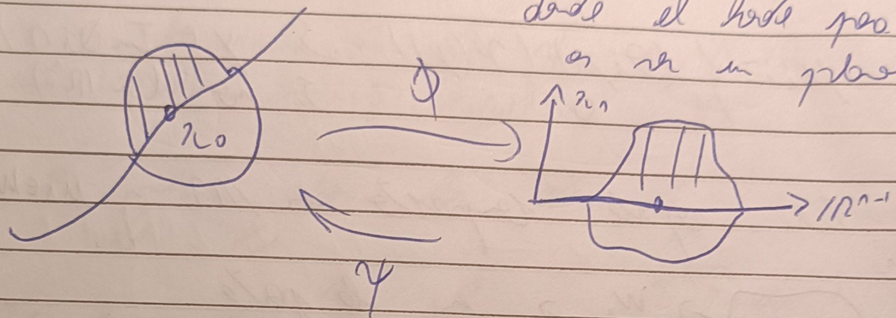
$T\hat{\mu}_n = \hat{\mu}_{n|\partial U} = 0 \longrightarrow T\mu$ $L^p(\partial U)$

$\Rightarrow T\mu = 0$

//

$(\Leftarrow)$ $T\mu = 0$

$\forall x_0 \in \partial U$ $\exists r > 0$ tal que $B(x_0, r)$ tiene 1 cambio de coord donde el borde pasa a ser un plano



Como ∂U compacto $\partial U \subseteq \bigcup_{i=1}^{N} B(x_i, r_i/2)$

donde las $B(x_i, r_i)$ tienen el cambio de coord y $x_i \in \partial U$

Sea $U \subseteq \bigcup_{i=0}^{N} W_i$ $\quad W_i := B(x_i, r_i/2)$ $\quad i = 1, \dots, N$

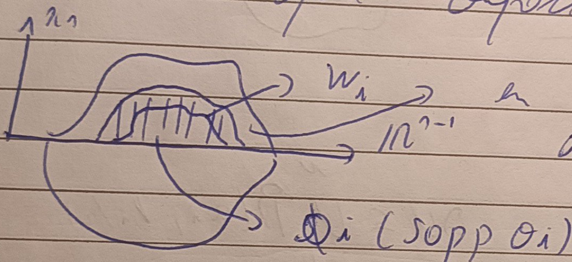
Tomamos la partición $\{\theta_i\}_{i=0}^{N}$ de la unidad con respecto a $\{W_i\}_{i=0}^{N}$. $\mathrm{sopp}\,\theta_i \subseteq W_i$.

Sean $\hat{\psi}_i$ y $\hat{\Phi}_i$ las cartas de coord. dadas por $B(x_i, r_i)$

$\psi_i := \hat{\psi}_i|_{\hat{\Phi}_i(W_i)}$ $\qquad \Phi_i := \hat{\Phi}_i|_{W_i}$

Defino $u_i := \begin{cases} (\theta_i u)(\psi(y)) & \text{si } y \in \operatorname{Im}\Phi_i \cap \mathbb{R}^n_+ \\ 0 & \text{si no (en } \mathbb{R}^n_+) \end{cases}$

u_i tiene soporte compacto en $\overline{\mathbb{R}^n_+}$ y $u_i \in W^{1,p}(\mathbb{R}^n_+)$



en la parte de afuera $u_i = 0$

Como $\overline{C^\infty(\bar{U})}^{W^{1,p}} = W^{1,p}(U)$

$\Rightarrow \exists \{u_n\} \subseteq C^\infty(U)$ tal que

$u_n \to u \quad W^{1,p}(U)$

Defino $u_n^i := \begin{cases} (\theta_i u_n)\psi(y) & y \in \operatorname{Im}\Phi_i \cap \overline{\mathbb{R}^n_+} \\ 0 & \text{si no} \end{cases}$

Sy $\mu_n^i \to \mu^i$ $W^{1,p}(\mathbb{R}^n_+)$:

$$\frac{\partial \mu_n^i}{\partial y_m} \circ \psi^i(y) = \sum_{j=1}^{n} \frac{\partial \psi^i_j(y)}{\partial y_m} \frac{\partial \mu_n^i}{\partial x_j}(\psi^i(y))$$

j-ésimo vector de ψ^i

ψ^i es la restricción de $\hat{\psi}_i$ a $B(x_i, r_i/2)$

$\Rightarrow |\psi_i(y)| \leq \|\hat{\psi}_i|_{\overline{B}(x_i, r_i/2)}\|_{sup} \quad \forall y$

