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# Stellar Interiors

## Physical Principles, Structure, and Evolution

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First we investigate the balance of forces within a star in equilibrium. From elementary physics, the local gravity at spherical radius $r$ is

$$g(r) = \frac{G\mathcal{M}_r}{r^2} = 2.74 \times 10^4 \left(\frac{\mathcal{M}_r}{\mathcal{M}_\odot}\right)\left(\frac{r}{\mathcal{R}_\odot}\right)^{-2} \quad \text{cm s}^{-2} \tag{1.1}$$

and

$$\mathcal{M}_{r+dr} - \mathcal{M}_r = d\mathcal{M}_r = 4\pi r^2 \rho(r)\, dr \tag{1.2}$$

is the mass contained within a spherical shell of infinitesimal thickness $dr$ at $r$. The integral of (1.2) yields the mass within $r$,

$$\mathcal{M}_r = \int_0^r 4\pi r^2 \rho\, dr\ . \tag{1.3}$$

Either (1.2) or (1.3) will be referred to as the *mass equation* or the *equation of mass conservation.*

Now consider a 1–cm$^2$ element of area on the surface of the shell at $r$. There is an inwardly directed gravitational force on a volume 1 cm$^2$ $\times$ $dr$ of

$$\rho g\, dr = \rho \frac{G\,\mathcal{M}_r}{r^2}\, dr\ . \tag{1.4}$$

To counterbalance this force we must rely on an imbalance of pressure forces; that is, the pressure $P(r)$ pushing outward against the inner side of the shell must be greater than the pressure acting inward on the outer face. The net pressure outward is $P(r) - P(r+dr) = -(dP/dr)\, dr$. Adding the gravitational and differential pressure forces then yields

$$\rho \ddot{r} = -\frac{dP}{dr} - \frac{G\,\mathcal{M}_r}{r^2}\rho \tag{1.5}$$

as the equation of motion, where $\ddot{r}$ is the local acceleration $d^2r/dt^2$.

By hypothesis, all net forces are zero, with $\ddot{r} = 0$, and we obtain the *equation of hydrostatic* (or mechanical) *equilibrium*:

$$\frac{dP}{dr} = -\frac{G\,\mathcal{M}_r}{r^2}\rho = -g\rho\ . \tag{1.6}$$

Since $g, \rho \geq 0$, then $dP/dr \leq 0$, and the pressure must decrease outward everywhere. If this condition is violated anywhere within the star, then hydrostatic equilibrium is impossible and local accelerations must occur.

We can obtain the hydrostatic equation in yet another way and, at the same time, introduce some new concepts.

## 1.2 An Energy Principle

The preceding was a local approach to mechanical equilibrium because only local quantities at $r$ were involved (although a gradient did appear). What we shall do now is take a global view wherein equilibrium is posed as an integral constraint on the structure of the entire star.

Imagine that the equilibrium star is only one of an infinity of possible configurations and the trick is to find the right one. (The wrong ones will not be in equilibrium and just won't do.) Each configuration will be specified by an integral function so constructed that the equilibrium star is represented by a stationary point in the series of possible functions. This begins to sound like a problem in classical mechanics and the calculus of variations—and it is. (We'll ease into the mathematics.) The function in question is the total stellar energy, and so let's see what it is.

The total gravitational potential energy, $\Omega$, of a self-gravitating body is defined as the *negative* of the total amount of energy required to disperse all mass elements of the body to infinity. The zero point of the potential is taken as the final state after dispersal. In other words, $\Omega$ is the energy required to assemble the star, in its current configuration, by collecting material from the outside universe. Thus $\Omega$ represents (negative) work done on, or by, the system and it must be accounted for when determining the total energy of the star.

We can get to the dispersed state by successively peeling off spherical shells from our spherical star. Suppose we have already done so down to an interior mass of $\mathcal{M}_r + d\mathcal{M}_r$ and we are just about to remove the next shell, which has a mass $d\mathcal{M}_r$. To move this shell outward from some radius $r'$ to $r' + dr'$ requires $(G\,\mathcal{M}_r/r'^2)\,d\mathcal{M}_r\,dr'$ units of work. To go from $r$ to infinity then gives a contribution to $\Omega$ of (remembering the minus sign for $\Omega$)

$$d\Omega = -\int_r^\infty \frac{G\,\mathcal{M}_r}{r'^2}\,d\mathcal{M}_r\,dr' = -\frac{G\,\mathcal{M}_r}{r}\,d\mathcal{M}_r .$$

To disperse the whole star requires that we do this for all $d\mathcal{M}_r$ or,

$$\Omega = -\int_0^{\mathcal{M}} \frac{G\,\mathcal{M}_r}{r}\,d\mathcal{M}_r \; . \tag{1.7}$$

The potential energy thus has the units of $G\,\mathcal{M}^2/\mathcal{R}$ and we shall often write it in the form

$$\Omega = -q\,\frac{G\,\mathcal{M}^2}{\mathcal{R}} \; . \tag{1.8}$$

For a uniform density sphere, with $\rho$ constant, it is easy to show that the pure number $q$ is equal to 3/5. (This should be familiar from electrostatics, where the energy required to disperse a uniformly charged sphere to infinity is $-3e^2/5\mathcal{R}$.) Because density almost always decreases outward for equilibrium stars, the value of 3/5 is, for all practical purposes, a lower limit with $q \geq 3/5$.

For the sun, $G\mathcal{M}_\odot{}^2/\mathcal{R}_\odot \approx 3.8 \times 10^{48}$ erg. If we divide this figure by the present solar luminosity, $\mathcal{L}_\odot$, we find a characteristic time (the Kelvin–Helmholtz time scale) of about $3 \times 10^7$ years. More will be said about this time scale later on.

