The Devil's Staircase

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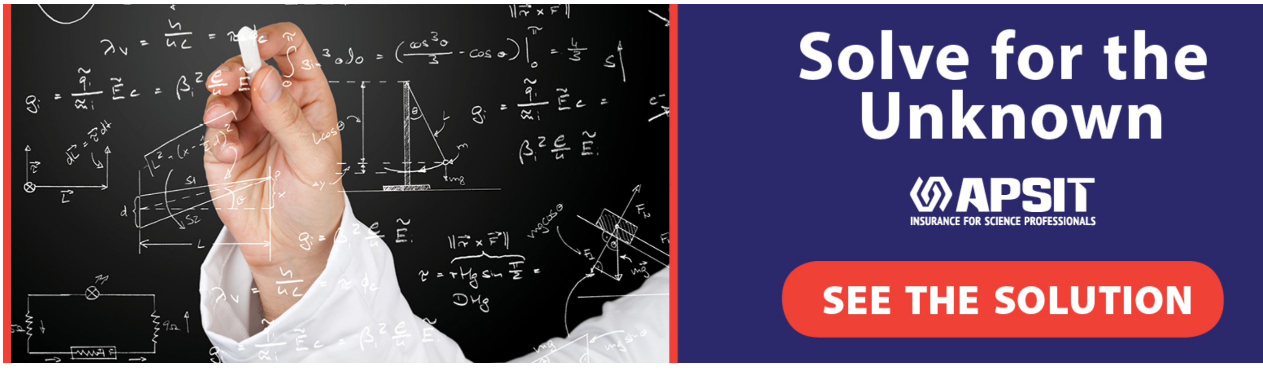

The Devil's staircase

BETTMANN ARCHIVE

'The swing,' a painting by Nicholas Lancret, 1690–1743. The swing and attendant illustrate phase locking in systems containing two competing frequencies. Figure 1

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When the interaction between an oscillator and its driver is strong enough, the oscillator will resonate at, or "lock" onto, an infinity of driving frequencies, giving rise to steps with a fractal dimension between 0 and 1.

Per Bak

In the 17th century the Dutch physicist Christian Huyghens observed[1] that two clocks hanging back to back on the wall tend to synchronize their motion. This phenomenon is known as phase locking, frequency locking or resonance, and is generally present in dynamical systems with two competing frequencies. The two frequencies may arise dynamically within the system, as with Huyghens's coupled clocks, or through the coupling of an oscillator to an external periodic force, as with the swing and attendant shown in figure 1. If some parameter is varied—the length of a pendulum or the frequency of the force that drives it, for instance—the system will pass through regimes that are phase locked and regimes that are not. When systems are phase locked the ratio between their frequencies is a rational number. For weak coupling the phase-locked intervals are narrow, so that even if there is an infinity of intervals, the motion is quasiperiodic for most driving frequencies; that is, the ratio between the two frequencies is more likely to be irrational. When the coupling increases, the phase-locked portions increase, and it becomes less likely that the motion is quasiperiodic. This is a unique situation, where it makes sense, despite experimental uncertainty, to ask whether a physical quantity is rational or irrational.

We shall see that if one plots the frequency of the oscillator against the frequency of the applied force the resulting curve may consist of an infinity of steps—the Devil's staircase. In this article I discuss the conditions under which such a staircase appears and how one can understand it through the modern theory of dynamical systems. The Devil's staircase emerges not only in dynamical systems but also in long-range spatially periodic solid structures, so many of my examples will come from condensed-matter physics.

Origins of staircases

Figure 2 shows dynamical behavior as a function of a frequency for a few systems of very different physical nature. The curves all show a characteristic staircase structure where the plateaus of the curves indicate locking at various rational frequency ratios. The current-driven Josephson junction (figure 2a) obeys an equation very similar to that of a damped driven pendulum; the voltage across the junction is a direct measure of the frequency, so the plot essentially shows[2] the current as a function of frequency. Figure 2b shows the frequency of oscillations in a complex *chemical* reaction, the Belusov–Zabotinsky reaction, measured[3] at the University of Texas by Jerzy Maselko and Harry Swinney. Figure 2c shows the frequency of voltage oscillations in the ionic conductor barium sodium niobate, as measured[4] at Frankfurt University by Samuel Martin and Werner Martienssen; in such a conductor the current is carried by ions rather than electrons. Other examples range from Rayleigh–Bénard hydrodynamic convection systems[5,6] and charge-density-wave systems[7] to periodically forced embryonic chicken hearts[8] and firing neurons subjected to external electrical pulses.[9]

As the interaction between two competing frequencies increases, the oscillations eventually begin to interfere with each other, and there is a transition to a state that features chaotic motion in addition to the periodic and quasiperiodic motion. Mogens Jensen, Tomas Bohr and I, working at the University of Copenhagen, recently investigated phase locking in great detail. We found[10] that at the transition to chaos the motion is always locked: As one changes the frequency of either oscillator—for example, by changing the length of a pendulum or the frequency of a driving force—the ratio between the two frequencies locks onto every single rational value p/q. If one subjects a pendulum to a fixed driving frequency and plots the actual frequency of the pendulum against the natural frequency or length of the pendulum, one obtains a curve consisting of an infinity of steps.

