Grupo de Lorentz (transf. propias de Lorentz)

$(a,b) = a^T b$ $\qquad x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ $\qquad \cancel{s^2 = x^T x}$ $\qquad g = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$

$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$ $\qquad g^2 = I_{4\times 4}$ $\qquad g_{\alpha\beta} = g^{\alpha\beta}$ $\qquad x'^2 = x^2$ transf. Lorentz

$x' = Lx$ $\qquad x'^\alpha = L^\alpha{}_\beta x^\beta$ $\qquad L^{-1\,\alpha}{}_\beta L^\beta{}_\delta = \delta^\alpha{}_\delta$

$$a\cdot b \equiv a^0 b^0 - \vec{a}\cdot\vec{b} = a^T g b \equiv (a, gb) = (ga, b) = (ga)^T b = a^T g^T b = a^T g b$$

$$\left.\begin{aligned} s'^2 &= (x'^0)^2 - \vec{x}'^2 = (x', gx') = x'^T g x' \\ s^2 &= x^{0\,2} - \vec{x}^2 = (x, gx) = x^T g x \end{aligned}\right\} \quad \begin{aligned} & s^2 = s'^2 \\ & x' = Lx \end{aligned} \qquad \begin{aligned} & (Lx)^T g\, Lx = x^T g x \\ & \underline{x^T (L^T g L) x = x^T g \cancel{x}} \end{aligned}$$

$$\boxed{L^T g L = g}$$

:

boost dir x, vel v ($\beta = v/c$) $\qquad L = (L^\alpha{}_\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\det(L^T g L) = \underbrace{\det L^T}_{\det L}\, \cancel{\det g}\, \det L = \cancel{\det g} \quad , \det g = -1$$

$\det L = \pm 1$
- $+1$ → t.L. propias / homogeneas
- -1 → t.L. impropias, contienen las inversiones

L : 16 parámetros

16 condiciones
$-3-2-1$

10 condiciones impuestas x $L^T g L = g$

6 parámetros libres!!
→ 3 boost indep. en 3 dir.
→ 3 ángulos Euler / rotaciones

T. propias,

$$L = e^{G} \qquad e^{G} = I + G + \frac{G^2}{2!} + \frac{G^3}{3!} + \cdots$$

$$\det L = \det e^{G} = e^{\mathrm{Tr}\,G} = 1 \quad , \; \mathrm{Tr}\,G = 0$$

$$L^{-1} = g L^{T} g = e^{-G} = g e^{G^T} g = e^{g G^T g} \qquad -G = g G^T g$$

$$L^T g L = g \qquad L^T g = g L^{-1} \qquad g L^T g = L^{-1} \qquad -gG = G^T g = (gG)^T$$

$$g^T = g = g^{-1}$$

gG antisimétrica!

$$\longrightarrow G = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

6 parámetros! $L_{01} \quad L_{02} \quad L_{03} \longrightarrow$ boosts

$L_{12} \quad L_{13} \quad L_{23} \longrightarrow$ rotaciones

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$S_i \longrightarrow$ generan rotaciones

$K_i \longrightarrow$ generan "boost"

$\left.\begin{matrix} S_i^2 \\ K_i^2 \end{matrix}\right\}$ diagonal $\qquad (a\cdot S)^3 = -(a\cdot S)$

$\qquad (b\cdot K)^3 = (b\cdot K)$

$$\boxed{G = -\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}} \qquad L = e^G$$

$\vec{\omega}$, $\vec{\zeta}$ vectores reales

ej $\vec{\omega} = 0$ $\quad \vec{\zeta} = \zeta_1 \hat{x}$ $\quad , G = -\zeta_1 K_1 \quad , L = e^G = \begin{pmatrix} \text{ch}\,\zeta_1 & -\text{sh}\,\zeta_1 & 0 & 0 \\ -\text{sh}\,\zeta_1 & \text{ch}\,\zeta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ boost

$$L = (1 - K_1^2) - K_1\,\text{sh}\,\zeta_1 + K_1^2\,\text{ch}\,\zeta_1$$

Boost arbitrario: $L = e^{-\vec{\xi}\cdot\vec{K}}$, $\vec{\xi} = \hat{\beta}\,\mathrm{th}^{-1}\beta$

$$L = e^{-\hat{\beta}\cdot\vec{K}\,\mathrm{th}^{-1}\beta} = \begin{pmatrix} \gamma & -\gamma\beta_1 & - & - \\ -\gamma\beta & \frac{1+(\gamma-1)\beta_1^2}{\beta^2} & \cdots & - \\ - & - & - & - \\ & - & - & \end{pmatrix}$$

