Chapter 7

Weighted Averages

This chapter addresses the problem of combining two or more separate and independent measurements of a single physical quantity. We will find that the best estimate of that quantity, based on the several measurements, is an appropriate *weighted average* of those measurements.

7.1 The Problem of Combining Separate Measurements

Often, a physical quantity is measured several times, perhaps in several separate laboratories, and the question arises how these measurements can be combined to give a single best estimate. Suppose, for example, that two students, A and B, measure a quantity x carefully and obtain these results:

$$\text{Student } A\text{:} \quad x = x_A \pm \sigma_A \tag{7.1}$$

and

$$\text{Student } B\text{:} \quad x = x_B \pm \sigma_B. \tag{7.2}$$

Each result will probably itself be the result of several measurements, in which case x_A will be the mean of all A's measurements and σ_A the standard deviation of that mean (and similarly for x_B and σ_B). The question is how best to combine x_A and x_B for a single best estimate of x.

Before examining this question, note that if the discrepancy $|x_A - x_B|$ between the two measurements is much greater than both uncertainties σ_A and σ_B, we should suspect that something has gone wrong in at least one of the measurements. In this situation, we would say that the two measurements are *inconsistent,* and we should examine both measurements carefully to see whether either (or both) was subject to unnoticed systematic errors.

Let us suppose, however, that the two measurements (7.1) and (7.2) are *consistent;* that is, the discrepancy $|x_A - x_B|$ is *not* significantly larger than both σ_A and σ_B. We can then sensibly ask what the best estimate x_{best} is of the true value X, based on the two measurements. Your first impulse might be to use the average $(x_A + x_B)/2$ of the two measurements. Some reflection should suggest, however, that this average is unsuitable if the two uncertainties σ_A and σ_B are unequal. The simple average $(x_A + x_B)/2$ gives equal importance to both measurements, whereas the more precise reading should somehow be given more weight.

Throughout this chapter, I will assume all systematic errors have been identified and reduced to a negligible level. Thus, all remaining errors are random, and the measurements of x are distributed normally around the true value X.

7.2 The Weighted Average

We can solve our problem easily by using the principle of maximum likelihood, much as we did in Section 5.5. We are assuming that both measurements are governed by the Gauss distribution and denote the unknown true value of x by X. Therefore, the probability of Student A's obtaining his particular value x_A is

$$Prob_X(x_A) \propto \frac{1}{\sigma_A} e^{-(x_A - X)^2/2\sigma_A{}^2}, \tag{7.3}$$

and that of B's getting his observed x_B is

$$Prob_X(x_B) \propto \frac{1}{\sigma_B} e^{-(x_B - X)^2/2\sigma_B{}^2}. \tag{7.4}$$

The subscript X indicates explicitly that these probabilities depend on the unknown actual value.

The probability that A finds the value x_A *and* B the value x_B is just the product of the two probabilities (7.3) and (7.4). In a way that should now be familiar, this product will involve an exponential function whose exponent is the sum of the two exponents in (7.3) and (7.4). We write this as

$$\begin{aligned} Prob_X(x_A, x_B) &= Prob_X(x_A)\, Prob_X(x_B) \\ &\propto \frac{1}{\sigma_A \sigma_B} e^{-\chi^2/2}, \end{aligned} \tag{7.5}$$

where I have introduced the convenient shorthand χ^2 (chi squared) for the exponent

$$\chi^2 = \left(\frac{x_A - X}{\sigma_A}\right)^2 + \left(\frac{x_B - X}{\sigma_B}\right)^2. \tag{7.6}$$

This important quantity is the sum of the squares of the deviations from X of the two measurements, each divided by its corresponding uncertainty.

The principle of maximum likelihood asserts, just as before, that our best estimate for the unknown true value X is that value for which the actual observations x_A, x_B are most likely. That is, the best estimate for X is the value for which the probability (7.5) is maximum or, equivalently, the exponent χ^2 is minimum. (Because maximizing the probability entails minimizing the "sum of squares" χ^2, this method for estimating X is sometimes called the "method of least squares.") Thus, to find the best estimate, we simply differentiate (7.6) with respect to X and set the derivative equal to zero,

$$2\frac{x_A - X}{\sigma_A{}^2} + 2\frac{x_B - X}{\sigma_B{}^2} = 0.$$

The solution of this equation for X is our best estimate and is easily seen to be

$$(\text{best estimate for } X) = \left(\frac{x_A}{\sigma_A{}^2} + \frac{x_B}{\sigma_B{}^2}\right) \Big/ \left(\frac{1}{\sigma_A{}^2} + \frac{1}{\sigma_B{}^2}\right). \tag{7.7}$$

This rather ugly result can be made tidier if we define *weights*

$$w_A = \frac{1}{\sigma_A{}^2} \quad \text{and} \quad w_B = \frac{1}{\sigma_B{}^2}. \tag{7.8}$$

With this notation, we can rewrite (7.7) as the *weighted average* (denoted x_{wav})

$$(\text{best estimate for } X) = x_{\text{wav}} = \frac{w_A x_A + w_B x_B}{w_A + w_B}. \tag{7.9}$$

If the original two measurements are equally uncertain ($\sigma_A = \sigma_B$ and hence $w_A = w_B$), this answer reduces to the simple average $(x_A + x_B)/2$. In general, when $w_A \neq w_B$, the weighted average (7.9) is *not* the same as the ordinary average; it is similar to the formula for the center of gravity of two bodies, where w_A and w_B are the actual weights of the two bodies, and x_A and x_B their positions. In (7.9), the "weights" are the inverse squares of the uncertainties in the original measurements, as in (7.8). If A's measurement is more precise than B's, then $\sigma_A < \sigma_B$ and hence $w_A > w_B$, so the best estimate x_{best} is closer to x_A than to x_B, just as it should be.

