Chapter 5. Oscillations

In this course, oscillations and waves are discussed in detail, because of their importance for fundamental and applied physics. This chapter starts with a discussion of the harmonic oscillator, whose differential equation of motion is linear and hence allows the full analytical solution, and then proceeds to so-called "nonlinear" and "parametric" systems whose dynamics may be only explored by either approximate analytical or numerical methods.

5.1. Free and forced oscillations

In Sec. 3.2 we briefly discussed oscillations in a keystone Hamiltonian system – a 1D *harmonic oscillator* described by a very simple Lagrangian[1]

$$L \equiv T(\dot{q}) - U(q) = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2, \tag{5.1}$$

whose Lagrange equation of motion,[2]

Harmonic oscillator: equation

$$m\ddot{q} + \kappa q = 0, \qquad \text{i.e. } \ddot{q} + \omega_0^2 q = 0, \qquad \text{with } \omega_0^2 \equiv \frac{\kappa}{m} \geq 0, \tag{5.2}$$

is a *linear homogeneous* differential equation. Its general solution is given by Eq. (3.16), which is frequently recast into another, amplitude-phase form:

Harmonic oscillator: motion

$$q(t) = u\cos\omega_0 t + v\sin\omega_0 t \equiv A\cos(\omega_0 t - \varphi), \tag{5.3a}$$

where A is the *amplitude* and φ is the *phase* of the oscillations, which are determined by the initial conditions. Mathematically, it is frequently easier to work with sinusoidal functions as complex exponents, by rewriting the last form of Eq. (3a) in one more form:[3]

$$q(t) = \mathrm{Re}\left[Ae^{-i(\omega_0 t - \varphi)}\right] = \mathrm{Re}\left[ae^{-i\omega_0 t}\right], \tag{5.3b}$$

Real and complex amplitudes

where a is the *complex amplitude* of the oscillations:

$$a \equiv Ae^{i\varphi}, \qquad |a| = A, \qquad \mathrm{Re}\,a = A\cos\varphi = u, \qquad \mathrm{Im}\,a = A\sin\varphi = v. \tag{5.4}$$

For an autonomous, Hamiltonian oscillator, Eqs. (3) give the full classical description of its dynamics. However, it is important to understand that this *free-oscillation* solution, with a constant amplitude A,

[1] For the notation brevity, in this chapter, I will drop indices "ef" in the energy components T and U, and in parameters like m, κ, etc. However, the reader should still remember that T and U do not necessarily coincide with the actual kinetic and potential energies (even if those energies may be uniquely identified) – see Sec. 3.1.

[2] ω_0 is usually called the *own frequency* of the oscillator. In quantum mechanics, the Germanized version of the same term, *eigenfrequency*, is used more. In this series, I will use either of the terms, depending on the context.

[3] Note that this is the so-called *physics convention*. Most engineering texts use the opposite sign in the imaginary exponent, $\exp\{-i\omega t\} \to \exp\{i\omega t\}$, with the corresponding sign implications for intermediate formulas but identical final results for real variables.

means the conservation of the energy $E \equiv T + U = \kappa A^2/2$ of the oscillator. If its energy changes for any reason, the description needs to be generalized.

First of all, if the energy leaks out of the oscillator to its environment (the effect usually called *energy dissipation*), the free oscillations decay with time. The simplest model of this effect is represented by an additional *linear drag* (or "kinematic friction") *force*, proportional to the generalized velocity and directed opposite to it:

$$F_v = -\eta \dot{q}\,, \tag{5.5}$$

where constant η is called the *drag coefficient*.[4] The inclusion of this force modifies the equation of motion (2) to become

$$m\ddot{q} + \eta\dot{q} + \kappa q = 0\,. \tag{5.6a}$$

This equation is frequently rewritten in the form

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0, \qquad \text{with } \delta \equiv \frac{\eta}{2m}, \tag{5.6b}$$

Free oscillator with damping

where the parameter δ is called the *damping coefficient* (or just "damping"). Note that Eq. (6) is still a linear homogeneous second-order differential equation, and its general solution still has the form of the sum (3.13) of two exponents of the type $\exp\{\lambda t\}$, with arbitrary pre-exponential coefficients. Plugging such an exponent into Eq. (6), we get the following algebraic characteristic equation for λ:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{5.7}$$

Solving this quadratic equation, we get

$$\lambda_\pm = -\delta \pm i\omega_0', \qquad \text{where } \omega_0' \equiv \left(\omega_0^2 - \delta^2\right)^{1/2}, \tag{5.8}$$

so for not very high damping ($\delta < \omega_0$) we get the following generalization of Eq. (3):[5]

$$q_{\text{free}}(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t} = \left(u_0 \cos\omega_0' t + v_0 \sin\omega_0' t\right) e^{-\delta t} = A_0 e^{-\delta t}\cos\left(\omega_0' t - \varphi_0\right). \tag{5.9}$$

The result shows that, besides a certain correction to the free oscillation frequency (which is very small in the most interesting *low damping limit*, $\delta << \omega_0$), the energy dissipation leads to an exponential decay of oscillation amplitude with the time constant $\tau = 1/\delta$:

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[4] Here Eq. (5) is treated as a phenomenological model, but in statistical mechanics, such dissipative term may be *derived* as an average force exerted upon a system by its environment, at very general assumptions. As will be discussed in detail later in this series (QM Chapter 7 and SM Chapter 5), due to the numerous degrees of freedom of a typical environment (think about the molecules of air surrounding a macroscopic pendulum), its force also has a random component; as a result, the *dissipation* is fundamentally related to *fluctuations*. The latter effect may be neglected (as it is in this course) only if the oscillator's energy E is much higher than the energy scale of its random fluctuations – in the thermal equilibrium at temperature T, the larger of $k_B T$ and $\hbar\omega_0/2$.

[5] Systems with high damping ($\delta > \omega_0$) can hardly be called oscillators, and though they are used in engineering and physical experiment (e.g., for shock and sound isolation), due to the lack of time/space, for their detailed discussion I have to refer the interested reader to special literature – see, e.g., C. Harris and A. Piersol, *Shock and Vibration Handbook*, 5th ed., McGraw Hill, 2002. Let me only note that according to Eq. (8), the dynamics of systems with very high damping ($\delta >> \omega_0$) has two very different time scales: a relatively short "momentum relaxation time" $1/|\lambda_-| \approx 1/2\delta = m/\eta$, and a much longer "coordinate relaxation time" $1/|\lambda_+| \approx 2\delta/\omega_0^2 = \eta/\kappa$.

Decaying free oscillations

$$A = A_0 e^{-t/\tau}, \quad \text{where } \tau \equiv \frac{1}{\delta} = \frac{2m}{\eta}. \tag{5.10}$$

A very popular dimensionless measure of damping is the so-called *quality factor* Q (or just the *Q-factor*) which is defined as $\omega_0/2\delta$, and may be rewritten in several other useful forms:

$$Q \equiv \frac{\omega_0}{2\delta} = \frac{m\omega_0}{\eta} = \frac{(m\kappa)^{1/2}}{\eta} = \pi\frac{\tau}{\mathcal{T}} = \frac{\omega_0\tau}{2}, \tag{5.11}$$

where $\mathcal{T} \equiv 2\pi/\omega_0$ is the oscillation period in the absence of damping – see Eq. (3.29). Since the oscillation energy E is proportional to A^2, i.e. decays as $\exp\{-2t/\tau\}$, i.e. with the time constant $\tau/2$, the last form of Eq. (11) may be used to rewrite the Q-factor in one more form:

$$Q = \omega_0 \frac{E}{(-\dot{E})} \equiv \omega_0 \frac{E}{\mathcal{P}}, \tag{5.12}$$

where $\mathcal{P}$ is the energy dissipation rate. (Other practical ways to measure Q will be discussed below.) The range of Q-factors of mechanical oscillators is very broad, from $Q \sim 10$ for a human leg (with relaxed muscles), to $Q \sim 10^4$ of the quartz crystals used in electronic clocks and watches, and all the way up to $Q > 10^{10}$ for nanoparticles suspended (by electrostatic forces) in high vacuum.

In contrast to the decaying free oscillations, *forced oscillations* induced by an external force $F(t)$, may maintain their amplitude (and hence energy) infinitely, even at non-zero damping. This process may be described using a still linear but now *inhomogeneous* differential equation

$$m\ddot{q} + \eta\dot{q} + \kappa q = F(t), \tag{5.13a}$$

or, more usually, the following generalization of Eq. (6b):

Forced oscillator with damping

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = f(t), \quad \text{where } f(t) \equiv F(t)/m. \tag{5.13b}$$

For a mechanical linear, dissipative 1D oscillator (6), under the effect of an additional external force $F(t)$, Eq. (13a) is just an expression of the 2nd Newton law. However, according to Eq. (1.41), Eq. (13) is valid for any dissipative, *linear*[6] 1D system whose Gibbs potential energy (1.39) has the form $U_G(q, t) = \kappa q^2/2 - F(t)q$.

The forced-oscillation solutions to Eq. (13) may be analyzed by two mathematically equivalent methods whose relative convenience depends on the character of function $f(t)$.

(i) *Frequency domain.* Representing the function $f(t)$ as a Fourier sum of sinusoidal harmonics:[7]

$$f(t) = \sum_{\omega} f_\omega e^{-i\omega t}, \tag{5.14}$$

and using the linearity of Eq. (13), we may represent its general solution as a sum of the decaying free oscillations (9) with the frequency ω_0', that are independent of the function $f(t)$, and forced oscillations due to each of the Fourier components of the force:[8]

[6] This is a very unfortunate, but common jargon, meaning "the system described by linear equations of motion".

[7] Here, in contrast to Eq. (3b), we may drop the operator Re, assuming that $f_{-\omega} = f_\omega^*$, so that the imaginary components of the sum compensate for each other.

$$q(t) = q_{\text{free}}(t) + q_{\text{forced}}(t), \quad q_{\text{forced}}(t) = \sum_{\omega} a_{\omega} e^{-i\omega t}. \tag{5.15}$$

General solution of Eq. (13)

Plugging Eq. (15) into Eq. (13), and requiring the factors before each $e^{-i\omega t}$ on both sides to be equal, we get

$$a_{\omega} = f_{\omega}\chi(\omega), \tag{5.16}$$

where the complex function $\chi(\omega)$, in our particular case equal to

$$\chi(\omega) = \frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta}, \tag{5.17}$$

is called either the *response function* or (especially for non-mechanical oscillators) the *generalized susceptibility*. From here, and Eq. (4), the amplitude of the oscillations under the effect of a sinusoidal force is

$$A_{\omega} \equiv |a_{\omega}| = |f_{\omega}||\chi(\omega)|, \qquad \text{with } |\chi(\omega)| = \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega\delta)^2\right]^{1/2}}. \tag{5.18}$$

Forced oscillations

This formula describes, in particular, an increase of the oscillation amplitude A_{ω} at $\omega \to \omega_0$ – see the left panel of Fig. 1. In particular, at the exact equality of these two frequencies,

$$|\chi(\omega)|_{\omega=\omega_0} = \frac{1}{2\omega_0\delta}, \tag{5.19}$$

so, according to Eq. (11), the ratio of the response magnitudes at $\omega = \omega_0$ and $\omega = 0$ ($|\chi(\omega)|_{\omega=0} = 1/\omega_0^2$) is exactly equal to the Q-factor of the oscillator. Thus, the response increase is especially strong in the low-damping limit ($\delta << \omega_0$, i.e. $Q >> 1$); moreover, at $Q \to \infty$ and $\omega \to \omega_0$, the response diverges. (This mathematical fact is very useful for the methods to be discussed later in this section.) This is the classical description of the famous phenomenon of *resonance*, so ubiquitous in physics.

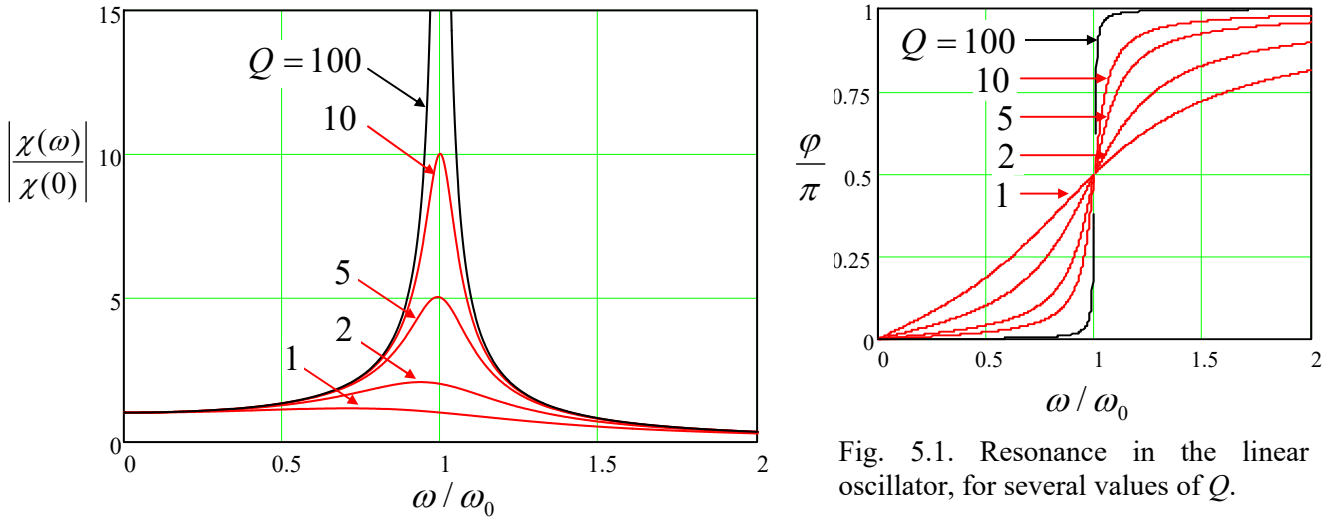

Fig. 5.1. Resonance in the linear oscillator, for several values of Q.

[8] In physics, this mathematical property of linear equations is frequently called the *linear superposition principle*.

Due to the increase of the resonance peak height, its width is inversely proportional to Q. Quantitatively, in the most interesting low-damping limit, i.e. at $Q >> 1$, the reciprocal Q-factor gives the normalized value of the so-called *full-width at half-maximum* (FWHM) of the resonance curve:[9]

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}. \tag{5.20}$$

Indeed, this $\Delta\omega$ is defined as the difference $(\omega_+ - \omega_-)$ between such two values of ω at that the modulus squared of the oscillator response function, $|\chi(\omega)|^2$ (which is proportional to the oscillation energy), equals a half of its resonance value (19). In the low damping limit, these points are very close to ω_0, so in the linear approximation in $|\omega - \omega_0| << \omega_0$, we may write $(\omega_0^2 - \omega^2) \equiv -(\omega+\omega_0)(\omega - \omega_0) \approx -2\omega\xi \approx -2\omega_0\xi$, where

$$\xi \equiv \omega - \omega_0 \tag{5.21}$$

is a convenient parameter called *detuning*, which will be repeatedly used later in this chapter, and beyond it. In this approximation, the second of Eqs. (18) is reduced to[10]

$$|\chi(\omega)|^2 = \frac{1}{4\omega_0^2\left(\delta^2 + \xi^2\right)}. \tag{5.22}$$

As a result, the points $\omega_\pm$ correspond to $\xi^2 = \delta^2$, i.e. $\omega_\pm = \omega_0 \pm \delta = \omega_0(1 \pm 1/2Q)$, so $\Delta\omega \equiv \omega_+ - \omega_- = \omega_0/Q$, thus proving Eq. (20).

(ii) *Time domain.* Returning to an arbitrary external force $f(t)$, one may argue that Eqs. (9), (15)-(17) provide a full solution of the forced oscillation problem even in this general case. This is formally correct, but this solution may be very inconvenient if the external force is far from a sinusoidal function of time, especially if it is not periodic at all. In this case, we should first calculate the complex amplitudes f_ω participating in the Fourier sum (14). In the general case of a non-periodic $f(t)$, this is actually the Fourier integral,[11]

$$f(t) = \int_{-\infty}^{+\infty} f_\omega e^{-i\omega t}\, dt, \tag{5.23}$$

so f_ω should be calculated using the reciprocal Fourier transform,

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t')e^{i\omega t'}\, dt'. \tag{5.24}$$

Now we may use Eq. (16) for each Fourier component of the resulting forced oscillations, and rewrite the last of Eqs. (15) as

[9] Note that the phase shift $\varphi \equiv \arg[\chi(\omega)]$ between the oscillations and the external force (see the right panel in Fig. 1) makes its steepest change, by $\pi/2$, within the same frequency interval $\Delta\omega$.

