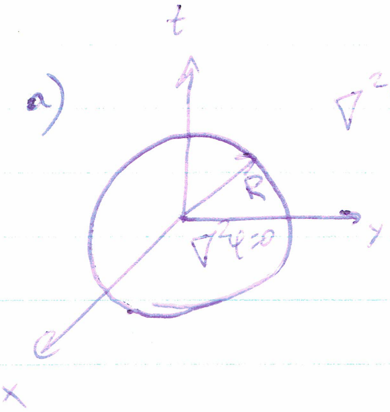
PARCIAL 1 - ELECTRO

1) a)



$\nabla^2\varphi = 0$

(I) $r < R$

(II) $r > R$

$$\varphi(\vec{r}) = \begin{cases} \varphi_I(\vec{r}) & si\ r<R \\ \varphi_{II}(\vec{r}) & si\ r>R \end{cases}$$

$$\varphi_I = \sum_{n=0}^{+\infty} \left(A_n^{(I)} r^n + \frac{B_n^{(I)}}{r^{n+1}} \right) P_n(\cos\theta)$$

$$\varphi_{II} = \sum_{n=0}^{+\infty} \left(A_n^{(II)} r^n + \frac{B_n^{(II)}}{r^{n+1}} \right) P_n(\cos\theta$$

C.B.

i) $\lim_{r\to\infty} \varphi(\vec{r}) = 0$

ii) $\lim_{r\to 0} \varphi(\vec{r}) < +\infty$

iii) $\varphi(R,\theta) = K\cos(3\theta) = 4K\cos^3(\theta) - 3K\cos\theta = \frac{K}{5}\left[-3P_1(\cos\theta) + 8P_3(\cos\theta)\right]$

i) $\lim_{r\to\infty} \varphi(\vec{r}) = \lim_{r\to\infty} \varphi_{II}(\vec{r}) = \lim \sum_{n=0}^{+\infty} \left(A_n^{(II)} r^n + \frac{B_n^{(II)}}{r^{n+1}} \right) P_n(\cos\theta) = 0$

$$\Rightarrow \boxed{A_n^{(II)} = 0 \quad \forall n}$$

ii) $\lim_{r\to 0} \varphi(\vec{r}) = \lim_{r\to 0} \varphi_I(\vec{r}) = \lim \sum_{n=0}^{+\infty} \left(A_n^{(I)} r^n + \frac{B_n^{(I)}}{r^{n+1}} \right) P_n(\cos\theta) < +\infty$

$$\Rightarrow \boxed{B_n^{(I)} = 0 \quad \forall n}$$

iii) $\varphi(R,\theta) = \frac{K}{5}\left[-3P_1(\cos\theta) + 8P_3(\cos\theta)\right] = \sum_{n=0}^{+\infty} \frac{B_n^{(II)}}{R^{n+1}} P_n(\cos\theta) = \sum_{n=0}^{+\infty} A_n^{(I)} R^n P_n(\cos\theta)$

$$\Rightarrow \boxed{A_n^{(I)} = B_n^{(II)} = 0 \quad \forall\, n \neq 1 \text{ y } n \neq 3}$$

$$\boxed{A_1^{(I)} = -\frac{3K}{5R}} \qquad \boxed{A_3^{(I)} = \frac{8K}{5R^3}}$$

$$\boxed{B_1^{(II)} = -\frac{3KR^2}{5}} \qquad \boxed{B_3^{(II)} = \frac{8KR^4}{5}}$$

$$\Rightarrow \boxed{\varphi(r,\theta) = \begin{cases} -\dfrac{3kr}{5R} P_1(\cos\theta) + \dfrac{8kr^3}{5R^3} P_3(\cos\theta) & \text{si } r<R \\[2mm] -\dfrac{3kR^2}{5r^2} P_1(\cos\theta) + \dfrac{8kR^4}{5r^4} P_3(\cos\theta) & \text{si } r>R \end{cases}}$$

$$\sigma(\theta) = -\varepsilon_0 \left(\frac{\partial\varphi}{\partial r}\Big|_{R^+} - \frac{\partial\varphi}{\partial r}\Big|_{R^-} \right) = -\varepsilon_0 \left(\frac{\partial\varphi_{II}}{\partial r}\Big|_{R} - \frac{\partial\varphi_{I}}{\partial r}\Big|_{R} \right)$$