Stated slightly differently, a simple pendulum that for weak coupling almost never locks onto the driving frequency (because the resonances are extremely narrow) will for strong enough coupling always lock onto one of the infinity of resonant frequencies. If one then slowly changes the driving frequency, the pendulum will lock onto each resonant frequency, jumping from one to the next, forming an infinite series of steps.

Between any two steps there is an infinity of steps, because between any two rational numbers there is an infinity of rational numbers. It is this property of the curve that has given rise to the name "the Devil's staircase." If part of the curve is blown up, the

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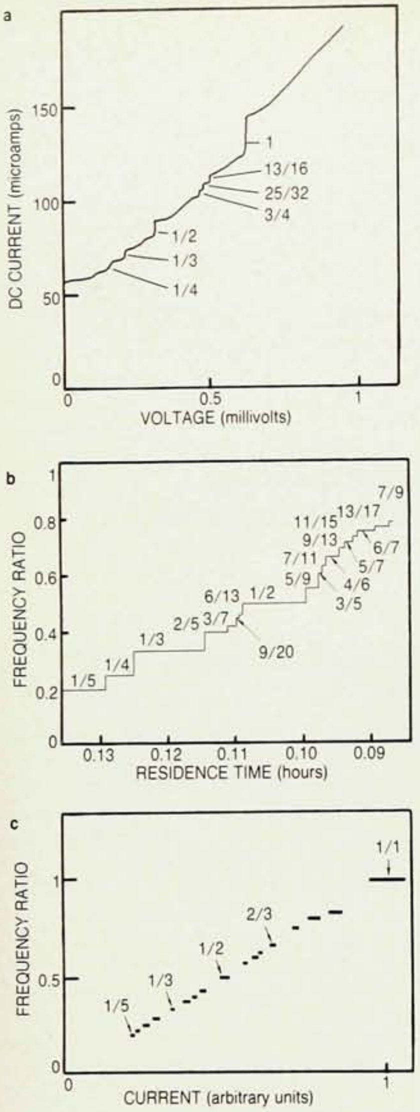


Dynamical-system staircases. The fractions are the ratios of competing frequencies that have locked to form the plateaus. **a:** Current–voltage characteristics of a niobium Josephson junction driven by a 295-GHz microwave current. The voltage is a measure of the frequency. (From reference 2.) **b:** Frequency of oscillations in a complex chemical reaction as a function of the reaction rate. (From reference 3.) **c:** Frequency of voltage oscillations in a superionic conductor, barium sodium niobate, driven by a direct current. (From reference 4.) Figure 2

resulting curve looks very much like the original curve. One can use a scaling index d to describe this self-similarity under magnification. The most striking property of the staircase is that this scaling index is "universal"—the same for all dynamical systems undergoing a mode-locking transition to chaos. The index d is thus a constant of nature. This view has been confirmed for several of the systems mentioned above.

The scaling index d has an interesting mathematical interpretation. Consider the horizontal frequency axis, and remove all the intervals where the frequency is locked. What remains is a set of points. This set of points, called a Cantor set after Georg Cantor, the mathematician who first constructed such sets, has measure zero because the frequency-locked intervals fill the entire axis. The total width of all the points is zero. However, the dimension of the set is not 0 as it would be for a countable set of points, nor is it 1 as it would be for a line segment or collection of line segments. In fact, the scaling index d is approximately 0.87 and can be interpreted as the Cantor set's dimension—in this case a "fractal" dimension between 0 and 1.

The fractal dimension is a generalization of our usual concept of dimension, and can be explained in the following way. Consider a circle or sphere of radius r around a point belonging to the Cantor set of points. If the number of points within the sphere scales as r^d, then d is the fractal dimension of the set. Clearly, for a line the number of points is directly proportional to the radius, so the fractal dimension d is 1, and for a plane the number of points goes as r^2, so d is 2.

The Cantor set has traditionally been thought of as an artificial mathematical construction with no physical application. It is thus quite fascinating that one can relate it directly to a system's mode locking and that one can measure its characteristic scaling dimension directly in rather simple experiments despite its zero measure. I return to the theory and the numerous experimental realizations below.

Solids. The Devil's staircase also shows up in an entirely different context—periodic structures with long spatial periods. Several intermetallic compounds, such as Ag_3Mg, CuAu, Cu_3Pt, Au_3Mn, Au_3Cd and Au_3Zn, form[11] crystals with extremely long periodicities along a unique direction. The long-range structures can be built from blocks of the form XY_3 by putting them together with or without stacking faults, as figure 3a indicates. The figure shows the structure 11111 . . . , which has distance 1 between stacking faults, and the structure 22222 . . . , which has distance 2 between stacking faults. The composition of the compound clearly depends on the density of stacking faults because each fault removes a layer of the majority Y atoms. If a crystal structure with period q has p stacking faults over a distance of q unit lengths, then the structure's average periodicity M is defined as q/p.