[10] Such function of frequency may be met in many branches of science, frequently under special names, including the "Cauchy distribution", "the Lorentz function" (or "Lorentzian line", or "Lorentzian distribution"), "the Breit-Wigner function" (or "the Breit-Wigner distribution"), etc.

[11] Let me hope that the reader knows that Eq. (23) may be used for periodic functions as well; in such a case, f_ω is a set of equidistant delta functions. (A reminder of the basic properties of the Dirac δ-function may be found, for example, in MA Sec. 14.)

$$q_{\text{forced}}(t) = \int_{-\infty}^{+\infty} a_\omega e^{-i\omega t} d\omega = \int_{-\infty}^{+\infty} \chi(\omega) f_\omega e^{-i\omega t} d\omega = \int_{-\infty}^{+\infty} d\omega\, \chi(\omega) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt'\, f(t') e^{i\omega(t'-t)}$$
$$\equiv \int_{-\infty}^{+\infty} dt'\, f(t') \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, \chi(\omega) e^{i\omega(t'-t)} \right], \tag{5.25}$$

with the response function $\chi(\omega)$ given, in our case, by Eq. (17). Besides requiring two integrations, Eq. (25) is conceptually uncomforting: it seems to indicate that the oscillator's coordinate at time t depends not only on the external force exerted at earlier times $t' < t$, but also at future times. This would contradict one of the most fundamental principles of physics (and indeed, science as a whole), *causality*: no effect may precede its cause.

Fortunately, a straightforward calculation (left for the reader's exercise) shows that the response function (17) satisfies the following rule:[12]

$$\int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega = 0, \qquad \text{for } \tau < 0. \tag{5.26}$$

This fact allows the last form of Eq. (25) to be rewritten in either of the following equivalent forms:

$$q_{\text{forced}}(t) = \int_{-\infty}^{t} f(t') G(t-t') dt' \equiv \int_{0}^{\infty} f(t-\tau) G(\tau) d\tau, \tag{5.27}$$

Linear system's response

where $G(\tau)$, defined as the Fourier transform of the response function,

$$G(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega, \tag{5.28}$$

Temporal Green's function

is called the (*temporal*) *Green's function* of the system. According to Eq. (26), $G(\tau) = 0$ for all $\tau < 0$.

While the second form of Eq. (27) is frequently more convenient for calculations, its first form is more suitable for physical interpretation of the Green's function. Indeed, let us consider the particular case when the force is a delta function

$$f(t) = \delta(t-t'), \qquad \text{with } t' < t, \ \text{ i.e. } \tau \equiv t - t' > 0, \tag{5.29}$$

representing an ultimately short pulse at the moment t', with unit "area" $\int f(t)dt$. Substituting Eq. (29a) into Eq. (27),[13] we get

$$q(t) = G(t-t'). \tag{5.30}$$

Thus Green's function $G(t - t')$ is just the oscillator's response, as measured at time t, to a short force pulse of unit "area", exerted at time t'. Hence Eq. (27) expresses the linear superposition principle in the time domain: the full effect of the force $f(t)$ on a linear system is a sum of the effects of short pulses of duration dt' and magnitude $f(t')$, each with its own "weight" $G(t - t')$ – see Fig. 2.

[12] Eq. (26) is true for any linear physical system in which $f(t)$ represents a cause, and $q(t)$ its effect. Following tradition, I discuss the frequency-domain expression of this causality relation (called the *Kramers-Kronig relations*) in the *Classical Electrodynamics* part of this lecture series – see EM Sec. 7.2.

[13] Technically, for this integration, t' in Eq. (27) should be temporarily replaced with another letter, say t''.

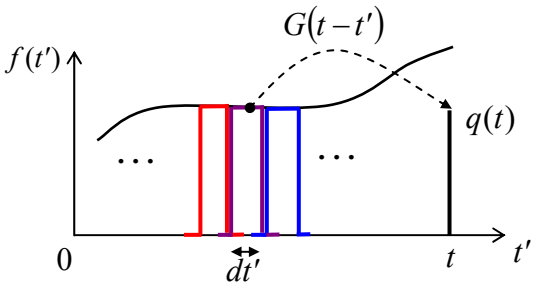

Fig. 5.2. A schematic, finite-interval representation of a force $f(t)$ as a sum of short pulses at all times $t' < t$, and their contributions to the linear system's response $q(t)$, as given by Eq. (27).

This picture may be used for the calculation of Green's function for our particular system. Indeed, Eqs. (29)-(30) mean that $G(\tau)$ is just the solution of the differential equation of motion of the system, in our case, Eq. (13), with the replacement $t \to \tau$, and a δ-functional right-hand side:

$$\frac{d^2G(\tau)}{d\tau^2} + 2\delta\frac{dG(\tau)}{d\tau} + \omega_0^2 G(\tau) = \delta(\tau). \tag{5.31}$$

Since Eqs. (27) describes only the second term in Eq. (15), i.e. only the forced, rather than free oscillations, we have to exclude the latter by solving Eq. (31) with zero initial conditions:

$$G(-0) = \frac{dG}{d\tau}(-0) = 0, \tag{5.32}$$

where $\tau = -0$ means the instant immediately preceding $\tau = 0$.

This problem may be simplified even further. Let us integrate both sides of Eq. (31) over an infinitesimal interval including the origin, e.g. $[-d\tau/2, +d\tau/2]$, and then follow the limit $d\tau \to 0$. Since Green's function has to be continuous because of its physical sense as the (generalized) coordinate, all terms on the left-hand side but the first one vanish, while the first term yields $dG/d\tau\big|_{+0} - dG/d\tau\big|_{-0}$. Due to the second of Eqs. (32), the last of these two derivatives has to equal zero, while the right-hand side of Eq. (31) yields 1 upon the integration. Thus, the function $G(\tau)$ may be calculated for $\tau > 0$ (i.e. for all times when it is different from zero) by solving the *homogeneous* version of the system's equation of motion for $\tau > 0$, with the following special initial conditions:

$$G(0) = 0, \qquad \frac{dG}{d\tau}(0) = 1. \tag{5.33}$$

This approach gives us a convenient way for the calculation of Green's functions of linear systems. In particular for the oscillator with not very high damping ($\delta < \omega_0$, i.e. $Q > ½$), imposing the boundary conditions (33) on the homogeneous equation's solution (9), we immediately get

Oscillator's Green's function

$$G(\tau) = \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau. \tag{5.34}$$

(The same result may be obtained directly from Eq. (28) with the response function $\chi(\omega)$ given by Eq. (19). This way is, however, a little bit more cumbersome, and is left for the reader's exercise.)

Relations (27) and (34) provide a very convenient recipe for solving many forced oscillations problems. As a very simple example, let us calculate the transient process in an oscillator under the effect of a constant force being turned on at $t = 0$, i.e. proportional to the Heaviside step function:

$$f(t) = f_0\theta(t) \equiv \begin{cases} 0, & \text{for } t < 0, \\ f_0, & \text{for } t > 0, \end{cases} \tag{5.35}$$

provided that at $t < 0$ the oscillator was at rest, so in Eq. (15), $q_{\text{free}}(t) \equiv 0$. Then the second form of Eq. (27), together with Eq. (34), yield

$$q(t) = \int_0^\infty f(t-\tau)\,G(\tau)\,d\tau = f_0 \int_0^t \frac{1}{\omega_0'} e^{-\delta\tau} \sin\omega_0'\tau \; d\tau. \tag{5.36}$$

The simplest way to work out such integrals is to represent the sine function under it as the imaginary part of $\exp\{i\omega_0't\}$, and merge the two exponents, getting

$$q(t) = f_0 \frac{1}{\omega_0'} \mathrm{Im}\left[\frac{1}{-\delta + i\omega_0'} e^{-\delta\tau + i\omega_0'\tau}\right]_0^t = \frac{F_0}{k}\left[1 - e^{-\delta t}\left(\cos\omega_0't + \frac{\delta}{\omega_0'}\sin\omega_0't\right)\right]. \tag{5.37}$$

This result, plotted in Fig. 3, is rather natural: it describes nothing more than the transient from the initial position $q = 0$ to the new equilibrium position $q_0 = f_0/\omega_0^2 = F_0/\kappa$, accompanied by decaying oscillations. For this particular simple function $f(t)$, the same result might be also obtained by introducing a new variable $\tilde{q}(t) \equiv q(t) - q_0$ and solving the resulting *homogeneous* equation for $\tilde{q}$ (with appropriate initial condition $\tilde{q}(0) = -q_0$). However, for more complicated functions $f(t)$, Green's function approach is irreplaceable.

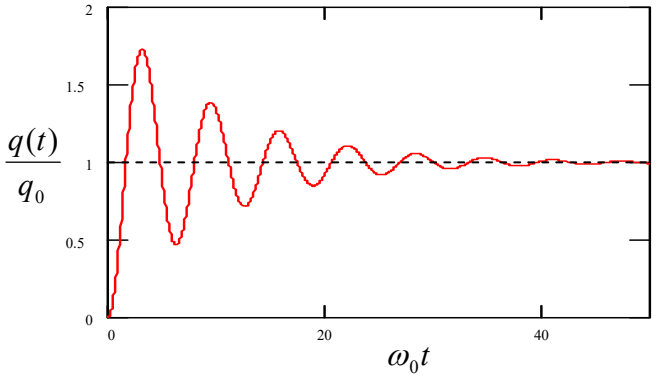

Fig. 5.3. The transient process in a linear oscillator, induced by a step-like force $f(t)$, for the particular case $\delta/\omega_0 = 0.1$ (i.e., $Q = 5$).

Note that for any particular linear system, its Green's function should be calculated only once, and then may be repeatedly used in Eq. (27) to calculate the system response to various external forces – either analytically or numerically. This property makes Green's function approach very popular in many other fields of physics – with the corresponding generalization or re-definition of the function.[14]

5.2. Weakly nonlinear oscillations

In comparison with systems discussed in the last section, which are described by linear differential equations with constant coefficients and thus allow a complete and exact analytical solution, oscillations in nonlinear systems (very unfortunately but commonly called *nonlinear oscillations*) present a complex and, generally, analytically intractable problem. However, much insight into possible

[14] See, e.g., Sec. 6.6, and also EM Sec. 2.7 and QM Sec. 2.2.

processes in such systems may be gained from a discussion of an important case of *weakly nonlinear* systems, which may be explored analytically. An important example of such systems is given by an *anharmonic oscillator* – a 1D system whose higher terms in the potential's expansion (3.10) cannot be neglected, but are small and may be accounted for approximately. If, in addition, damping is low (or negligible), and the external harmonic force exerted on the system is not too large, the equation of motion is a slightly modified version of Eq. (13):

Weakly nonlinear oscillator

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}, ...), \tag{5.38}$$

where $\omega \approx \omega_0$ is the anticipated frequency of oscillations (whose choice may be to a certain extent arbitrary – see below), and the right-hand side f is small (say, scales as some small dimensionless parameter $\varepsilon << 1$), and may be considered as a small *perturbation*.

Since at $\varepsilon = 0$, this equation has the sinusoidal solution given by Eq. (3), one might naïvely think that at a nonzero but small ε, the approximate solution to Eq. (38) should be sought in the form

Perturbative solution

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} + ..., \quad \text{where } q^{(n)} \propto \varepsilon^n, \tag{5.39}$$

with $q^{(0)} = A\cos(\omega_0 t - \varphi) \propto \varepsilon^0$. This is a good example of apparently impeccable mathematical reasoning that would lead to a very inefficient procedure. Indeed, let us apply it to the problem we already know the exact solution for, namely free oscillations in a linear but damped oscillator, for this occasion assuming the damping to be very low, $\delta/\omega_0 \sim \varepsilon << 1$. The corresponding equation of motion, Eq. (6), may be represented in form (38) if we take $\omega = \omega_0$ and

$$f = -2\delta\dot{q}, \qquad \text{with } \delta \propto \varepsilon. \tag{5.40}$$

The naïve perturbation theory based on Eq. (39) would allow us to find *small* corrections, of the order of δ, to the free, non-decaying oscillations $A\cos(\omega_0 t - \varphi)$. However, we already know from Eq. (9) that the main effect of damping is a gradual decrease of the free oscillation amplitude to zero, i.e. a very *large* change of the amplitude, though at low damping, $\delta << \omega_0$, this decay takes large time $t \sim \tau >> 1/\omega_0$. Hence, if we want our approximate method to be productive (i.e. to work at all time scales, in particular for forced oscillations with stationary amplitude and phase), we need to account for the fact that even a *small* right-hand side of Eq. (38) may eventually lead to *large* changes of oscillation's amplitude A (and sometimes, as we will see below, also of oscillation's phase φ) at large times, because of the *slowly accumulating* effects of the small perturbation.[15]

This goal may be achieved[16] by the account of these slow changes already in the "0th approximation", i.e. the basic part of the solution in the expansion (39):

[15] The same flexible approach is necessary for approximations used in quantum mechanics. The method discussed here is closer in spirit (though not completely identical) to the *WKB approximation* (see, e.g., QM Sec. 2.4) rather than most perturbative approaches (QM Ch. 6).

[16] This approach has a long history and, unfortunately, does not have a commonly accepted name. It had been gradually developed in celestial mechanics, but its application to 1D systems (on which I am focusing) was clearly spelled out only in 1926 by Balthasar van der Pol. So, I will follow several authors who call it the *van der Pol method*. Note, however, that in optics and quantum mechanics, this method is commonly called the *Rotating Wave Approximation* (RWA). In math-oriented texts, this approach, and especially its extensions to higher approximations, is usually called either the *small parameter method* or the *asymptotic method*. The list of other

$$q^{(0)} = A(t)\cos[\omega t - \varphi(t)], \qquad \text{with } \dot{A}, \dot{\varphi} \to 0 \text{ at } \varepsilon \to 0. \tag{5.41}$$

0th approximation

(It is evident that Eq. (9) is a particular case of this form.) Let me discuss this approach using a simple but representative example of a dissipative (but high-Q) pendulum driven by a weak sinusoidal external force with a nearly resonant frequency:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \sin q = f_0 \cos \omega t, \tag{5.42}$$

with $|\omega - \omega_0|$, $\delta << \omega_0$, and the force amplitude f_0 so small that $|q| << 1$ at all times. From what we know about the forced oscillations from Sec. 1, in this case, it is natural to identify ω on the left-hand side of Eq. (38) with the force's frequency. Expanding $\sin q$ into the Taylor series in small q, keeping only the first two terms of this expansion, and moving all small terms to the right-hand side, we can rewrite Eq. (42) in the following popular form (38):[17]

$$\ddot{q} + \omega^2 q = -2\delta\dot{q} + 2\xi\omega q + \alpha q^3 + f_0 \cos \omega t \equiv f(t, q, \dot{q}). \tag{5.43}$$

Duffing equation

Here $\alpha = \omega_0^2/6$ in the case of the pendulum (though the calculations below will be valid for any α), and the second term on the right-hand side was obtained using the approximation already employed in Sec. 1: $(\omega^2 - \omega_0^2)\, q \approx 2\omega(\omega - \omega_0)\, q = 2\omega\xi q$, where $\xi \equiv \omega - \omega_0$ is the detuning parameter that was already used earlier – see Eq. (21).

Now, following the general recipe expressed by Eqs. (39) and (41), in the 1st approximation in $f \propto \varepsilon$ we may look for the solution to Eq. (43) in the following form:

$$q(t) = A\cos\Psi + q^{(1)}(t), \quad \text{where } \Psi \equiv \omega t - \varphi, \qquad q^{(1)} \sim \varepsilon. \tag{5.44}$$

Let us plug this solution into both parts of Eq. (43), keeping only the terms of the first order in ε. Thanks to our (smart :-) choice of ω on the left-hand side of that equation, the two zero-order terms in that part cancel each other. Moreover, since each term on the right-hand side of Eq. (43) is already of the order of ε, we may drop $q^{(1)} \propto \varepsilon$ from the substitution into that part at all, because this would give us only terms $O(\varepsilon^2)$ or higher. As a result, we get the following approximate equation:

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f^{(0)}(t) \equiv -2\delta\frac{d}{dt}(A\cos\Psi) + 2\xi\omega(A\cos\Psi) + \alpha(A\cos\Psi)^3 + f_0\cos\omega t. \tag{5.45}$$

According to Eq. (41), generally, A and φ should be considered (slow) functions of time. However, let us leave the analyses of the transient process and system's stability until the next section, and use Eq. (45) to find stationary oscillations in the system, that are established after an initial transient process. For that limited task, we may take A = const, φ = const, so $q^{(0)}$ represents sinusoidal oscillations of frequency ω. Sorting the terms on the right-hand side according to their time dependence,[18] we see that it has terms with frequencies ω and 3ω:

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scientists credited for the development of this method, its variations, and extensions includes, most notably, N. Krylov, N. Bogolyubov, and Yu. Mitroplolsky.

