12
Strain

12.1 Introduction

Deforming a circle into an ellipse clearly demonstrates that the orientations and lengths of lines and the angles between pairs of lines generally change. With suitable material the stretch and angle of shear associated with a line may be determined from measurements made on deformed objects whose original shape or size are known. It may then be possible to determine something of the shape, size and orientation of the strain ellipse. For example, measurement of the deformed shape of an originally spherical oolite yields the shape of the ellipse and its orientation directly. This chapter deals with some additional techniques for extracting two-dimensional strain information from deformed rocks. Many more examples, including some excellent photographs, can be found in the book by Ramsay and Huber (1983). Lisle (1994) gives a good review of more recent developments.

Before describing the full analytical method it will be useful to show that in some situations the shape and orientation of the strain ellipse can be obtained simply and directly using purely graphical means.

12.2 Deformed grains

The center points of individual grains in a section through a rock form a grid. In terms of the *center-to-center* distances the possible geometrical patterns have two end-members. If the distribution is random, the minimum distance between centers is zero and such pattern exhibits *clustering*. If all the grains are perfectly uniform circles and are closely packed then all distances between centers will be equal in the undeformed state (Fig. 12.1a), and thus are radii of a circle. After a homogeneous deformation these are systematically altered; they are now radii of an ellipse (Fig. 12.1b). Rocks commonly display patterns between these two and thus show degrees of *anticlustering*.

If the grain centers in the undeformed rock have a pronounced degree of anticlustering and the pattern is *isotropic*, that is, the spacing in all directions is the same, then the shape and orientation of the strain ellipse can be recovered from the deformed grid (Fry, 1979a; 1979b; Hanna & Fry, 1979; also Ramsay & Huber, 1983, p. 111–113; Simpson, 1988, p. 352). The analysis is usually performed on the tracing of a photomicrograph.

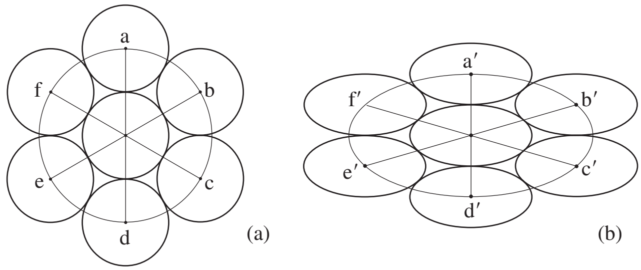

Figure 12.1 Center-to-center distances: (a) before strain; (b) after strain.

Problem

- From a section of deformed grains, determine the shape and orientation of the strain ellipse.

Construction

1. Plot the centers of all individual grains, numbering each (Fig. 12.2a).
2. On an overlay sheet establish a reference mark representing the coordinate origin and position it over center No. 1. Mark all the other center points on this sheet.
3. Without rotating the overlay, translate the reference mark to center No. 2 and again mark all the other centers (Fig. 12.2b). Repeat this procedure for all grain centers.
4. After a few of these steps a vacancy about the reference mark should begin to take shape. Then grain centers at significantly greater distances need not be marked and this speeds the work considerably.

Result

- The vacancy with the reference mark at its center defines the shape but not size of the strain ellipse and its orientation (Fig. 12.2c). Because we do not know the initial center-to-center distances the magnitude of the principal stretches can not be obtained, only their ratio.

Usually the recognizable pattern starts to emerge after about 25 points but, depending on the strength of the initial anticlustering, up to several hundred points may be required to adequately define the ellipse. If no such vacancy develops then the initial distribution was not sufficiently anticlustered and the determination of the strain is not possible.

This repetitive plotting procedure is an ideal computer application. The coordinates of each center can recorded by hand using graph paper or better with the aid of a digitizing tablet. In some cases, such as deformed oolites, the centers may be clearly defined. In

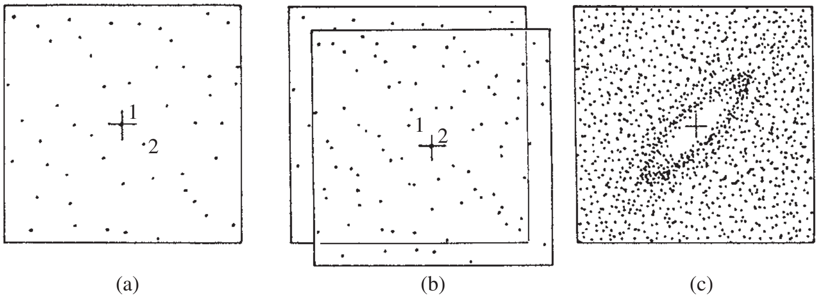

Figure 12.2 Deformed grains: (a) centers; (b) plotting; (c) ellipse (from Ramsay & Huber, 1983, p. 113 with permission of Elsevier.)

other cases the locations of the centers may be estimated or calculated from points along the boundaries of the grains. Several programs are available to process this data file. De Paor (1989) lists a BASIC program which perform the necessary calculations and plots the results interactively.

Several refinements in the basic technique have been suggested. Crespi (1986) examined real and artificial patterns and discussed the sources of possible errors. Ghaleb and Fry (1995) describe a computer program to produce center-to-center models. The technique has been mostly used for grain aggregates whose shapes are the result of strain, but it has been extended to grain shapes due to pressure solution (Onasch, 1986a, b; Bhattacharyya & Longiaru, 1986).

