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5.3.d). $W = \{A \in M_2(\mathbb{R}) : A^t = A\}$

$W^{\perp} = \{B \in M_2(\mathbb{R}) : \langle B, A\rangle = 0 \quad \forall A \in W\}$

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$

$\langle B, A\rangle = \sum_{i,j=1}^{2} b_{ij} a_{ij}$

$= b_{11} a_{11} + b_{12} \overline{a_{12}} + b_{21} \overline{a_{12}} + b_{22} a_{22} = 0$

$= \underbrace{b_{11}}_{=0} \overline{a_{11}} + \underbrace{[b_{12} + b_{21}]}_{=0} \overline{a_{12}} + \underbrace{b_{22}}_{=0} \overline{a_{22}} = 0$

$B = \begin{pmatrix} 0 & b_{12} \\ -b_{12} & 0 \end{pmatrix} \rightarrow W^{\perp} = \{B \in M_2(\mathbb{R}) : B^t = -B\}$

7.1.b)

$V = \mathbb{R}_1[x]$, $T(p) = p'$

$T(ax+b) = a$

Sea $p = ax+b,\ q = cx+d \in V$,

$\langle T(p), q \rangle = \langle a, cx+d \rangle = \int_0^1 acx + ad \quad dx$

$= \frac{ac}{2} + ad$

$\langle p, T^*(q) \rangle = \langle ax+b, T^*(q) \rangle = \int_0^1 ax\, T^*(q) + b\, T^*(q) \quad dx$

Si $\{v_1, v_2\}$ es b.o.n. $\Rightarrow$ $T^*(v) = \sum_{i=1}^{2} \langle T(v), v_i \rangle v_i$

Sea $\mathcal{C} = \{1, x\}$ la base canonica.

$\langle I, x \rangle = \int_0^1 \cancel{1}\, x \, dx = \frac{1}{2} \neq 0 \Rightarrow \mathcal{C}$ no es ortogonal

Aplicamos G-S: $w_1 = v_1 = I$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

$= x - \frac{\overset{1/2}{\langle x, 1 \rangle}}{\underset{1}{\|1\|^2}} I$

$w_2 = x - \frac{1}{2} I$

$\|1\|^2 = \left[\int_0^1 1 \, dx\right]^2 = 1^2 = 1$

$\mathcal{C}' = \{I, x - \frac{1}{2} I\}$

$$\langle I, x-\tfrac{1}{2} I\rangle = \int_0^1 I\cdot x\,dx - \tfrac{1}{2}\int_0^1 I^2\,dx$$

$$= \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$\|x-\tfrac{1}{2} I\| = \sqrt{\langle x-\tfrac{1}{2} I, x-\tfrac{1}{2} I\rangle} = \left[\int_0^1 x^2 + \tfrac{1}{4} I^2 - x\cdot I\,dx\right]^{1/2}$$

$$= \left[\tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{2}\right]^{1/2}$$

$$= \left(\frac{4+3-6}{12}\right)^{1/2} = \sqrt{\frac{1}{12}}$$

Luego, $\mathcal{B} = \left\{ I, \dfrac{x-\frac{1}{2} I}{\sqrt{1/12}} \right\}$ es una b.o.n.

Recordar $T(ax+b) = a$.

$$T^*(ax+b) = \sum_{i=1}^{2} \langle T(ax+b), v_i\rangle v_i$$

$$= \langle a, I\rangle I + \cancel{\left\langle a, \frac{x-\frac{1}{2} I}{\sqrt{1/12}}\right\rangle} \frac{x-\frac{1}{2} I}{\sqrt{1/12}}$$

$$\langle a, I\rangle = \int_0^1 a\cdot I\,dx = a$$

$$\left\langle a, \frac{x-\frac{1}{2} I}{\sqrt{1/12}}\right\rangle = \frac{a}{\sqrt{1/12}}\int_0^1 x - \tfrac{1}{2} I\,dx$$

$$= \frac{a}{\sqrt{1/12}} \left[\frac{1}{2} - \frac{1}{2} \right] = 0$$

$\boxed{T^*(cx+d) = 0}$ es en $\beta = \left\{ 1, \frac{x - \frac{1}{2}}{\sqrt{1/12}} \right\}$

$$\langle T(ax+b), cx+d \rangle = \int_0^1 acx + da = \frac{ac}{2} + da$$

$$\neq$$

$$\langle ax+b, T^*(cx+d) \rangle = \int_0^1 acx + ac = \frac{ac}{2} + ac$$

<u>Revisar:</u>

6.1.6)