[17] This equation is frequently called the *Duffing equation* (or the equation of the *Duffing oscillator*), after Georg Duffing who carried out its first (rather incomplete) analysis in 1918.

[18] Using the second of Eqs. (44), $\cos\omega t$ may be rewritten as $\cos(\Psi + \varphi) \equiv \cos\Psi\cos\varphi - \sin\Psi\sin\varphi$. Then using the identity given, for example, by MA Eq. (3.4): $\cos^3\Psi = (3/4)\cos\Psi + (1/4)\cos 3\Psi$, we get Eq. (46).

$$f^{(0)} = \left(2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi\right)\cos\Psi + \left(2\delta\omega A - f_0\sin\varphi\right)\sin\Psi + \frac{1}{4}\alpha A^3\cos 3\Psi. \qquad (5.46)$$

Now comes the main punch of the van der Pol approach: mathematically, Eq. (45) may be viewed as the equation of oscillations in a linear, *dissipation-free* harmonic oscillator of own frequency ω (not ω_0!) under the action of an external force $f^{(0)}(t)$. In our particular case, this force is given by Eq. (46) and has three terms: two "quadrature" components with that very frequency ω, and the third one with frequency 3ω. As we know from our analysis of this problem in Sec. 1, if any of the first two components is not equal to zero, $q^{(1)}$ grows to infinity – see Eq. (19) with $\delta = 0$. At the same time, by the very structure of the van der Pol approximation, $q^{(1)}$ has to be finite – moreover, small! The only way to avoid these infinitely growing (so-called *secular*) terms is to require that the amplitudes of both quadrature components of $f^{(0)}$ with frequency ω are equal to zero:

$$2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi = 0, \qquad 2\delta\omega A - f_0\sin\varphi = 0. \qquad (5.47)$$

These two *harmonic balance equations* enable us to find both parameters of the forced oscillations: their amplitude A and phase φ. The phase may be readily eliminated from this system (most easily, by expressing $\sin\varphi$ and $\cos\varphi$ from Eqs. (47) and then requiring the sum $\sin^2\varphi + \cos^2\varphi$ to equal 1), and the solution for A recast in the following implicit but convenient form:

$$A^2 = \frac{f_0^2}{4\omega^2}\frac{1}{\xi^2(A) + \delta^2}, \qquad \text{where } \xi(A) \equiv \xi + \frac{3}{8}\frac{\alpha A^2}{\omega} = \omega - \left(\omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}\right). \qquad (5.48)$$

This expression differs from Eq. (22) for the linear resonance in the low-damping limit only by the replacement of the detuning ξ with its effective amplitude-dependent value $\xi(A)$ – or, equivalently, the replacement of the frequency ω_0 of the oscillator with its effective, amplitude-dependent value

$$\omega_0(A) = \omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}. \qquad (5.49)$$

The physical meaning of $\omega_0(A)$ is simple: this is just the frequency of free oscillations of amplitude A in a similar nonlinear system but with zero damping.[19] Indeed, for $\delta = 0$ and $f_0 = 0$, we could repeat our calculations, assuming that ω is an amplitude-dependent eigenfrequency $\omega_0(A)$. Then the second of Eqs. (47) is trivially satisfied, while the second of them gives Eq. (49). The implicit relation (48) enables us to draw the curves of this *nonlinear resonance* just by bending the linear resonance plots (Fig. 1) according to the so-called *skeleton curve* expressed by Eq. (49). Figure 4 shows the result of this procedure. Note that at small amplitude, $\omega(A) \to \omega_0$, i.e. we return to the usual, "linear" resonance (22).

To bring our solution to its logical completion, we should still find the first perturbation $q^{(1)}(t)$ from what is left of Eq. (45). Since the structure of this equation is identical to Eq. (13) with zero damping and the force of frequency 3ω, we may use Eqs. (16)-(17) to obtain

[19] The effect of the pendulum's frequency dependence on its oscillation amplitude was observed as early as 1673 by Christiaan Huygens – who by the way had invented the pendulum clock, increasing the timekeeping accuracy by about three orders of magnitude. (He also discovered the largest of Saturn's moons, Titan).

$$q^{(1)}(t) = -\frac{1}{32\omega^2}\alpha A^3 \cos 3(\omega t - \varphi). \qquad (5.50)$$

Adding this perturbation (note the negative sign!) to the sinusoidal oscillation (41), we see that as the amplitude A of oscillations in a system with $\alpha > 0$ (e.g., a pendulum) grows, their waveform becomes a bit more "blunt" near the largest deviations from the equilibrium.

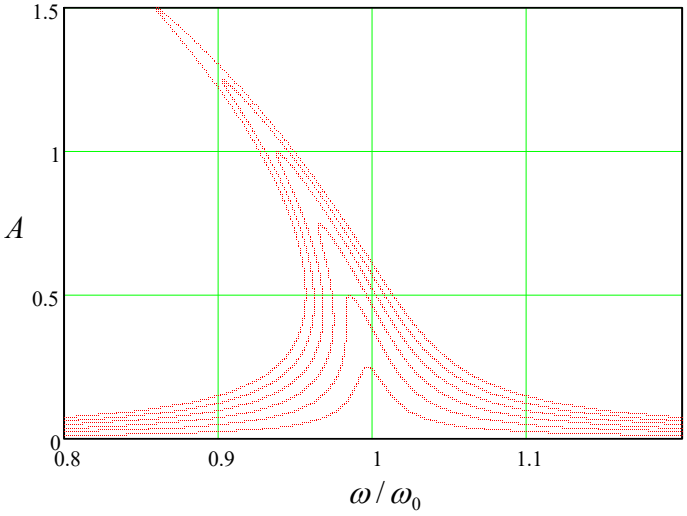


Fig. 5.4. The nonlinear resonance in the Duffing oscillator, as described by Eq. (48), for the particular case $\alpha = \omega_0^2/6$, $\delta/\omega = 0.01$ (i.e. $Q = 50$), and several values of the parameter f_0/ω_0^2, increased by equal steps of 0.005 from 0 to 0.03.

The same Eq. (50) also enables an estimate of the range of validity of our first approximation: since it has been based on the assumption $|q^{(1)}| << |q^{(0)}| \le A$, for this particular problem we have to require $\alpha A^2/32\omega^2 << 1$. For a pendulum (i.e. for $\alpha = \omega_0^2/6$), this condition becomes $A^2 << 192$. Though numerical coefficients in such strong inequalities should be taken with a grain of salt, the large magnitude of this particular coefficient gives a good hint that the method may give very accurate results even for relatively large oscillations with $A \sim 1$. In Sec. 7 below, we will see that this is indeed the case.

From the mathematical viewpoint, the next step would be to write the next approximation as

$$q(t) = A\cos\Psi + q^{(1)}(t) + q^{(2)}(t), \qquad q^{(2)} \sim \varepsilon^2, \qquad (5.51)$$

and plug it into the Duffing equation (43), which (thanks to our special choice of $q^{(0)}$ and $q^{(1)}$) would retain only the sum $\ddot{q}^{(2)} + \omega^2 q^{(2)}$ on its left-hand side. Again, requiring the amplitudes of two quadrature components of the frequency ω on the right-hand side to vanish, we may get second-order corrections to the values of A and φ. Then we may use the remaining part of the equation to calculate $q^{(2)}$, and then go after the third-order terms, etc. [20] However, for most purposes, the sum $q^{(0)} + q^{(1)}$, and sometimes even the crudest approximation of $q^{(0)}$ alone, are completely sufficient. For example, according to Eq. (50), in the case of a simple pendulum swinging as much as between the opposite horizontal positions ($A = \pi/2$), the 1st order correction $q^{(1)}$ is of the order of 0.5%. (Soon beyond this value, completely new dynamic phenomena start – see Sec. 7 below – but they cannot be described by these successive approximations

[20] For a mathematically rigorous treatment of higher approximations, see, e.g., Yu. Mitropolsky and N. Dao, *Applied Asymptotic Methods in Nonlinear Oscillations*, Springer, 2004. A more layman (and, by today's standards, somewhat verbose) discussion of various oscillatory phenomena may be found in the classical text A. Andronov, A. Vitt, and S. Khaikin, *Theory of Oscillators*, first published in the 1960s and still available online as Dover's republication in 2011.

at all.) Due to such reasons, for the analysis of particular systems, higher approximations are rarely pursued.

5.3. Reduced equations

A much more important issue is the stability of the solutions described by Eq. (48). Indeed, Fig. 4 shows that within a certain range of parameters, these equations give three different values for the oscillation amplitude (and phase), and it is important to understand which of them are stable. Since these solutions are not the fixed points in the sense discussed in Sec. 3.2 (each point in Fig. 4 represents a nearly-sinusoidal oscillation), their stability analysis needs a more general approach that would be valid for oscillations with amplitude and phase slowly evolving in time. This approach will also enable the analysis of non-stationary (especially the initial transient) processes, which are of importance for some dynamic systems.

First of all, let us formalize the way the harmonic balance equations, such as Eqs. (47), should be obtained for the general case (38) – rather than for the particular Eq. (43) considered in the last section. After plugging in the 0^{th} approximation (41) into the right-hand side of equation (38) we have to require the amplitudes of both quadrature components of frequency ω to vanish. From the standard Fourier analysis, we know that these requirements may be represented as

Harmonic balance equations

$$\boxed{\overline{f^{(0)}\sin\Psi}=0,\quad \overline{f^{(0)}\cos\Psi}=0,} \tag{5.52}$$

where the top bar means the time averaging – in our current case, over the period $2\pi/\omega$ of the right-hand side of Eq. (52), with the arguments calculated in the 0^{th} approximation:

$$f^{(0)}\equiv f(t,q^{(0)},\dot{q}^{(0)},...)\equiv f\left(t,A\cos\Psi,-A\omega\sin\Psi,...\right),\qquad \text{with } \Psi=\omega t-\varphi\,. \tag{5.53}$$

Now, for a transient process the contribution of $q^{(0)}$ to the left-hand side of Eq. (38) is not zero any longer, because its amplitude and phase may be both slow functions of time – see Eq. (41). Let us calculate this contribution. The exact result would be

$$\begin{aligned}\ddot{q}^{(0)}+\omega^2 q^{(0)}&\equiv\left(\frac{d^2}{dt^2}+\omega^2\right)A\cos(\omega t-\varphi)\\&=\left(\ddot{A}+2\dot{\varphi}\omega A-\dot{\varphi}^2A\right)\cos(\omega t-\varphi)-2\dot{A}(\omega-\dot{\varphi})\sin(\omega t-\varphi).\end{aligned} \tag{5.54}$$

However, in the first approximation in ε, we may neglect the second derivative of A, and also the squares and products of the first derivatives of A and φ (which are all of the second order in ε), so Eq. (54) is reduced to

$$\ddot{q}^{(0)}+\omega^2 q^{(0)}\approx 2A\dot{\varphi}\omega\cos(\omega t-\varphi)-2\dot{A}\omega\sin(\omega t-\varphi)\,. \tag{5.55}$$

On the right-hand side of Eq. (53), we can neglect the time derivatives of the amplitude and phase at all, because this part is already proportional to the small parameter. Hence, in the first order in ε, Eq. (38) becomes

$$\ddot{q}^{(1)}+\omega^2 q^{(1)}=f_{\text{ef}}^{(0)}\equiv f^{(0)}-\left(2A\dot{\varphi}\omega\cos\Psi-2\dot{A}\omega\sin\Psi\right). \tag{5.56}$$

Now, applying Eqs. (52) to the function $f_{ef}^{(0)}$, and taking into account that the time averages of $\sin^2\Psi$ and $\cos^2\Psi$ are both equal to ½, while the time average of the product $\sin\Psi\cos\Psi$ vanishes, we get a pair of so-called *reduced equations* (alternatively called either "truncated", or "RWA", or "van der Pol" equations) for the time evolution of the amplitude and phase:

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi}, \qquad \dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi}. \tag{5.57a}$$

Reduced (RWA) equations

Extending the definition (4) of the complex amplitude of oscillations to their slow evolution in time, $a(t) \equiv A(t)\exp\{i\varphi(t)\}$, and differentiating this relation, the two equations (57a) may be also rewritten in the form of either one equation for a:

$$\dot{a} = \frac{i}{\omega}\overline{f^{(0)}e^{i(\Psi+\varphi)}} \equiv \frac{i}{\omega}\overline{f^{(0)}e^{i\omega t}}, \tag{5.57b}$$

Reduced equations: alternative forms

or two equations for the real and imaginary parts of $a(t) = u(t) + iv(t)$:

$$\dot{u} = -\frac{1}{\omega}\overline{f^{(0)}\sin\omega t}, \qquad \dot{v} = \frac{1}{\omega}\overline{f^{(0)}\cos\omega t}. \tag{5.57c}$$

The first-order harmonic balance equations (52) are evidently just the particular case of the reduced equations (57) for stationary oscillations ($\dot{A} = \dot{\varphi} = 0$).[21]

Superficially, the system (57a) of two coupled, first-order differential equations may look more complex than the initial, second-order differential equation (38), but actually, it is usually much simpler. For example, let us spell them out for the easy case of free oscillations a linear oscillator with damping. For that, we may reuse the ready Eq. (46) by taking $\alpha = f_0 = 0$, and thus turning Eqs. (57a) into

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi} \equiv -\frac{1}{\omega}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\sin\Psi} \equiv -\delta A, \tag{5.58a}$$

$$\dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi} \equiv \frac{1}{\omega A}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\cos\Psi} \equiv \xi. \tag{5.58b}$$

The solution of Eq. (58a) gives us the same "envelope" law $A(t) = A(0)e^{-\delta t}$ as the exact solution (10) of the initial differential equation, while the elementary integration of Eq. (58b) yields $\varphi(t) = \xi t + \varphi(0) \equiv \omega t - \omega_0 t + \varphi(0)$. This means that our approximate solution,

$$q^{(0)}(t) = A(t)\cos[\omega t - \varphi(t)] = A(0)e^{-\delta t}\cos[\omega_0 t - \varphi(0)], \tag{5.59}$$

agrees with the exact Eq. (9), and misses only the correction (8) of the oscillation frequency. (This correction is of the second order in δ, i.e. of the order of ε^2, and hence is beyond the accuracy of our first approximation.) It is remarkable how nicely do the reduced equations recover the proper frequency of free oscillations in this autonomous system – in which the very notion of ω is ambiguous.

[21] One may ask why we cannot stick to just one, most compact, complex–amplitude form (57b) of the reduced equations. The main reason is that when the function $f(q,\dot{q},t)$ is nonlinear, we cannot replace its real arguments, such as $q = A\cos(\omega t - \varphi)$, with their complex-function representations like $a\exp\{-i\omega t\}$ (as could be done in the linear problems considered in Sec. 5.1), and need to use real variables, such as either $\{A, \varphi\}$ or $\{u, v\}$, anyway.

The result is different at forced oscillations. For example, for the (generally, nonlinear) Duffing oscillator described by Eq. (43) with $f_0 \neq 0$, Eqs. (57a) yield the reduced equations,

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \qquad A\dot{\varphi} = \xi(A)\,A + \frac{f_0}{2\omega}\cos\varphi, \tag{5.60}$$

which are valid for an arbitrary function $\xi(A)$, provided that this nonlinear detuning remains much smaller than the oscillation frequency. Here (after a transient), the amplitude and phase tend to the stationary states described by Eqs. (47). This means that φ becomes a constant, so $q^{(0)} \to A\cos(\omega t -$ const), i.e. the reduced equations again automatically recover the correct frequency of the solution, in this case, equal to the external force frequency.

Note that each stationary oscillation regime, with certain amplitude and phase, corresponds to a fixed point of the reduced equations, so the stability of those fixed points determines that of the oscillations. In the next three sections, we will carry out such analyses for several simple systems of key importance for physics and engineering.