Rocks are, of course, three-dimensional aggregates of grains, and a section through such an array will not pass through all grain centers. Erslev (1988) suggested a normalization procedure which improves the definition of the ellipse-shaped vacancy by compensating for this effect. McNaught (1994, 2002) described an alternative method and a way of estimating uncertainty. Erslev and Ge (1990) and Ailleres and Campenois (1994) described how to calculate a best-fit ellipse. Dunne, *et al.* (1990) noted that if the post-deformation grain centers do not coincide with their pre-deformation centers, the strain will be underestimated.

Recently, Waldron and Wallace (2007) described a method for objectively fitting ellipses to the center-to-center method.

Alternatives to the center-to-center approaches have also been suggested. Panozzo (1984, 1987) developed methods by treating the traces of grain boundaries as reoriented lines. Srivastava (1995) described a quick and easy way to estimate strain by counting the number of grains intersected along a series of radiating lines.

12.3 Deformed fossils

Fossils often possess planes of symmetry, known angular relationships, or proportions which are constant in individuals of a given species. They are, therefore, common objects of known original shape. Wellman (1962) showed how the shape and orientation of the strain ellipse can be obtained from a collection of such forms in a simple way.

Problem

- From a collection of deformed brachiopods on a plane, construct the strain ellipse (Fig. 12.3a).

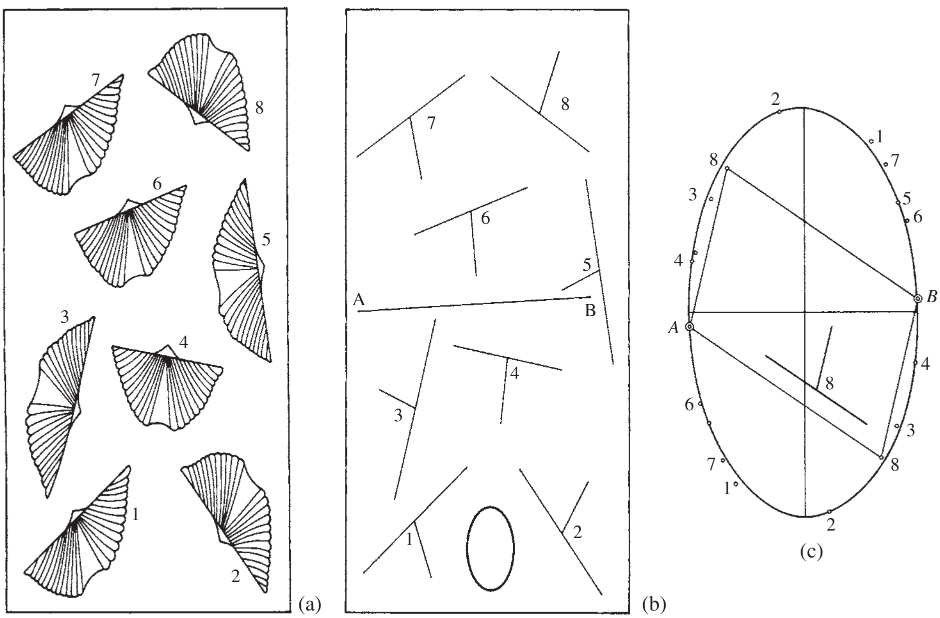

Figure 12.3 Brachiopods: (a) slab of shells; (b) deformed lines; (c) strain ellipse.

Construction

1. Transfer the hinge and symmetry lines of each deformed fossil to a tracing sheet (Fig. 12.3b).
2. On this tracing draw a line of arbitrary orientation and length; it should be at least 10 cm long and preferably not parallel to any fossil line (see line AB in Fig. 12.3b).
3. For each deformed shell in turn, draw a pair of lines parallel to the hinge and another pair parallel to the median line through the points A and B giving a parallelogram (see example drawn for Shell No. 8 in Fig. 12.3c).
4. Through all the pairs of fossil points determined in this way, including points A and B, sketch a best-fit ellipse and add the major and minor axes. This represents the strain ellipse.
5. Measure the orientation of the principal semi-axes, and their lengths.

Answer

- Because the size of the constructed ellipse depends entirely on the arbitrary length of line AB, the absolute lengths of its semi-axes have no meaning. However, their ratio is independent of the size of the ellipse, and is found to be $R_s = 1.7$. The S_1 direction makes an angle of 10° with the hinge of Shell No. 5. These results can be checked against the small ellipse, which was a circle before deformation.

In order to see why this method works, imagine having made the same construction before deformation. Because each pair of hinge and symmetry lines was originally perpendicular, rectangles rather than parallelograms would have resulted. Collectively, the corners of all these rectangles would have defined a circle with AB as diameter. This is the circle from which the constructed strain ellipse is derived.

Now reexamine the deformed brachiopods. The shape of each deformed shell is a function of orientation. Strictly, all right angles have been eliminated. However, Shell No. 3 is still nearly symmetrical; this is also the *narrow form*. Shell No. 4 also retains close to a 90° angle, but it is deformed into a *broad form*. Because the principal axes are the only pair of lines which remain perpendicular, the S_1 direction must nearly coincide with the hinge line of Shell No. 3, and the median line of Shell No. 4. Thus one can estimate the orientation of the strain ellipse by inspection.

12.4 Deformed pebbles

Before deformation, the shapes of the constituent grains in many rocks are approximately elliptical. After a homogeneous deformation, these shapes are systematically changed and from these the state of strain can be determined. Because these situations are common, the methods that have been developed are widely used and reliable. Lisle (1985a) gives a comprehensive description.[1]

The way elliptical grains deform is geometrically similar to the results of superimposed deformations (see §11.6). The role of the strain ellipse after the first deformation is replaced by the shape of the elliptical grain.