Figure 3a also shows a plot of the average periodicity M as a function of temperature for the compounds $Ti_{1+x}Al_{3-x}$, as measured[11] by a French group. Note again the staircaselike dependence where M assumes only rational values. The stairs appear even though the quantity M plotted here is the ratio between two *spatial* periodicities—that of the ordered structure and that of the lattice—in contrast to the dynamical systems such as the pendulum, where we plotted the ratio between two *temporal* periods or frequencies. The analog in a structurally modulated system of the frequency in a dynamical system is the wavevector.

This phenomenon of compounds crystallizing in a multitude of long-range periodic structures is known as polytypism. Structures arising from the stacking of individual atomic layers that have hexagonal or triangular symmetry are another example of polytypism. Hexagonal close-packed structures can be formed by stacking hexagonal layers in the pattern ABABAB . . . ; face-centered-cubic phases have the pattern ABCABC . . . . Using a notation where h means that near neighbors are identically stacked and c means opposite stacking, these structures can be written hhhhhh . . . and cccccc . . . . Certain magnesium-based ternary alloys crystallize in long-range periodic structures with a mixed stacking pattern hchchc . . . . Figure 3b shows the length of the period divided by the number of c's, or stacking faults, as a function of composition in these alloys, as measured[12] at the University of Hiroshima, Japan.

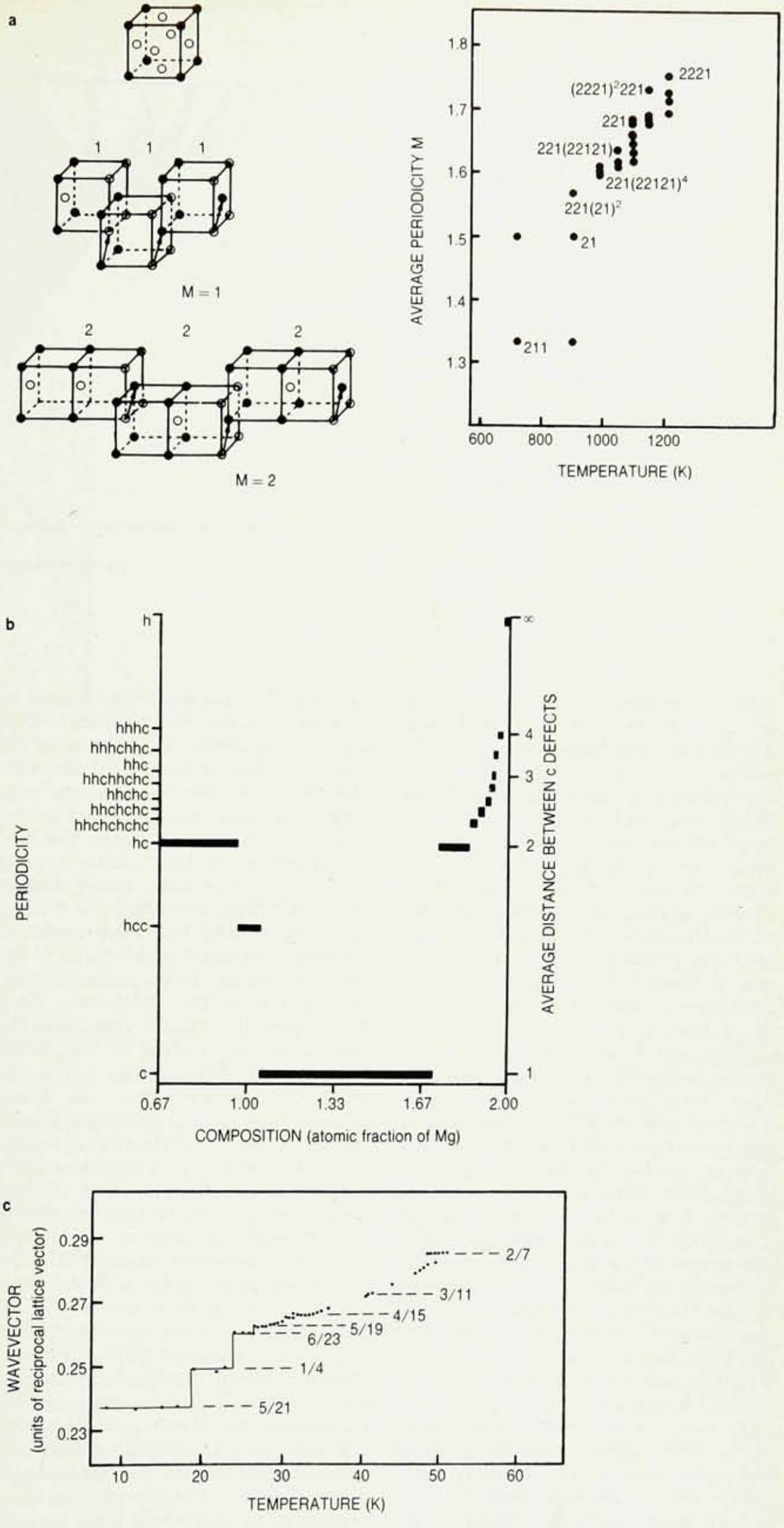
Figure 3c shows the periodicity of the magnetic structure of the rare-earth element erbium as a function of temperature. Physicists at Brookhaven measured[13] this curve by the novel technique of magnetic x-ray scattering. Erbium has a modulated magnetic structure with aligned spins. The various spin configurations are formed by periodic sequences of layers of up spins and layers of down spins. The structure with period 4 (and wavevector ¼), for example, has two layers of up spins followed by two layers of down spins and so on. Similar structures have been found[14] in the rare-earth element holmium and in cerium antimonide. Other systems exhibiting[15] structural staircases are ferroelectrics and the stacking structures of graphite intercalation compounds.