$T: \mathbb{C}^2 \to \mathbb{C}^2$

$$T(x,y) = \left(2x + (3-3i)y,\ (3+3i)x + 5y \right)$$

$$\langle T(x,y), (z,w) \rangle =$$

$$= 2x\bar{z} + (3-3i)y\bar{z} + (3+3i)x\bar{w} + 5y\bar{w}$$

$$\langle (x,y), T(z,w) \rangle = 2x\bar{z} + x\overline{(3-3i)w} + \overline{(3+3i)z}\,y + 5y\bar{w}$$

$$= 2x\bar{z} + (3+3i)x\bar{w} + (3-3i)\bar{z}y + 5y\bar{w}$$

8) $T_1 = \frac{1}{2}(T + T^*)$

Queremos $\rightarrow \langle v, T_1(w) \rangle$

$$\langle T_1(v), w \rangle = \langle \tfrac{1}{2}(T + T^*)(v), w \rangle$$

$$= \langle \tfrac{1}{2} T(v) + \tfrac{1}{2} T^*(v), w \rangle$$

$$= \langle \tfrac{1}{2} T(v), w \rangle + \langle \tfrac{1}{2} T^*(v), w \rangle$$

$$= \langle \tfrac{1}{2} v, T^*(w) \rangle + \langle \tfrac{1}{2} v, \underbrace{(T^*)^*}_{T} (w) \rangle$$

$T: V \rightarrow V$
$T^*: V \rightarrow V$

$$= \langle \tfrac{1}{2} v, T^*(w) + T(w) \rangle$$

$$= \tfrac{1}{2} \langle v, T^*(w) + T(w) \rangle$$

$$= \langle v, \overline{\tfrac{1}{2}} [T^*(w) + T(w)] \rangle$$

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$$= \langle v, \tfrac{1}{2} [T(w) + T^*(w)] \rangle$$

$$= \langle v, \tfrac{1}{2}(T + T^*)(w) \rangle$$

$$= \langle v, T_1(w) \rangle$$

10) $\lambda \in \mathbb{C}$, $T_\lambda : \mathbb{C} \to \mathbb{C}$, $T_\lambda(z) = \lambda \cdot z$

a) Hallar para cuales λ, T_λ es autoadjunto.

$$\langle T_\lambda(z), w \rangle = \langle \lambda z, w \rangle = \lambda z \overline{w}$$

$\parallel \leftarrow$ Queremos

$$\langle z, T_\lambda(w) \rangle = \langle z, \lambda w \rangle = \overline{\lambda} z \overline{w}$$

$\lambda = \overline{\lambda}$ sii $\lambda \in \mathbb{R}$

b) T_λ es positivo si: • $\langle T(v), v \rangle > 0$: $\forall v \neq 0$
• Autoadjunto

$$\langle T(z), z \rangle = \langle \lambda z, z \rangle = \lambda \langle z, z \rangle = \lambda \underbrace{|z|^2}_{\underset{z \neq 0}{> 0}}$$

$\langle T(z), z \rangle > 0$ sii $\lambda > 0$.

c) ¿Para cuales λ, T_λ es normal?