Given a sufficient number of homogeneously deformed two-dimensional pebbles subject to the conditions that the initial shapes were identical and the pebbles were initially without preferred orientation, we may determine the orientation of the principal strain axes in the deformed state, the strain ratio R_s and the initial shape ratio R_i. The basic method relies on the fact that two of these pebbles will be oriented coaxially with the strain ellipse and these will be deformed into *narrow* and *broad* forms. There are two general cases.

[1] Mulchrone and Meere (2001) describe a computer program which performs the analysis of passively deformed elliptical markers. Meere and Mulchrone (2003) examine the role of sample size in several different analytical techniques.

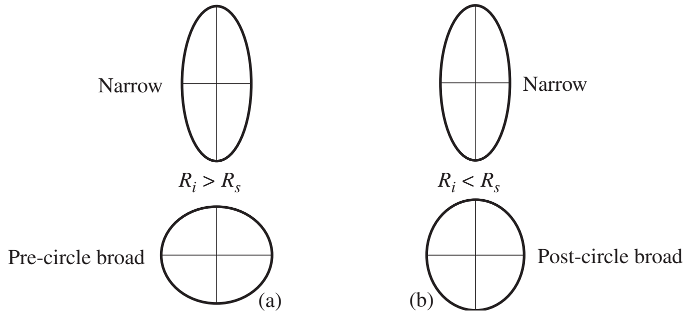

Figure 12.4 Narrow and broad forms: (a) $R_i > R_s$; (b) $R_i < R_s$.

1. If $R_i > R_s$ the axes of the resulting two extreme shapes will be perpendicular and the deformed ellipses will have the *narrow* and *pre-circle broad forms* (Fig. 12.4a). The axial ratios of these are related by

$$R_{max} = R_s R_i \quad \text{and} \quad R_{min} = R_i / R_s.$$

Solving these two equations for the two unknown ratios gives

$$R_s = \sqrt{R_{max}/R_{min}} \quad \text{and} \quad R_i = \sqrt{R_{max} R_{min}}. \tag{12.1}$$

In the example $R_{max} = 2.24$ and $R_{min} = 1.14$, then $R_s = 1.4$ and $R_i = 1.6$.

2. If $R_i < R_s$ the axes of the two extreme shapes will be parallel and the deformed ellipses will have the *narrow* and *post-circle broad forms* (Fig. 12.4b). The axial ratios

$$R_{max} = R_s R_i \quad \text{and} \quad R_{min} = R_s / R_i.$$

Solving these for the two unknown ratios gives

$$R_s = \sqrt{R_{max} R_{min}} \quad \text{and} \quad R_i = \sqrt{R_{max}/R_{min}}. \tag{12.2}$$

In the example, $R_{min} = 2.24$ and $R_{min} = 1.14$, then $R_s = 1.6$ and $R_i = 1.4$.

Problem

- From a section through a suite of deformed pebbles, determine the final shape ratios R_f of the narrow and broad forms, and from these determine R_i, R_s and the orientation of the principal strain axes (Fig. 12.5a).

Procedure

1. Measure the orientational angle θ' which the long axis of each pebble makes with arbitrary reference direction.
2. Measure the axial length of each pebble and calculate the final shape ratio R_f.
3. Plot each pair of values (θ', R_f) as a point on a graph (Fig. 12.5b).

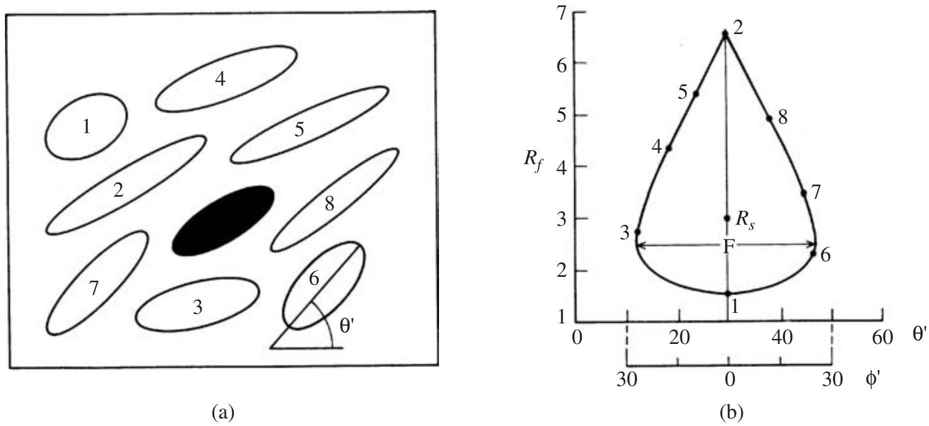

Figure 12.5 Deformed pebbles: (a) angle θ'; (b) R_f vs. θ'.

Results

1. These eight points lie on a closed tear-drop shaped curve, symmetrical about a fixed value of θ', which defines the S_1 direction. Because both the narrow and broad forms have the same orientation, this is an example where $R_i < R_s$ (see Fig. 12.5b). Having identified this direction, it is then convenient to adopt it as the reference direction and to relate the orientation of the deformed pebbles to it by the angles $\pm\phi'$.
2. The two points on the line $\phi' = 0$ are $R_{max} = 6.5$ and $R_{min} = 1.5$.

