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Citation: The Journal of the Acoustical Society of America **49**, 841 (1971); doi: 10.1121/1.1912424
View online: https://doi.org/10.1121/1.1912424
View Table of Contents: https://asa.scitation.org/toc/jas/49/3B
Published by the Acoustical Society of America

Received 28 September 1970 13.11t; 5.18

The Time-Dependent Force and Radiation Impedance on a Piston in a Rigid Infinite Planar Baffle*

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An approach is presented to compute the time-dependent force acting on a piston in a rigid infinite planar baffle as a result of the specified velocity of the piston. The approach to computing the force is applicable to both sinusoidal and nonsinusoidal velocity pulses and is valid for all piston shapes. The approach, which is based on a Green's-function solution to the time-dependent boundary value problem, utilizes a transformation of coordinates to simplify the evaluation of the double surface integrals. An impulse response function is defined such that the time-dependent force can be obtained by differentiating the convolution of the impulse response and piston velocity time functions. A closed-form expression for the impulse response of a circular piston is derived and discussed. Numerical results for the impulse response and the forces on large square pistons resulting from sinusoidal piston velocities are then presented and discussed to compare the transient and steady-state behavior of the forces. Finally, an approach is presented to compute the radiation impedance as a function of normalized frequency from the impulse response data, and the approach is used to obtain the normalized radiation resistance and reactance for square pistons.

INTRODUCTION

The area of acoustic transient phenomena has undergone considerable recent investigation. Previous investigators have considered several transient problems of interest and have obtained solutions to the appropriate boundary value problems consisting of the time-dependent wave equation with initial and boundary value conditions. Among these problems of interest, the transient acoustic loading on a baffled circular piston[1,2] and on a baffled strip[3] have been investigated using Laplace and Fourier transform techniques. In addition, the study of transient acoustic pressures generated by impulsively accelerated three-dimensional bodies has recently been investigated[4] along with the energy exchange between the near- and farfield.[5] In a still more recent article, Freedman investigated the time-dependent sound field from radiators in large rigid planar baffles[6] and also published a tutorial paper on the transient fields of acoustic radiators.[7]

This paper presents an extension of an approach developed in an earlier paper[8] to computing time-dependent acoustic interaction forces among pistons in a planar array. The present paper is concerned with the transient and steady-state acoustic loading on a baffled piston of any shape. An approach is now presented to compute the time-dependent force acting on a piston in a planar baffle as a result of the time-dependent velocity of the piston. Unlike the earlier works, the present approach is based on the use of the Green's-function solution to the time-dependent boundary value problem for an impulsive piston motion. The solution to the time-dependent boundary value problem is then utilized to obtain the steady-state radiation impedance of the piston, and the results are then compared to those of earlier investigators.[9]

I. THEORY

In this section the development of an approach to compute the time-dependent acoustic force on a piston as a result of the time-dependent velocity of the piston is presented. The development is based on results presented in an earlier paper[8]; however, for the sake of completeness, a detailed development of the approach from basic considerations is now presented.

Initially consider the problem of determining the pressure versus time at a spatial point in the half-space $x_3 \geq 0$ resulting from a specified velocity of the piston in Fig. 1. The baffle is considered to be an infinite planar rigid baffle, i.e., the normal velocity is zero, and the problem is formulated as a classical boundary value problem in the velocity potential, $\phi(\mathbf{x},t)$, where $\mathbf{x}$ denotes the spatial point of interest in the half-space.

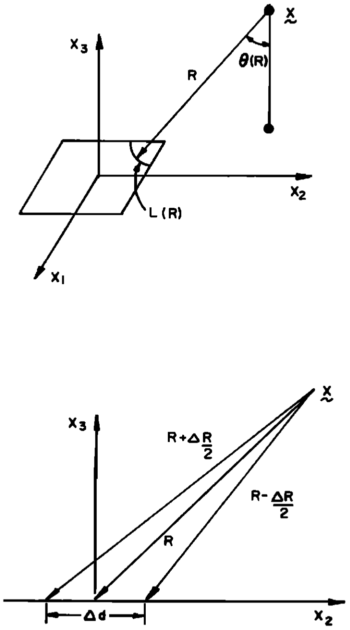

FIG. 1. Geometrical variables used to compute $h(\mathbf{x},t)$.

From the velocity potential, the pressure $p(\mathbf{x},t)$ and the velocity $v(\mathbf{x},t)$ are obtained using the following equations:

$$p(\mathbf{x},t)=\rho\frac{\partial\phi(\mathbf{x},t)}{\partial t} \tag{1}$$

and

$$\mathbf{v}(\mathbf{x},t)=-\nabla\phi(\mathbf{x},t), \tag{2}$$

where ρ is the density of the medium. The mathematical specification of the boundary value problem yields the following system of equations:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=0, \quad \mathbf{x} \text{ in } B, \tag{3}$$

$$\frac{\partial\phi}{\partial x_3}=v(\mathbf{x},t) \;\; \mathbf{x} \text{ on } \sigma', \quad t>0,$$

and

$$\phi(\mathbf{x},t)\Big|_{t=0}=\frac{\partial}{\partial t}\phi(\mathbf{x},t)\Big|_{t=0}=0, \tag{4}$$

where B is the half-space $x_3>0$, σ' is the $x_3=0$ plane, and $\mathbf{x}$ is a point in B and σ'. It is noted $v(\mathbf{x},t)=0$ for $\mathbf{x}$ not on σ, the piston area of the piston.