$$= -\varepsilon_0 \left(\frac{6kR^2}{5R^3} P_1(\cos\theta) - \frac{32kR^4}{5R^5} P_3(\cos\theta) + \frac{3k}{5R} P_1(\cos\theta) - \frac{24kR^2}{5R^3} P_3(\cos\theta) \right)$$

$$\Rightarrow \boxed{\sigma(\theta) = -\frac{\varepsilon_0 k}{5R} \left(9 P_1(\cos\theta) - 56 P_3(\cos\theta) \right)}$$

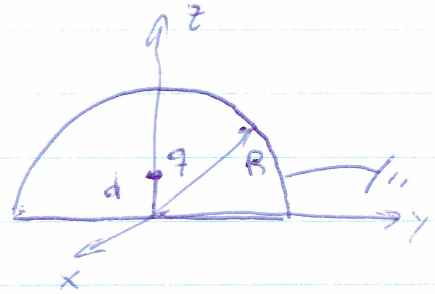
b) Debido al desarrollo multipolar, el potencial a puntos lejanos es

$$\varphi(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \ldots$$

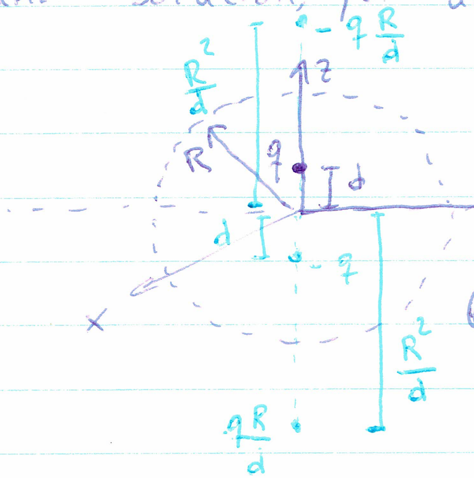
Por simetría $\vec{p}$ apunta según $\hat{k}$ (o $-\hat{k}$)

$\Rightarrow$ comparando con $\varphi_{II}(\vec{r})$ es inmediato $\boxed{Q=0}$ y $\boxed{\vec{p} = -\dfrac{12\pi k\varepsilon_0 R^2}{5}\hat{k}}$

2)



pruebo resolver el problema con imágenes, si encuentro una solución, por unicidad es la solución.



$$\vec{d} = d\hat{k}$$

$$(*)\quad \varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0|\vec{r}-\vec{d}|} - \frac{q}{4\pi\varepsilon_0|\vec{r}+\vec{d}|} - \frac{q\frac{R}{d}}{4\pi\varepsilon_0|\vec{r}-\frac{R^2}{d}\hat{k}|} + \frac{q\frac{R}{d}}{4\pi\varepsilon_0|\vec{r}+\frac{R^2}{d}\hat{k}|}$$

$$\Rightarrow \varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}} - \frac{R}{d}\frac{1}{\sqrt{r^2+\frac{R^4}{d^2}-2r\frac{R^2}{d}\cos\theta}} + \frac{R}{d}\frac{1}{\sqrt{r^2+\frac{R^4}{d^2}+2r\frac{R^2}{d}\cos\theta}}\right]$$

$$\varphi(R,\theta,\phi) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+d^2-2Rd\cos\theta}} - \frac{1}{\sqrt{R^2+d^2+2Rd\cos\theta}} - \frac{1}{\sqrt{\frac{d^2}{R^2}R^2+\frac{d^2}{R^2}\frac{R^4}{d^2}-\frac{d^2}{R^2}\frac{2R^3}{d}\cos\theta}} + \frac{1}{\sqrt{\frac{d^2}{R^2}R^2+\frac{d^2}{R^2}\frac{R^4}{d^2}+\frac{d^2}{R^2}\frac{2R^3}{d}\cos\theta}}\right]$$

