3

Boundary-Value Problems in Electrostatics: II

In this chapter the discussion of boundary-value problems is continued. Spherical and cylindrical geometries are first considered, and solutions of Laplace's equation are represented by expansions in series of the appropriate orthonormal functions. Only an outline is given of the solution of the various ordinary differential equations obtained from Laplace's equation by separation of variables, but an adequate summary of the properties of the different functions is presented.

The problem of construction of Green's functions in terms of orthonormal functions arises naturally in the attempt to solve Poisson's equation in the various geometries. Explicit examples of Green's functions are obtained and applied to specific problems, and the equivalence of the various approaches to potential problems is discussed.

3.1 Laplace's Equation in Spherical Coordinates

In spherical coordinates (r, θ, ϕ), shown in Fig. 3.1, Laplace's equation can be written in the form:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.1}$$

If a product form for the potential is assumed, then it can be written:

$$\Phi = \frac{U(r)}{r}P(\theta)Q(\phi) \tag{3.2}$$

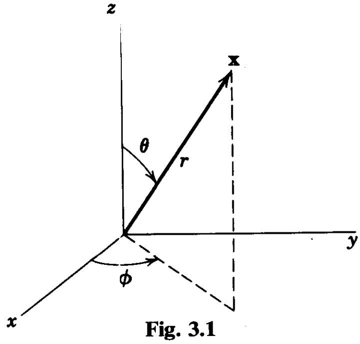

Fig. 3.1

When this is substituted into (3.1), there results the equation:

$$PQ\frac{d^2U}{dr^2} + \frac{UQ}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0$$

If we multiply by $r^2 \sin^2 \theta / UPQ$, we obtain:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \quad (3.3)$$

The ϕ dependence of the equation has now been isolated in the last term. Consequently that term must be a constant which we call $(-m^2)$:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \quad (3.4)$$

This has solutions

$$Q = e^{\pm im\phi} \quad (3.5)$$

In order that Q be single valued, m must be an integer. By similar considerations we find separate equations for $P(\theta)$ and $U(r)$:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \quad (3.6)$$

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \quad (3.7)$$

where $l(l + 1)$ is another real constant.

From the form of the radial equation it is apparent that a single power of r (rather than a power series) will satisfy it. The solution is found to be:

$$U = Ar^{l+1} + Br^{-l} \quad (3.8)$$

but l is as yet undetermined.

3.2 Legendre Equation and Legendre Polynomials

The θ equation for $P(\theta)$ is customarily expressed in terms of $x = \cos\theta$, instead of θ itself. Then it takes the form:

$$\frac{d}{dx}\left((1 - x^2)\frac{dP}{dx}\right) + \left(l(l + 1) - \frac{m^2}{1 - x^2}\right)P = 0 \tag{3.9}$$

This equation is called the generalized Legendre equation, and its solutions are the associated Legendre functions. Before considering (3.9) we will outline the solution by power series of the ordinary Legendre differential equation with $m^2 = 0$:

$$\frac{d}{dx}\left((1 - x^2)\frac{dP}{dx}\right) + l(l + 1)P = 0 \tag{3.10}$$

The desired solution should be single valued, finite, and continuous on the interval $-1 \leq x \leq 1$ in order that it represents a physical potential. The solution will be assumed to be represented by a power series of the form:

$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j \tag{3.11}$$

where α is a parameter to be determined. When this is substituted into (3.10), there results the series:

$$\sum_{j=0}^{\infty} \{(\alpha + j)(\alpha + j - 1)a_j x^{\alpha+j-2} - [(\alpha + j)(\alpha + j + 1) - l(l + 1)]a_j x^{\alpha+j}\} = 0 \tag{3.12}$$

In this expansion the coefficient of each power of x must vanish separately. For $j = 0, 1$ we find that

$$\left.\begin{aligned} &\text{if } a_0 \neq 0, \quad \text{then } \alpha(\alpha - 1) = 0 \\ &\text{if } a_1 \neq 0, \quad \text{then } \alpha(\alpha + 1) = 0 \end{aligned}\right\} \tag{3.13}$$

while for a general j value

$$a_{j+2} = \left[\frac{(\alpha + j)(\alpha + j + 1) - l(l + 1)}{(\alpha + j + 1)(\alpha + j + 2)}\right]a_j \tag{3.14}$$

A moment's thought shows that the two relations (3.13) are equivalent and that it is sufficient to choose *either* a_0 *or* a_1 different from zero, but not both. Making the former choice, we have $\alpha = 0$ or $\alpha = 1$. From (3.14) we see that the power series has only even powers of $x (\alpha = 0)$ or only odd powers of $x (\alpha = 1)$.