Quick Check 7.1. Workers from two laboratories report the lifetime of a certain particle as 10.0 ± 0.5 and 12 ± 1, both in nanoseconds. If they decide to combine the two results, what will be their respective weights as given by (7.8) and their weighted average as given by (7.9)?

Our analysis of two measurements can be generalized to cover any number of measurements. Suppose we have N separate measurements of a quantity x,

$$x_1 \pm \sigma_1, \quad x_2 \pm \sigma_2, \ldots, \quad x_N \pm \sigma_N,$$

with their corresponding uncertainties $\sigma_1, \sigma_2, \ldots, \sigma_N$. Arguing much as before, we find that the best estimate based on these measurements is the weighted average

$$\boxed{x_{\text{wav}} = \frac{\sum w_i x_i}{\sum w_i},} \tag{7.10}$$

where the sums are over all N measurements, $i = 1, \ldots, N$, and the *weight* w_i of each measurement is the reciprocal square of the corresponding uncertainty,

$$\boxed{w_i = \frac{1}{\sigma_i^2}} \tag{7.11}$$

for $i = 1, 2, \ldots, N$.

Because the weight $w_i = 1/\sigma_i^2$ associated with each measurement involves the *square* of the corresponding uncertainty σ_i, any measurement that is much less precise than the others contributes *very* much less to the final answer (7.10). For example, if one measurement is four times less precise than the rest, its weight is 16 times less than the other weights, and for many purposes this measurement could simply be ignored.

Because the weighted average x_{wav}, is a function of the original measured values $x_1, x_2, \ldots, x_N$, the uncertainty in x_{wav} can be calculated using error propagation. As you can easily check (Problem 7.8), the uncertainty in x_{wav} is

$$\boxed{\sigma_{\text{wav}} = \frac{1}{\sqrt{\Sigma w_i}}.} \tag{7.12}$$

This rather ugly result is perhaps a little easier to remember if we rewrite (7.11) as

$$\sigma_i = \frac{1}{\sqrt{w_i}}. \tag{7.13}$$

Paraphrasing Equation (7.13), we can say that the uncertainty in each measurement is the reciprocal square root of its weight. Returning to Equation (7.12), we can paraphrase it similarly to say that the uncertainty in the grand answer x_{wav} is the reciprocal square root of *the sum of all the individual weights;* in other words, the total weight of the final answer is the sum of the individual weights w_i.

Quick Check 7.2. What is the uncertainty in your final answer for Quick Check 7.1?

7.3 An Example

Here is an example involving three separate measurements of the same resistance.

Example: Three Measurements of a Resistance

Each of three students measures the same resistance several times, and their three final answers are (all in ohms):

$$\begin{aligned}
\text{Student 1:}\quad R &= 11 \pm 1 \\
\text{Student 2:}\quad R &= 12 \pm 1 \\
\text{Student 3:}\quad R &= 10 \pm 3
\end{aligned}$$

Given these three results, what is the best estimate for the resistance R?

The three uncertainties σ_1, σ_2, σ_3 are 1, 1, and 3. Therefore, the corresponding weights $w_i = 1/\sigma_i^2$ are

$$w_1 = 1, \qquad w_2 = 1, \qquad w_3 = \tfrac{1}{9}.$$

The best estimate for R is the weighted average, which according to (7.10) is

$$R_{\text{wav}} = \frac{\sum w_i R_i}{\sum w_i}$$
$$= \frac{(1 \times 11) + (1 \times 12) + (\frac{1}{9} \times 10)}{1 + 1 + \frac{1}{9}} = 11.42 \text{ ohms.}$$

The uncertainty in this answer is given by (7.12) as

$$\sigma_{\text{wav}} = \frac{1}{\sqrt{\sum w_i}} = \frac{1}{\sqrt{1 + 1 + \frac{1}{9}}} = 0.69.$$

Thus, our final conclusion (properly rounded) is

$$R = 11.4 \pm 0.7 \text{ ohms.}$$

For interest, let us see what answer we would get if we were to ignore completely the third student's measurement, which is three times less accurate and hence nine times less important. Here, a simple calculation gives $R_{\text{best}} = 11.50$ (compared with 11.42) with an uncertainty of 0.71 (compared with 0.69). Obviously, the third measurement does not have a big effect.

Principal Definitions and Equations of Chapter 7

If $x_1, x_2, \ldots, x_N$ are measurements of a single quantity x, with known uncertainties $\sigma_1, \sigma_2, \ldots, \sigma_N$, then the best estimate for the true value of x is the *weighted average*

$$x_{\text{wav}} = \frac{\sum w_i x_i}{\sum w_i}, \qquad \text{[See (7.10)]}$$

where the sums are over all N measurements, $i = 1, \ldots, N$, and the weights w_i are the reciprocal squares of the corresponding uncertainties,

$$w_i = \frac{1}{\sigma_i^2}.$$

The uncertainty in x_{wav} is

$$\sigma_{\text{wav}} = \frac{1}{\sqrt{\sum w_i}}, \qquad \text{[See (7.12)]}$$

where, again, the sum runs over all of the measurements $i = 1, 2, \ldots, N$.