5.4. Self-oscillations and phase locking

B. van der Pol's motivation for developing his method was the analysis of one more oscillatory motion type: the so-called *self-oscillations*. Several systems, e.g., electronic rf amplifiers with positive feedback and optical media with quantum-level population inversion, provide convenient means for the compensation and even over-compensation of the intrinsic energy losses in oscillators. Phenomenologically, this effect may be described as the change of sign of the coefficient δ from positive to negative. Since for small oscillations, the equation of motion is still linear, we may use Eq. (9) to describe its general solution. This equation shows that at $\delta < 0$, even infinitesimal deviations from equilibrium (say, due to unavoidable fluctuations) lead to oscillations with exponentially growing amplitude. Of course, in any real system such growth cannot persist infinitely, and shall be limited by this or that effect – e.g., in the above examples, respectively, by the amplifier's saturation and the quantum level population's exhaustion.

In many cases, the amplitude limitation may be described reasonably well by making the following replacement:

$$2\delta\dot{q} \to 2\delta\dot{q} + \beta\dot{q}^3, \tag{5.61}$$

with $\beta > 0$. Let us analyze the effects of such *nonlinear damping*, applying the van der Pol's approach[22] to the corresponding differential equation:

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2 q = 0. \tag{5.62}$$

Carrying out the dissipative and detuning terms to the right-hand side, and taking them for f in the canonical Eq. (38), we can easily calculate the right-hand sides of the reduced equations (57a), getting[23]

$$\dot{A} = -\delta(A)\,A, \qquad \text{where } \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2, \tag{5.63a}$$

[22] In his original work, B. van der Pol considered a very similar equation (frequently called the van *der Pol oscillator*) that differs from Eq. (62) only by the nonlinear term: $\dot{q}^3 \to q^2\dot{q}$, and has very similar properties.

[23] For that, one needs to use the trigonometric identity $\sin^3\Psi = (3/4)\sin\Psi - (1/4)\sin 3\Psi$ – see, e.g., MA Eq. (3.4).

$$A\dot{\varphi} = \xi A\,. \tag{5.63b}$$

The last of these equations has exactly the same form as Eq. (58b) for the case of decaying oscillations and hence shows that the self-oscillations (if they happen, i.e. if $A \neq 0$) have the own frequency ω_0 of the oscillator – cf. Eq. (59). However, Eq. (63a) is more substantive. If the initial damping δ is positive, it has only the trivial fixed point, $A_0 = 0$ (that describes the oscillator at rest), but if δ is negative, there is also another fixed point,

$$A_1 = \frac{2}{\sqrt{3}} q_0, \quad \text{where } q_0 \equiv \left(\frac{2|\delta|}{\beta\omega^2} \right)^{1/2}, \qquad \text{for } \delta < 0\,, \tag{5.64}$$

which describes steady self-oscillations with a non-zero amplitude A_1.

To understand which of these points is stable, let us apply the general approach discussed in Sec. 3.2, the linearization of equations of motion, to Eq. (63a). For the trivial fixed point $A_0 = 0$, its linearization is reduced to discarding the nonlinear term in the definition of the amplitude-dependent damping $\delta(A)$. The resulting linear equation evidently shows that the system's equilibrium point, $A = A_0 = 0$, is stable at $\delta > 0$ and unstable at $\delta < 0$. (This *self-excitation condition* was already discussed above.) On the other hand, the linearization of near the non-trivial fixed point A_1 requires a bit more math: in the first order in $\tilde{A} \equiv A - A_1 \to 0$, we get

$$\dot{\tilde{A}} \equiv \dot{A} = -\delta(A_1 + \tilde{A}) - \frac{3}{8}\beta\omega^2 (A_1 + \tilde{A})^3 \approx -\delta\,\tilde{A} - \frac{3}{8}\beta\omega^2 3A_1^2 \tilde{A} = \left(-\delta + 3\delta\right)\tilde{A} = 2\delta\tilde{A}\,, \tag{5.65}$$

where Eq. (64) has been used to eliminate A_1. We see that the fixed point A_1 (and hence the self-oscillation process) is stable as soon as it exists ($\delta < 0$) – similarly to the situation in our "testbed problem" (Fig. 2.1), besides that in our current, dissipative system, the stability is "actual" rather than "orbital" – see Sec. 6 for more on this issue.

Now let us consider another important problem: the effect of an external oscillating force on a self-excited oscillator. If the force is sufficiently small, its effects on the self-excitation condition and the oscillation amplitude are negligible. However, if the frequency ω of such a weak force is close to the own frequency ω_0 of the oscillator, it may lead to *phase locking*[24] – also called "synchronization", though the latter term also has a much broader meaning. At this effect, the oscillation frequency deviates from ω_0, and becomes exactly equal to the external force's frequency ω, within a certain range

$$-\Delta \le \omega - \omega_0 < +\Delta\,. \tag{5.66}$$

To prove this fact, and also to calculate the phase-locking range width 2Δ, we may repeat the calculation of the right-hand sides of the reduced equations (57a), adding the term $f_0 \cos\omega t$ to the right-hand side of Eq. (62) – cf. Eqs. (42)-(43). This addition modifies Eqs. (63) as follows:[25]

$$\dot{A} = -\delta(A)\,A + \frac{f_0}{2\omega}\sin\varphi, \tag{5.67a}$$

$$A\dot{\varphi} = \xi A + \frac{f_0}{2\omega}\cos\varphi. \tag{5.67b}$$

[24] Apparently, the phase locking was first noticed by the same C. Huygens for pendulum clocks.

[25] Actually, this result should be evident, even without calculations, from the comparison of Eqs. (60) and (63).

If the system is self-excited, and the external force is weak, its effect on the oscillation amplitude is small, and in the first approximation in f_0 we can take A to be constant and equal to the value A_1 given by Eq. (64). Plugging this approximation into Eq. (67b), we get a very simple equation[26]

Phase locking equation

$$\dot{\varphi} = \xi + \Delta\cos\varphi, \tag{5.68}$$

where in our current case

$$\Delta \equiv \frac{f_0}{2\omega A_1}. \tag{5.69}$$

Within the range $-|\Delta| < \xi < +|\Delta|$, Eq. (68) has two fixed points on each 2π-segment of the variable φ:

$$\varphi_\pm = \pm\cos^{-1}\left(-\frac{\xi}{\Delta}\right) + 2\pi n. \tag{5.70}$$

It is easy to linearize Eq. (68) near each point to analyze their stability in our usual way; however, let me use this case to demonstrate another convenient way to do this in 1D systems, using the *phase plane* $[\varphi, \dot{\varphi}]$ – see Fig. 5, where the red line shows the right-hand side of Eq. (68).

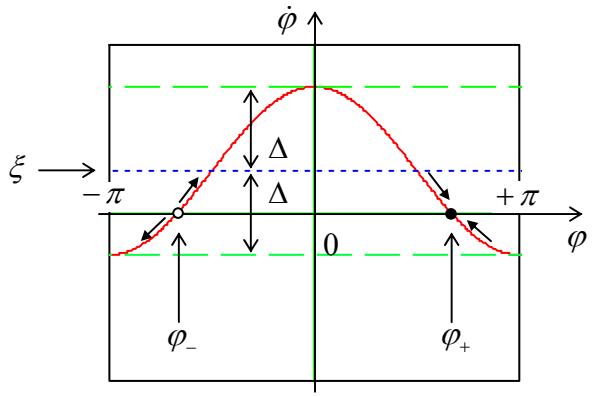


Fig. 5.5. The phase plane of a phase-locked oscillator, for the particular case $\xi = \Delta/2$, $f_0 > 0$.

Since according to Eq. (68), positive values of the plotted function correspond to the growth of phase φ in time and vice versa, we may draw the arrows showing the direction of phase evolution. From this graphics, it is clear that one of these fixed points (for $f_0 > 0$, φ_+) is stable, while its counterpart (in this case, φ_-) is unstable. Hence the magnitude of Δ given by Eq. (69) is indeed the phase-locking range (or rather its half) that we wanted to find. Note that the range is proportional to the phase-locking signal's amplitude – perhaps the most important quantitative feature of this effect.

To complete our simple analysis, based on the assumption of fixed oscillation amplitude, we need to find the condition of its validity. For that, we may linearize Eq. (67a), for the stationary case, near the value A_1, just as we have done in Eq. (65) for the transient process. The stationary result,

$$\tilde{A} \equiv A - A_1 = \frac{1}{2|\delta|}\frac{f_0}{2\omega}\sin\varphi_\pm \approx A_1\left|\frac{\Delta}{2\delta}\right|\sin\varphi_\pm, \tag{5.71}$$

shows that our assumption, $|\tilde{A}| << A_1$, and hence the final result (69), are valid if the calculated phase-locking range 2Δ is much smaller than $4|\delta|$.

[26] This equation is ubiquitous in phase-locking system descriptions, including even some digital electronic circuits used for that purpose – at the proper re-definition of the phase difference φ.

5.5. Parametric excitation

In both problems solved in the last section, the stability analysis was easy because it could be carried out for just one slow variable, *either* amplitude *or* phase. More generally, such an analysis of the reduced equations involves both of these variables. A classical example of such a situation is provided by one important physical effect – the *parametric excitation* of oscillations. A simple example of such excitation is given by a pendulum with a variable parameter, for example, the suspension length $l(t)$ – see Fig. 6. Experiments[27] and numerical simulations show that if the length is changed periodically (*modulated*) with some frequency 2ω that is close to $2\omega_0$, and a sufficiently large depth Δl, the equilibrium position of the pendulum becomes unstable, and it starts oscillating with frequency ω equal *exactly* to the half of the modulation frequency – and hence only *approximately* equal to the average frequency ω_0 of the oscillator.

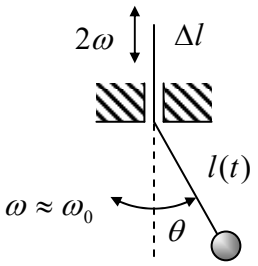


Fig. 5.6. Parametric excitation of a pendulum.

For an elementary analysis of this effect, we may consider the simplest case when the oscillations are small. At the lowest point ($\theta = 0$), where the pendulum moves with the highest velocity v_{max}, the suspension string's tension $\mathcal{T}$ is *higher* than mg by the centripetal force: $\mathcal{T}_{max} = mg + mv_{max}^2/l$. On the contrary, at the maximum deviation of the pendulum from the equilibrium, the force is *lower* than mg, because of the string's tilt: $\mathcal{T}_{min} = mg\cos\theta_{max}$. Using the energy conservation, $E = mv_{max}^2/2 = mgl(1 - \cos\theta_{max})$, we may express these values as $\mathcal{T}_{max} = mg + 2E/l$ and $\mathcal{T}_{min} = mg - E/l$. Now, if during each oscillation period, the string is pulled up slightly by Δl (with $|\Delta l| << l$) at each of its two passages through the lowest point, and is let to go down by the same amount at each of two points of the maximum deviation, the net work of the external force per period is positive:

$$\mathcal{W} \approx 2(\mathcal{T}_{max} - \mathcal{T}_{min})\Delta l \approx 6\frac{\Delta l}{l}E, \tag{5.72}$$

and hence increases the oscillator's energy. If the parameter modulation depth Δl is sufficient, this increase may overcompensate the energy drained out by damping during the same period. Quantitatively, Eq. (10) shows that low damping ($\delta << \omega_0$) leads to the following energy decrease,

$$\Delta E \approx -4\pi\frac{\delta}{\omega_0}E, \tag{5.73}$$

per oscillation period. Comparing Eqs. (72) and (73), we see that the net energy flow into the oscillations is positive, $\mathcal{W} + \Delta E > 0$, i.e. oscillation amplitude has to grow if[28]

[27] The simplest experiments of this kind may be done with the usual playground swings, where moving your body up and down moves the system's c.o.m. position, and hence the effective length l_{ef} of the support – see Eq. (4.41).

[28] Modulation of the pendulum's mass (say, by periodic pumping water in and out of a suspended bottle) gives a qualitatively similar result. Note, however, that parametric oscillations cannot be excited by modulating *every*

$$\frac{\Delta l}{l} > \frac{2\pi\delta}{3\omega_0} \equiv \frac{\pi}{3Q}. \tag{5.74}$$

Since this result is independent of the oscillation energy E, the growth of energy and amplitude is exponential (until E becomes so large that some of our assumptions fail), so Eq. (74) is the parametric excitation's condition – in this simple model.

However, this result does not account for a possible difference between the oscillation frequency ω and the eigenfrequency ω_0, and also does not clarify whether the best phase shift between the oscillations and parameter modulation, assumed in the above calculation, may be sustained automatically. To address these issues, we may apply the van der Pol approach to a simple but reasonable model:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(1 + \mu\cos 2\omega t)q = 0, \tag{5.75}$$

describing the parametric excitation in a linear oscillator with a sinusoidal modulation of the parameter $\omega_0^2(t)$. Rewriting this equation in the canonical form (38),

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}) \equiv -2\delta\dot{q} + 2\xi\omega q - \mu\omega_0^2 q\cos 2\omega t, \tag{5.76}$$

and assuming that the dimensionless ratios δ/ω and $|\xi|/\omega$, and the modulation depth μ are all much less than 1, we may use general Eqs. (57a) to get the following reduced equations:

$$\begin{aligned} \dot{A} &= -\delta A - \frac{\mu\omega}{4} A\sin 2\varphi, \\ A\dot{\varphi} &= A\xi - \frac{\mu\omega}{4} A\cos 2\varphi. \end{aligned} \tag{5.77}$$

These equations evidently have a fixed point, with $A_0 = 0$, but its stability analysis (though possible) is not absolutely straightforward, because the phase φ of oscillations is undetermined at that point. In order to avoid this (technical rather than conceptual) difficulty, we may use, instead of the real amplitude and phase of oscillations, either their complex amplitude $a = A\exp\{i\varphi\}$, or its components u and v – see Eqs. (4). Indeed, for our function f, Eq. (57b) gives

$$\dot{a} = (-\delta + i\xi)a - i\frac{\mu\omega}{4}a^*, \tag{5.78}$$

while Eqs. (57c) yield

$$\begin{aligned} \dot{u} &= -\delta u - \xi v - \frac{\mu\omega}{4} v, \\ \dot{v} &= -\delta v + \xi u - \frac{\mu\omega}{4} u. \end{aligned} \tag{5.79}$$

We see that in contrast to Eqs. (77), in the "Cartesian coordinates" $\{u, v\}$ the trivial fixed point $A_0 = 0$ (i.e. $u_0 = v_0 = 0$) is absolutely regular. Moreover, equations (78)-(79) are already linear, so they do not require any additional linearization. Thus we may use the same approach as was already used in Secs. 3.2 and 5.1, i.e. look for the solution of Eqs. (79) in the exponential form $\exp\{\lambda t\}$. However, now

oscillator's parameter – for example, the oscillator's damping coefficient (at least if it stays positive at all times), because this does not change the system's energy, just the energy drain rate.

we are dealing with two variables and should allow them to have, for each value of λ, a certain ratio u/v. For that, we may take the partial solution in the form

$$u = c_u e^{\lambda t}, \quad v = c_v e^{\lambda t}. \tag{5.80}$$

where the constants c_u and c_v are frequently called the *distribution coefficients*. Plugging this solution into Eqs. (79), we get from them the following system of two linear algebraic equations:

$$\begin{aligned} \left(-\delta - \lambda\right)c_u + \left(-\xi - \frac{\mu\omega}{4}\right)c_v = 0, \\ \left(+\xi - \frac{\mu\omega}{4}\right)c_u + \left(-\delta - \lambda\right)c_v = 0. \end{aligned} \tag{5.81}$$

The characteristic equation of this system, i.e. the condition of compatibility of Eqs. (81),

$$\begin{vmatrix} -\delta - \lambda & -\xi - \frac{\mu\omega}{4} \\ \xi - \frac{\mu\omega}{4} & -\delta - \lambda \end{vmatrix} \equiv \lambda^2 + 2\delta\lambda + \delta^2 + \xi^2 - \left(\frac{\mu\omega}{4}\right)^2 = 0, \tag{5.82}$$

has two roots:

$$\lambda_{\pm} = -\delta \pm \left[\left(\frac{\mu\omega}{4}\right)^2 - \xi^2\right]^{1/2}. \tag{5.83}$$

Requiring the fixed point to be unstable, $\mathrm{Re}\lambda_+ > 0$, we get the parametric excitation condition

$$\frac{\mu\omega}{4} > \left(\delta^2 + \xi^2\right)^{1/2}. \tag{5.84}$$

Thus the parametric excitation may indeed happen without any external phase control: the arising oscillations self-adjust their phase to pick up energy from the external source responsible for the periodic parameter variation.