The solution of the preceding system of equations is easily obtained using a Green's-function development. The solution for the velocity potential may then be expressed as a function of the source coordinates as follows:

$$\phi(\mathbf{x},t)=\int_0^t dt_0\int_\sigma dS\, v(\mathbf{x}_0,t_0)g(\mathbf{x},t|\mathbf{x}_0,t_0), \tag{5}$$

where $g(\mathbf{x},t|\mathbf{x}_0,t_0)$ is the time-dependent Green's function for the problem and $v(\mathbf{x},t)$ is the specified velocity of the piston. Using the method of images, the well-known Green's function[10] for the problem is

$$g(\mathbf{x},t|\mathbf{x}_0,t_0)=\frac{\delta(t-t_0-|\mathbf{x}-\mathbf{x}_0|/c)}{2\pi|\mathbf{x}-\mathbf{x}_0|}. \tag{6}$$

Substituting Eq. 6 into Eq. 5, $\phi(\mathbf{x},t)$ may then be expressed as

$$\phi(\mathbf{x},t)=\int_0^t dt_0 v(t_0)\int_\sigma \frac{\delta(t-t_0-|\mathbf{x}-\mathbf{x}_0|/c)}{2\pi|\mathbf{x}-\mathbf{x}_0|}dS, \tag{7}$$

where the piston is assumed to be rigid, i.e., $v(\mathbf{x},t)=v(t)$. Equation 7 may then be expressed as

$$\phi(\mathbf{x},t)=\int_0^t v(t_0)h(\mathbf{x},t-t_0)dt_0, \tag{8}$$

where

$$h(\mathbf{x},t-t_0)=\int_\sigma \frac{\delta(t-t_0-|\mathbf{x}-\mathbf{x}_0|/c)}{2\pi|\mathbf{x}-\mathbf{x}_0|}dS. \tag{9}$$

It is noted that Eq. 8 is a familiar convolution integral and may be expressed as

$$\phi(\mathbf{x},t)=v(t)*h(\mathbf{x},t), \tag{10}$$

where $*$ denotes the convolution of the indicated time functions.

A simple expression for $h(\mathbf{x},t)$ may be obtained by evaluating the surface integral indicated in Eq. 9. Consider the piston and geometrical variables shown in Fig. 1. The piston which is located in the $x_3=0$ plane may be any shape and the spatial point of interest is indicated by $\mathbf{x}$ in the figure. Locating a spherical coordinate system at the spatial point of interest, the transformation

$$R=|\mathbf{x}-\mathbf{x}_0|$$

is noted, where R may be considered as the radius of a sphere centered at $\mathbf{x}$. Also indicated in the figure is the arc length of intersection $L(R)$ of the piston with the surface of a sphere of radius R centered at $\mathbf{x}$. The angle $\theta(R)$ indicated in the figure is defined by points on the arc length of intersection $L(R)$ and the normal to the $x_3=0$ plane passing through the spatial point of interest. It is further noted $\theta(R)$ is a constant for all points on $L(R)$; however, both $\theta(R)$ and $L(R)$ vary as R varies.

To simplify the evaluation of $h(\mathbf{x},t)$ the transformation is applied to Eq. 9. Since a relationship exists between the incremental surface area ΔS in the $x_3=0$ plane and R, the surface integral may be simplified. In Fig. 1, Δd is shown as the width of the incremental surface element in the $x_3=0$ plane resulting from the two radii, $R-\frac{1}{2}\Delta R$ and $R+\frac{1}{2}\Delta R$. From geometric considerations Δd may be expressed as

$$\Delta d\approx\Delta R/\sin\theta(R). \tag{11}$$

Since the incremental surface area ΔS is equal to

$$\Delta S = L(R)\Delta d, \tag{12}$$

then, utilizing Eq. 11, ΔS may be expressed as

$$\Delta S = L(R)\Delta R/\sin\theta(R). \tag{13}$$

Taking the limit as $\Delta R \rightarrow 0$ in Eq. 13 and utilizing the coordinate transformation, then $h(\mathbf{x},t)$ may be expressed as

$$h(\mathbf{x},t) = \int_0^\infty \frac{\delta(t-R/c)}{2\pi R \sin\theta(R)} L(R)dR. \tag{14}$$

Making the substitution, $\tau = R/c$,

$$h(\mathbf{x},t) = \int_0^\infty \frac{\delta(t-\tau)L(c\tau)}{2\pi\tau \sin\theta(c\tau)} d\tau. \tag{15}$$