Lo mismo con las derivadas parciales de ψ_i

Usando $\left|\sum_{i=1}^{n} x_i\right|^p \leq n^{p-1} \sum_{i=1}^{n} |x_i|^p \quad p \geq 1$

tengo: $\left|\frac{\partial \mu_n^i \circ \psi^i}{\partial y_m}(y)\right|^p \leq C(\psi^i) \sum_{j=1}^{n} \left|\frac{\partial \mu_n^i}{\partial x_j}\right|^p (\psi^i(y))$

$$\Rightarrow \int_{Q_i(W_i) \cap \mathbb{R}^n_+} |\nabla(\theta_i(\mu^i_n - \mu) \circ \psi(y)|^p \, dy$$

$$\leq \left[C \sum_{j=1}^{n} \int_{Q_i(W_i) \cap \mathbb{R}^n_+} \left|\frac{\partial(\theta_i(\mu_n - \mu)) \circ \psi_i(y)}{\partial y_j}\right|^p\right]$$

y $|\operatorname{Det} D Q_i| = 1$

$$\Rightarrow = C \sum_{j=1}^{n} \int_{W_i \cap U} \left|\frac{\partial(\theta_i(\mu_n - \mu))(x)}{\partial x_j}\right|^p dx$$

$\longrightarrow 0$ ya que $\mu_n \xrightarrow[W^{1,p}(U)]{} \mu$

Lo mismo con:

$$\int_{\Phi_i(W_i)\cap \mathbb{R}^n_+} |\theta_i(u_n - u)\circ\psi_i(y)|^p \, dy$$

$\Rightarrow \quad u_n^i \longrightarrow u^i \quad W^{1,p}(\mathbb{R}^n_+)$

Tenemos $\quad Tu_n \longrightarrow Tu \quad L^p(\partial U)$

$\Rightarrow \quad \theta_i \, Tu_n \circ \psi_i \longrightarrow \theta_i \, Tu \circ \psi_i \quad L^p(\partial \mathbb{R}^n_+)$

Como $\quad \psi_i(\partial\mathbb{R}^n_+ \cap \operatorname{Im}\Phi_i) = \partial U \cap W_i$

$\Rightarrow \quad \underbrace{\theta_i \, Tu_n \circ \psi_i}_{\substack{\| \\ u_n^i|_{\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}}}} \longrightarrow 0 \qquad (Tu = 0)$

TFC

Si $\quad x' \in \mathbb{R}^{n-1} \quad , x_n \geq 0$

$$\Rightarrow |u_m^i(x', x_n)| \leq |u_m^i(x', 0)| + \int_0^{x_n} |u^i_{m,x_n}(x', t)| \, dt$$

$\Rightarrow$ Usando $\quad (a+b)^p \leq 2^{p-1}a^p + 2^{p-1}b^p$:

$$|u_m^i(x', x_n)|^p \leq C\left[|u_m^i(x', 0)|^p + \left(\int_0^{x_n} |u^i_{m,x_n}(x', t)| \, dt\right)^p \right]$$

Ahora usando Holder:

$$\int_0^{x_n} |u_{m,x_n}^{i}(x',t)|\,dt \leq \left(\int_0^{x_n} 1^{p/p-1}\,dt\right)^{\frac{p-1}{p}} \left(\int_0^{x_n} |u_{m,x_n}^{i}(x',t)|^p\,dt\right)^{1/p}$$

$$= (x_n)^{\frac{p-1}{p}} \left(\int_0^{x_n} |u_{m,x_n}^{i}(x',t)|^p\,dt\right)^{1/p}$$

Entonces juntando todo:

$$\int_{\mathbb{R}^{n-1}} |u_m^i(x',x_n)|^p\,dx'$$

$$\leq C\left[\int_{\mathbb{R}^{n-1}} |u_m^i(x',0)|^p\,dx' + x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |u_{m,x_n}^{i}(x',t)|^p\,dx'dt\right]$$

$$\leq C\left[\int_{\mathbb{R}^{n-1}} |u_m^i(x',0)|^p\,dx' + x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |Du_m^i(x',t)|^p\,dx'dt\right.$$

Ahora usando:

$u_m^i|_{\partial\mathbb{R}^n_+} \longrightarrow 0 \quad L^p(\mathbb{R}^{n-1})$

$u_m^i \longrightarrow u^i \quad W^{1,p}(\mathbb{R}^n_+)$

$$\xrightarrow[n\to\infty]{} C x_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |Du^i|^p\,dx'dt$$

$$\Longrightarrow \int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p \, dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p \, dx' dt \quad (*)$$

Para ③

Sea $\varphi \in C^\infty(\mathbb{R}_+)$ tal que:

$\varphi \equiv 1$ en $[0,1]$
$\varphi \equiv 0$ en $\mathbb{R}_+^n \setminus [0,2]$
$0 \leq \varphi \leq 1$