Our key result (84) may be compared with two other calculations. First, in the case of negligible damping ($\delta = 0$), Eq. (84) turns into the condition $\mu\omega/4 > |\xi|$. This result may be compared with the well-developed theory of the so-called *Mathieu equation*, whose canonical form is

$$\frac{d^2y}{dv^2} + \left(a - 2b\cos 2v\right)y = 0. \tag{5.85}$$

With the substitutions $y \to q$, $v \to \omega t$, $a \to (\omega_0/\omega)^2$, and $b/a \to -\mu/2$, this equation is just a particular case of Eq. (75) for $\delta = 0$. In terms of Eq. (85), our result (84) may be rewritten just as $b > |a - 1|$, and is supposed to be valid for $b << 1$. The boundaries given by this condition are shown with dashed lines in Fig. 7 together with the numerically calculated[29] stability boundaries for the Mathieu equation. One can see that the van der Pol approximation works just fine within its applicability limit (and a bit beyond :-), though it fails to predict some other important features of the Mathieu equation, such as the existence of higher, more narrow regions of parametric excitation (at $a \approx n^2$, i.e. $\omega_0 \approx \omega/n$, for all integer n), and some

[29] Such calculations are substantially simplified by the use of the so-called *Floquet theorem*, which is also the mathematical basis for the discussion of wave propagation in periodic media – see the next chapter.

spill-over of the stability region into the lower half-plane $a < 0$.[30] The reason for these failures is the fact that, as can be seen in Fig. 7, these phenomena do not appear in the first approximation in the parameter modulation amplitude $\mu \propto b$, which is the realm of the reduced equations (79).

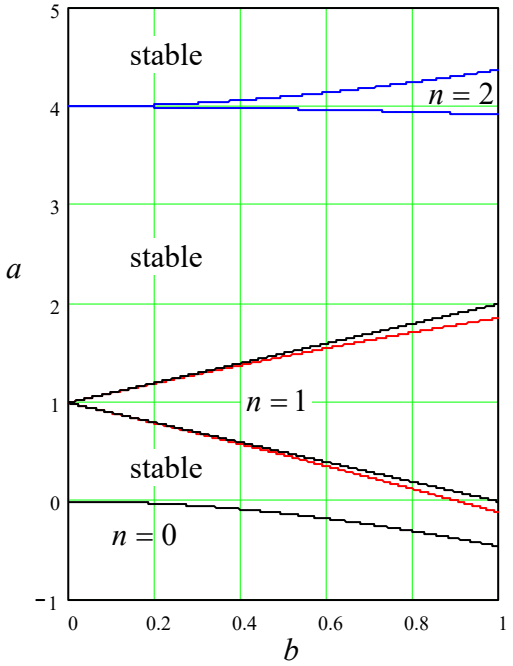


Fig. 5.7. Stability boundaries of the Mathieu equation (85), as calculated: numerically (solid curves) and using the reduced equations (79) (dashed straight lines). In the regions numbered by various n, the trivial solution $y = 0$ of the equation is unstable, i.e. its general solution $y(v)$ includes an exponentially growing term.

In the opposite case of non-zero damping but exact tuning ($\xi = 0$, $\omega \approx \omega_0$), Eq. (84) becomes

$$\mu > \frac{4\delta}{\omega_0} \equiv \frac{2}{Q}. \tag{5.86}$$

This condition may be compared with Eq. (74) by taking $\Delta l/l = 2\mu$. The comparison shows that while the structure of these conditions is similar, the numerical coefficients are different by a factor close to 2. The first reason for this difference is that the instant parameter change at optimal moments of time is more efficient than the smooth, sinusoidal variation described by (75). Even more significantly, the change of the pendulum's length modulates not only its frequency $\omega_0 \equiv (g/l)^{1/2}$ as Eq. (75) implies but also its *mechanical impedance* $Z \equiv (gl)^{1/2}$ – the notion to be discussed in detail in the next chapter. (The analysis of the general case of the simultaneous modulation of ω_0 and Z is left for the reader's exercise.)

To conclude this section, let me summarize the most important differences between the excitation of parametric and forced oscillations:

(i) Parametric oscillations completely disappear outside of their excitation range, while the forced oscillations have a non-zero amplitude for any frequency and amplitude of the external force – see Eq. (18).

(ii) While the parametric excitation may be described by linear equations such as Eq. (75), such equations cannot predict a finite oscillation amplitude within the excitation range, even at finite

[30] This region (for $b << 1$, $-b^2/2 < a < 0$) describes, in particular, the counter-intuitive stability of the so-called *Kapitza pendulum* – an inverted pendulum with the suspension point oscillated fast in the vertical direction – the effect first observed by Andrew Stephenson in 1908.

damping. In order to describe stationary parametric oscillations, some nonlinear effects have to be taken into account. (I am leaving analyses of such effects for the reader's exercise.)

One more important feature of parametric oscillations will be discussed in the next section.

5.6. Fixed point classification

The reduced equations (79) give us a good pretext for a brief discussion of an important general topic of dynamics: classification and stability of the fixed points of a system described by two time-independent, first-order differential equations with time-independent coefficients.[31] After their linearization near a fixed point, the equations for deviations can always be expressed in a form similar to Eq. (79):

$$\begin{aligned} \dot{\tilde{q}}_1 &= M_{11}\tilde{q}_1 + M_{12}\tilde{q}_2, \\ \dot{\tilde{q}}_2 &= M_{21}\tilde{q}_1 + M_{22}\tilde{q}_2, \end{aligned} \tag{5.87}$$

where $M_{jj'}$ (with $j,j' = 1, 2$) are some real scalars, which may be viewed as the elements of a 2×2 matrix M. Looking for an exponential solution of the type (80),

$$\tilde{q}_1 = c_1 e^{\lambda t}, \quad \tilde{q}_2 = c_2 e^{\lambda t}, \tag{5.88}$$

we get a general system of two linear equations for the distribution coefficients $c_{1,2}$:

$$\begin{aligned} (M_{11} - \lambda)c_1 + M_{12}c_2 &= 0, \\ M_{21}c_1 + (M_{22} - \lambda)c_2 &= 0. \end{aligned} \tag{5.89}$$

These equations are consistent if

$$\begin{vmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{vmatrix} = 0, \tag{5.90}$$

giving us a quadratic characteristic equation:

$$\lambda^2 - \lambda(M_{11} + M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0. \tag{5.91}$$

Its solution,[32]

$$\lambda_\pm = \frac{1}{2}(M_{11} + M_{22}) \pm \frac{1}{2}\left[(M_{11} - M_{22})^2 + 4M_{12}M_{21}\right]^{1/2}, \tag{5.92}$$

shows that the following situations are possible:

A. The expression under the square root, $(M_{11} - M_{22})^2 + 4M_{12}M_{21}$, is positive. In this case, both characteristic exponents $\lambda_\pm$ are real, and we can distinguish three sub-cases:

---

[31] Autonomous systems described by a single, second-order homogeneous differential equation, say $F(q, \dot{q}, \ddot{q}) = 0$, also belong to this class, because we may always treat the generalized velocity $\dot{q} \equiv v$ as a new variable, and use this definition as one first-order differential equation, while the initial equation, in the form $F(q, v, \dot{v}) = 0$, as the second first-order equation.

[32] In the language of linear algebra, $\lambda_\pm$ are the *eigenvalues*, and the corresponding sets of the distribution coefficients $[c_1, c_2]_\pm$ are the *eigenvectors* of the matrix M with elements $M_{jj'}$.

(i) Both λ_+ and λ_- are negative. As Eqs. (88) show, in this case the deviations $\tilde{q}$ tend to zero at $t \to \infty$, i.e. the fixed point is stable. Because of generally different magnitudes of the exponents $\lambda_\pm$, the process represented on the phase plane $[\tilde{q}_1, \tilde{q}_2]$ (see Fig. 8a, with the solid arrows, for an example) may be seen as consisting of two stages: first, a faster (with the rate $|\lambda_-| > |\lambda_+|$) relaxation to a linear *asymptote*,[33] and then a slower decline, with the rate $|\lambda_+|$, along this line, i.e. at a virtually fixed ratio of the variables. Such a fixed point is called the *stable node*.

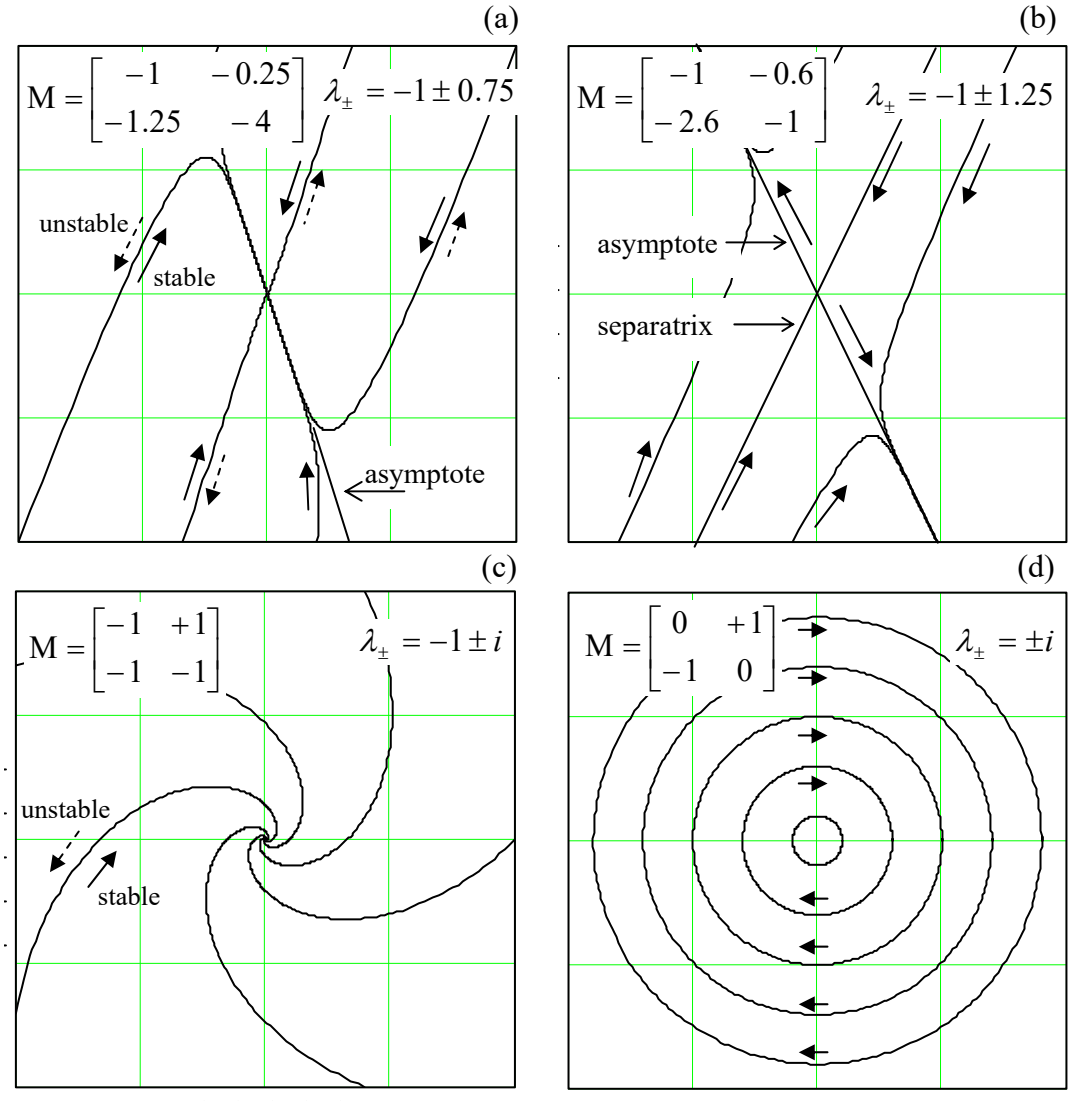


Fig. 5.8. Typical trajectories on the phase plane $[\tilde{q}_1, \tilde{q}_2]$ near fixed points of different types: (a) node, (b) saddle, (c) focus, and (d) center. The particular matrices M used for the first three panels correspond to Eqs. (81) for the parametric excitation, with $\xi = \delta$ and three different values of the ratio $\mu\omega/4\delta$: (a) 1.25, (b) 1.6, and (c) 0.

[33] The asymptote direction may be found by plugging the value λ_+ back into Eq. (89) and finding the corresponding ratio c_1/c_2. Note that the separation of the system's evolution into the two stages is conditional, being most vivid in the case of a large difference between the exponents λ_+ and λ_-.

(ii) Both λ_+ and λ_- are positive. This case of an *unstable node* differs from the previous one only by the direction of motion along the phase plane trajectories – see the dashed arrows in Fig. 8a. Here the variable ratio is also approaching a constant soon, now the one corresponding to $\lambda_+ > \lambda_-$.

(iii) Finally, in the case of a *saddle* ($\lambda_+ > 0$, $\lambda_- < 0$), the system's dynamics is different (Fig. 8b): after the rate-$|\lambda_-|$ relaxation to an asymptote, with the perturbation starts to grow, with the rate λ_+, along one of two opposite directions. (The direction is determined on which side of another straight line, called the *separatrix*, the system has been initially.) So the saddle[34] is an unstable fixed point.

B. The expression under the square root in Eq. (92), $(M_{11} - M_{22})^2 + 4M_{12}M_{21}$, is negative. In this case, the square root is imaginary, making the real parts of both roots equal, $\mathrm{Re}\lambda_\pm = (M_{11} + M_{22})/2$, and their imaginary parts equal but opposite. As a result, here there can be just two types of fixed points:

(i) *Stable focus*, at $(M_{11} + M_{22}) < 0$. The phase plane trajectories are spirals going to the origin (i.e. toward the fixed point) – see Fig. 8c with the solid arrow.

(ii) *Unstable focus*, taking place at $(M_{11} + M_{22}) > 0$, differs from the stable one only by the direction of motion along the phase trajectories – see the dashed arrow in the same Fig. 8c.

C. Frequently, the border case, $M_{11} + M_{22} = 0$, corresponding to the orbital ("indifferent") stability already discussed in Sec. 3.2, is also distinguished, and the corresponding fixed point is referred to as the *center* (Fig. 8d). Considering centers as a separate category makes sense because such fixed points are typical for Hamiltonian systems, whose first integral of motion may be frequently represented as the distance of the representing point from a certain center. For example, by introducing new variables $\tilde{q}_1 \equiv \tilde{q}$ and $\tilde{q}_2 \equiv m\dot{\tilde{q}}_1$, we may rewrite Eq. (3.12) of a harmonic oscillator without dissipation (again, with indices "ef" dropped for brevity), as a system of two first-order differential equations:

$$\dot{\tilde{q}}_1 = \frac{1}{m}\tilde{q}_2, \qquad \dot{\tilde{q}}_2 = -\kappa\tilde{q}_1, \tag{5.93}$$

i.e. as a particular case of Eq. (87), with $M_{11} = M_{22} = 0$, and $M_{12}M_{21} = -\kappa/m \equiv -\omega_0^2 < 0$, and hence $(M_{11} - M_{22})^2 + 4M_{12}M_{21} = -4\omega_0^2 < 0$, and $M_{11} + M_{22} = 0$. On the symmetrized phase plane $[\tilde{q}_1, \tilde{q}_2/Z]$, where the parameter $Z \equiv (\kappa m)^{1/2} \equiv m\omega_0$ is the oscillator's impedance, the sinusoidal oscillations of amplitude A are represented by a circle of radius A about the center-type fixed point $A = 0$. In the case when $\tilde{q}_1 \equiv \tilde{q}$ is the linear coordinate q of an actual mechanical oscillator, so $\tilde{q}_2 \equiv m\dot{\tilde{q}}_1$ is its linear momentum $p = m\dot{q}$, such a circular trajectory corresponds to the conservation of the oscillator's energy

$$E \equiv T + U \equiv \frac{p^2}{2m} + \frac{\kappa q^2}{2} \equiv \frac{\kappa}{2}\left[\tilde{q}_1^2 + \left(\frac{\tilde{q}_2}{Z}\right)^2\right] = \frac{\kappa A^2}{2} = \text{const}. \tag{5.94}$$

This is a convenient moment for a brief discussion of the so-called *Poincaré* (or "slow-variable", or "stroboscopic") *plane*.[35] From the point of view of the basic Eq. (41), the sinusoidal oscillations $q(t)$

[34] The term "saddle" is due to the fact that in this case, the system's dynamics is qualitatively similar to that of a heavily damped motion in a 2D potential $U(\tilde{q}_1, \tilde{q}_2)$ having the shape of a horse saddle (or a mountain pass).