All these structures arise from competition between spatial periodicities. Despite the seemingly enormous complexity of these structures and the wide range of physics that underlies them, two simple models explain the main features of them all. The first[16] is a simple Ising system with long-range repulsive interactions; such a system has a complete Devil's staircase with all possible rational periodicities. The second is the "axial next-nearest-neighbor Ising model," which exhibits[17] the most spectacular phase diagram of any model studied so far; shown in figure 4, the diagram is a "Devil's flower" consisting of an infinity of leaves that represent periodic phases and that spring from a "multiphase point."

Experiments with dynamical systems

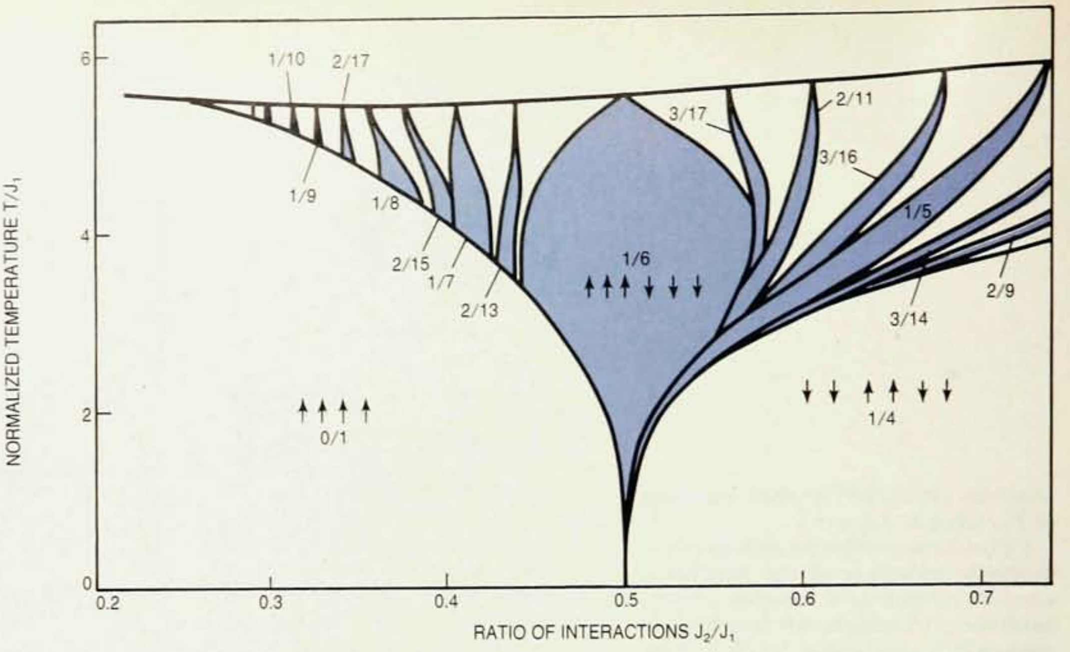
Theoretical analysis of dynamical systems leads to the conjecture (see the box on page 44) that the Devil's staircase is universal at the transition to chaos. It is interesting to examine in this light the real physical systems considered above and illustrated in figure 2.

For the driven pendulum, one can show[10] numerically that (in the termin-



Structural staircases in periodic lattice systems. **a:** Building blocks for long-range periodic crystal structures in $Ti_{1+x}Al_{3-x}$, and a plot of the average periodicity as a function of temperature for an alloy with 72% aluminum. Open circles represent aluminum atoms; closed circles, titanium. (From reference 11.) **b:** Periodicity as a function of composition for ternary magnesium-based compounds. The h phase is a hexagonal close-packed structure; the c phase is a face-centered cubic structure. (From reference 12.) **c:** Modulation wavevector as a function of temperature for the magnetic structure of erbium. (From reference 13.) Figure 3

'**Devil's flower.**' In this phase diagram for the axial next-nearest-neighbor Ising model, the horizontal axis is the ratio of antiferromagnetic interactions to ferromagnetic interactions. The white areas contain yet more leaves, representing infinities of periodic phases with incommensurate phases between. Arrows indicate the sequences of ferromagnetic layers. (From ref. 17.) Figure 4



ology of the theoretical analysis given in the box) the return map indeed develops an inflection point at a critical surface in the parameter space. Numerical simulations by groups at Stony Brook[18] and the University of Copenhagen[19] confirm the existence of a complete Devil's staircase along a line where the map has an inflection point. Preben Alstrom and Mogens Levinsen at the University of Copenhagen found that the fractal dimension of the staircase is about 0.87, in agreement with prediction. Stewart Brown, George Mozurkewich and George Grüner at UCLA also found[7] indications of a complete Devil's staircase, with dimension about 0.92, for charge-density waves in niobium triselenide driven by the combination of a constant current and an oscillating current, but they made little attempt to verify that the system was indeed at the transition line. The most accurate experiments are probably the Rayleigh–Bénard experiment[6] by Albert Libchaber's group at the University of Chicago and the experiments[4] on the ionic conductor BSN by Martin and Martienssen at Frankfurt University (figure 2c).