Recordar: T_λ es normal si $T_\lambda \circ T_\lambda^* = T_\lambda^* \circ T_\lambda$

$$\langle T_\lambda(z), w \rangle \overset{\text{sabemos}}{=} \langle z, T_\lambda^*(w) \rangle$$

$\parallel$

$\underline{\underline{\lambda z \overline{w}}}$

Vale sii $\boxed{T_\lambda^*(w) = \overline{\lambda} w}$

$$\langle z, T^*_\lambda(w) \rangle = \langle z, \overline{\lambda} w \rangle$$
$$= \overline{\overline{\lambda}} \langle z, w \rangle$$
$$= \lambda z \overline{w}$$

$T_\lambda \circ T_\lambda^*(z) = T_\lambda(\bar{\lambda} z) = \lambda \bar{\lambda} z = |\lambda|^2 z$

"Normal"

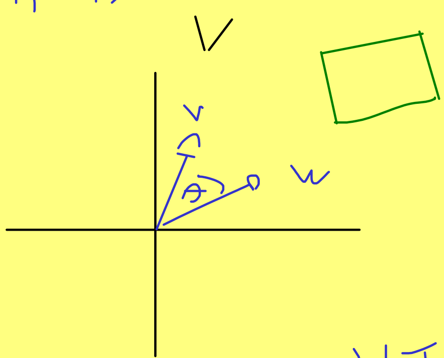
$T_\lambda^* \circ T_\lambda(z) = T_\lambda^*(\lambda z) = \bar{\lambda} \lambda z = |\lambda|^2 z$

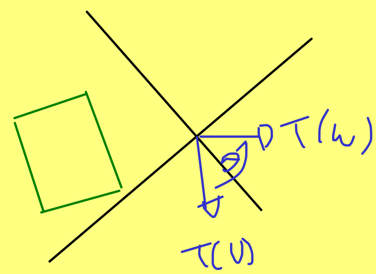
Entonce T_λ es normal $\forall \lambda \in \mathbb{C}$.

Esto muestra que normal no implica autoadjunto.

d) T_λ isométria.

$\langle T_\lambda(z), T_\lambda(w) \rangle = \langle z, w \rangle$

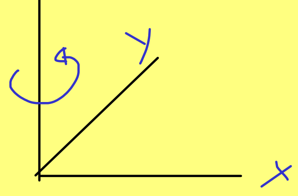


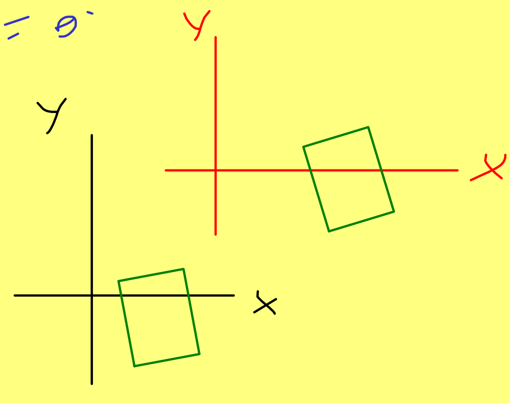


$\|T(v)\| = \|v\|$

$\|T(w)\| = \|w\|$

$\theta = \theta'$

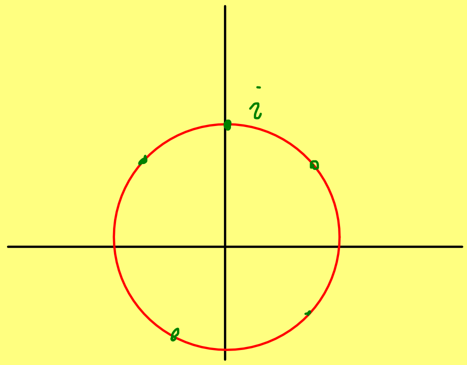




$\langle T_\lambda(z), T_\lambda(w) \rangle = \langle \lambda z, \lambda w \rangle = \lambda \bar{\lambda} \langle z, w \rangle$