[35] Named after Jules Henri Poincaré (1854-1912), who is credited, among many other achievements in physics and mathematics, for his contributions to special relativity (see, e.g., EM Chapter 9), and the basic idea of unstable trajectories responsible for the deterministic chaos – to be discussed in Chapter 9 of this course.

$= A\cos(\omega t - \varphi)$, described by a circular trajectory on the actual (symmetrized) phase plane, correspond to a fixed point $\{A, \varphi\}$, which may be conveniently represented by a stationary geometric point on the plane with these polar coordinates – see Fig. 9a. (As follows from Eq. (4), the point's Cartesian coordinates on that plane are just the variables $u \equiv A\cos\varphi$ and $v \equiv A\cos\varphi$ that were used, in particular, in the last section.) The quasi-sinusoidal process (41), with slowly changing A and φ, may be represented by slow motion of that point on this Poincaré plane.

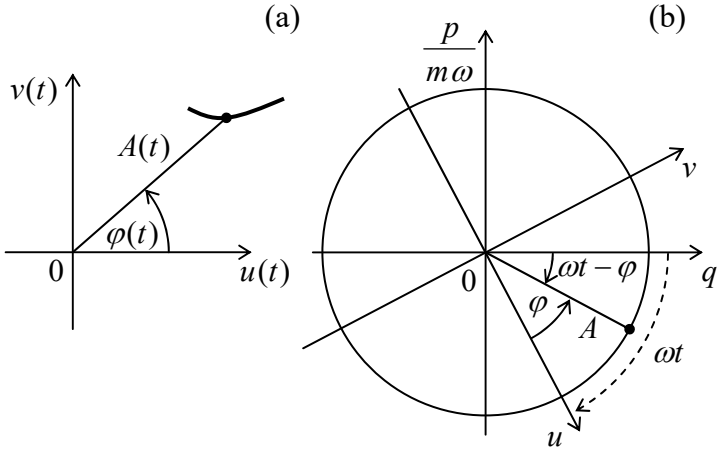


Fig. 5.9. (a) Representation of a sinusoidal oscillation (point) and a slow transient process (line) on the Poincaré plane, and (b) the relation between the usual ("fast") phase plane and the "slow" (Poincaré) plane.

Figure 9b shows a convenient way to visualize the relation between the actual phase plane of an oscillator, with the "fast" symmetrized coordinates q and $p/m\omega$, and the Poincaré plane with the "slow" coordinates u and v: the latter plane rotates relative to the former one, about the origin, clockwise, with the angular velocity ω.[36] Another, "stroboscopic" way to generate the Poincaré plane pattern is to have a fast glance at the "real" phase plane just once during the oscillation period $\mathcal{T} = 2\pi/\omega$.

In many cases, the representation on the Poincaré plane is more convenient than that on the "real" phase plane. In particular, we have already seen that the reduced equations for such important phenomena as phase locking and parametric oscillations, whose original differential equations include time explicitly, are time-independent – cf., e.g., Eqs. (75) and (79) describing the latter effect. This simplification brings the equations into the category considered earlier in this section and enables an easy classification of their fixed points, which may shed additional light on their dynamic properties.

In particular, Fig. 10 shows the classification of the only (trivial) fixed point $A_1 = 0$ on the Poincaré plane of the parametric oscillator, which follows from Eq. (83). As the parameter modulation depth μ is increased, the type of this fixed point changes from a stable focus (pertinent to a simple oscillator with damping) to a stable node and then to a saddle describing the parametric excitation. In the last case, the two directions of the perturbation growth, so prominently featured in Fig. 8b, correspond to the two possible values of the oscillation phase φ, with the phase choice determined by initial conditions.

This double degeneracy of the parametric oscillation's phase could already be noticed from Eqs. (77), because they are evidently invariant with respect to the replacement $\varphi \to \varphi + \pi$. Moreover, the degeneracy is not an artifact of the van der Pol approximation, because the initial equation (75) is already invariant with respect to the corresponding replacement $q(t) \to q(t - \pi/\omega)$. This invariance

[36] This notion of phase plane rotation is the origin of the term "Rotating Wave Approximation", mentioned above. (The word "wave" is an artifact of this method's wide application in classical and quantum optics.)

means that all other characteristics (including the amplitude) of the parametric oscillations excited with either of the two phases are *exactly* similar. At the dawn of the computer age (in the late 1950s and early 1960s), there were substantial attempts, especially in Japan, to use this property for storage and processing digital information coded in the binary-phase form. Though these attempts have not survived the competition with simpler approaches based on binary-voltage coding, some current trends in the development of prospective reversible and quantum computers may be traced back to that idea.

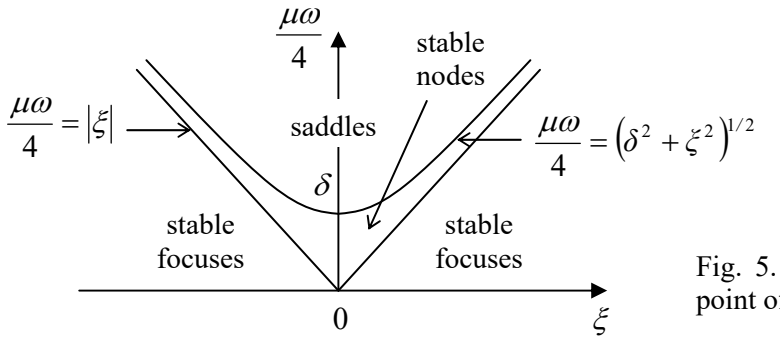


Fig. 5.10. Types of the trivial fixed point of a parametric oscillator.

5.7. Numerical approaches

If the amplitude of oscillations, for whatever reason, becomes so large that nonlinear terms in the equation describing an oscillator become comparable with its linear terms, numerical methods are virtually the only avenue available for their theoretical studies. In Hamiltonian 1D systems, such methods may be applied directly to Eq. (3.26), but dissipative and/or parametric systems typically lack such first integrals of motion, so the initial differential equation has to be solved.

Let us discuss the general idea of such methods on the example of what mathematicians call the *Cauchy problem* (finding the solution for all moments of time, starting from the known initial conditions) for the first-order differential equation

$$\dot{q} = f(t,q). \tag{5.95}$$

(The generalization to a system of several such equations is straightforward.) Breaking the time axis into small equal steps h (Fig. 11) we can reduce the equation integration problem to finding the function's value at the next time point, $q_{n+1} \equiv q(t_{n+1}) \equiv q(t_n + h)$ from the previously found value $q_n = q(t_n)$ – and, if necessary, the values of q at other previous time steps.

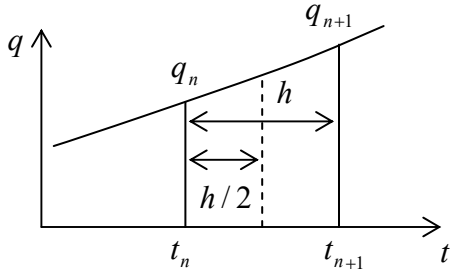


Fig. 5.11. The basic notions used at numerical integration of ordinary differential equations.

In the simplest approach (called the *Euler method*), q_{n+1} is found using the following formula:

$$\begin{aligned} q_{n+1} &= q_n + k, \\ k &\equiv h\, f(t_n, q_n). \end{aligned} \tag{5.96}$$

This approximation is equivalent to the replacement of the genuine function $q(t)$, on the segment $[t_n, t_{n+1}]$, with the two first terms of its Taylor expansion in point t_n:

$$q(t_n + h) \approx q(t_n) + \dot{q}(t_n)h \equiv q(t_n) + hf(t_n, q_n). \tag{5.97}$$

This approximation has an error proportional to h^2. One could argue that by making the step h sufficiently small, the Euler method's error might be made arbitrarily small, but even with all the number-crunching power of modern computer platforms, the CPU time necessary to reach sufficient accuracy may be too large for big problems.[37] Besides that, the increase of the number of time steps, which is necessary at $h \to 0$ at a fixed total time interval, increases the total rounding errors and eventually may cause an increase, rather than the reduction of the overall error of the computed result.

A more efficient way is to modify Eq. (96) to include the terms of the second order in h. There are several ways to do this, for example using the *2nd-order Runge-Kutta* method:

$$\begin{aligned} &q_{n+1} = q_n + k_2, \\ &k_2 \equiv h f\left(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}\right), \quad k_1 \equiv h f(t_n, q_n). \end{aligned} \tag{5.98}$$

One can readily check that this method gives the exact result if the function $q(t)$ is a quadratic polynomial, and hence in the general case its errors are of the order of h^3. We see that the main idea here is to first break the segment $[t_n, t_{n+1}]$ in half (see Fig. 11 again), evaluate the right-hand side of the differential equation (95) at the point intermediate (in both t and q) between the points number n and $(n + 1)$, and then use this information to evaluate q_{n+1}.

The advantage of the Runge-Kutta approach over other second-order methods is that it may be readily extended to the 4th order, without an additional breakup of the interval $[t_n, t_{n+1}]$:

$$\begin{aligned} &q_{n+1} = q_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ &k_4 \equiv h f(t_n + h, q_n + k_3), \ k_3 \equiv h f\left(t_n + \frac{h}{2}, q_n + \frac{k_2}{2}\right), \ k_2 \equiv h f\left(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}\right), \ k_1 \equiv h f(t_n, q_n). \end{aligned} \tag{5.99}$$

This method has a much lower error, $O(h^5)$, without being too cumbersome. These features have made the 4th-order Runge-Kutta the default method in most numerical libraries. Its extension to higher orders is possible, but requires more complex formulas, and is justified only for some special cases, e.g., very abrupt functions $q(t)$.[38] The most frequent enhancement of the method is an automatic adjustment of the step h to reach the pre-specified accuracy, but not make more calculations than necessary.

Figure 12 shows a typical example of an application of that method to the very simple problem of a damped linear oscillator, for two values of the fixed time step h – expressed in terms of the number (N) of such steps per oscillation period. The black straight lines connect the adjacent points obtained by

[37] In addition, the Euler method is not time-reversible. This handicap that may be essential for Hamiltonian systems described by systems of second-order differential equations. However, this drawback may be partly overcome by the so-called *leapfrogging* – the overlap of time steps h for a generalized coordinate and the corresponding generalized velocity.

[38] The most popular approaches in such cases are the *Richardson extrapolation*, the *Bulirsch-Stoer algorithm*, and a set of so-called *prediction-correction techniques*, e.g. the *Adams-Bashforth-Moulton method* – see the literature recommended in MA Sec. 16(iii).

the 4th-order Runge-Kutta method, while the points connected with the green straight lines represent the exact analytical solution (22). The plots show that a-few-percent errors start to appear only at as few as ~10 time steps per period, so the method is indeed very efficient.

Let me hope that the discussion in the next section will make the conveniences and the handicaps of the numerical approach to problems of nonlinear dynamics very clear.

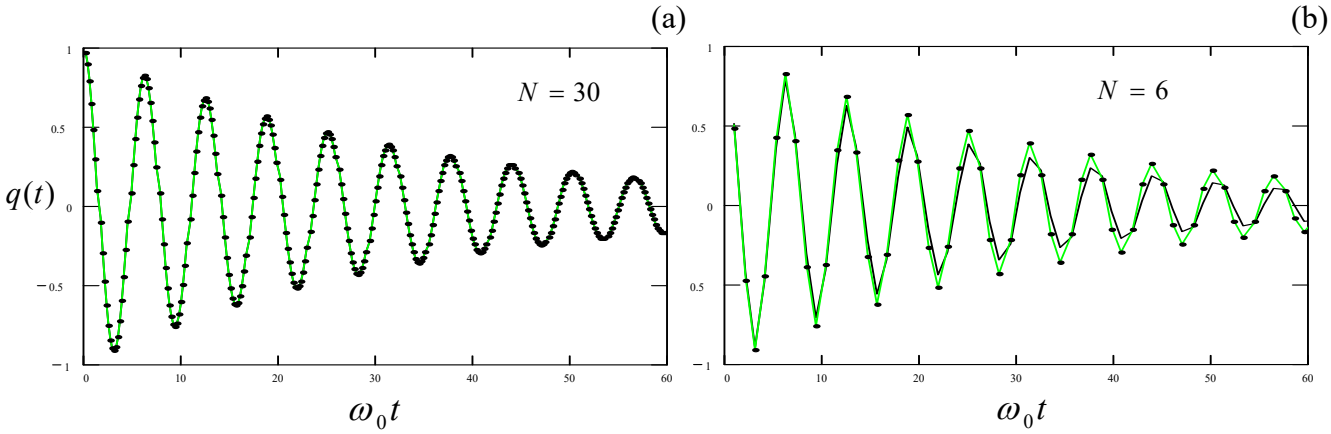


Fig. 5.12. Results of the Runge-Kutta solution of Eq. (6) (with $\delta/\omega_0 = 0.03$) for: (a) 30 and (b) 6 points per oscillation period. The results are shown by points; the black and green lines are only the guides for the eye.

5.8. Higher-harmonic and subharmonic oscillations

Figure 13 shows the numerically calculated[39] transient process and stationary oscillations in a linear oscillator and a very representative nonlinear system, the pendulum described by Eq. (42), both with the same ω_0. Both systems are driven by a sinusoidal external force of the same amplitude and frequency – in this illustration, equal to the own small-oscillation frequency ω_0 of both systems. The plots show that despite a very substantial amplitude of the pendulum oscillations (the angle amplitude of about one radian), their waveform remains almost exactly sinusoidal.[40] On the other hand, the nonlinearity affects the oscillation amplitude very substantially. These results imply that the corresponding reduced equations (60), which are based on the assumption (41), may work very well far beyond its formal restriction $|q| << 1$.

Still, the waveform of oscillations in a nonlinear system always differs from that of the applied force – in our case, from the sine function of frequency ω. This fact is frequently formulated as the generation, by the system, of *higher harmonics*. Indeed, the Fourier theorem tells us that any non-sinusoidal periodic function of time may be represented as a sum of its basic harmonic of frequency ω, plus higher harmonics with frequencies $n\omega$, with integer $n > 1$.

Note that an effective generation of higher harmonics is only possible with adequate nonlinearity of the system. For example, consider the nonlinear term αq^3 used in the equations explored in Secs. 2

[39] All numerical results shown in this section have been obtained by the 4th-order Runge-Kutta method with the automatic step adjustment that guarantees the relative error of the order of 10^{-4} – much smaller than the pixel size in the shown plots.