The Chicago group used a cell filled with mercury and heated from below. At a critical point above the onset of convection there is an oscillatory instability into a time-dependent state involving an ac vertical vorticity in the fluid. The period of this oscillation defines one frequency. The experimenters generated the second oscillator by applying a small horizontal dc magnetic field parallel to the axis of the convection cells in the fluid and by applying a vertical sheet of current through the fluid. The Lorenz force induces an ac vorticity in the fluid's velocity. This is the second oscillator. By varying the amplitude and frequency of the current, they were able to scan a very large range of winding numbers and amplitudes, resulting in a phase diagram with numerous Arnold tongues, similar to the one shown for a circle map in figure 5b. They identified the critical curve by tuning the experiment to quasiperiodic states and looking to see when broadband noise, indicating chaos, appears in the spectrum. They determined the fractal dimension by looking at the scaling of the mode-locked steps at the critical curve. Indeed, they verified that the mode-locked steps form a complete Devil's staircase, and found the fractal dimension to be $0.86 \pm 3\%$, in agreement with theoretical predictions. The Chicago experiment is a spectacular example of the physical relevance of the Cantor set. The universal scaling behavior allows one to extrapolate to the limiting set in a situation where one can only measure a finite number of steps.

The ionic conductor BSN is unique in that a constant driving current gives rise to an oscillating voltage. In the experiment by Martin and Martienssen the periodic voltage oscillations define one frequency. An additional ac current defines the second frequency, and again one can scan a large range of winding numbers and amplitudes by varying the frequency and strength of the ac current. The measured return map indeed appears to develop an inflection point at a transition to chaotic voltage oscillations. Thus the theory described in the box applies to this system. Martin and Martienssen studied the scaling behavior of the staircase formed by the mode-locked steps (figure 2c) following our procedures described in the box. They confirmed that the staircase is complete and has fractal dimension 0.93, in reasonable agreement with theory.

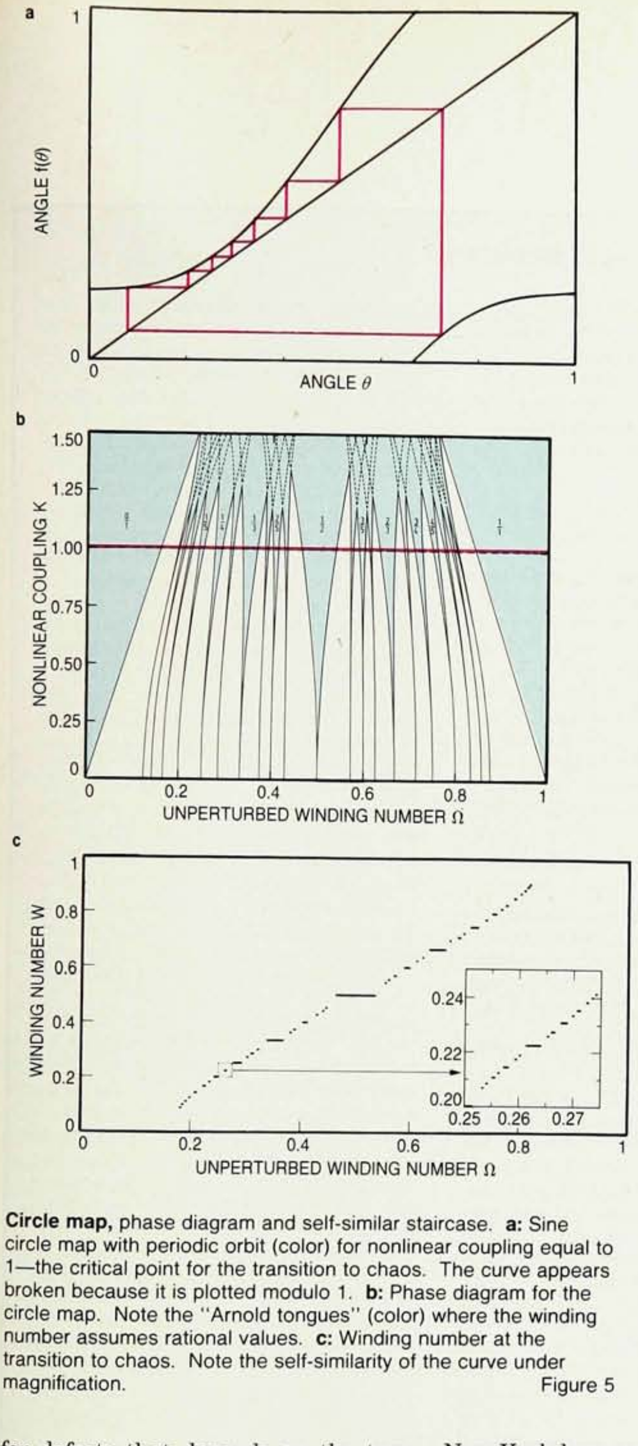
Long-range periodic structures

We can understand the essential features of the magnetic structures, ferroelectric structures and crystal structures, whether long-range periodic or incommensurate, in terms of two very simple models. The first[16] is the one-dimensional Ising model with long-range repulsive interactions whose energy E is given by

