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Práctica 8

Def: Una forma bilineal es una función

$\varphi : V \times V \to \mathbb{K}$ lineal en ambas coordenadas.

$(v,w) \mapsto$

Ej: $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ es bilineal

1.a) $V = \mathbb{K}$, $\varphi(x,y) = x + 2y$

i) $\varphi(ax+z, y) = a\varphi(x,y) + \varphi(z,y)$

ii) $\varphi(x, by+z) = b\varphi(x,y) + \varphi(x,z)$

Hay que verificar.

i) $\varphi(ax+z, y) = (ax+z) + 2y$

$a\varphi(x,y) = a[x+2y]$

$\varphi(z,y) = z + 2y$

$a\varphi(x,y) + \varphi(z,y) = ax + z + 2ya + 2y$

Molesta

$\neq 0$

No es bilineal

5) Dado V espacio vectorial,

$V^* = \{ \alpha : V \to \mathbb{K} \text{ lineal} \}$

Fijamos $\alpha, \beta \in V^*$, y definimos

$\varphi : V \times V \to \mathbb{K}$

$$\varphi(u,v) = \underbrace{\alpha(u)\cdot\beta}(v)$$

operarios en el cuerpo

Hay que probar: $\varphi(au+w,v) = a\varphi(u,v) + \varphi(w,v)$

$\varphi(u, bw+v) = b\varphi(u,w) + \varphi(u,v)$

g) $X, Y \in M_{m\times n}(\mathbb{K})$ $\quad A \in M_m(\mathbb{K})$ fija

$\varphi(X,Y) = tr(X^t A Y)$

Hay que ver: $\varphi(aX+Z, Y) = a\varphi(X,Y) + \varphi(Z,Y)$

$\varphi(X, bZ+Y) = b\varphi(X,Z) + \varphi(X,Y)$

$$\varphi(aX+Z,Y) = tr([aX+Z]^T A\ Y)$$

$$= tr([a(X)^T + Z^T] A\ Y)$$

$$= tr(aX^T AY + Z^T AY)$$

$$= tr(aX^T AY) + tr(Z^T AY)$$

$$= a\,tr(X^T AY) + tr(Z^T AY)$$

$$= a\varphi(X,Y) + \varphi(Z,Y)$$

Def: Dados $\mathcal{B} = \{v_1, \dots, v_n\}$ base de V y

$\varphi: V \times V \to \mathbb{K}$ una forma bilineal.

Definimos la matriz asociada a φ

en la base $\mathcal{B}$ como:

$$M_{\mathcal{B}}(\varphi) := (\varphi(v_i, v_j))_{ij} = \begin{pmatrix} \varphi(v_1,v_1) & \cdots & \varphi(v_1,v_n) \\ \vdots & & \vdots \\ \varphi(v_n,v_1) & \cdots & \varphi(v_n,v_n) \end{pmatrix} \in M_n(\mathbb{K})$$

Ej:

Sea $\langle \, , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ el P.I. usual.

Sea $\mathcal{B} = \{(1,0), (0,1)\}$

$$M_{\mathcal{B}}(\langle \, , \rangle) = \begin{pmatrix} \langle (1,0), (1,0) \rangle & \langle (1,0), (0,1) \rangle \\ \langle (0,1), (1,0) \rangle & \langle (0,1), (0,1) \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Si $\mathcal{B} = \{(1,1), (0,-1)\}$

$$M_{\mathcal{B}}(\langle \, , \rangle) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

2.6) $X, Y \in M_2(\mathbb{K})$, $\varphi(X,Y) = tr(X)\,tr(Y)$

$$\mathcal{B} = \left\{ \overset{m_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{m_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{m_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{m_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

$\varphi(M_1, M_1) = tr(M_1)tr(M_1) = 1$

$\varphi(M_1, M_2) = tr(M_1)tr(M_2) = 0$

$\varphi(M_1, M_3) = 0$

$\varphi(M_1, M_4) = 1$

$$M_{\mathcal{E}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & & & \end{pmatrix}$$

$\varphi(M_2, M_1) = 0$

$\varphi(M_2, M_2) = 0$

$\varphi(M_2, M_3) = 0$

$\varphi(M_2, M_4) = 0$

$$M_{\mathcal{E}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\varphi(M_3, M_i) = 0 \quad \forall i \in \{1,2,3,4\}$, porque $tr(M_3) = 0$

$\varphi(M_4, M_1) = 1$

$\varphi(M_4, M_2) = 0$

$\varphi(M_4, M_3) = 0$

$\varphi(M_4, M_4) = 1$

$$M_{\mathcal{E}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Obs: $M_{\mathcal{E}}(\varphi)$ es simétrica.