[40] In this particular case, the higher harmonic content is about 0.5%, dominated by the 3rd harmonic, whose amplitude and phase are in very good agreement with Eq. (50).

and 3. If the waveform $q(t)$ is sinusoidal, such term will have only the basic (1st) and the 3rd harmonics – see, e.g., Eq. (50). As another example, the "pendulum nonlinearity" sin$q$ cannot produce, without a time-independent component ("bias") in $q(t)$, any even harmonic, including the 2nd one. The most efficient generation of harmonics may be achieved using systems with the sharpest nonlinearities – e.g., semiconductor diodes whose current may follow an exponential dependence on the applied voltage through several orders of magnitude. [41]

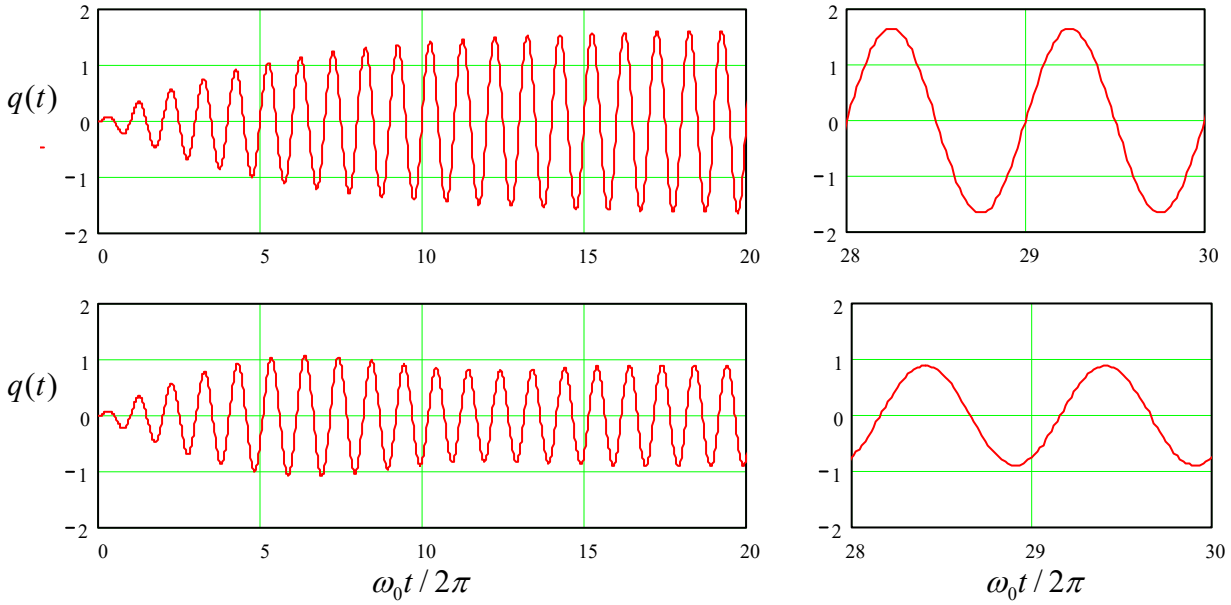


Fig. 5.13. The oscillations induced by a similar sinusoidal external force (turned on at $t = 0$) in two systems with the same small-oscillation frequency ω_0 and low damping: a linear oscillator (two top panels) and a pendulum (two bottom panels). In all cases, $\delta/\omega_0 = 0.03$, $f_0 = 0.1$, and $\omega = \omega_0$.

Another way to increase the contents of an n^{th} higher harmonic in a nonlinear oscillator is to reduce the excitation frequency ω to $\sim\omega_0/n$, so the oscillator resonated at the frequency $n\omega \approx \omega_0$ of the desired harmonic. For example, Fig. 14a shows the oscillations in a pendulum described by the same Eq. (42), but driven at frequency $\omega = \omega_0/3$. One can see that the 3rd harmonic amplitude may be comparable with that of the basic harmonic, especially if the external frequency is additionally lowered (Fig. 14b) to accommodate for the deviation of the effective frequency $\omega_0(A)$ of own oscillations from its small-oscillation value ω_0 – see Eq. (49), Fig. 4, and their discussion in Sec. 2 above.

However, numerical modeling of nonlinear oscillators, as well as experiments with their physical implementations, bring more surprises. For example, the bottom panels of Fig. 15 show oscillations in a pendulum under the effect of a strong sinusoidal force with a frequency ω close to $3\omega_0$. One can see that at some parameter values and initial conditions, the system's oscillation spectrum is heavily contributed (almost dominated) by the 3rd *sub*harmonic, i.e. the Fourier component of frequency $\omega/3 \approx \omega_0$.

This counter-intuitive phenomenon of such *subharmonic generation* may be explained as follows. Let us assume that subharmonic oscillations of frequency $\omega/3 \approx \omega_0$ have somehow appeared, and coexist with the forced oscillations of frequency 3ω:

[41] This method is used in practice, for example, for the generation of electromagnetic waves with frequencies in the terahertz range (10^{12}-10^{13} Hz), which is still in wait for efficient electronic self-oscillators.

$$q(t) \approx A\cos\Psi + A_{\text{sub}}\cos\Psi_{\text{sub}}, \qquad \text{where } \Psi \equiv \omega t - \varphi, \quad \Psi_{\text{sub}} \equiv \frac{\omega t}{3} - \varphi_{\text{sub}}. \tag{5.100}$$

Then the leading nonlinear term, αq^3, of the Taylor expansion of the pendulum's nonlinearity $\sin q$, is proportional to

$$\begin{aligned} q^3 &= (A\cos\Psi + A_{\text{sub}}\cos\Psi_{\text{sub}})^3 \\ &\equiv A^3\cos^3\Psi + 3A^2A_{\text{sub}}\cos^2\Psi\cos\Psi_{\text{sub}} + 3AA_{\text{sub}}^2\cos\Psi\cos^2\Psi_{\text{sub}} + A_{\text{sub}}^3\cos^3\Psi_{\text{sub}}. \end{aligned} \tag{5.101}$$

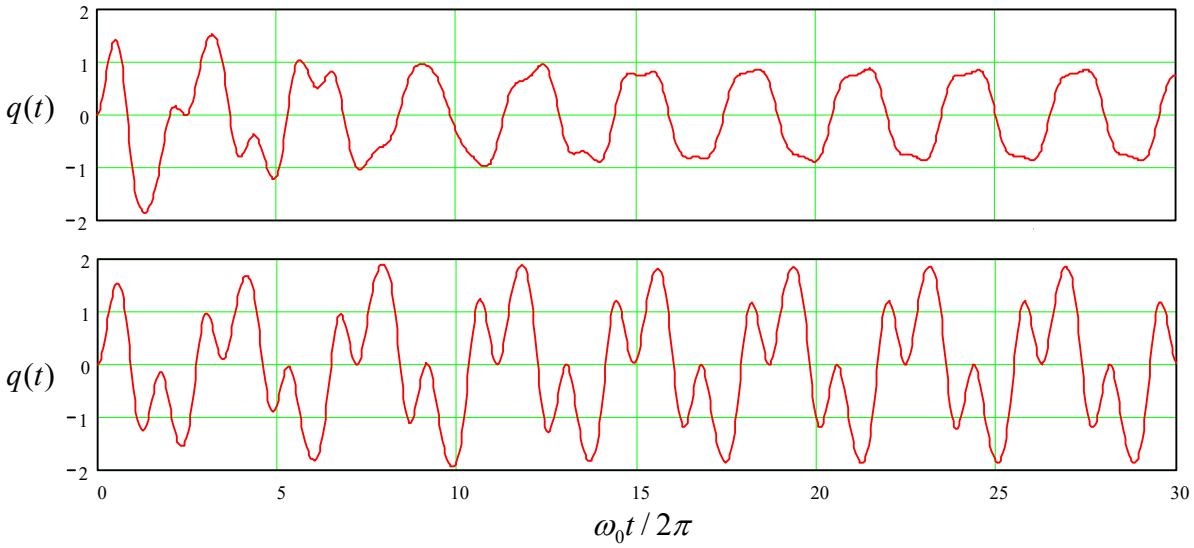


Fig. 5.14. The oscillations induced in a pendulum, with damping $\delta/\omega_0 = 0.03$, by a sinusoidal external force of amplitude $f_0 = 0.75$, and frequencies $\omega_0/3$ (top panel) and $0.8\times\omega_0/3$ (bottom panel).

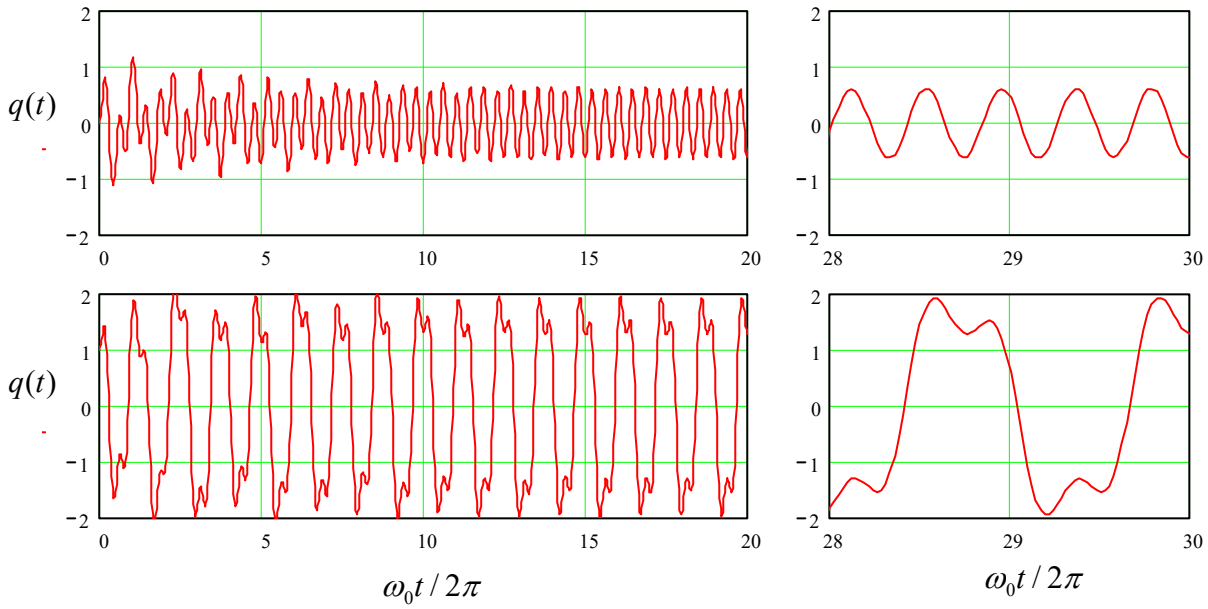


Fig. 5.15. The oscillations of a pendulum with $\delta/\omega_0 = 0.03$, driven by a sinusoidal external force of amplitude $f_0 = 3$ and frequency $0.8\times3\omega_0$, at initial conditions $q(0) = 0$ (the top panels) and $q(0) = 1$ (the bottom panels), with $dq/dt\ (0) = 0$ in both cases.

While the first and the last terms of the last expression depend only on the amplitudes of the individual components of oscillations, the two middle terms are more interesting, because they produce so-called *combinational frequencies* of the two components. In our case, the third term,

$$3A\,A_{\rm sub}^2\cos\Psi\cos^2\Psi_{\rm sub}=\frac{3}{4}AA_{\rm sub}^2\cos(\Psi-2\Psi_{\rm sub})+\ldots, \tag{5.102}$$

is of special importance, because it produces (besides other combinational frequencies) the subharmonic component with the total phase

$$\Psi-2\Psi_{\rm sub}=\frac{\omega t}{3}-\varphi+2\varphi_{\rm sub}. \tag{5.103}$$

Thus, this nonlinear contribution is synchronous with the subharmonic oscillations, and describes the interaction that can, within a certain range of the mutual phase shift between the Fourier components, deliver to them energy from the external force, so that the oscillations may be sustained. Note, however, that the amplitude of the term describing this energy exchange is proportional to the square of $A_{\rm sub}$, and vanishes at the linearization of the equations of motion near the trivial fixed point. This means that the point is always stable, i.e., the 3rd subharmonic cannot be self-excited and always needs an initial "kick-off" – compare the two panels of Fig. 15. The same is true for higher-order subharmonics.

Only the second subharmonic is a special case. Indeed, let us make a calculation similar to Eq. (102), by replacing Eq. (101) with

$$q(t)\approx A\cos\Psi+A_{\rm sub}\cos\Psi_{\rm sub},\qquad \text{where } \Psi\equiv\omega t-\varphi,\quad \Psi_{\rm sub}\equiv\frac{\omega t}{2}-\varphi_{\rm sub}, \tag{5.104}$$

for a nonlinear term proportional to q^2:

$$q^2=(A\cos\Psi+A_{\rm sub}\cos\Psi_{\rm sub})^2=A^2\cos^2\Psi+2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub}+A_{\rm sub}^2\cos^2\Psi_{\rm sub}. \tag{5.105}$$

Here the combinational-frequency term capable of supporting the 2nd subharmonic,

$$2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub}=AA_{\rm sub}\cos\left(\Psi-\Psi_{\rm sub}\right)=AA_{\rm sub}\cos\left(\omega t-\varphi+\varphi_{\rm sub}\right)+\ldots, \tag{5.106}$$

is *linear* in the subharmonic's amplitude, i.e. survives the linearization near the trivial fixed point. This means that the second subharmonic may arise spontaneously, from infinitesimal fluctuations.

Moreover, such excitation of the second subharmonic is very similar to the parametric excitation that was discussed in detail in Sec. 5, and this similarity is not coincidental. Indeed, let us redo the expansion (106) making a somewhat different assumption – that the oscillations are a sum of the forced oscillations at the external force's frequency ω and an *arbitrary but weak* perturbation:

$$q(t)=A\cos(\omega t-\varphi)+\tilde{q}(t),\qquad \text{with } \left|\tilde{q}\right|<<A. \tag{5.107}$$

Then, neglecting the small term proportional to $\tilde{q}^2$, we get

$$q^2\approx A^2\cos^2(\omega t-\varphi)+2\tilde{q}(t)A\cos(\omega t-\varphi). \tag{5.108}$$

Besides the inconsequential phase shift φ, the second term in the last formula is *exactly* similar to the term describing the parametric effects in Eq. (75). This fact means that for a weak perturbation, a system with a quadratic nonlinearity in the presence of a strong "pumping" signal of frequency ω is equivalent to a system with parameters changing in time with frequency ω. This fact is broadly used for the

parametric excitation at high (e.g., optical) frequencies, where the mechanical means of parameter modulation (see, e.g., Fig. 5) are not practicable. The necessary quadratic nonlinearity at optical frequencies may be provided by a *non-centrosymmetric nonlinear crystal*, e.g., the β-phase barium borate (BaB_2O_4).

Before finishing this section, let me elaborate a bit on a general topic: the relation between the numerical and analytical approaches to problems of dynamics – and to physics as a whole. We have just seen that sometimes numerical solutions, like those shown in Fig. 15b, may give vital clues for previously unanticipated phenomena such as the excitation of subharmonics. (The phenomenon of deterministic chaos, which will be discussed in Chapter 9 below, presents another example of such "numerical discoveries".) One might also argue that for problems without exact analytical solutions, the numerical simulation may be an equally productive theoretical tool. These hopes are, however, muted by the general problem that is frequently called the *curse of dimensionality*,[42] in which the last word refers to the number of parameters of the problem to be solved.[43]

Indeed, let us have one more look at Fig. 15. OK, we have been lucky to find a new phenomenon, the 3rd subharmonic generation, for a particular set of parameters – in that case, five of them: $\delta/\omega_0 = 0.03$, $\omega/\omega_0 = 2.4$, $f_0 = 3$, $q(0) = 1$, and $dq/dt\ (0) = 0$. Could we tell anything about how common this effect is? Are subharmonics with different n possible in this system? The only way to address these questions computationally is to carry out similar numerical simulations at many points of the d-dimensional (in this case, $d = 5$) space of parameters. Say, we have decided that breaking the reasonable range of each parameter to $N = 100$ points is sufficient. (For many problems, even more points are necessary – see, e.g., Sec. 9.1.) Then the total number of numerical experiments to carry out is $N^d = (10^2)^5 = 10^{10}$ – not a simple task even for the powerful modern computing facilities. (Besides the pure number of required CPU cycles, consider the storage and analysis of the results.) For many important problems of nonlinear dynamics, e.g., turbulence, the parameter dimensionality d is substantially larger, and the computer resources necessary even for one numerical experiment, are much greater.

In view of the curse of dimensionality, approximate analytical considerations, like those outlined above for the subharmonic excitation, are invaluable. More generally, physics used to stand on two legs: experiment and analytical theory. The enormous progress of computer performance during the few last decades has provided it with one more support point (a tail? :-) – numerical simulation. This does not mean we can afford to discard any of the legs we are standing on.

5.9. Relaxation oscillations

Such synthesis of the analytical and numerical approaches is also beneficial for the discussion of the last subject of this chapter: nonlinear oscillators with high damping. Perhaps the most interesting effect in such systems is the so-called *relaxation oscillations*, a type of self-oscillations with highly non-sinusoidal waveforms. Let me demonstrate them using our old friend, Eq. (62) with $\delta < 0$, whose

[42] This term had been coined in 1957 by Richard Bellman in the context of the optimal control theory (where the dimensionality means the number of parameters affecting the system under control) but gradually has spread all over quantitative sciences using numerical methods.

[43] In EM Sec. 1.2, I discuss the implications of this "curse" for a different case, when both analytical and numerical solutions to the same problem are possible.

properties at $|\delta| << \omega_0$ were discussed in Sec. 4, because it will enable us to follow the crossover from the harmonic oscillations to the relaxation ones.