$$E = -\sum_i HS_i + \sum_{\langle i,j\rangle} J(i-j)(S_i + \tfrac{1}{2})(S_j + \tfrac{1}{2}) \quad (1)$$

Here S_i are the spins, which are $\pm \tfrac{1}{2}$, H is the magnetic field and the function $J(i-j)$ is the antiferromagnetic interaction between up spins at sites i and j. The antiferromagnetic interaction could, for instance, decay as a power law or exponentially.

It is easy to relate the spins and their interactions to physical quantities for the systems discussed at the beginning of this article. For long-range periodic crystal structures such as $Ti_{1+x}Al_{3-x}$, the spin-up state represents[11] the presence of a defect (figure 3a) and the spin-down state the absence of a defect. In this case, the interaction $J(i-j)$ is the interaction between defects, and the background H a "chemical potential"



Circle map, phase diagram and self-similar staircase. **a:** Sine circle map with periodic orbit (color) for nonlinear coupling equal to 1—the critical point for the transition to chaos. The curve appears broken because it is plotted modulo 1. **b:** Phase diagram for the circle map. Note the "Arnold tongues" (color) where the winding number assumes rational values. **c:** Winding number at the transition to chaos. Note the self-similarity of the curve under magnification. Figure 5

for defects that depends on the temperature and composition. For the magnesium-based ternary alloys[12] (figure 3b), which crystallize in long-range periodic structures with mixed stacking hhchchc . . . , spin $-\frac{1}{2}$ could represent h stacking, so that a chain of down spins would represent a hexagonal close-packed structure, and spin $+\frac{1}{2}$ could represent c stacking, so that a chain of up spins would represent a face-centered cubic structure. The interaction $J(i-j)$ would be the interaction between the c stacking "defects." Robijn Bruinsma at UCLA and Andy Zangwill at the Polytechnic Institute of New York have constructed[20] a microscopic theory for the origin of these interactions based on the electronic structure and elastic properties of the compounds. An interpretation similar to the one for long-range periodic structures would apply to the stacking sequences in graphite intercalation compounds; Roy Clarke and M. J. Winokur have studied[15] these sequences at the University of Michigan.

Bruinsma and I originally constructed[16] the Ising model with long-range interactions (equation 1) to study neutral–ionic transitions in organic charge-transfer compounds, where the parameter H represents the electron ionization energy and J the interaction between ionized layers. For a given fraction q of up spins, the position x_i of the ith up spin in the stable configuration is given by the simple formula

$$x_i = \text{integer}(i/q) \qquad (2)$$

Hence, for $q = \frac{1}{2}$ the structure is

$$- + - + - + \ldots$$

For $q = \frac{10}{23}$ the ground state is

$$- + - + - + - - + - + - + - - + - + \ldots$$

Trivially, these structures translate into 222 and 2223223223, respectively, for the ternary alloys, and hchc and hchchchhchchchhchchchhc for the $TiAl_3$ structures. All the structures that have been found experimentally can be expressed in terms of equation 2. When the fraction q of up spins is irrational—for instance, for q equal to the golden mean, $(\sqrt{5}-1)/2$—the sequence given by equation 2 is not periodic, and the resulting crystal structures have been called[21] "quasicrystalline." The Ising model of equation 1 thus is an extremely simple microscopic model for quasicrystals.

It turns out that for every rational concentration q equal to m/n there exists an interval $\Delta H(m/n)$ in which the structure is stable, and these intervals fill the entire H interval. Hence the concentration of up spins plotted against the parameter H constitutes a complete Devil's staircase. Figure 6 shows the staircase for an interaction of the form $J(n) = 1/n^2$. Note the self-similarity of the staircase under magnification. The dimension d for the Cantor set can easily be determined

Theory for dynamical systems

Consider first the simple pendulum driven by a periodic force. A second-order differential equation describes the variation of the angle θ as a function of time t:

$$\alpha\ddot{\theta} + \beta\dot{\theta} + \gamma \sin\theta = A + B\cos(2\pi t)$$

Here α is the inertia, β the damping, γ the gravitation, A the amplitude of a constant torque and B the amplitude of an external periodic force. Despite its simplicity, this equation cannot be solved analytically; it has a richness of periodic, quasiperiodic and chaotic solutions. For the other systems mentioned in the article, the situation is even worse: We do not even know with any confidence the equations that govern their behavior. However, we shall see that this ignorance does not prevent us from making quantitative predictions about mode locking in the systems.

