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Def: i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

ii) $\langle \alpha v, w\rangle = \alpha \langle v, w\rangle$

iii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

iv) $\langle v, v\rangle > 0$ si y solo si $v \neq 0_v$

$\Rightarrow \langle v, \alpha u\rangle = \overline{\langle \alpha u, v\rangle} = \bar{\alpha}\,\overline{\langle u, v\rangle} = \bar{\alpha}\langle v, u\rangle$

6) $\langle\ ,\ \rangle_{usual}$, $\langle\ ,\ \rangle_{\varphi}$,

$\langle u, v\rangle_{\varphi} = \alpha \langle u, v\rangle_{usual}$ $\quad \alpha \in \mathbb{R}^+$

$\langle u, v\rangle_1 = \sum_{i=1}^{n} |u_i|\,|v_i|$ $\quad (\mathbb{R}^n)$

$\|u\|_1 = \sum_{i=1}^{n} |u_i|^2$ $\leftarrow$ Pensarla

$\mathcal{C}^0(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \text{ continuas} \}$

$\langle f, g\rangle = \int_{-\infty}^{\infty} f g\, dx$, $\quad \langle f+h, g\rangle = \int (f+h)g$

$= \int fg + hg$

$= \int fg + \int hg$

Si $f \neq 0$, $\quad \langle f, f\rangle = \int_{-\infty}^{\infty} f^2\, dx$

Si $f^2 > 0 \Rightarrow \int_{-\infty}^{\infty} f^2 dx > 0$

$\|f+g\| \leq \|f\| + \|g\|$

$\|f+g\| = \langle f+g, f+g\rangle^{1/2} = \left[\int_{-\infty}^{\infty} (f+g)(f+g)\, dx\right]^{1/2}$

$\underset{\sqrt{\langle f, f\rangle}}{\|f\|} + \|g\| = \left[\int_{-\infty}^{\infty} f^2 dx\right]^{1/2} + \left[\int_{-\infty}^{\infty} g^2 dx\right]^{1/2}$

$\left\{\int_{-\infty}^{\infty} f^2 + g^2 + 2fg \, dx\right\}^{1/2} = \left[\int_{-\infty}^{\infty} f^2 dx + \int_{-\infty}^{\infty} g^2 dx + \int_{-\infty}^{\infty} 2fg\, dx\right]^{1/2}$

$\left[\int_{-\infty}^{\infty} f^2 dx + \int_{-\infty}^{\infty} g^2 dx + 2\int_{-\infty}^{\infty} fg\, dx\right]^{1/2} \leq \left[\int_{-\infty}^{\infty} f^2 dx\right]^{1/2} + \left[\int_{-\infty}^{\infty} g^2 dx\right]^{1/2}$

$\sqrt{a} \leq \sqrt{b} \iff a \leq b$, apliquemos cuadrado:

$\|f+g\|^2 \leq [\|f\| + \|g\|]^2$

$\cancel{\int f^2} + \cancel{\int g^2} + \cancel{2}\int fg \leq \cancel{\int f^2} + \cancel{\int g^2} + \cancel{2}\left[\int f^2\right]^{1/2}\left[\int g^2\right]^{1/2}$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$

$¿\int fg \leq \underbrace{\left[\int f^2\right]^{1/2}\left[\int g^2\right]^{1/2}}_{\left(\int f^2 g^2\right)^{1/2}}? \qquad \int f^2 \leq \left(\int f^4\right)^{1/2}$

Observar: A veces probar abstractamente es más fácil, que un caso particular.

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2) Cauchy-Schwarz : $|\langle v, w\rangle| \leq \|v\|\,\|w\|$

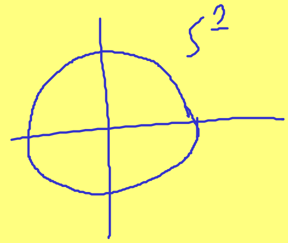
$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^* = (\bar{x}, \bar{y})$ ← Para el ejercicio 3

Si $z=(z_1, z_2) \in \mathbb{C}^2$, entonces

$|z| = z\bar{z}$, Ojo: $i^2 = -1$

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$

S^1



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3) $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$, $x, y \in \mathbb{C}^2$

$$\langle x, y \rangle = x A y^*$$

$$\left.\begin{array}{l} X = (x_1, x_2) \\ Y = (y_1, y_2) \end{array}\right\} \langle x, y \rangle = (x_1, x_2) \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

5) Gram-Schmidt.