For either of the series $\alpha = 0$ or $\alpha = 1$ it is possible to prove the following properties:

(*a*) the series converges for $x^2 < 1$, regardless of the value of l;

(*b*) the series diverges at $x = \pm 1$, unless it terminates.

Since we want a solution that is finite at $x = \pm 1$, as well as for $x^2 < 1$, we demand that the series terminate. Since α and j are positive integers or zero, the recurrence relation (3.14) will terminate only if *l is zero or a positive integer*. Even then only one of the two series converges at $x = \pm 1$. If l is even (odd), then only the $\alpha = 0$ ($\alpha = 1$) series terminates.* The polynomials in each case have x^l as their highest power of x, the next highest being x^{l-2}, and so on, down to x^0 (x) for l even (odd). By convention these polynomials are normalized to have the value unity at $x = +1$ and are called the *Legendre polynomials* of order l, $P_l(x)$. The first few Legendre polynomials are:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}\right\} \qquad (3.15)$$

By manipulation of the power series solutions (3.11) and (3.14) it is possible to obtain a compact representation of the Legendre polynomials, known as *Rodrigues' formula*:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l \qquad (3.16)$$

[This can be obtained by other, more elegant means, or by direct l-fold integration of the differential equation (3.10).]

The Legendre polynomials form a complete orthogonal set of functions on the interval $-1 \leq x \leq 1$. To prove the orthogonality we can appeal directly to the differential equation (3.10). We write down the differential equation for $P_l(x)$, multiply by $P_{l'}(x)$, and then integrate over the interval:

$$\int_{-1}^{1} P_{l'}(x)\left[\frac{d}{dx}\left((1 - x^2)\frac{dP_l}{dx}\right) + l(l+1)P_l(x)\right] dx = 0 \qquad (3.17)$$

* For example, if $l = 0$ the $\alpha = 1$ series has a general coefficient $a_j = a_0/j + 1$ for $j = 0, 2, 4, \ldots$. Thus the series is $a_0(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots)$ This is just the power series expansion of a function $Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, which clearly diverges at $x = \pm 1$. For each l value there is a similar function $Q_l(x)$ with logarithms in it as the partner to the well-behaved polynomial solution. See Magnus and Oberhettinger, p. 59.

Integrating the first term by parts, we obtain

$$\int_{-1}^{1}\left[(x^2-1)\frac{dP_l}{dx}\frac{dP_{l'}}{dx}+l(l+1)P_{l'}(x)P_l(x)\right]dx=0 \tag{3.18}$$

If we now write down (3.18) with l and l' interchanged and subtract it from (3.18), the result is the orthogonality condition:

$$[l(l+1)-l'(l'+1)]\int_{-1}^{1}P_{l'}(x)P_l(x)\,dx=0 \tag{3.19}$$

For $l\neq l'$, the integral must vanish. For $l=l'$, the integral is finite. To determine its value it is necessary to use an explicit representation of the Legendre polynomials, e.g., Rodrigues' formula. Then the integral is explicitly:

$$\int_{-1}^{1}[P_l(x)]^2\,dx=\frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{d^l}{dx^l}(x^2-1)^l\frac{d^l}{dx^l}(x^2-1)^l\,dx$$

Integration by parts l times yields the result:

$$\int_{-1}^{1}[P_l(x)]^2\,dx=\frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}(x^2-1)^l\frac{d^{2l}}{dx^{2l}}(x^2-1)^l\,dx$$

The differentiation of $(x^2-1)^l$ $2l$ times yields the constant $(2l)!$, so that

$$\int_{-1}^{1}[P_l(x)]^2\,dx=\frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^l\,dx \tag{3.20}$$