Answer

- Using these values of R_{max} and R_{min} in Eqs. 12.2 gives $R_s = 3.12$ and $R_i = 2.08$.

In Fig. 12.5a, the black ellipse was initially a circle, and therefore represents the strain. Note that its ratio R_s plots near the center of the R_f/ϕ' curve (Fig. 12.5b).

Suites of deformed elliptical grains have characteristic ranges of orientations, called the *fluctuation* F (Cloos, 1947, p. 861), and it can be determined from the R_f/ϕ' curve as the angle between the extreme orientations. Depending on the relative values of R_i and R_s there are three characteristic types of fluctuation.

1. If $R_i > R_s$ then $F = 180°$.
2. If $R_i = R_s$ then the "broad" form is a circle and $F = 90°$.
3. If $R_i < R_s$ then $F < 90°$ (in Fig. 12.5b, $F = 56°$).

The important feature of this evolving fluctuation is that F remains constant at 180° until $R_s = R_i$ at which point a preferred orientation suddenly appears, and thereafter strengthens as R_s increases. In many deformed terranes, slaty cleavage appears quite abruptly and Elliott (1970, p. 2232) suggested that this *cleavage front* marks such a sudden onset of preferred orientation.

With a constant initial shape ratio this example is not very realistic. If a variety of distinct initial shapes are present, the R_f/ϕ' graphs consist of a series of nested curves, one for each R_i.

More generally yet, real data will not plot on such distinct curves but will appear as a scatter of points, reflecting a continuous variation of initial shapes. Then a quasi-statistical graphical technique is usually used (see Lisle, 1985a). Several computer programs are available to accomplish this analysis (Peach & Lisle, 1979; Kutty & Joy, 1994; Mulchrone & Meere, 2001).

As described, this procedure needs a range of original orientations for a complete analysis, but Borradaile and McArthur (1991), following Yu and Zheng (1984), linearized the R_f/ϕ' curves, thus facilitating the analysis of initially non-random fabrics.

In applying any of these techniques to naturally deformed materials, there are several factors which may limit their use. Most naturally occurring sedimentary fabrics show some degree of preferred orientation. In a study of the simulated deformation of such fabrics, Seymour and Boulter (1979) showed that large errors may result if it is mistakenly assumed that they were originally uniform. Also, if there is a ductility contrast between the elliptical objects and the matrix material, an additional component of rotation will be present which may invalidate the strain analysis (De Paor, 1980).

12.5 Geometry of the strain ellipse

With only a few strained objects we need a different approach and this requires a more fundamental description of the way lengths and angles change as the result of a deformation. To do this we refer the initial state to a set of xy axes which are parallel to the principal directions in the unstrained state (Fig. 12.6a). The equation of the reference circle of unit radius in these *material coordinates* is then

$$x^2 + y^2 = 1. \tag{12.3}$$

Similarly, we refer the strained state to a set of $x'y'$ axes with the same orientation (Fig. 12.6b). The equation of the strain ellipse in these *spatial coordinates* is then

$$\frac{x'^2}{S_1^2} + \frac{y'^2}{S_3^2} = 1. \tag{12.4}$$

By this choice of axes and directions we have eliminated from consideration any translation or rotation and we do this to concentrate on the properties of the stretch component of the deformation.

Within this framework we now examine the geometrical changes associated with a particular material line. At a typical point P on the unit circle, we identify the direction of the *radius vector* $\mathbf{r} = OP$ by the angle ϕ it makes with the x axis (Fig. 12.6a). The components of this vector are (x, y) and its direction cosines are $(\cos\phi, \sin\phi)$.

The corresponding point on the ellipse is P' and we identify the direction of the radius vector $\mathbf{r}' = OP'$ by the angle ϕ' it makes with the x' axis (Fig. 12.6b). The components of this vector are (x', y') and its direction cosines are $(\cos\phi', \sin\phi')$.

As a result of strain, radius vector $\mathbf{r}$ of the circle is transformed into radius vector $\mathbf{r}'$ of the ellipse. Three separate geometrical features are associated with this transformation and all these changes can be observed within a circle-to-ellipse card-deck experiment.

1. Orientational angle ϕ changes to ϕ'.
2. The length of a radius of a circle $r = 1$ changes to the radius of an ellipse $r' = S$.
3. The right angle between $\mathbf{r}$ and the tangent T at point $P(x, y)$ changes and the measure of this change is the angle of shear ψ. This is represented by the angle between the tangent T' and the line perpendicular to $\mathbf{r}'$ at point $P'(x', y')$. Note that in the first and third quadrants the tangent rotates in an anticlockwise sense hence ψ is positive. In the two other quadrants ψ is negative.

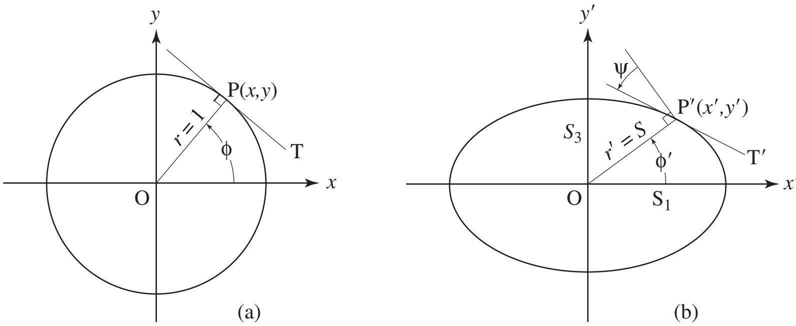

Figure 12.6 Circle and ellipse: (a) *xy* coordinates; (b) *x′y′* coordinates.