Defino
$$\begin{cases} \varphi_m(x) = \varphi(m x_n) & (x \in \mathbb{R}_+^n) \\ w_m(x) = u^i(x)(1-\varphi_m) \end{cases}$$

$$\Rightarrow w_{m, x_n}(x) = u^i_{x_n}(1-\varphi_m) - m u^i \varphi'$$

$$D_{x'} w_m = D_{x'} u^i (1-\varphi_m)$$

$$\Rightarrow \int_{\mathbb{R}_+^n} |Dw_m - Du^i|^p \, dx$$

$$= \int_{\mathbb{R}_+^n} |Du^i(x)\varphi(m x_n) - m u^i(x) \varphi'(m x_n)|^p \, dx$$

$$\leq C \int_{\mathbb{R}_+^n} |\varphi_m|^p |Du^i|^p + m^p |\varphi'|^p |u^i|^p \, dx$$

$\downarrow$

$(a+b)^p \leq 2^{p-1}(a^p + b^p)$

$$\leq C \int_{\mathbb{R}^n_+} |\varphi_m|^p \, |D\mu^i|^p + C \int_0^{2/m} \int_{\mathbb{R}^n_+} m^p \, |\mu^i|^p \, dx' \, dt$$

$\varphi' \in C_c^\infty \Rightarrow \varphi' \in L^\infty$

$$\int_{\mathbb{R}^n_+} = \int_{\mathbb{R}^{n-1}} \int_0^\infty$$

y $\operatorname{sopp} \varphi' \subseteq [0, 2/m]$

$$A = \int_{\mathbb{R}^n_+} |\varphi_m|^p \, |D\mu^i|^p \xrightarrow[m \to +\infty]{} 0$$

$$B = C m^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |\mu^i|^p \, dx' \, dt$$

Ge $\varphi_m \neq 0$ solo en $0 \leq x_n \leq 2/m$

Usando $\circledast$

$$B \leq C m^p \Big(\overbrace{\int_0^{2/m} t^{p-1} \, dt}^{\frac{1}{1}\left(\frac{2}{m}\right)^p \frac{1}{p}} \Big) \left(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |D\mu^i|^p \, dx' \, dt \right)$$

$$= C m^p \left(\frac{2}{m}\right)^p \frac{1}{p} \left(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |D\mu^i|^p \, dx' \, dt \right) \xrightarrow[m \to \infty]{} 0$$

$\Rightarrow D w_m \to D\mu^i \quad L^p(\mathbb{R}^n_+)$

y $w_m \to \mu^i \quad L^p(\mathbb{R}^n_+)$

$\Rightarrow w_m \to \mu^i \quad W^{1,p}(\mathbb{R}^n_+)$

Tenemos $w_m = 0$ si $0 < x_n < 1/m$

Modificamos w_m tal que exista $u_m \in C_c^\infty(\mathbb{R}^n_+)$ y $u_m \to u^i$ $W^{1,p}(\mathbb{R}^n_+)$

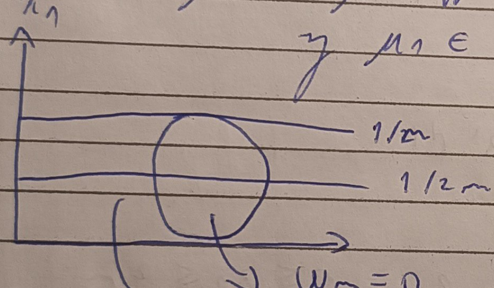
Sea $m \in \mathbb{N}$ $\quad w_m * \eta_\varepsilon \xrightarrow{\varepsilon \to \infty} w_m$ $L^p(\mathbb{R}^n_+)$

tomamos ε_m tal que:

$\varepsilon_m < 1/2m$ y $\underset{|\alpha| \le 1}{\|D^\alpha w_m - \eta_{\varepsilon_m} * D^\alpha w_m\| < 1/m}$

$\Rightarrow u_n \to u^i$ $W^{1,p}(\mathbb{R}^n_+)$

y $u_n \in C_c^\infty(\mathbb{R}^n_+)$ (ya que u^i y w_n, η_ε tienen sop compacto)



$w_m = 0 \Rightarrow \operatorname{sop} u_n \subseteq \{x_n > 1/2m\} \subseteq \mathbb{R}^n_+$

$Q_{1/2m} = 0$

$\Rightarrow \cancel{u_m \in C_c^\infty(\mathbb{R})}$

Pasamos u_n a W_i

$$\hat{u}_m(x) = \begin{cases} \theta_i(x)\, u(\Phi^i(x)) & x \in U \cap W_i \\ 0 & \text{si no} \end{cases}$$

$\Rightarrow \hat{u}_m \to u_i$ $W^{1,p}(U \cap W_i)$

$u_m = \sum_{i=1}^{N} u_m^i \Rightarrow u_m \to u$ $W^{1,p}(U)$ //