Now, using the sifting property of the δ function, Eq. 15 reduces to

$$h(\mathbf{x},t) = L(ct)/2\pi t \sin\theta(ct). \tag{16}$$

For the piston shown in Fig. 1, the time-dependent force acting on the piston is simply written

$$f(t) = \int_\sigma p(\mathbf{x},t)dS. \tag{17}$$

Alternatively, $f(t)$ may be expressed as

$$f(t) = \rho \frac{d}{dt} \int_\sigma \phi(\mathbf{x},t)dS. \tag{18}$$

Now, using the results of Eq. 10, $f(t)$ may be expressed as

$$f(t) = \rho \frac{d}{dt} \int_\sigma h(\mathbf{x},t) * v(t)dS. \tag{19}$$

Since $v(t)$ is independent of σ, $f(t)$ may be expressed as

$$f(t) = \rho \frac{d}{dt}[h^*(t) * v_i(t)], \tag{20}$$

where $h^*(t)$ is defined as

$$h^*(t) = \int_\sigma h(\mathbf{x},t)dS. \tag{21}$$

By performing the indicated differentiation, the time-dependent force may also be expressed in the following forms:

$$f(t) = \rho \left\{ h^*(0)v(t) + \int_0^t \dot{h}^*(t-\tau)v(\tau)d\tau \right\} \tag{22}$$

or

$$f(t) = \rho \left\{ h^*(t)v(0) + \int_0^t h^*(t-\tau)\dot{v}(\tau)d\tau \right\}. \tag{23}$$

To investigate the time-dependent force acting on the piston as a result of the piston velocity, the impulse response $h^*(t)$ is required. A simple expression for $h^*(t)$ is now presented. Substituting the closed-form expression for $h(\mathbf{x},t)$ shown in Eq. 16 into Eq. 9, the following expression is obtained:

$$h^*(t) = \int_\sigma \frac{L(\mathbf{x},ct)dS}{2\pi t \sin\theta(\mathbf{x},ct)}. \tag{24}$$

For points on the piston area $\theta(\mathbf{x},ct) \equiv 90°$; thus $\sin\theta(\mathbf{x},ct) \equiv 1$. The impulse response function $h^*(t)$ can then be simply expressed as

$$h^*(t) = \frac{1}{2\pi t} \int_\sigma L(\mathbf{x},ct)dS, \tag{25}$$

where the $1/t$ variation is independent of the surface integral. In general, a closed-form expression of $h^*(t)$ for a specified piston shape cannot be obtained.

An approach to evaluate the normalized mutual impedance coefficients from the impulse response data $h^*(t)$ is now presented. The approach utilizes a numerical evaluation of the Fourier transform of $h^*(t)$. The Fourier transform of a function $g(t)$ is now defined as follows:

$$G(j\omega) = F\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt, \tag{26}$$

where an upper-case letter denotes the transform. Thus, the inverse transform is defined as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)e^{+j\omega t}d\omega. \tag{27}$$

From Eq. 20 the following equation is easily obtained:

$$F(j\omega) = \rho(j\omega)H^*(j\omega)V(j\omega). \tag{28}$$

The radiation impedance of a rigid piston is commonly defined as follows:

$$Z_R(j\omega) = F(j\omega)/V(j\omega), \tag{29}$$

where it is noted that the force, $F(j\omega)$, is the force acting on the piston and is thus identical to the force in Eq. 28. From Eqs. 28 and 29 the radiation impedance may then be expressed as follows:

$$Z_R(j\omega) = \rho j\omega H^*(j\omega). \tag{30}$$

The evaluation of $Z_R(j\omega)$ is a simple task once $H^*(j\omega)$ is known. To compute $H^*(j\omega)$ we make use of the relationship:

$$H^*(j\omega) = F\{h^*(t)\}. \tag{31}$$

It is noted that $h^*(t)$ is a time-limited function; i.e., the

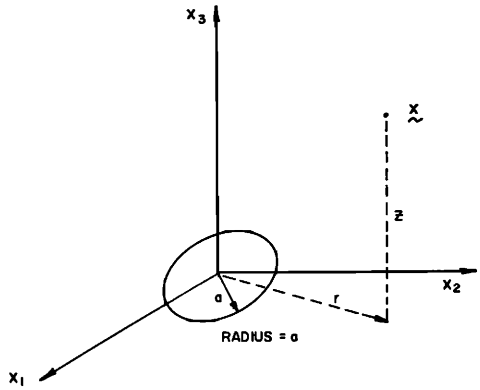

FIG. 2. Circular piston in an infinite rigid planar baffle.

time duration of the function is finite. Now $h^*(t)$ can be approximated as a sum of appropriately weighted impulse response functions and then expressed as follows:

$$h^*(t) \approx \sum_{n=0}^{N} \{h^*(nT)T\}\delta(t-nT), \tag{32}$$

where T is the sampling interval. Substituting Eq. 32 into Eq. 26, the following approximation for $H^*(j\omega)$ is obtained:

$$H^*(j\omega) \approx \sum_{n=0}^{N} \{Th^*(nT)\}e^{-jn\omega T}. \tag{33}$$

Since $h^*(t)$ is a time-limited function, the number of samples can be selected to yield a specified accuracy at the frequency of interest. Now $Z_R(j\omega)$ may be expressed as

$$Z_R(j\omega) \approx \rho(j\omega) \sum_{n=0}^{N} \{h^*(nT)T\}e^{-jn\omega T}. \tag{34}$$

