RADIATION RESEARCH **109**, 184–189 (1987)

On Fano's and O'Connor's Theorems

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BJÄRNGARD, B. E. On Fano's and O'Connor's Theorems. *Radiat. Res.* **109**, 184–189 (1987).

Fano's theorem states that the fluence of particles, emitted uniformly per unit mass, is constant throughout an infinite medium of uniform composition but varying density. O'Connor's scaling theorem says that the ratio of the fluence of secondary particles to that of primary particles, caused by an external source irradiating a medium in a collimated beam, is the same in two uniform media of the same composition but different density, provided geometrical distances are scaled inversely to density. These two theorems are proved by one line of reasoning. The scaling theorem is given a more general formulation. © 1987 Academic Press, Inc.

INTRODUCTION

Fano's theorem was originally formulated in 1954 (*1*): "In a medium of given composition exposed to a uniform flux of primary radiation (such as X rays or neutrons) the flux of secondary radiation is also uniform and independent of the density of the medium as well as of the density variations from point to point." More specific conditions for the validity of the theorem were provided by Failla (*2*): "the interactions of the primary radiation and the secondary radiation with the atoms of the medium are both independent of its density." Harder (*3*) pointed out that the uniform primary flux has to extend throughout a volume which is large compared to the range of the secondary particles.

O'Connor's scaling theorem (*4*), on the other hand, compares two situations where media of uniform density but finite dimensions are irradiated by photons from an external source in a collimated beam. The media differ only in density. The geometries are such that all linear measures (source positions, beam edges) relate with a single scaling factor, inversely proportional to density. Two points are said to be "corresponding" if they occupy the same relative position in the two systems, and the theorem states that the ratio of scattered to primary photon fluences is the same in corresponding points.

Fano's theorem addresses a situation with what he called "flux balance," i.e., "radiation equilibrium" (*5*, *6*), while O'Connor's theorem deals with conditions where such equilibrium does not exist. In spite of this difference, one may anticipate that the two theorems have a common foundation, since both derive from the fact that the probabilities of interaction between both the primary and the secondary radiation

0033-7587/87 $3.00

and the atoms of the medium are independent of density. This is explored in this study.

THEORETICAL

General

Consider a medium of varying density ρ in which particles of various energies E are emitted and move in different directions $\mathbf{u}$. The particles interact with the medium with probabilities per unit mass that do not vary from point to point. (All particles do not have to be of the same kind. We may for instance consider photons and electrons. In the following we will omit this aspect. The generalization follows easily.) We divide the particles into groups, characterized by energy and direction, and single out for the initial discussion those belonging to the energy interval $(E, E + \Delta E)$ and the directional interval $(\mathbf{u}, \mathbf{u} + \Delta\mathbf{u})$.

We will first demonstrate that the fluence $\psi(E, \mathbf{u})$ of this particular group of particles is uniform if its mass–source function $S(E, \mathbf{u})$, i.e., the number of particles emitted per unit mass, is constant. When these particles undergo interactions, they contribute to the mass–source functions for particles of other energies and directions. If $\psi(E, \mathbf{u})$ is uniform, these mass–source functions for subsequent generations of particles are also uniform, since the interaction probabilities per unit mass do not depend on position r. Hence, if the initial (primary) particle has a uniform mass–source function or fluence, so do all subsequently generated particle groups.

This approach differs from Fano's proof only in minor details, primarily that Fano considered isotropic fluences and source functions (*1*). However, to derive O'Connor's theorem with the same formalism, we start with an expression that describes particle radiance, i.e., the fluence $\psi = \psi(E, \mathbf{u})$ of paricles of energy $(E, E + \Delta E)$ coming in toward the point of observation from a given solid angle $\Delta\mathbf{\Omega}$ $(\Delta\mathbf{\Omega} = \Delta\mathbf{u})$ in the direction $\mathbf{\Omega} = -\mathbf{u}$.