S_i, K_i 6 generadores de la T.L. propias

rel. de conmutación entre los generadores

$$\begin{cases} [S_i, S_j] = S_i S_j - S_j S_i = \epsilon_{ijk} \underbrace{S_k}_{\text{sumado}} \\ [K_i, K_j] = -\epsilon_{ijk} S_k \\ [S_i, K_j] = \epsilon_{ijk} K_k \end{cases}$$

Levi Civita tensor

→ "K trasf. como un vector"

"álgebra de generadores"

→ $SL(2, \mathbb{C})$, $O(3,1)$

2x2 complejas det = 1

→ dejan invariantes $x^2+y^2+z^2-tc^2$!

L propias son un grupo? $\to I \in L$ ✓

$L^T g L = g$ $\to L^{-1}$ es TL $L = e^{G}$, $L^{-1} = e^{-G}$ ✓

$L^{-1}: (L^{-1})^T g L^{-1} = g$?

$(e^{-G})^T g (e^{-G}) = g$ $\leftarrow$ $(e^{G})^T g e^{G} = g$

$\to L_1 = e^{G_1}$, $L_2 = e^{G_2}$

$L_1 L_2$ es TL

$(L_1)^T g L_1 = g$ $(L_1 L_2)^T g (L_1 L_2) = g$ ✓

$\quad\;\; 2 \qquad\;\; 2$

Covariancia electrodinámica y fza. Lorentz

Fza Lorentz $\quad \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times B\right)$ (unidades gaussianas)

$$p^\alpha = (p^0, \vec{p}) = m(U^0, \vec{U}) \quad , \quad U^0 = \frac{dx^0}{d\tau} = \gamma c \quad , \quad \vec{U} = \frac{d\vec{x}}{d\tau} = \gamma\vec{v}$$

$$d\tau = \frac{dt}{\gamma} = dt\sqrt{1-\beta^2} \quad , \quad \underbrace{\frac{d\vec{p}}{d\tau}}_{\text{parte espacial de un 4-vector}} = \frac{q}{c}\left(U^0\vec{E} + \vec{v}\times\vec{B}\right)$$

1) parte temporal $\quad \frac{dp^0}{dt} = \frac{q}{c}\vec{U}\cdot\vec{E}$

q es invariante Lorentz

$U\cdot E$ es la parte temporal de un 4vector

$$\vec{U}\cdot\vec{E} = \underbrace{\cdots}_{\text{índice temporal}}{}^{\alpha\beta}\, U_\beta \qquad \overline{U\cdot E} \longrightarrow \underbrace{F^{0\beta}}_{\vec{E}\text{ está acá!}} U_\beta$$

$F^{01} \to -E_x$

$F^{02} \to -E_y$

$F^{03} \to -E_z$

Usando $\overset{F^{\alpha\beta}\quad F^{0\beta}}{\frac{dp^i}{d\tau}}$, obtengo $F^{i0} \to U^0\vec{E}$, B idem.

Ecs Maxwell $\to$ TL $\qquad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ si $J^\alpha = (c\rho, \vec{J})$ es un 4-vector

$\Rightarrow$ la ec. cont. es una 4-div. $\partial^\alpha J_\alpha = 0$

Carga δq ref. reposo, ρ / $\delta q = \rho\, d^3x$ en otro ref $\rho' d^3x' = \rho\, d^3x$

$d^4x' = d^4x$ ante T.L.