We now need algebraic expressions for each of these changes associated with the material line in terms of its orientation and the principal stretches.[2]

Change in orientation

The relationship between vector $\mathbf{r}(x, y)$ which marks a material line in the reference circle and the vector $\mathbf{r}'(x', y')$ which marks the same material line in the strain ellipse is

$$x' = S_1 x \quad \text{and} \quad y' = S_3 y. \tag{12.5}$$

That is, the x component of $\mathbf{r}$ is stretched to become the x' component of $\mathbf{r}'$ and the y component is stretched to become y'. Dividing the second of these equations by the first gives

$$\frac{y'}{x'} = \frac{S_3}{S_1}\frac{y}{x}. \tag{12.6}$$

[2]On a first reading you may wish to skip the details of the derivations and go directly to §12.6 for the results.

From Figs. 12.6a and 12.6b

$$\tan\phi = y/x \quad \text{and} \quad \tan\phi' = y'/x'.$$

Using these in Eq. 12.6 and using the definition of the strain ratio $R_s = S_1/S_3$ we have the useful result, first obtained by Harker (1885, p. 822),

$$\tan\phi' = \frac{\tan\phi}{R_s} \quad \text{or} \quad R_s = \frac{\tan\phi}{\tan\phi'}. \tag{12.7}$$

This result may be used in two ways. The first version gives ϕ' when ϕ and R_s are known. By definition $R_s > 1$ and therefore $\phi' < \phi$, that is, the angle a material line makes with the S_1 direction is generally reduced. The only exceptions are when $\phi = 0°$ or $\phi = 90°$ and the orientation is unchanged. The second version can be used to determine the strain ratio R_s if both ϕ and ϕ' are known.

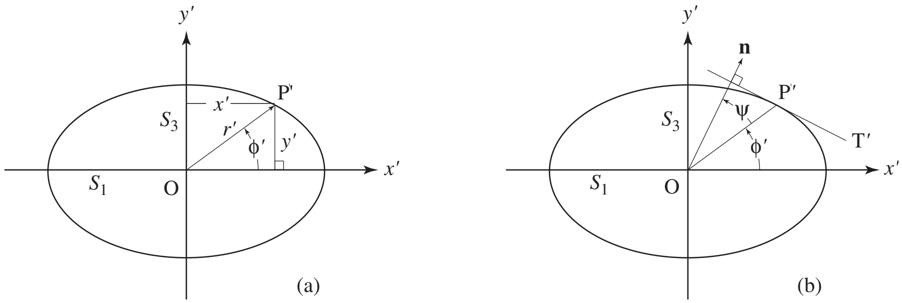

Figure 12.7 Strain ellipse: (a) stretch; (b) angle of shear.

Change in length

Next we need an expression for the stretch associated with the transformation of $\mathbf{r}$ into $\mathbf{r}'$. From Fig. 12.7a, where the magnitude of the radius vector $\mathbf{r}'$ is the stretch S,

$$x' = S\cos\phi' \quad \text{and} \quad y' = S\sin\phi'. \tag{12.8}$$

Using these expressions for x' and y' in the equation of the ellipse of Eq. 12.4 we have

$$\frac{S^2\cos^2\phi'}{S_1^2} + \frac{S^2\sin^2\phi'}{S_3^2} = 1 \quad \text{or} \quad \frac{1}{S^2} = \frac{\cos^2\phi'}{S_1^2} + \frac{\sin^2\phi'}{S_3^2}. \tag{12.9}$$

We now introduce a new parameter of longitudinal strain. The *reciprocal quadratic elongation* λ' is defined as the reciprocal of the square of the stretch

$$\boxed{\lambda' = 1/S^2.} \tag{12.10}$$

The *principal reciprocal quadratic elongations* are then $\lambda_1' = 1/S_1^2$ and $\lambda_3' = 1/S_3^2$ and Eq. 12.9 becomes

$$\boxed{\lambda' = \lambda_1' \cos^2\phi' + \lambda_3' \sin^2\phi'.} \tag{12.11}$$

and this is the desired result. The equation of the strain ellipse can now be written as (see Eq. 12.4)

$$\lambda_1' x'^2 + \lambda_3' y'^2 = 1. \tag{12.12}$$

Change of a right angle

To find the angle of shear associated with the direction OP' graphically, draw vector $\mathbf{n}$ normal to the line T' tangent to the ellipse at point P' (Fig. 12.7b). Then ψ is the angle between $\mathbf{r}'$ and $\mathbf{n}$.

We can also find an expression for ψ from the dot product of vectors $\mathbf{r}'$ and $\mathbf{n}$ (see §7.3). This is simple in principle but unfortunately a little messy in execution because of the need to normalize the components of $\mathbf{n}$ and to convert this angle to a strain parameter.