Since $Z_R(j\omega)$ may also be written as

$$Z_R(j\omega) = \rho cA\{R(j\omega)+jX(j\omega)\}, \tag{35}$$

the normalized radiation resistance and reactance may then be written as

$$R(j\omega) \approx \frac{\omega T}{cA} \sum_{n=0}^{N} h^*(nT)\sin(n\omega T)$$

and

$$X(j\omega) \approx \frac{\omega T}{cA} \sum_{n=0}^{N} h^*(nT)\cos(n\omega T). \tag{36}$$

Now the sample period T may also be expressed as a fraction of the travel time across the piston width, i.e., $T=mW/c$, where $0<m\ll 1$. Then both the normalized resistance and reactance can be expressed as a function of the kW of the piston, where $k=\omega/c$ is the wavenumber. The resistance and reactance functions may then be written as follows:

$$R(kW) \approx mkW \sum_{n=0}^{N} h^*_N(nT)\sin(nmkW)$$

and

$$X(kW) \approx mkW \sum_{n=0}^{N} h^*_N(nT)\cos(nmkW), \tag{37}$$

where NT is equivalent to the time duration of $h^*(t)$ and $h^*_N(nT) \equiv h(nT)/cA$.

II. THE CIRCULAR PISTON

The time-dependent force on a circular piston as shown in Fig. 2 vibrating in an infinite rigid baffle is now investigated. A derivation of a simple analytic expression for the impulse response $h^*(t)$, defined as a surface integral in Eq. 25, is presented. As a result of the circular symmetry of the radiation from the circular piston, the surface integral representation of $h^*(t)$ in Eq. 25 may now be simplified to obtain

$$h^*(t) = \int_0^a h(r,ct)2\pi r\,dr. \tag{38}$$

An analytic expression for $h(\mathbf{x},t)$ for a circular piston is developed in Appendix A, and for points on the surface of the piston the expression for $h(\mathbf{x},t)$ reduces to

$$\begin{aligned} h(r,ct) &= c, && (ct<a-r),\\ &= \frac{c}{\pi}\cos^{-1}\left(\frac{(ct)^2+r^2-a^2}{2rct}\right), && (a-r<ct<a+r),\\ &= 0, && (a+r<ct). \end{aligned} \tag{39}$$

Figure 3 shows a pictorial representation of $h(r,t)$ as a function of r and ct. It is noted that the surface $h(r,ct)$ may be used to obtain the behavior of $h(r,ct)$ easily for fixed ct and r variable, or vice versa. From Eq. 39 and Fig. 3 it is thus easily seen that $h^*(t)$ may be expressed

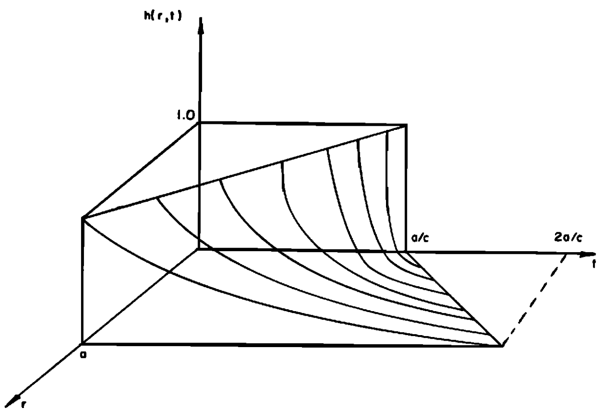

FIG. 3. The surface $h(r,ct)$ for a circular piston.

as follows:

$$h^*(t)=2\pi\left\{\int_0^{a-ct} crdr + \int_{a-ct}^{a} \frac{c}{\pi}\cos^{-1}\left[\frac{(ct)^2+r^2-a^2}{2rct}\right]rdr\right\}, \quad (40)$$

where at $t=0$ it is noted that $h^*(0)=\pi a^2c$.

To evaluate $h^*(t)$ in Eq. 40, a somewhat indirect route is followed. Differentiating $h^*(t)$ with care to include the endpoint contributions of the integrals, $\dot{h}^*(t)$ may be expressed as

$$\dot{h}^*(t)=2\pi\int_{a-ct}^{a}\frac{c}{\pi t} \times\frac{\{-r^2+[(ct)^2+a^2]\}rdr}{\{-r^4+2[(ct)^2+a^2]-[(ct)^2-a^2]^2\}^{\frac{1}{2}}}. \quad (41)$$

Utilizing the transformation $u=r^2$, $\dot{h}^*(t)$ may be expressed as

$$\dot{h}^*(t)=2\pi\int_{(a-ct)^2}^{a^2}\frac{c}{\pi t} \times\frac{\{-u+[(ct)^2+a^2]\}du}{\{-u^2+2[(ct)^2+a^2]u-[(ct)^2-a^2]\}^{\frac{1}{2}}}. \quad (42)$$

Equation 42 can now be readily evaluated using standard integral tables to obtain the expression

$$\dot{h}^*(t)=-c^2[(2a)^2-(ct)^2]^{\frac{1}{2}}H(2a-ct), \quad (43)$$

where $H(2a-ct)$ is the familiar Heaviside function, i.e.,

$$\begin{aligned} H(2a-ct)&=1, \quad 2a>ct,\\ &=0, \quad 2a<ct. \end{aligned} \quad (44)$$