The probability that such a particle can travel a distance R without undergoing such an interaction that it loses energy and/or changes direction sufficiently to leave this group $(E, E + \Delta E)$, $(\mathbf{u}, \mathbf{u} + \Delta\mathbf{u})$ is

$$\exp[-\int_0^R k(E)\rho(r)dr]. \tag{1}$$

Here k, the probability, per unit mass per unit area, of the interaction does not depend on the density and hence not on the position r. This expression, Eq. (1), is justified if ΔE and $\Delta\mathbf{u}$ are small enough.

Using Eq. (1) the fluence ψ of particles of energy $(E, E + \Delta E)$ and direction $(\mathbf{u}, \mathbf{u} + \Delta\mathbf{u})$ at a point can then be written

$$\psi = \int_0^\infty \{S(R)\rho(R)\exp[-\int_0^R k\rho(r)dr]\}dR. \tag{2}$$

The notation has been simplified in that it is not shown that k is a function of E and that ψ and S are functions of E and $\mathbf{u}$.

Fano's Theorem

If the number of particles emitted per unit mass is constant and does not depend on position, i.e., $S(R) = S$, Eq. (2) becomes

$$\psi = S \int_0^\infty \{\rho(R)\exp[-\int_0^R k\rho(r)dr]\}dR. \tag{3}$$

This can be written as

$$\psi = \frac{S}{k}\int_\infty^0 \frac{d}{dR}[\exp-\int_0^R k\rho(r)dr]dR = \frac{S}{k} \tag{4}$$

if $\int_0^\infty k\rho(r)dr = \infty$, which is fulfilled in an infinite medium, independently of how $\rho(R)$ varies. Hence, if S does not depend on R, neither does ψ, and ψ is the same as in a medium of uniform density with the same mass–source function. If S or ψ is constant for an original (primary) group of particles, so are these functions for all subsequent generations of particles. This proves Fano's theorem. Since Eq. (4) is formulated for the fluence differentiated in energy and direction, it follows that the energy spectrum and the directional distribution do not vary from point to point either.

In the particular case of isotropic source–function we can integrate over all directions with the same conclusion, which is how Fano formulated the theorem (*1*).

O'Connor's Theorem

O'Connor's theorem compares two situations with media of uniform densities, ρ and ρ'. The source functions are no longer constant but are $S(R)$ and $S'(R')$, respectively. Using Eq. (2), the fluences ψ and ψ' of particles of energy $(E, E + \Delta E)$ and direction $(\mathbf{u}, \mathbf{u} + \Delta\mathbf{u})$ in the two geometries are

$$\psi = \int_0^\infty S(R)\,\rho(R)\exp[-\int_0^R k\rho(r)dr]dR \tag{5a}$$

$$\psi' = \int_0^\infty S'(R')\,\rho'(R')\exp[-\int_0^{R'} k'\rho'(r')dr']dR'. \tag{5b}$$

It is obvious that

$$\psi' = \psi \text{ if } k'\rho'(r')dr' = k\rho(r)dr$$

and

$$S'(R')\rho'(R')dR' = S(R)\rho(R)dR.$$

These conditions are met if

$$k' = k \tag{6a}$$

$$S'(R') = S(R) \tag{6b}$$

$$R/R' = r/r' = \rho'(R')/\rho(R). \tag{6c}$$

Since ψ and ψ' contribute to the source functions for particles of lower energies and/or different directions, it follows that if $\psi = \psi'$ for the original (primary) particles this is also true for subsequent generations of particles. Hence the energy spectra, the directional distributions, and the total fluences are equal in corresponding points, defined by Eq. (6c), if $k'(E, \mathbf{u}) = k(E, \mathbf{u})$ and if Eq. (6b) is satisfied for the primary particles.

For collimated beams, the situation that O'Connor addressed (*4*), $S(R) = 0$ for $R > R_m(\mathbf{\Omega})$, where R_m is the field edge in the direction $\mathbf{\Omega}$. Equation (5a) can then be written

$$\psi = \int_0^{R_m} \{S(R)\rho(R)\exp[-\int_0^R k\rho(r)dr]\}dR \tag{7}$$

The corresponding expression can be written for ψ'. In the two systems $\psi' = \psi$ if $R_m/R'_m = \rho'/\rho$ as a special case of Eq. (6c). As before, if this is fulfilled for the primary particles, it is true for subsequent generations of particles of any energy E and direction $\mathbf{u}$.