The remaining integral is easily shown to be $2^{2l+1}(l!)^2/(2l+1)!$ Consequently the orthogonality condition can be written:

$$\int_{-1}^{1}P_{l'}(x)P_l(x)\,dx=\frac{2}{2l+1}\,\delta_{l'l} \tag{3.21}$$

and the orthonormal functions in the sense of Section 2.9 are

$$U_l(x)=\sqrt{\frac{2l+1}{2}}\,P_l(x) \tag{3.22}$$

Since the Legendre polynomials form a complete set of orthogonal functions, any function $f(x)$ on the interval $-1\leq x\leq 1$ can be expanded in

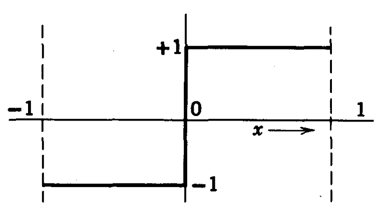

Fig. 3.2

terms of them. The Legendre series representation is:

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{3.23}$$

where

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x)\, dx \tag{3.24}$$

As an example, consider the function shown in Fig. 3.2:

$$\begin{aligned} f(x) &= +1 \text{ for } x > 0 \\ &= -1 \text{ for } x < 0 \end{aligned}$$

Then

$$A_l = \frac{2l+1}{2}\left[\int_0^1 P_l(x)\, dx - \int_{-1}^{0} P_l(x)\, dx\right]$$

Since $P_l(x)$ is odd (even) about $x = 0$ if l is odd (even), only the odd l coefficients are different from zero. Thus, for l odd,

$$A_l = (2l+1)\int_0^1 P_l(x)\, dx \tag{3.25}$$

By means of Rodrigues' formula the integral can be evaluated, yielding

$$A_l = \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!} \tag{3.26}$$

where $(2n+1)!! \equiv (2n+1)(2n-1)(2n-3)\cdots \times 5 \times 3 \times 1$. Thus the series for $f(x)$ is:

$$f(x) = \tfrac{3}{2}P_1(x) - \tfrac{7}{8}P_3(x) + \tfrac{11}{16}P_5(x) - \cdots \tag{3.27}$$

Certain recurrence relations among Legendre polynomials of different order are useful in evaluating integrals, generating higher-order polynomials from lower-order ones, etc. From Rodrigues' formula it is a straightforward matter to show that

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0 \tag{3.28}$$

This result, combined with differential equation (3.10), can be made to yield various recurrence formulas, some of which are:

$$\left.\begin{aligned} (l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} &= 0 \\ \frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l &= 0 \\ (x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} &= 0 \end{aligned}\right\} \tag{3.29}$$

As an illustration of the use of these recurrence formulas consider the evaluation of the integral:

$$I_1 = \int_{-1}^{1} xP_l(x)P_{l'}(x)\,dx \tag{3.30}$$

From the first of the recurrence formulas (3.29) we obtain an expression for $xP(x)$. Therefore (3.30) becomes

$$I_1 = \frac{1}{2l+1}\int_{-1}^{1} P_{l'}(x)[(l+1)P_{l+1}(x) + lP_{l-1}(x)]\,dx$$

The orthogonality integral (3.21) can now be employed to show that the integral vanishes unless $l' = l \pm 1$, and that, for those values,

$$\int_{-1}^{1} xP_l(x)P_{l'}(x)\,dx = \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)}, & l' = l+1 \\[2ex] \dfrac{2l}{(2l-1)(2l+1)}, & l' = l-1 \end{cases} \tag{3.31}$$

These are really the same result with the roles of l and l' interchanged. In a similar manner it is easy to show that

$$\int_{-1}^{1} x^2P_l(x)P_{l'}(x)\,dx = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}, & l' = l+2 \\[2ex] \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)}, & l' = l \end{cases} \tag{3.32}$$

where it is assumed that $l' \geq l$.

3.3 Boundary-Value Problems with Azimuthal Symmetry

From the form of the solution of Laplace's equation in spherical coordinates (3.2) it will be seen that, for a problem possessing azimuthal symmetry, $m = 0$ in (3.5). This means that the general solution for such a problem is:

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta) \tag{3.33}$$

The coefficients A_l and B_l can be determined from the boundary conditions. Suppose that the potential is specified to be $V(\theta)$ on the surface of a sphere of radius a, and it is required to find the potential inside the sphere. If there are no charges at the origin, the potential must be finite there. Consequently $B_l = 0$ for all l. The coefficients A_l are found by evaluating