To obtain $h^*(t)$ from $\dot{h}^*(t)$, Eq. 43 must be integrated, i.e.,

$$h^*(t)=\int^t \dot{h}^*(\tau)d\tau. \quad (45)$$

Performing the indicated integration and noting the initial value of $h^*(t)$, $h^*(0)=\pi a^2c$, then $h^*(t)$ may be expressed as

$$h^*(t)=(c\pi a^2)\frac{2}{\pi}\left\{\cos^{-1}\left(\frac{ct}{2a}\right) -\frac{ct}{2a}\left[1-\left(\frac{ct}{2a}\right)^2\right]^{\frac{1}{2}}\right\}H(2a-ct). \quad (46)$$

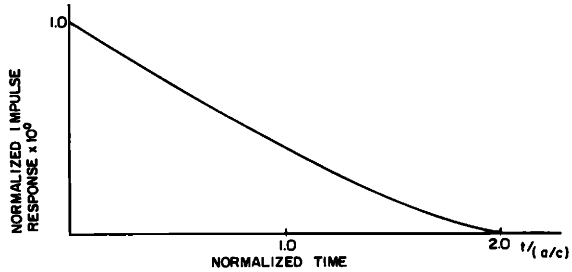

FIG. 4. The normalized impulse response of a circular piston.

Normalizing the impulse response by the piston area times the velocity of propagation in the medium, a normalized impulse response can be defined as follows:

$$h^*_N(t)=\frac{2}{\pi}\left\{\cos^{-1}\left(\frac{ct}{2a}\right) -\frac{ct}{2a}\left[1-\left(\frac{ct}{2a}\right)^2\right]^{\frac{1}{2}}\right\}H(2a-ct). \quad (47)$$

The force on the piston can thus be obtained from

$$f(t)=(\rho cA)\frac{d}{dt}[h^*_N(t)*v(t)], \quad (48)$$

where ρcA is the plane-wave resistance of the piston.

In an earlier paper, Miles[1] solved for the indicial impedance of a circular piston, i.e., the normalized time-dependent force required to produce a unit step change in the piston velocity, using both Laplace transforms and integral properties of Bessel functions. It is noted that $h^*_N(t)$ as defined in Eq. 47 is identical to the indicial impedance of a circular piston derived by Miles, and thus the impulse response approach represents an alternative formulation and solution to the problem. In addition, Miles has also shown that the time-dependent force resulting from a specified piston velocity may be expressed as the derivative of a convolution integral in agreement with Eq. 48.

In a more recent paper, Mangulis[2] utilized a Fourier-transform approach to compute the time-dependent force on a circular piston assuming the velocity is zero for $t<0$ and sinusoidal for $t\geq 0$. Unlike the work by Mangulis, the impulse-response approach is readily applicable to compute the time-dependent force resulting from a piston velocity of any bandwidth or spectral content. Also, in contrast to the results of the impulse-response approach, where the force may be expressed as a convolution or derivative of a convolution integral, Mangulis expresses the force as a product of a time-dependent radiation impedance and the piston velocity.

To conclude the discussion of the circular piston, Fig. 4 presents a curve of the normalized impulse response, $h^*_N(t)$, where the time scale is normalized by the travel

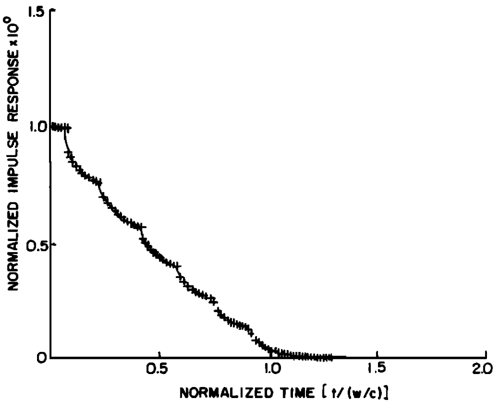

FIG. 5. The normalized impulse response of a square piston.

time across the piston radius. Several interesting observations concerning the time-dependent force resulting from a piston velocity can be obtained from the figure. Since the length of the impulse response is noted to be equal to the maximum travel time across the piston, the force resulting from a sinusoidal piston velocity reaches steady state after turn-on in a time corresponding to the maximum travel across the piston. Another point of interest is the initial impedance or force presented to the piston after turn-on. As a result of the step discontinuity in $h^*_N(t)$ at $t=0$, Eq. 22 may be utilized to show the initial impedance is the plane-wave resistance and the force and velocity are related by

$$f(t_+)=\rho c A v(t_+), \tag{49}$$

where t_+ refers to the turn-on time. This result has also been previously obtained by Junger and Thompson[11] using the initial value theorem of Laplace transform theory.

III. THE SQUARE PISTON

Unlike the circular piston, a closed-form expression for the impulse response of a square piston cannot be

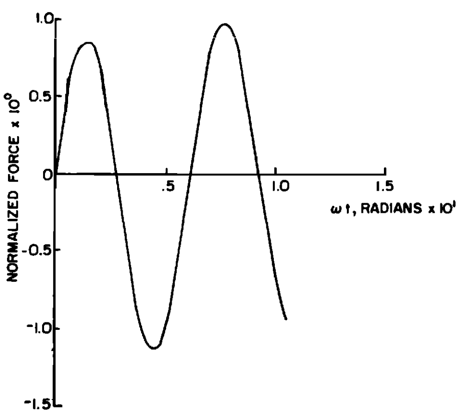

FIG. 6. Normalized reaction force for $W/\lambda=1.0$.