The equation of tangent T' can be written down directly from the equation of the ellipse using a simple recipe: replace one x' and one y' in Eq. 12.12 with the corresponding coordinates of the point of tangency $P'(x', y')$ in the second of Eqs. 12.9. After dividing through by S the result is

$$(\lambda_1' \cos\phi')x' + (\lambda_3' \sin\phi')y' = 1/S.$$

The direction cosines of the normal vector $\mathbf{n}$ are proportional to the coefficients of x' and y' in this equation. Normalizing both by dividing each by the square root of the sum of their squares gives

$$\frac{\lambda_1' \cos\phi'}{\sqrt{\lambda_1'^2 \cos^2\phi' + \lambda_3'^2 \sin^2\phi'}} \quad \text{and} \quad \frac{\lambda_3' \sin\phi'}{\sqrt{\lambda_1'^2 \cos^2\phi' + \lambda_2'^2 \sin^2\phi'}},$$

and these are the required direction cosines of $\mathbf{n}$. With these direction cosines of $\mathbf{n}$ and the direction cosines of $\mathbf{r}'$ from Eq. 12.8, the dot product gives an expression for $\cos\psi$

$$\cos\psi = \frac{(\lambda_1' \cos\phi')(\cos\phi') + (\lambda_3' \sin\phi')(\sin\phi')}{\sqrt{\lambda_1'^2 \cos^2\phi' + \lambda_3'^2 \sin^2\phi'}},$$

which, after expanding and squaring, becomes

$$\cos^2\psi = \frac{(\lambda_1' \cos^2\phi' + \lambda_3' \sin^2\phi')^2}{\lambda_1'^2 \cos^2\phi' + \lambda_3'^2 \sin^2\phi'}.$$

Substituting the identities $\cos\psi = 1/\sec\psi$ and $\sec^2\psi = 1+\tan^2\psi$, together with the definition $\gamma = \tan\psi$, this can be rearranged to give

$$\gamma^2 = \frac{\lambda_1'^2\cos^2\phi' + \lambda_3'^2\sin^2\phi'}{(\lambda_1'\cos^2\phi' + \lambda_3'\sin^2\phi')^2} - 1. \tag{12.13}$$

Observing that the denominator is equal to λ'^2 (see Eq. 12.11), and defining a new measure of shear strain

$$\gamma' = \gamma\lambda' \quad \text{or} \quad \gamma = \gamma'/\lambda', \tag{12.14}$$

we then write Eq. 12.13 as

$$\gamma'^2 = \lambda_1'^2\cos^2\phi' + \lambda_3'^2\sin^2\phi' - (\lambda_1'\cos^2\phi' + \lambda_3'\sin^2\phi')^2.$$

Expanding and combining terms gives

$$\gamma'^2 = \lambda_1'^2\cos^2\phi'(1-\cos^2\phi') - 2\lambda_1'\lambda_3'\cos^2\phi'\sin^2\phi' + \lambda_3'^2\sin^2\phi'(1-\sin^2\phi'). \tag{12.15}$$

From the identity $\cos^2\phi' + \sin^2\phi' = 1$ we obtain two relationships

$$\cos^2\phi' = (1-\sin^2\phi') \quad \text{and} \quad \sin^2\phi' = (1-\cos^2\phi').$$

Using these in Eq. 12.15 and again rearranging yields

$$\gamma'^2 = (\lambda_1'^2 - 2\lambda_1'\lambda_3' + \lambda_3'^2)\cos^2\phi'\sin^2\phi' = (\lambda_1' - \lambda_3')^2\cos^2\phi'\sin^2\phi'.$$

Taking the square root we finally obtain the desired result

$$\boxed{\gamma' = (\lambda_1' - \lambda_3')\cos\phi'\sin\phi'.} \tag{12.16}$$

12.6 Mohr Circle for finite strain

The introduction of the new strain parameters λ' (Eq. 12.10) and γ' (Eq. 12.14) was aimed at obtaining Eq. 12.11 and Eq. 12.16 in these particular forms. It is useful to convert them by substituting the double angle identities

$$\cos^2\phi' = \tfrac{1}{2}(1+\cos 2\phi'), \qquad \sin^2\phi' = \tfrac{1}{2}(1-\cos 2\phi'), \qquad \cos\phi'\sin\phi' = \tfrac{1}{2}\sin 2\phi',$$

with the result

$$\lambda' = \tfrac{1}{2}(\lambda_1' + \lambda_3') + \tfrac{1}{2}(\lambda_1' - \lambda_3')\cos 2\phi', \tag{12.17a}$$
$$\gamma' = \tfrac{1}{2}(\lambda_1' - \lambda_3')\sin 2\phi'. \tag{12.17b}$$

These should look familiar. Their form is identical to the equations for the normal and shearing components of the traction vector (see Eqs. 9.17). Just as in that case, these expressions for λ' and γ' can be represented graphically by a *Mohr Circle for finite strain*. The main feature of this construction is a circle on the horizontal λ' axis (Fig. 12.8a). The distance to the center c and its radius r are given by

$$c = \tfrac{1}{2}(\lambda_1' + \lambda_3') \quad \text{and} \quad r = \tfrac{1}{2}(\lambda_3' - \lambda_1').$$

This circle has a number of features in common with the Mohr Circle for stress, but there are also some important differences. Because the lengths of the semi-axes of the strain ellipse are never negative, the circle lies wholly to the right of the origin.

Because $S_1 > S_3$, by definition $\lambda_1' < \lambda_3'$. Therefore $(\lambda_1' - \lambda_3')$, which appears in both of these expressions, is always a negative quantity. This has two consequences.