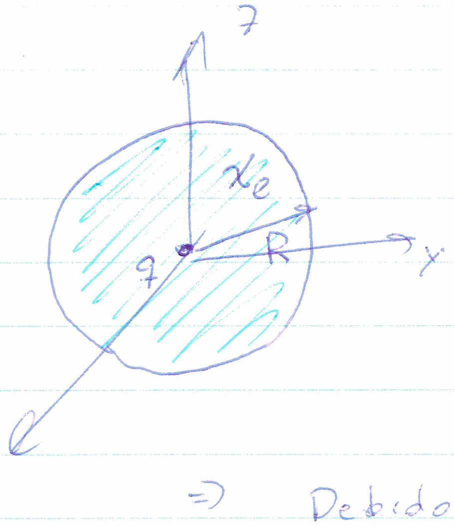
y por simetría es evidente que $\varphi(x,y,0)=0$

$\Rightarrow$ (*) es la solución $\Rightarrow$ $$\boxed{\vec{F}_{/q} = \vec{E}_q q = \frac{q^2\hat{k}}{4\pi\varepsilon_0}\left[-\frac{1}{4d^2} + \frac{R/d}{(\frac{R^2}{d}-d)^2} + \frac{R/d}{(\frac{R^2}{d}+d)^2}\right]}$$

con $\vec{E}_q$ el campo que generan todas las otras cargas menos q

3)

a)



$$\vec{D} = \varepsilon_0 \vec{E} + P$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \Rightarrow \vec{D} = \varepsilon \vec{E}$$

si $r < R$ con $\varepsilon = \begin{cases} \varepsilon_0 & \text{si } r > R \\ \varepsilon_0(1+\chi_e) & \text{si } r < R \end{cases}$

$\Rightarrow$ Debido a la simetría del problema puedo aplicar Gauss $\left(\vec{E}(\vec{r}) = E(r)\,\hat{e}_r \right)$

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{q_{enc}}{\varepsilon_0}$$

$$\oint_S \vec{E}\cdot\hat{n}\,da = 4\pi r^2 E(r)$$

siendo S la superficie de una esfera de radio r centrada en el origen

$$\Rightarrow E(r) = \frac{q_{enc}}{4\pi\varepsilon_0 r^2}$$

si $r > R$ $\boxed{q_{enc} = q}$ porque el dieléctrico es neutro

si $r < R$ es

$$\frac{\rho_{total}}{\varepsilon_0} = \nabla\cdot\vec{E} = \frac{1}{\varepsilon}\nabla\cdot\vec{D} = \frac{\rho_{libre}}{\varepsilon} = \frac{q\,\delta^{(3)}(\vec{r})}{\varepsilon}$$

$$\Rightarrow \rho_{total} = q\frac{\varepsilon_0}{\varepsilon}\delta^{(3)}(\vec{r})$$

$$\Rightarrow \boxed{q_{enc} = q\frac{\varepsilon_0}{\varepsilon}}$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon r^2}\hat{e}_r & \text{si } r < R \\ \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{e}_r & \text{si } r > R \end{cases}$$

$$\Rightarrow \vec{P} = \begin{cases} \dfrac{q\chi_e\varepsilon_0}{4\pi\varepsilon r^2}\hat{e}_r & \text{si } r < R \\ \vec{0} & \text{si } r > R \end{cases}$$

b) si $r < R$ $\qquad \nabla \cdot \vec{P} = -\rho_p = \varepsilon_0 \chi_e \nabla \cdot \vec{E} = \dfrac{\varepsilon_0 \chi_e q \delta(\vec{r})}{\varepsilon_0}$

$$\vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{e}_r = \sigma_p = \frac{q \chi_e \varepsilon_0}{4\pi \varepsilon R^2}$$

c) $Q_{sup} = \int \sigma \, da = \dfrac{q \chi_e \varepsilon_0}{4\pi \varepsilon R^2} \displaystyle\int_0^{2\pi}\int_0^{\pi} R^2 \operatorname{sen}\theta \, d\theta \, d\phi = \dfrac{q \chi_e \varepsilon_0}{\varepsilon} = \dfrac{q \chi_e}{1+\chi_e}$

d) como se respondió en b) el resto de la carga del dieléctrico rodea a la carga puntual apantallándola.