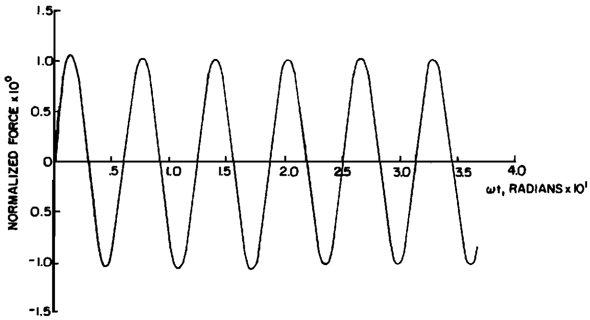

FIG. 7. Normalized reaction force for $W/\lambda=3.0$.

easily obtained. A direct numerical evaluation of the surface integral in Eq. 35 was thus required to obtain $h^*(t)$ for square pistons. The results of numerically evaluating the surface integral for the normalized impulse response, $h^*_N(t)$, are shown in Fig. 5. The integral was approximated using a finite sum of terms corresponding to weighted values of the integral evaluated at the midpoints of a square grid over the piston area. It is noted that the grid was required only over a quarter of the piston area as a result of symmetry.

Similar observations to those made for the circular piston can also be readily noted for the square piston. Again the time duration of the impulse response corresponds to the maximum travel time across the piston, i.e., the time required for a wave to transverse the piston diagonal. The normalized impulse response for the

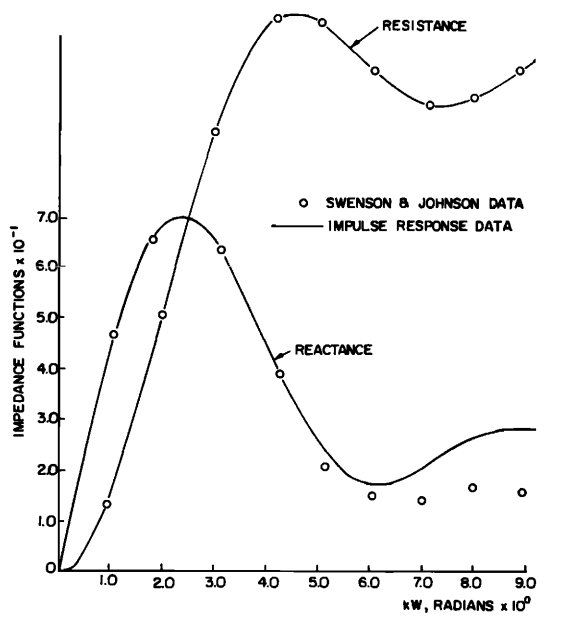

FIG. 8. Self-radiation resistance and reactance of a square piston.

square piston also has a step discontinuity at turn-on which leads to a purely resistive loading at turn-on equal to the plane-wave impedance of the piston. From the equations for the impulse response, it is easily noted that the preceding observations are valid for *all* piston sizes and shapes.

The time-dependent force acting on a square piston of width W is now investigated for sinusoidal piston velocities of different frequency turned on at time zero. Similar to steady-state analyses, the force may be conveniently normalized by the plane-wave resistance of the piston. In addition, the forces can also be investigated for different W/λ ratios, where λ is the acoustic wavelength in the medium, thus extending the usefulness of the numerical results.

Initially, the time-dependent forces on small pistons, where $W/\lambda \ll 1$, are discussed. From results presented earlier, the time duration of $h^*_N(t)$ is thus considerably less than a period of the driving frequency. It can be easily seen from a graphical evaluation of the convolution of $h^*_N(t)$ with the piston velocity that the time duration of the transient portion of the force is identical to the time duration of $h^*_N(t)$. Since the radiation impedance of the square piston is, for the most part, mass reactance at small W/λ as shown in Fig. 6, the initial peak values of the steady-state force would thus occur at approximately a quarter of a cycle after turn-on. Thus, for cases where $W/\lambda \ll 1$ the peak value of the transient force is smaller than the initial peak value of the force which is equivalent to the steady-state value obtained from the radiation impedance. It can be easily seen that as the time duration of $h^*(t)$ approaches a quarter of a cycle of the drive frequency, i.e., $\sqrt{2}W \rightarrow \frac{1}{4}\lambda$, the peak transient force approaches the corresponding steady-state value for the piston. For pistons of any W/λ, the normalized force can be computed by evaluating the convolution integral as indicated in Eq. 48 for a sinusoidal piston velocity. The velocity is zero at turn-on, and the normalized force is simply the convolution of the impulse response shown in Fig. 5 and the piston acceleration. Figure 6 presents the normalized force resulting from a sinusoidal piston velocity for a piston where $W/\lambda = 1.0$, and Fig. 7 presents a similar curve for the case where $W/\lambda = 3.0$. The forces were computed for a time period equivalent to the time duration of $h_N(t)$ plus a half-period of the driving frequency. Therefore, the peak value occurring within the last half-cycle of each figure is the peak value of the steady-state force.