1. By Eq. 2.17a, if $\phi' = 0$ $(2\phi' = 0)$ then $\lambda' = \lambda_1'$ and $r\cos 2\phi' < 0$ so that λ_1' plots to the left of the center. If $\phi' = 90°$ $(2\phi' = 180°)$ then $\lambda' = \lambda_3'$ and $r\cos 2\phi' > 0$ and λ_3' plots to the right of the center. This reversal arises from the definition $\lambda' = 1/S^2$.
2. As we have noted in Figs. 12.6 and 12.7, the angle of shear ψ, and therefore also the shear strain γ, is positive in the first and third quadrants $(0 < \phi' < 90°$ and $180° < \phi' < 270°)$ and negative in the other two. By Eq. 12.17b, however, the parameter γ' has the opposite sign in each of these quadrants. Because of this switch in signs, negative values of γ' are plotted above the horizontal axis and positive values are plotted below it. This is the important *clockwise-up* convention used for constructing this Mohr Circle (Treagus, 1987).[3] Now both $2\phi'$ (Fig. 12.8a) and ϕ' (Fig. 12.8b) are measured in the same sense on the Mohr Circle and physical planes.

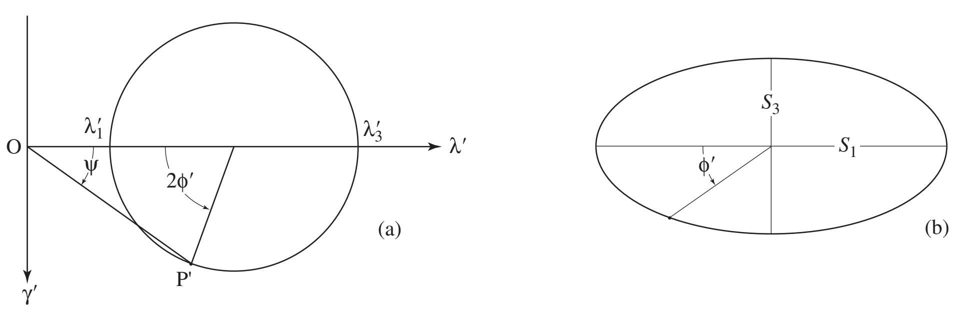

Figure 12.8 Finite strain: (a) Mohr Circle plane; (b) physical plane.

There is an auxiliary construction which greatly increases the usefulness of this diagram. The slope angle of line OP' is the angle of shear ψ associated with this particular direction (Fig. 12.8a). This fact follows directly from the definition $\tan\psi = \gamma'/\lambda'$ (see

[3]This same convention is used in the Mohr Circle for stress when tension is reckoned positive (see Fig. 9.15b).

Eq. 12.14), and it bypasses the mathematically convenient but otherwise obscure parameter γ', which is therefore little used in graphical work.

12.7 Pole of the Mohr Circle

The geometry of the physical plane and Mohr Circle plane can be even more closely related with the aid of a special point on the circle called the *pole* or *origin of lines*. This point, here denoted O_L, has a very useful property: A line through O_L which intersects the circle at P' is parallel to the corresponding line on the physical plane whose strain parameters are given by the coordinates of point P'.

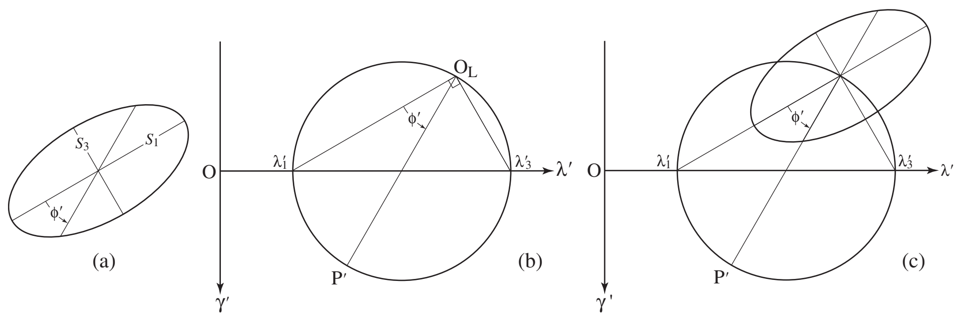

Figure 12.9 Pole construction: (a) physical plane; (b) Mohr Circle plane; (c) combination.

The pole is the point on the Mohr Circle through which all lines parallel to the corresponding lines on the physical plane pass just as they radiate from the center of the ellipse (Fig. 12.9a). This is the meaning of the phrase *origin of lines* which is used here as a short definition. It is analogous to the *origin of normals* of the Mohr Circle for stress. The pole may be located in several ways, most simply by drawing either a line parallel to the S_1 axis of the ellipse through the point λ'_1 to intersect the circle at O_L, or a line parallel to the S_3 axis of the ellipse through the point λ'_3 to intersect the circle at O_L (Fig. 12.9b). Note that these two lines at O_L are orthogonal, as are the axes of the ellipse.

Having located the pole we may now combine the two representations of the state of finite strain by drawing the ellipse centered at O_L (Fig. 12.9c). Note that this construction would not be possible without the clockwise-up convention for shear. In practice it is not necessary to draw an accurate ellipse because all the quantitative information is contained on the circle, but a sketch is a useful aid, especially for beginners.

We now may easily determine the strain parameters associated with any general line. For example, the line through O_L parallel to any general radius in the strain ellipse making an angle ϕ' with the S_1 direction intersects the circle at P' and the coordinates of this point are the required strain parameters associated with this radius (Fig. 12.9c).

We can then see that the Mohr Circle represents the locus of all possible values of the two strain parameters for a given ellipse and the pole represents its particular orientation on the physical plane. As the ellipse rotates on the physical plane, the pole moves along the circumference of the Mohr Circle. There are two special cases: If the S_1 direction is vertical, pole O_L coincides with the point representing λ'_1 (Fig. 12.10a), and if the S_3 direction is vertical, pole O_L coincides with the point representing λ'_3 (Fig. 12.10b).