In its original and usual formulation (*4*), O'Connor's theorem addresses the *ratio* of scattered and primary photon fluences. The derivation above is somewhat more restrictive in that conditions were defined for $\psi = \psi'$ and expressed as Eqs. (6a)–(6c). If conditions for $\psi/\psi' =$ constant are sought instead, it easily can be shown that it is sufficient that the quantities in Eqs. (6a)–(6c) are proportional rather than equal.

O'Connor's theorem is usually restricted to the comparison of two media of uniform densities and the same atomic composition. In this case, $k = k'$ as required by Eq. (6a) if the probabilities of the interactions are independent of density. There may, however, be other situations when $k = k'$ even when the compositions are different. One such case (Compton scattering of photons) is briefly discussed below.

DISCUSSION

Equations (5a) and (5b) with the conditions Eqs. (6a)–(6c) are more general than O'Connor's formulation in that the two media do not have to be of uniform density and the sources do not have to be external point sources. The fluences in two corresponding points remain the same if the mass–source functions $S(R) = S'(R')$ and the ratio of densities in any pair of corresponding points is constant and inversely proportional to the scaling factor for the linear distances. The "external source" that O'Connor discussed can be thought of as internal and located in a portion of the volume of vanishingly low density. It is then obvious that for the fluences to remain the same, the source-to-surface distance must scale as other linear dimension (Eq. 6c) and the total source emission I must scale as $I'/I = (\rho/\rho')^2$. This is the way Spencer formulated the scaling theorem (*7*).

One may note that Eq. (5a) can be written, by integration-in-parts,

$$\psi = \int_0^{\infty} \left\{\frac{S(R)}{k}\frac{d}{dR}\exp[-\int_0^R k\rho(r)dr]\right\}dR$$

$$= \frac{S(0)}{k} - \int_0^{\infty} \left\{\frac{dS}{dR}k^{-1}\exp[-\int_0^R k\rho(r)dr]\right\}dR. \tag{8}$$

This formulation expresses the need for a "gradient correction factor" when considering the fluence in a situation which does not satisfy the conditions for Fano's theorem. The "gradient correction factor" was introduced in a recent protocol for the determination of absorbed dose from high-energy photon and electron beams (*8*). This protocol separates the "replacement correction factor" into a "gradient correction factor" and an "electron fluence correction factor." Equation (8) shows that this separation is somewhat artificial (*9*) and that the electron fluence changes only when gradients are present in the primary field, i.e., when $dS/dR \neq 0$. This minor point does not, however, detract from the value of this protocol (*8*).

Both theorems require that the probabilities for particle interactions per unit mass (k) are independent of density. For electrons, k is the mass-stopping power and the polarization effect introduces a density dependence for electrons of high energies, as pointed out, e.g., by Roesch (*5*) and recently examined by O'Connor (*10*). With this exception, Fano's theorem and the scaling theorem are applicable to photons, to neutrons, and to electrons and other charged particles.

Under certain conditions it is possible to relax the requirement of *uniform* composition (Fano's theorem) or the *same* composition (O'Connor's theorem). For instance, for photons of such energies that Compton scattering dominates the interaction, one may apply the scaling theorem, provided the density ρ in Eq. (6c) is replaced by the electron density. This was utilized by Pruitt and Loevinger (*11*) when discussing the scaling of measurement of ^{60}Co γ-radiation doses in different media.

CONCLUSION

Fano's theorem and O'Connor's scaling theorem can be derived using the same approach. The latter relation can be given a more general formulation. A particularly interesting situation occurs when comparing two systems of uniform and the same composition but varying density, for which a linear scaling can be found such that in any two corresponding points in the two geometries the ratio of densities is constant, while the mass–source functions of primary particles are equal. The fluences of particles are then the same in corresponding points and so are their energy spectra and directional distributions, if the interaction probabilities are independent of density.

ACKNOWLEDGMENTS

The author appreciates the helpful discussions with Robert L. Siddon and Paula L. Petti. This study was supported in part by PHS Grant CA 34964, awarded by the National Cancer Institute.

RECEIVED: May 5, 1986; REVISED: October 21, 1986

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