Let us look at the pendulum with a stroboscopic light that flashes at the discrete times $t = n$, where n is an integer. We are in a sense using the external force as a clock. Let θ_n be the phase, or angle, of the pendulum, and let $\dot{\theta}_n$ be its derivative at time $t = n$. Because the equation is of second order the phase at time $n+1$ is an (unknown) function h of the phase and its derivative at time n:

$$\theta_{n+1} = h(\theta_n, \dot{\theta}_n)$$

The function h is called a Poincaré map, or return map, of the system. Damping may cause the derivative of the phase to become a "slave" of the phase after a transient period: $\dot{\theta}_n = g(\theta_n)$. Then

$$\theta_{n+1} = h(\theta_n, g(\theta_n)) = f(\theta_n)$$

The two-dimensional map h has thus collapsed into a one-dimensional map, which is called a "circle map" because it maps one point θ_n on the circle $0 < \theta < 2\pi$ onto another point θ_{n+1} on the circle.

For given values of the parameters we do not know whether or not this dimensional reduction actually takes place. The best we can do is to generate the return map either by solving the differential equation numerically or by measuring it experimentally. Figure 5a shows an example of a one-dimensional map $f(\theta)$ where we have chosen the period to be 1. Numerical calculations show[10] that the return map for the damped driven pendulum indeed reduces to such a one-dimensional map. Measurements[4] on the ionic conductor BSN, a system for which the underlying equations are not very well known, have also yielded a one-dimensional map. The advantage of studying simple maps of this form is obvious. It is much easier to identify periodic, quasiperiodic and chaotic solutions by iterating the map than by a cumbersome numerical integration of the underlying differential equation. We shall see that the qualitative behavior of the system does not depend on the details of the map as long as it is a circle map. This universality allows us to make predictions with confidence even for systems where we do not know the underlying equations.

Let us therefore study the simple sine circle map

$$\theta_{n+1} = f_\Omega(\theta_n) = \theta_n + \Omega + (K/2\pi)\sin(2\pi\theta_n)$$

The map has a linear term θ_n and a bias term Ω representing the frequency of the system in the absence of nonlinear coupling K. (The factor 2π is used so that the transition to chaos occurs at $K = 1$.) To study the mode locking in the circle map we consider iterations of the map f: θ, $f(\theta)$, $f^2(\theta)\ldots$, or $\theta_1, \theta_2, \theta_3 \ldots$. The frequency of the dynamical system is given by the mapping's "winding number" W, which is the average phase increase per unit time:

$$W = \lim_{n\to\infty} (f^n(\theta_1) - \theta_1)/n$$

In the absence of nonlinear coupling, the winding number W is equal to the bias Ω, so that Ω is the frequency of the unperturbed system. Under iteration the variable θ_n may converge to a series that is periodic, quasiperiodic or chaotic. If the series is periodic, then $\theta_{n+q} = \theta_n + p$ and the frequency of the system is given by a rational winding number p/q. If the series is quasiperiodic, the winding number is irrational. If the series is chaotic, the winding number is not defined. When the nonlinear coupling K is less than 1, the map is strictly monotonic and has only periodic and quasiperiodic winding numbers. When the nonlinear coupling is 1 (figure 5a), the map develops a cubic inflection point at $\theta = 0$. And when the nonlinear coupling is greater than 1, the map has a local maximum.

The transition to chaos takes place precisely at $K = 1$. Figure 5a shows a periodic orbit with a winding number W of $\frac{1}{8}$, a nonlinear coupling K of 1 and a bias Ω of 0.2. Figure 5b shows the phase diagram for the circle map. The regimes where the winding number W assumes rational values are called "Arnold tongues" after the Soviet mathematician V. I. Arnold. When the nonlinear coupling K is close to zero, all intervals of resonance are quite small, so the probability that the winding number is rational is almost zero (as for the Huyghens clocks), and the probability of hitting an irrational winding number is almost 1. However, with increasing nonlinear coupling the widths of all intervals increase. Clearly, the widths of the resonances cannot grow indefinitely; at some point they will interact and overlap.