The results in Figs. 6 and 7 illustrate an interesting point to be noted. Although the time duration of the transient exceeds the time period of the drive frequency, there is little peak-amplitude change during the transient, and the peak amplitude agrees well with the steady-state value. We can thus conclude that the transient loading of the square piston is quite similar to the steady-state loading on the piston, i.e., the plane-wave resistance, and thus excessive transient overshoots do not occur for large pistons. Since the normalized frequency-dependent radiation impedance for the square and equivalent circular piston are quite similar,[12] the normalized force on the square and circular pistons would also be expected to be quite similar for identical velocities; the observations for the square-piston case are thus applicable to the circular piston.

Although the total force on a large piston does not exhibit any sizable overshoot characteristics during the transient, the pressure and local force at points on the piston can and do exhibit large transient overshoots above the corresponding steady-state value. This problem is easily studied by substituting $h(\mathbf{x},t)$ for $h^*(t)$ in the convolution relationships where $\mathbf{x}$ is the point of interest; it is discussed in a later paper. The lack of overshoot of the transient portions of the total force is, of course, the result of the surface averaging process.

Although the main emphasis in the present paper has been the investigation of time-dependent phenomena, an approach was also presented to obtain the radiation impedance of a piston from the impulse response $h^*_N(t)$. Since $h^*_N(t)$ for a square piston is shown in Fig. 5, the normalized piston resistance and reactance functions can be obtained by utilizing the relationships in Eq. 37.

The results of evaluating the square piston resistance and reactance functions are shown in Fig. 8, along with the results of an earlier solution by Swenson and Johnson.[9] A comparison of the two solutions for the piston resistance and reactance functions shows both approaches to be in excellent agreement for small kW.

Although the impulse-response approach is in error at the higher kW values, these errors can be reduced by improving the approximations leading to Eq. 37. The errors in the resistance and reactance can thus be reduced at the higher kW value by decreasing the sample period T with a corresponding increase in N. Since $h^*_N(t)$ shown in Fig. 5 for a square piston resulted from the numerical evaluation of a surface integral, it is also evident that the numerical errors in the resistance and reactance are thus dependent on the accuracy of the surface integration. The accuracy of the resistance and reactance is thus dependent on both the sampling intervals used in the spatial and time integrations, unless $h^*_N(t)$ is known exactly as in the case of the circular piston.

* A portion of this paper formed part of the author's doctoral thesis at Pennsylvania State University, December 1969.

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Appendix A. Evaluation of h(x,t) for a Circular Piston

The object of the present Appendix is to evaluate $h(\mathbf{x},ct)$ for a circular piston from the more general expression

$$h(\mathbf{x},t)=L(ct)/2\pi t\sin\theta(ct). \qquad \text{(A1)}$$

As a result of the circular symmetry, the transformation to circular coordinates shown in Fig. 2 is utilized and we can now study $h(r,z,ct)$. To evaluate $h(r,z,ct)$, an expression for $L(ct)$ which is implicitly a function of the field point (r,z) is required. Initially, the case where $r>a$ is considered. From Fig. A-1(a) and the use of a little geometry, is it easily seen that

$$L(r,z,ct)=L(r,0,ct). \qquad \text{(A2)}$$

For the case of $z=0$, the evaluation of the arc length of intersection reduces to the two-dimensional problem shown in Fig. A-1(b). It is easily seen from the figure that

$$L(r,0,ct)=2r^1\cos^{-1}(d/r^1), \qquad \text{(A3)}$$

where d is determined by the intersection of the edge of the piston with a circle of radius r^1 centered at r on

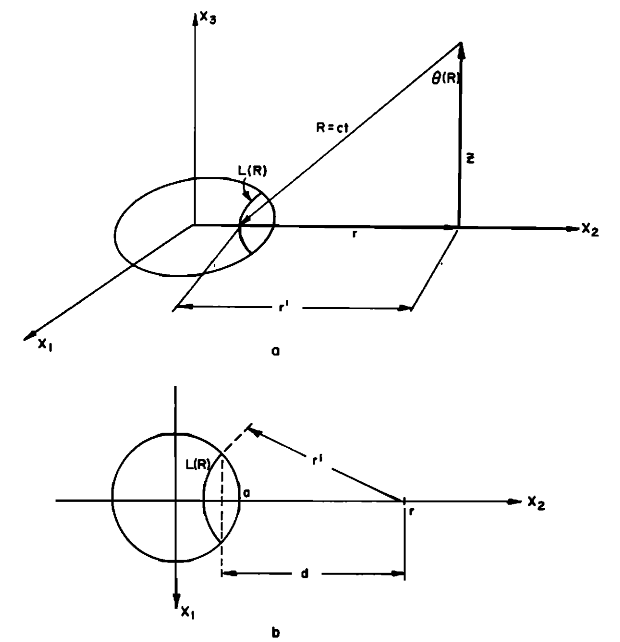

FIG. A-1. Geometrical variables used to compute $h(r,z,ct)$ for $r>a$.