Un conjunto L.I. $\longrightarrow$ $\boxed{Gr-S}$ $\longrightarrow$ Conjunto ortogonal

$S = \{v_1, v_2, \dots, v_n\}$. $\qquad$ $T = \{w_1, w_2, \dots, w_n\}$

Además $\qquad [S] = [T]$.

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\vdots$$

$$w_n = v_n - \sum_{i=1}^{n-1} \frac{\langle v_n, w_i \rangle}{\|w_i\|^2} w_i$$

Método Gram-Schmidt

1) $S = \{ (1,0,1), (0,1,1), (1,3,3) \}$, Base de $V = \mathbb{R}^3$

$$(\lambda, 0, \lambda) + (0, \mu, \mu) + (t, 3t, 3t) = (0,0,0)$$

$$\begin{cases} \lambda + t = 0 \\ \mu + 3t = 0 \\ \lambda + \mu + 3t = 0 \end{cases} \leadsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

S es una base de $\mathbb{R}^3$, pues es L.I. con 3 elementos.

Aplicando G-S, obtenemos una base ortogonal.

$w_1 = v_1 = (1,0,1)$

$w_2 = v_2 - \dfrac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

$S = \{ (1,0,1), (0,1,1), (1,3,3) \}$

$= (0,1,1) - \dfrac{1}{2}(1,0,1) = \left(-\dfrac{1}{2}, 1, \dfrac{1}{2}\right)$

$\| w_2 \| = \sqrt{\dfrac{1}{4} + 1 + \dfrac{1}{4}} = \sqrt{2 + \dfrac{2}{4}}$
$= \sqrt{\dfrac{3}{2}}$

$\langle v_3, w_2 \rangle = -\dfrac{1}{2} + 3 + \dfrac{3}{2} =$
$= 3 + \dfrac{2}{2} = 4$

$w_3 = v_3 - \dfrac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \dfrac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$

$= (1,3,3) - \dfrac{4}{2}(1,0,1) - \dfrac{4}{3/2}\left(-\dfrac{1}{2}, 1, 1/2\right)$

$\dfrac{4}{2} = 2 \qquad \dfrac{4}{3/2} = \dfrac{8}{3}$

$w_3 = \left(1 - 2 + \dfrac{8}{6}, 3 - \dfrac{8}{3}, 3 - 2 - \dfrac{8}{6}\right) = \left(\dfrac{1}{3}, \dfrac{1}{3}, -\dfrac{1}{3}\right)$

$T = \{ (1,0,1), (-\tfrac{1}{2}, 1, 1/2), (1/3, 1/3, -1/3) \}$

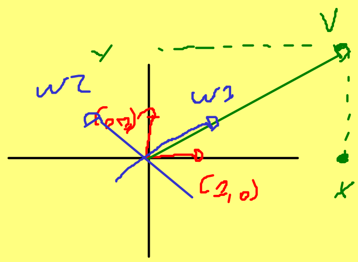
Base ortogonal, queremos ortonormal

Para eso dividimos cada vector por su norma.

Coef. de Fourier :

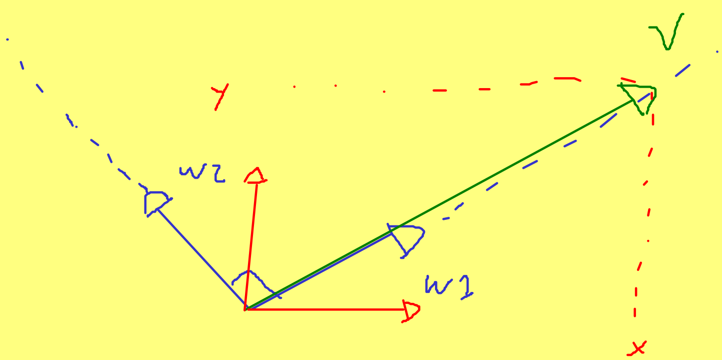
Si $\mathcal{B} = \{w_1, \dots, w_n\}$ es una base ortonormal, entonces los coeficientes de Fourier de $v \in V$ son $\langle w_i, v \rangle$

$$v = \sum_{i=1}^{n} \langle w_i, v \rangle \, w_i$$



$\langle (1,0), v \rangle = x$

$\langle (0,1), v \rangle = y$



$\langle w_2, v \rangle = 0$

$\langle w_1, v \rangle = \|v\|$

$$\langle 1, x \rangle = \int_0^1 x \, dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

Obs: $\{1, x, x^2\}$ no es ortogonal

$$\langle p, q \rangle = \int_0^1 p q \, dx$$