If we neglect gross mass motions or phenomena such as turbulence, then the total energy of the star is $\Omega$ plus the total internal energy arising from microscopic processes. Let $E$ be the local specific internal energy in units of ergs per gram of material. It is to be multiplied by $\rho$ if you want energy per unit volume. (Thus $E$ will sometimes have the units of erg cm$^{-3}$ but you will either be forewarned by a statement or the appearance of those units.) The total energy, $W$, is then the sum of $\Omega$ and the mass integral of $E$,

$$W = \int_{\mathcal{M}} E \, d\mathcal{M}_r + \Omega = U + \Omega \tag{1.9}$$

which also defines the total internal energy

$$U = \int_{\mathcal{M}} E \, d\mathcal{M}_r \; . \tag{1.10}$$

The statement now is that the equilibrium state of the star corresponds to a stationary point with respect to $W$. This means that $W$ for the star in hydrostatic equilibrium is an extremum (a maximum or minimum) relative to all other possible configurations the star could have (with the possible exception of other extrema). What we are going to do to test this idea is to perturb the star away from its original state in an *adiabatic* but otherwise arbitrary and infinitesimal fashion. The adiabatic part can be satisfied if the perturbation is performed sufficiently rapidly that heat transfer between mass elements does not take place (as in an adiabatic sound wave). We shall show later that energy redistribution in normal stars takes place on time scales longer than mechanical response times. On the other hand, we also require that the perturbation be sufficiently slow that kinetic energies of mass motions can be ignored.

If $\delta$ represents either a local or global perturbation operator (think of it as taking a differential), then the stellar hydrostatic equilibrium state is that for which

$$(\delta W)_{\text{ad}} = 0$$

where the "ad" subscript denotes "adiabatic." Thus if arbitrary, but small, adiabatic changes result in no change in $W$, then the initial stellar state is in hydrostatic equilibrium. To show this, we have to look how $U$ and $\Omega$ change when $\rho$, $T$, etc., are varied adiabatically. We thus have to look at the pieces of

$$(\delta W)_{\text{ad}} = (\delta U)_{\text{ad}} + (\delta \Omega)_{\text{ad}} \; .$$

A perturbation $\delta$ causes $U$ to change by $\delta U$ with

$$U \longrightarrow U + \delta U = U + \delta \int_{\mathcal{M}} E \, d\mathcal{M}_r = U + \int_{\mathcal{M}} \delta E \, d\mathcal{M}_r \; .$$

The last step follows because we choose to consider the change in specific internal energy of a particular mass element $d\mathcal{M}_r$. (This is a *Lagrangian* description of the perturbation about which more will be said in Chap. 8.) Now consider $\delta E$. We label each mass element of $d\mathcal{M}_r$ worth of matter and see what happens to it (and $E$) when its position $r$, and $\rho$, and $T$ are changed.

For an infinitesimal and reversible change (it would be nice to be able to put the star back together again), the combined first and second laws of thermodynamics state that

$$dQ = dE + P\,dV_\rho = T\,dS\ . \tag{1.11}$$

Here $dQ$ is the heat added to the system, $dE$ is the increase in internal specific energy, and $P\,dV_\rho$ is the work done by the system on its surroundings if the "volume" changes by $dV_\rho$. This volume is the *specific volume*, with

$$V_\rho = 1/\rho \tag{1.12}$$

and is that associated with a given gram of material. It has the units of $\text{cm}^3\ \text{g}^{-1}$. (The symbol $V$ will be reserved for ordinary volume with units of $\text{cm}^3$.) The entropy $S$, and $Q$, are also mass-specific quantities. If we replace the differentials in the preceding by $\delta$s, then the requirement of adiabaticity ($\delta S = 0$) immediately yields $(\delta E)_{\text{ad}} = -P\,\delta V_\rho$. Thus,

$$(\delta U)_{\text{ad}} = -\int_{\mathcal{M}} P\,\delta V_\rho\,d\mathcal{M}_r\ .$$

What is $\delta V_\rho$? From the definition of the specific volume (1.12) and the mass equation (1.2),

$$V_\rho = \frac{1}{\rho} = \frac{4\pi r^2\,dr}{d\mathcal{M}_r} = \frac{d(4\pi r^3/3)}{d\mathcal{M}_r}\ . \tag{1.13}$$

To make life easy, we restrict all perturbations to those that maintain spherical symmetry. Thus if the mass parcel $d\mathcal{M}_r$ moves at all, it moves only in the radial direction to a new position $r + \delta r$. Perturbing $V_\rho$ in (1.13) is then equivalent to perturbing $r$ or

$$V_\rho \longrightarrow V_\rho + \delta V_\rho = \frac{d[4\pi(r+\delta r)^3/3]}{d\mathcal{M}_r} = V_\rho + \frac{d(4\pi r^2 \delta r)}{d\mathcal{M}_r} \tag{1.14}$$

to first order in $\delta r$, where we assume that $|\delta r/r| \ll 1$. (Later we will call this sort of thing "linearization.") The variation in total internal energy is then

$$(\delta U)_{\text{ad}} = -\int_{\mathcal{M}} P\,\frac{d(4\pi r^2\,\delta r)}{d\mathcal{M}_r}\,d\mathcal{M}_r\ . \tag{1.15}$$

We now introduce two boundary conditions. The first is obvious: we don't allow the center of our spherically symmetric star to move. This amounts to requiring that $\delta r(\mathcal{M}_r = 0) = 0$. The second is called the "zero boundary condition on pressure" and it requires that the pressure at the surface vanish.