the x_1 axis. From the equation for the circle of radius r^1 centered at r and the equation of the circle describing the piston, it can then be shown that

$$d=r-[r^2+a^2-(r^1)^2]/2r. \qquad \text{(A4)}$$

From Fig. A-1 it can be seen that r^1 may be expressed as

$$r^1=(R^2-z^2)^{\frac{1}{2}} \qquad \text{(A5)}$$

and

$$r^1=R\sin\theta(R). \qquad \text{(A6)}$$

Noting the space–time equivalence $R=ct$, r^1 may also be expressed in the forms

$$r^1=[(ct)^2-z^2]^{\frac{1}{2}} \qquad \text{(A7)}$$

and

$$r^1=ct\sin\theta(ct). \qquad \text{(A8)}$$

Substituting Eq. A7 into Eq. A4, d can then be expressed as

$$d=[(ct)^2-z^2+r^2-a^2]/2r. \qquad \text{(A9)}$$

Substituting Eqs. A9 and A8 into Eq. A3 and noting Eq. A2:

$$L(r,z,ct)=2ct\sin\theta(ct)\cos^{-1}\left\{\frac{(ct)^2-z^2+r^2-a^2}{2r[(ct)^2-z^2]^{\frac{1}{2}}}\right\}. \qquad \text{(A10)}$$

Finally from Eqs. A10 and A1, it is readily seen that for the case $a<r$

$$h(r,z,t)=\frac{c}{\pi}\cos^{-1}\left(\frac{(ct)^2-z^2+r^2-a^2}{2r[(ct)^2-z^2]^{\frac{1}{2}}}\right) \qquad \text{(A11)}$$

for

$$R_1<ct<R_2,$$

where

$$R_1\equiv[z^2+(r-a)^2]^{\frac{1}{2}}$$

and

$$R_2\equiv[z^2+(r+a)^2]^{\frac{1}{2}}.$$

The limits are, of course, the result of $L(ct)=0$ for $R_1>ct$ and $R_2<ct$ as seen from Fig. A-1.

The case of $a>r$ may be handled in a similar manner. Unlike the case $a<r$, the projection of the point onto the $z=0$ plane is now on the piston area. From Fig. A-2 the arc length of intersection is now the circumference of a circle of radius $R\sin[\theta(R)]$ for $z<R<[z^2+(a-r)^2]^{\frac{1}{2}}$.

Once again accounting for the space–time equivalence, Eq. A1 can be easily evaluated to obtain

$$h(r,z,ct)=0, \quad z>ct,$$
$$=c, \quad z<ct<[z^2+(a-r)^2]^{\frac{1}{2}}. \qquad \text{(A12)}$$

From Fig. A-2 it can be seen that for $ct>[z^2+(a-r)^2]^{\frac{1}{2}}$ the arc length of intersection is no longer a circle. Following the same approach used to obtain $h(r,z,ct)$ for $r>a$, it is easily shown that

$$h(r,z,ct)=\frac{c}{\pi}\cos^{-1}\left\{\frac{(ct)^2-z^2+r^2-a^2}{2r[(ct)^2-z^2]^{\frac{1}{2}}}\right\}, \qquad \text{(A13)}$$

when

$$[z^2+(a-r)^2]^{\frac{1}{2}}<ct<[z^2+(a+r)^2]^{\frac{1}{2}}.$$

Combining Eqs. A12 and A13, $h(r,z,ct)$ for $r<a$ may be expressed as follows:

$$\begin{aligned} h(r,z,ct)&=0, && ct<z,\\ &=c, && z<ct<R_1,\\ &=\frac{c}{\pi}\cos^{-1}\left\{\frac{(ct)^2-z^2+r^2-a^2}{2r[(ct)^2-z^2]^{\frac{1}{2}}}\right\}, && R_1<ct<R_2,\\ &=0, && R_2<ct, \qquad \text{(A14)}\end{aligned}$$

where R_1 and R_2 were defined earlier.

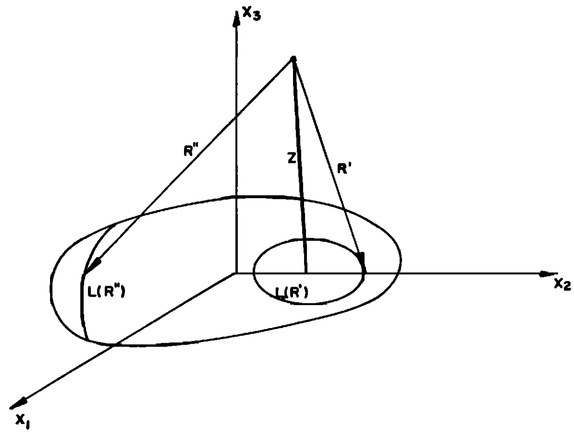

FIG. A-2. Geometrical variables used to compute $h(r,z,ct)$ for $r<a$.

Equations A11 and A14 express $h(r,z,ct)$ for the two cases $a<r$ and $r<a$. It is noted that the expressions are valid for all field points, and no restrictions on piston size have been assumed. It is also noted that the equations are in agreement with results of an earlier analysis by Oberhettinger[A1] using integral transforms and properties of Bessel functions.

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