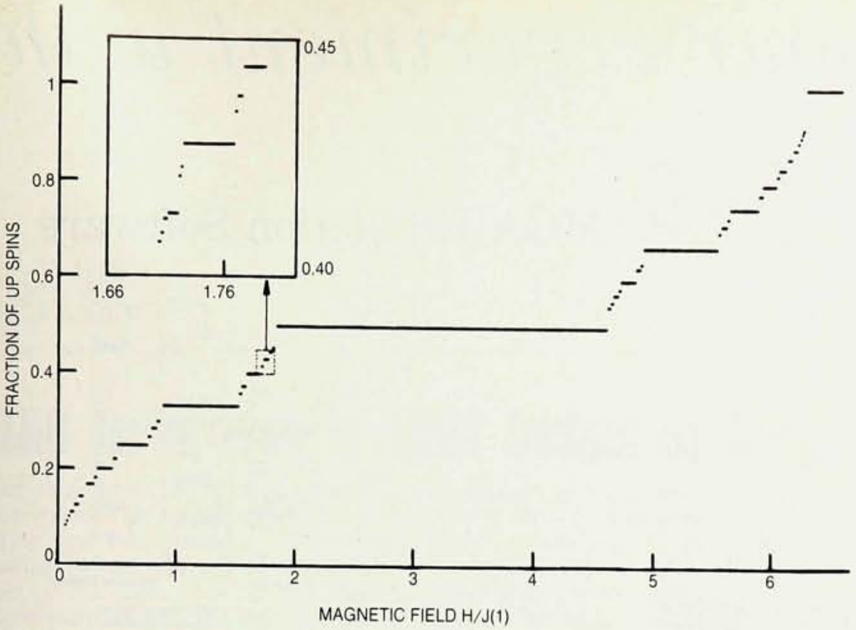
Avoiding unending calculations. One might speculate that when the nonlinear coupling is 1 there is a resonance for all values of the unperturbed frequency Ω. How is it possible to check this without calculating an infinity of intervals? To do so, we calculated[10] the widths $\Delta(p/q)$ of about 1400 steps for a nonlinear coupling of 1. Figure 5c shows the "staircase" formed by plotting the winding number W against the bias Ω. As one includes more and more steps the Ω axis becomes more and more filled. To investigate whether or not the mode-locked steps will eventually cover the entire Ω axis, we calculated the total width $S(r)$ of all steps wider than a given scale r. The quantity of interest is the space between the steps, $1 - S(r)$, which eventually shrinks to a Cantor set. We measured this quantity on the scale r to find the number N of "holes," given by $N(r) = (1 - S(r))/r$. The points in a log–log plot of $N(r)$ fall excellently on a straight line, indicating a power law

$$N(r) \approx (1/r)^d$$

From the slope of the straight line one finds the exponent d to be 0.8700 ± 0.0004. This result means that the space between the steps vanishes as r^{1-d} as the scale r goes to 0. Thus there is no room for quasiperiodic motion and the Devil's staircase is complete. We can interpret the exponent d as the dimension of the Cantor set that is the complementary set to the mode-locked intervals on the Ω axis.

When the nonlinear coupling K passes beyond 1, the widths of the steps continue to increase. Because they fill the entire Ω axis at $K = 1$, they must necessarily overlap for $K > 1$, as figure 5b indicates. The transition to chaos is basically caused by the overlap of resonances, and one can visualize the chaotic motion as an erratic jumping between resonances. Most nonlinear systems perturbed by an external periodic field will probably exhibit a transition to chaos caused by the overlap of resonances that follow the Devil's staircase as described here.

An important question is whether or not the critical behavior at the transition is universal, that is, whether or not it depends on the specific function f defined above. If the behavior is not universal, we cannot predict with confidence how any specific system will behave in an experiment because the function f is generally not known. To check universality, we studied a broad class of circle maps with more complicated nonlinear terms. Generally the details are different from those of the staircase in figure 5c. Some steps become narrower, some wider. The scaling, however, remains the same, with the dimension about 0.870.



Density of up spins as a function of magnetic field for an Ising model with long-range interactions. (From reference 16.) Figure 6

analytically. For power-law interactions of the form $n^{-\alpha}$, the dimension d is $2/(1+\alpha)$. This dimension is obviously not universal because it depends on the exponent α, and it is not possible to predict the dimension from a simple general theory.

For the magnetic structures of cerium antimonide and erbium (figure 3c), and for many of the ferroelectric structures with staircase behavior, the temperature and hence the entropy play an essential role in determining the sequence of the periodic phases. Also, there often appear to be incommensurate phases between the high-order periodic ones. It is essential to construct a real three-dimensional thermodynamic model for these systems. To accomplish this, Juhani von Boehm and I, then at the Bohr Institute in Copenhagen, constructed[17] a simple Ising model with competing nearest- and next-nearest-neighbor interactions. The model was later named the axial next-nearest-neighbor Ising model. Within each layer the model has nearest-neighbor ferromagnetic interactions J_1. In the axial direction the model has ferromagnetic nearest-neighbor interactions J_1, but antiferromagnetic next-nearest-neighbor interactions J_2. When the ratio J_2/J_1 is small, it is energetically favorable for the system to order ferromagnetically into a structure of the form + + + +, where the pluses indicate a sequence of up layers. When the ratio J_2/J_1 is large, it is energetically favorable for the system to order into a structure of period 4, or wavevector $1/4$, of the form + + − − + + − − + + − −.

Figure 4 shows the resulting phase diagram of the model. The diagram is possibly the most spectacular found for any statistical-mechanical model whatsoever. At zero temperature only the + + + and + + − − + + phases can exist. At finite temperatures there is an infinity of periodic phases with wavevectors between 0 and $1/4$; these spring out of a multiphase point at $T=0$. It can be shown that near the transition line to the paramagnetic phase all possible rational phases become stable, so there is a Devil's staircase. The white areas between the leaves of the Devil's flower in figure 4 indicate regimes with yet more leaves, leaves of infinities of long-range periodic phases with incommensurate phases between.

* * *

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