Figure 16 shows the results of the numerical solution of this equation for three characteristic values of its only substantial parameter[44]

$$\mathcal{D} \equiv \frac{2|\delta|}{\omega_0} > 0. \qquad (5.109)$$

(Indeed, if we introduce the natural dimensionless variables: time $\tau \equiv \omega_0 t$, displacement $x \equiv q/q_0$, where q_0 is the scale defined in Eq. (64), and velocity $y \equiv dx/d\tau$, then the second-order differential equation (62) may be rewritten as the following system of two first-order equations:[45]

$$\frac{dx}{d\tau} = y, \qquad \frac{dy}{d\tau} = \mathcal{D}\left(1 - y^2\right)y - x, \qquad (5.110)$$

with $\mathcal{D}$ being its only parameter.) The left panels show phase planes $[x, dx/d\tau]$ of the oscillator, with their axes swapped[46] for the comparison with the right panels showing the displacement x as a function of time.

If the damping is low (top two panels), the system, launched from any initial state, gradually approaches the "limit cycle" of nearly sinusoidal oscillations. Note that even for this, not extremely small value $\mathcal{D} = 0.2$, deviations of the waveform $x(\tau)$ from a purely sinusoidal function of time are

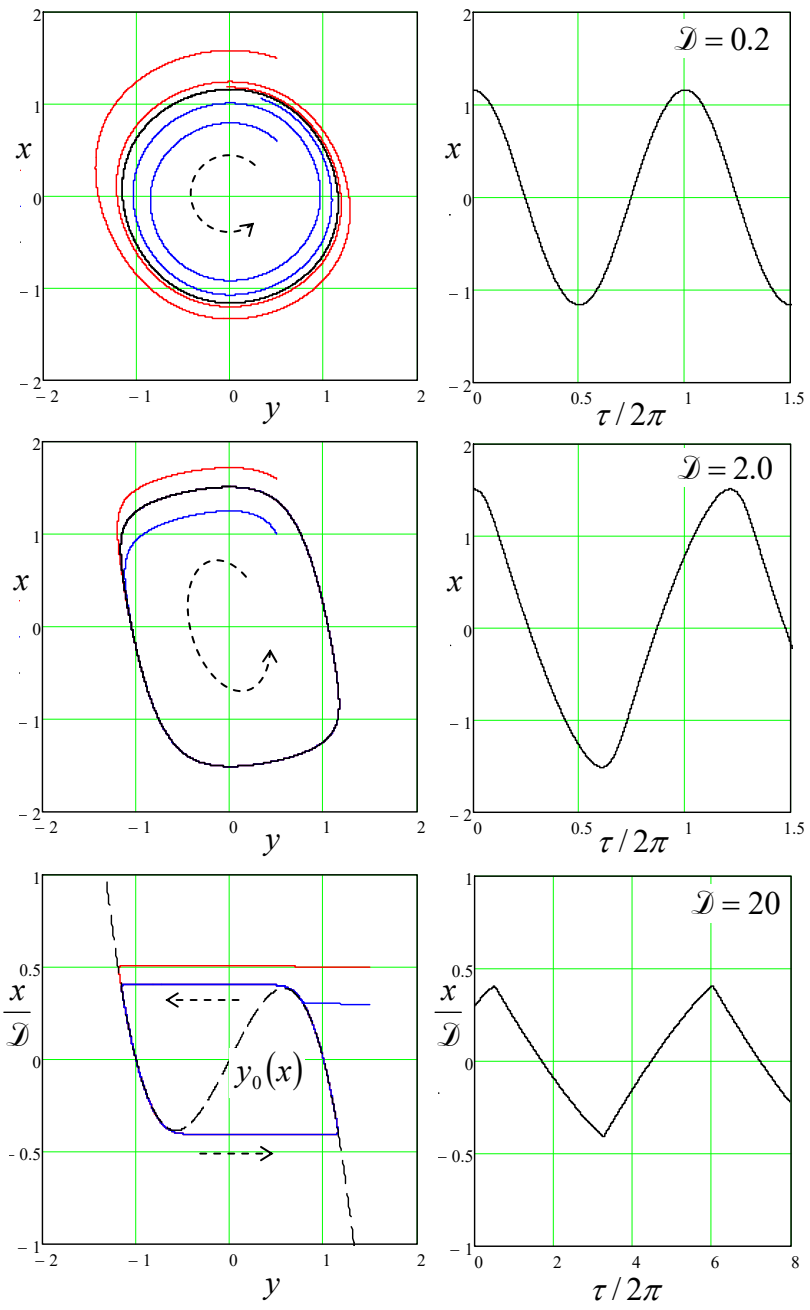


Fig. 5.16. The phase plane and time evolution of the self-oscillator described by Eqs. (62) and (110), for three values of the normalized damping (109). The red and blue lines show the system's dynamics for two representative initial conditions, while the black lines, its asymptotic behavior (the "limit cycles").

[44] As Eq. (11) shows, for positive damping, this parameter is just the reciprocal Q-factor.

[45] A somewhat different equation used in 1926 by B. van der Pol to trace the harmonic-to-relaxation-oscillation crossover for the first time may be also reduced to Eq. (110) by using the so-called *Liénard's transformation*.

[46] Note that while on the usual phase plane, the free-oscillation process corresponds to a clockwise rotation of the representation point (see, e.g., Fig. 9), the axes' swap in Fig. 16 makes the rotation counterclockwise.

very small, its period (in the normalized time τ) is very close to the small-oscillation value 2π, and its amplitude is also very close to the value $2/\sqrt{3} \approx 1.15$ predicted by the van der Pol method – see Eq. (64).

As the damping is increased to $\mathcal{D} = 2$ (middle panels), the limit cycle's deviations from the circle, and hence the deviations of the waveform $x(\tau)$ from a sinusoidal function become obvious. Note also that while the oscillation period becomes somewhat *longer* than its small-oscillation value, the transient processes of approaching the limit cycle become *faster*.

The trend of these changes becomes evident on the bottom panels, showing case $\mathcal{D} = 20$. (The further increase of the damping does not change the results noticeably, only rescaling the displacements as $x \propto \mathcal{D}$ – note the vertical scale of the bottom panels of Fig. 16.) It shows that the oscillation period is dominated by two similar parts, of equal duration. During these two intervals of relatively slow evolution, the limit cycle closely follows the declining branches of the function

$$x = \mathcal{D}\left(1 - y_0^2\right) y_0, \tag{5.111}$$

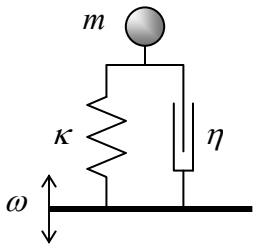
corresponding to the zero value of the first (and nominally, the largest) term in the second equation of the system (110) – see the dashed line on the left bottom panel. During these intervals, the displacement x grows in accordance with the first of these equations, with its right-hand part virtually equal to the y_0 corresponding to Eq. (111). Even without solving the resulting differential equation exactly,[47] we see that at these brunches, with $y_0 \approx \pm 1$, $x(\tau)$ changes with a speed of the order of $\mathcal{D}$, and hence the path from the initial and final points of each branch, of a length $\Delta x \sim \mathcal{D}$, takes a time interval $\Delta\tau$ of the order of 1 – exactly as the right panel shows.

As soon as the system reaches the branch's endpoint $x = \pm(2/3\sqrt{3})\mathcal{D} \approx \pm 0.385\mathcal{D}$, where the derivative dy_0/dx diverges, the balance of the terms on the right-hand part of the second Eq. (110) is not more possible, and its magnitude abruptly becomes of the order of $\mathcal{D} >> 1$. As a result, the system jumps from this point to the opposite branch of the curve (111) very rapidly, during a time interval $\Delta\tau \sim \Delta y_0/\mathcal{D} \sim 1/\mathcal{D} << 1$, insufficient for x to change much. (The initial transient processes, i.e. the approaches to the limit cycle from almost arbitrary initial conditions, are equally fast, also with $x \approx$ const.) Upon reaching the new branch, the system "relaxes" to a relatively slow evolution in the opposite direction (hence the term "relaxation oscillations"), and the process repeats again and again.

Such oscillations take place in a large number of practical mechanical systems and electronic devices, ranging from bowed string musical instruments (including those of the violin family), to usual mechanical clocks, to car light blinkers. Many of them allow for simple analyses; to save time/space, let me leave a couple of problems of this type for the reader's exercise.

5.10. Exercise problems

5.1. A body of mass m is connected to its support not only with an elastic spring but also with a *damper* (say, an air brake) that provides a drag force obeying Eq. (5) – see the figure on the right.



[47] Its integration leads to an elementary function for $\tau(y)$, but transcendental equations for $y(\tau)$ and $x(\tau)$.

(i) How to select the constants κ and η to minimize the body's vibrations caused by vertical oscillations of its support with frequency ω?

(ii)* What if the oscillations are random?

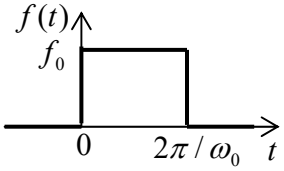
5.2. For a system with the response function given by Eq. (17):

(i) prove Eq. (26), and

(ii) use an approach different from the one used in Sec. 1, to derive Eq. (34).

Hint: You may like to use the *Cauchy integral theorem* and the *Cauchy integral formula* for analytical functions of a complex variable.[48]

5.3. A square-wave pulse of force (see the figure on the right) is exerted on a damping-free linear oscillator of frequency ω_0, initially at rest. Calculate the law of motion $q(t)$, sketch it, and interpret the result.



5.4. A linear oscillator with frequency ω_0 and damping δ was at rest at $t \le 0$. At $t = 0$, an external force $F(t) = F_0 \cos \omega t$ starts to be exerted on it.

(i) Derive the general expression for the time evolution of the oscillator's displacement, and interpret the result.

(ii) Spell out the result for the exact resonance ($\omega = \omega_0$) in a system with low damping ($\delta << \omega_0$) and explore the limit $\delta \to 0$.

5.5. A pulse of external force $F(t)$, with a finite duration $\mathcal{T}$, is exerted on a linear oscillator with negligible damping, initially at rest in its equilibrium position. Use two different approaches to calculate the resulting change of the oscillator's energy.

5.6. A bead may slide, without friction, in a vertical plane along a parabolic curve $y = \alpha x^2/2$, in a uniform gravity field $\mathbf{g} = -g\mathbf{n}_y$. Calculate the change its free oscillations' frequency as a function of their amplitude A, in the first nonvanishing approximation in $A \to 0$, by using two different approaches.

5.7. For a system with the Lagrangian function

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \varepsilon \dot{q}^4,$$

with small parameter ε, use the harmonic balance method to find the frequency of free oscillations as a function of their amplitude.

5.8. Use a different approach to derive Eq. (49) for the frequency of free oscillations of the system described by the Duffing equation (43) with $\delta = 0$, in the first nonvanishing approximation in the small parameter $\alpha A^2/\omega_0^2 << 1$.

5.9. On the plane $[a_1, a_2]$ of two real parameters a_1 and a_2, find the regions in which the fixed point of the following system of equations,

[48] See, e.g., MA Eq. (15.1).

$$\dot{q}_1 = a_1(q_2 - q_1),$$
$$\dot{q}_2 = a_2 q_1 - q_2,$$

is unstable, and sketch the regions of each fixed point type – stable and unstable nodes, focuses, etc.

5.10. Solve Problem 4(ii) by using the reduced equations (57), and compare the result with the exact solution.

5.11. Use the reduced equations to analyze forced oscillations in an oscillator with weak nonlinear damping, described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \beta\dot{q}^3 = f_0 \cos\omega t,$$

with $\omega \approx \omega_0$; $\beta, \delta > 0$; and $\beta\omega A^2 << 1$. In particular, find the stationary amplitude of the forced oscillations and analyze their stability. Discuss the effect(s) of the nonlinear term on the resonance.

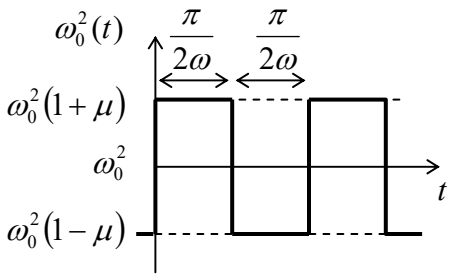
5.12. Within the approach discussed in Sec. 4, calculate the average frequency of a self-oscillator outside of the range of its phase locking by a weak sinusoidal force.

5.13.* Use the reduced equations to analyze the stability of the forced nonlinear oscillations described by the Duffing equation (43). Relate the result to the slope of the resonance curves (Fig. 4).

5.14. Use the van der Pol method to find the condition of parametric excitation of an oscillator described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(t) q = 0,$$

where $\omega_0^2(t)$ is the square-wave function shown in the figure on the right, with $\omega \approx \omega_0$.



5.15. Use the van der Pol method to analyze the parametric excitation of an oscillator with weak nonlinear damping, described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2(1 + \mu\cos 2\omega t)q = 0,$$

with $\omega \approx \omega_0$; $\beta, \delta > 0$; and $\mu, \beta\omega A^2 << 1$. In particular, find the amplitude of stationary oscillations and analyze their stability.

5.16. Upon adding the nonlinear term αq^3 to the left-hand side of Eq. (75),

(i) find the corresponding addition to the reduced equations,
(ii) calculate the stationary amplitude A of the parametric oscillations,
(iii) find the type and stability of each fixed point of the reduced equations,
(iv) sketch the Poincaré phase plane of the system in major parameter regions.

5.17. Use the van der Pol method to find the condition of parametric excitation of a linear oscillator with simultaneous weak modulation of the effective mass $m(t) = m_0(1 + \mu_m\cos 2\omega t)$ and the effective spring constant $\kappa(t) = \kappa_0[1 + \mu_\kappa\cos(2\omega t - \psi)]$, with the same frequency $2\omega \approx 2\omega_0$, for arbitrary

modulation depths ratio μ_m/μ_κ and phase shift ψ. Interpret the result in terms of modulation of the oscillator's instantaneous frequency $\omega(t) \equiv [\kappa(t)/m(t)]^{1/2}$ and impedance $Z(t) \equiv [\kappa(t)m(t)]^{1/2}$.

5.18.* Find the condition of parametric excitation of a nonlinear oscillator described by the following equation:

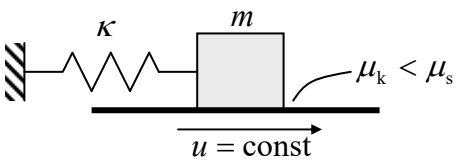
$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \gamma q^2 = f_0 \cos 2\omega t,$$

with sufficiently small δ, γ, f_0, and $\xi \equiv \omega - \omega_0$.

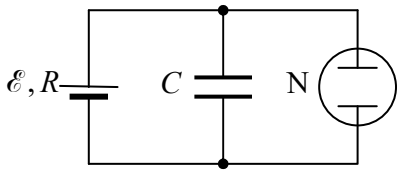
5.19. Find the condition of stability of the equilibrium point $q = 0$ of a parametric oscillator described by Eq. (75), in the limit when $\delta << |\omega_0| << \omega$ and $\mu << 1$. Use the result to analyze the stability of the Kapitza pendulum mentioned in Sec. 5.

5.20.* Use numerical simulation to explore phase-plane trajectories $[q, \dot{q}]$ of an autonomous pendulum described by Eq. (42) with $f_0 = 0$, for both low and high damping, and discuss their most significant features.

5.21. Analyze relaxation oscillations of the system shown in the figure on the right. Here an elastic spring prevents a block of mass m from being carried away by a horizontal conveyor belt moving with a constant velocity u. Assume that the coefficient μ_k of the kinematic friction between the block and the belt is lower than the static friction coefficient μ_s.



5.22. The figure on the right shows the circuit of the simplest electronic relaxation oscillator. N is a bistable circuit element that switches very rapidly from its very-high-resistance state to a very-low-resistance state as the voltage across it is increased beyond some value V_t, and switches back as the voltage is decreased below another value $V_t' < V_t$.[49] Calculate the waveform and the time period of voltage oscillations in the circuit.



Hint: The solution of this problem requires a very basic understanding of electric circuits, including such notions as the e.m.f. $\mathscr{E}$ and the internal resistance R of a dc current source – e.g., of an electric battery.

[49] This is a reasonable model for many two-terminal gas-discharge devices (such as *glow lamps*), whose effective resistance may drop by up to 5 orders of magnitude when the discharge has been ignited by voltage $V > V_t$. In usual neon glow lamps, the discharge stops at a voltage V_t' that is about 30% lower than V_t.

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