Def: Una forma bilineal $\varphi : V \times V \to \mathbb{K}$ es simétrica si $\varphi(v, w) = \varphi(w, v) \quad \forall v, w \in V$

En el ejercicio: $\varphi(X, Y) = tr(X)tr(Y) = tr(Y)tr(X) = \varphi(Y, X)$

Def: Dada $\varphi: V\times V \to \mathbb{K}$ una forma bilineal definimos la forma cuadrática asociada a φ como la función

$\Phi: V \to \mathbb{K}$, $\Phi(v) = \varphi(v,v)$

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left] & V\times V \arrow[r, "\varphi"] & \mathbb{K} \end{tikzcd}$$

$$v \longmapsto (v,v)$$

Ej: Si φ es $\langle\, ,\rangle$, ¿$\Phi(v)$ quién es?

$\Phi(v) = \varphi(v,v) = \langle v,v\rangle = \|v\|^2$

- Si $\varphi(X,Y) = tr(X)\,tr(Y) \Rightarrow \Phi(X) = tr(X)^2$

- $\varphi((x,y,z),(x',y',z')) = xx' - 2xy' + yx' - zz'$

$\Phi(x,y,z) = \varphi((x,y,z),(x,y,z)) = x^2 - 2xy + yx - z^2$
$= x^2 - xy - z^2$

Def: Dada $\varphi: V\times V \to \mathbb{K}$ bilineal, el radical de φ es $V_0 = \{v\in V : \varphi(v,w) = 0 \ \forall w \in W\}$

- φ se dice no degenerada si $V_0 = \{0_V\}$

3) $\varphi(u,w) = \dfrac{\Phi(u+w) - \Phi(u) - \Phi(w)}{2}$

$\hookrightarrow \operatorname{car} \mathbb{K} \neq 2$

Conocemos Φ $\Rightarrow$ Podemos hallar φ

$\Rightarrow$ Podemos calcular el radical

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4) $\operatorname{Bil} = \{ \varphi : V \times V \to \mathbb{K} \text{ bilineal} \}$

$\operatorname{Bil}_S = \{ \varphi : V \times V \to \mathbb{K} \text{ bilineal simétrica} \}$

$\varphi(v,w) = \varphi(w,v)$
$\forall v,w \in V.$

$\varphi \in \operatorname{Bil}_S(M_2(\mathbb{K}))$, $\varphi(X,Y) = tr(X)\,tr(Y)$

Sea $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_V$

$\varphi(A,Y) = tr(A)\,tr(Y) = 0 \quad \forall Y \Rightarrow A \in \operatorname{rad}(\varphi)$

Entonces φ es degenerada.

$\operatorname{rad} \varphi = \{ B \in M_2(\mathbb{K}) : \varphi(B,Y) = 0 \quad \forall Y \}$

$= \{ B \in M_2(\mathbb{K}) : tr(B) = 0 \}$

$= \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{K} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathbb{K} \right\}$

$\oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{K} \right\}$

$\text{rad}\,\varphi = \{ B \in M_2(\mathbb{K}) : B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{K} \}$

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5) $\varphi : M_2(\mathbb{K}) \times M_2(\mathbb{K}) \to \mathbb{K}$ $\quad \varphi(X, Y) = tr(X\,Y)$

a) $\varphi(X, Y) = tr(X\,Y)$

$\quad\; ||$

$\varphi(Y, X) = tr(YX)$

La manera corta es hallar $M_\beta(\varphi)$ en alguna base y ver si es simétrica.

b) Probar que φ es no - degenerada.

$\to \text{rad}(\varphi) = \{ B \in M_2(\mathbb{K}) : tr(BX) = 0 \;\; \forall X \in M_2(\mathbb{K}) \}$

$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$