$d^4x' = (\det L)\, d^4x$ ✓

$\underbrace{\det L}_{1}$

$\underbrace{dx_0}\, d^3x$

ρ transf. como comp. temporal de un 4-vector

$\vec{J}$ es la comp. espacial de un 4-vector. $(\vec{J} = \vec{v}\rho)$

$J^\alpha = (c\rho, \vec{J})$ es 4-vector

Ecs Maxwell en el gauge Lorenz $\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0$ $\partial^\alpha A_\alpha = 0$

ecs. $\begin{cases} \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J} \\ \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 4\pi\rho \end{cases}$

$\Box \rightarrow$ operador escalar $\Box \vec{A} = \frac{4\pi}{c}\vec{J}$ $(c\rho, \vec{J})$ 4 vect

$\Box \phi = 4\pi\rho$ $\Downarrow$

(ϕ, A) 4 vector

y el gauge Lorenz es simplemente la 4-div. de $(\phi, A) \longrightarrow A^\alpha = (A^0, \vec{A})$

$\begin{cases} \Box A^\alpha = \frac{4\pi}{c} J^\alpha \\ \partial_\alpha J^\alpha = 0 \end{cases}$ en forma explícita vemos que las ecs. de M son covariantes

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$
$$\vec{B} = \nabla \times \vec{A}$$

$$A^\alpha = (A^0, \vec{A})$$

$$E_x = -(\partial^0 A^1 - \partial^1 A^0)$$
$$= -\frac{1}{c}\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x}$$

$$B_x = -(\partial^2 A^3 - \partial^3 A^2) \qquad \partial^i = \frac{\partial}{\partial x_i}$$

$\vdots$

$$\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$$

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

tensor 2^{do} rango antisimétrico !!

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F'^{\alpha\beta} = \underbrace{\frac{\partial x'^\alpha}{\partial x^\gamma}}_{} \underbrace{\frac{\partial x'^\beta}{\partial x^\delta}}_{} F^{\gamma\delta} \longrightarrow F' = L F L^T$$

$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} \longrightarrow F_{\alpha\beta} \text{ "="} F^{\alpha\beta}(\vec{E} \to -\vec{E})$

Tensor dual $\mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$

$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{perm. pares de } (0123) \\ 0 & \text{se repite indice} \\ -1 & \text{perm impares} \end{cases}$

$\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$, pseudotensor

$\mathcal{F}^{\alpha\beta} \text{ "="} F^{\alpha\beta}\begin{pmatrix} E \to B \\ B \to -E \end{pmatrix} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$

caso especial de las trasf. dualidad.

$F^{\alpha\beta} F_{\alpha\beta} \propto E^2 - B^2$

$\mathcal{F}^{\alpha\beta} F_{\alpha\beta} \propto \vec{E} \cdot \vec{B}$

$$\left\{\begin{array}{l}\nabla\cdot E = 4\pi\rho \\ \nabla\times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}\end{array}\right. \longrightarrow \underbrace{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta}_{\text{(hacerlo)}}$$ 4 ecs para $\beta = 0,1,2,3$

$$\left\{\begin{array}{l}\nabla\cdot B = 0 \\ \nabla\times E + \frac{1}{c}\frac{\partial E}{\partial t} = 0\end{array}\right. \longrightarrow \underbrace{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0}_{\text{(hacerlo)}}$$, 4 ecs , $\beta = 0,1,2,3$

es equivalente $\underbrace{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}_{\text{4 ecuaciones}}$

(hacerlo)

Fza Lorentz : $\boxed{\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta}$, $\alpha = 0,1,2,3$

Macroscópico : $\begin{array}{l}E\to D \\ B\to H\end{array}$ $\quad G^{\alpha\beta}(D,H) = \text{"}F^{\alpha\beta}\left(\begin{array}{l}E\to D\\ B\to H\end{array}\right)\text{"}$ $\quad \underbrace{\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c}J^\beta}$, $\underbrace{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0}$

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$\vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$$

dir x

$E'_x = E_x \qquad B'_x = B_x$

$E'_y = \gamma\left(E_y - \beta E_z\right) \qquad B'_y = \gamma\left(B_y + \beta E_z\right)$

$E'_z = \gamma\left(E_z + \beta B_y\right) \qquad B'_z = \gamma\left(B_z - \beta E_y\right)$