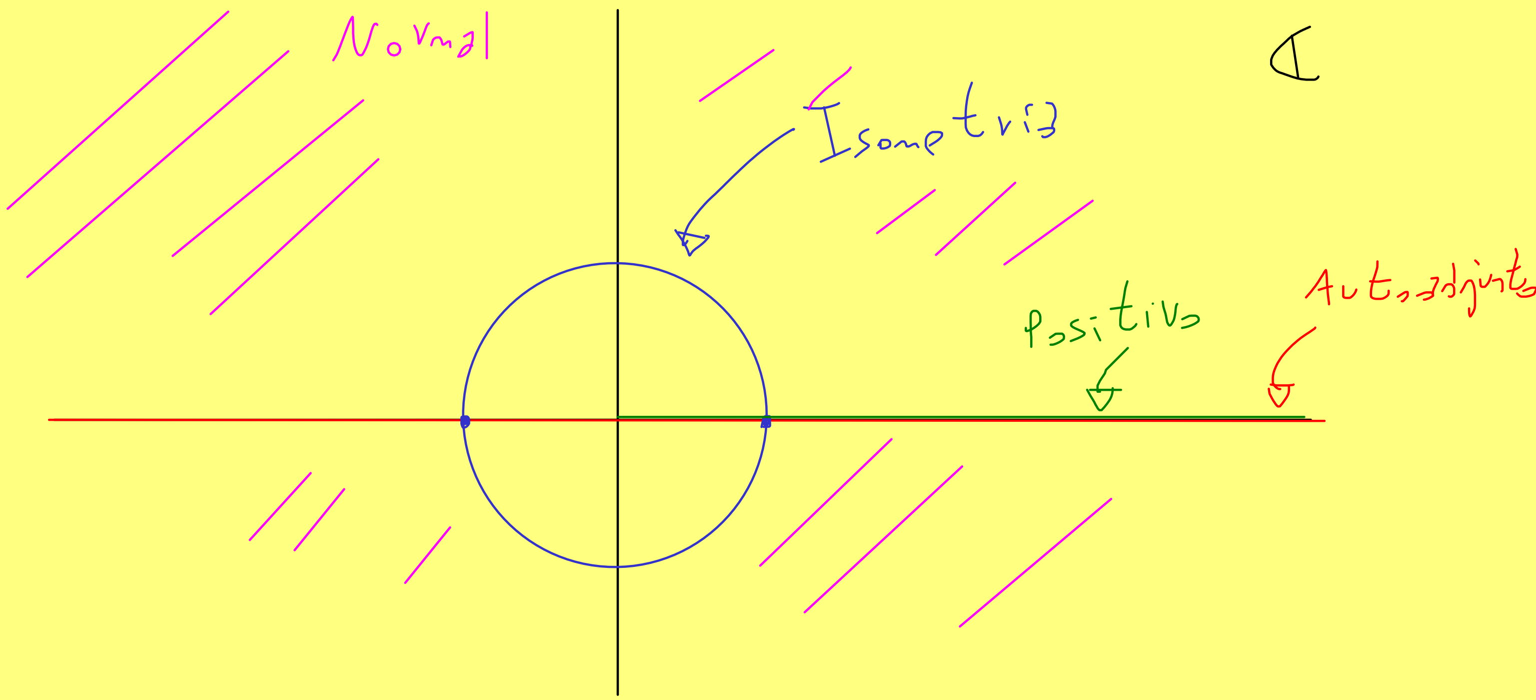
$= |\lambda|^2 \langle z, w \rangle$

Entonces para que T_λ sea isometria, $\lambda \bar{\lambda} = 1$



$\lambda = \pm 1$

$$\left.\begin{aligned} \lambda &= i \\ \bar{\lambda} &= -i \end{aligned}\right\} \lambda\bar{\lambda} = -i^2 = -(-1) = 1$$



6.3.6)

$\mathbb{K} = \mathbb{C}$, T autoadjunto

$$\|\underbrace{T \pm i\,Id}(v)\|^2 = \|T(v)\|^2 + \|v\|^2$$

$T(v) \pm iv$.

6.4) S, T autoadjuntos $\Rightarrow$ S, T normales.

Si $\mathbb{K}=\mathbb{R} \to$ S, T autoadjunto $\Rightarrow$ S, T diago

$\mathbb{K}=\mathbb{C} \to$ S, T normal $\Rightarrow$ S, T diago

Por hipotesis $S \circ T = T \circ S$, entonces

por un ejercicio del practico 3 ✓

$$\langle u, v \rangle = \|u\| \|v\| \cos\theta$$

$$\langle u, v_0 \rangle = \|u\|' \|v_0\|' \cos\alpha$$

$$\langle u, v \rangle' = \langle u, v \rangle$$

$$\|u\|' = \sqrt{\langle u, u \rangle'}$$