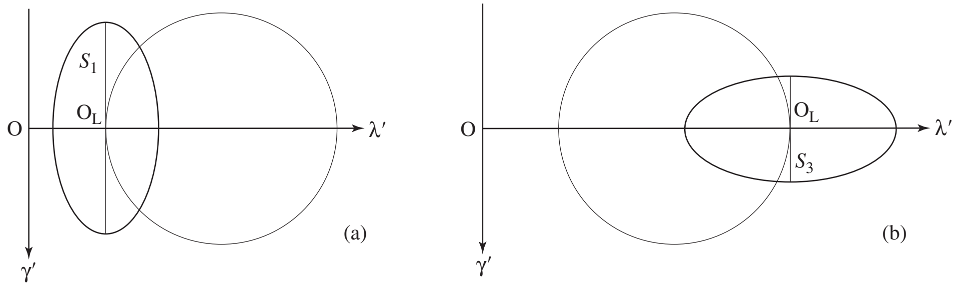

Figure 12.10 Special cases: (a) $O_L = \lambda'_1$; (b) $O_L = \lambda'_3$.

An even more important use of the Mohr Circle construction is to determine the principal stretches and their orientation from measurements of deformed angles or lines, and the pole plays a crucial role in this procedure.

The important first step is to form a *strain rosette* by drawing the stretched lines radiating from a single point, just as the radius vectors radiate from the center of the strain ellipse. In all these applications we are free to rotate the rosette into any orientation.

12.8 Strain from measured angles

Features from which angular changes can be determined are relatively common. As we have seen in the Wellman construction (Fig. 12.3), the angle of shear can be determined directly from a single deformed bilaterally symmetrical fossil. There are two main cases and both solutions utilize the pole and both follow closely the method described by Lisle (1991). The first involves one angle of shear associated with a line in known angular relation to the principal axes.

Problem

- A deformed trilobite is exposed on the plane of slaty cleavage and its median line m makes an angle of $\phi'_m = 20°$ with a lineation marking the S_1 direction. The angle of shear associated with this line is $\psi_m = +36°$ (Fig. 12.11a). Determine the shape of the strain ellipse.

Construction

1. Form a rosette by assembling the strained median line m and the line representing the S_1 direction radiating from a common point (Fig. 12.11b).
2. On a set of $\lambda'\gamma'$ axes, draw a line through the origin with a slope angle $\psi_m = +36°$ (with the clockwise-up convention this line slopes downward).

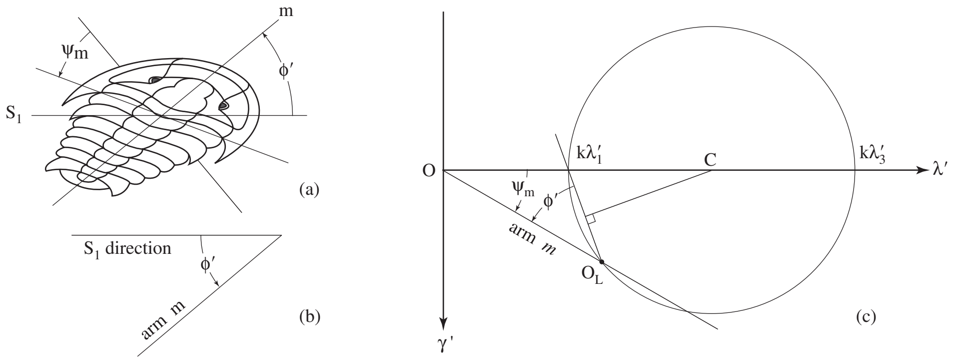

Figure 12.11 Trilobite: (a) physical plane; (b) rosette; (c) Mohr Circle.

3. At a convenient but arbitrary distance along the sloping ψ_m line locate the pole O_L, which is then a first point on the circle.
4. At O_L construct the rosette with arm m along the ψ_m line. The arm representing the S_1 direction then intersects the horizontal axis at λ_1', which is a second point on the Mohr Circle.
5. The perpendicular bisector of the segment $O_L\lambda_1'$ locates the center C on the horizontal axis.
6. Then with radius $C\lambda_1'$ complete the circle (Fig. 12.11c).

Answer

- Because the pole O_L is arbitrarily located in this procedure, the two principal values can not be uniquely determined. We can, however, determine the strain ratio from

$$R_s = \sqrt{k\lambda_3'/k\lambda_1'}, \tag{12.18}$$

where k is an unknown scale factor. This calculation, based on measurement of the two intercepts, yields $R_s = 2.0$. An analytical solution for this problem is given by Ramsay (1967, p. 234).

For some purposes it is convenient to represent the strain derived from such angular measurements by a specific ellipse, and the ellipse with the same area as the unit circle is the most appropriate. From the definition of the strain ratio (Eq. 11.4)

$$R_s = S_1/S_3,$$

and from the condition for no area change $\Delta = 0$ (Eq. 11.15)

$$S_3 = 1/S_1,$$

we can express the principal stretches $\tilde{S}_1$ and $\tilde{S}_3$ of the constant-area ellipse as

$$\tilde{S}_1 = \sqrt{R_s} \quad \text{and} \quad \tilde{S}_3 = 1/\sqrt{R_s}. \tag{12.19}$$

The second problem involves known angles of shear associated with two lines and the angle between these two lines. Then both the shape of the ellipse and its orientation can be found.

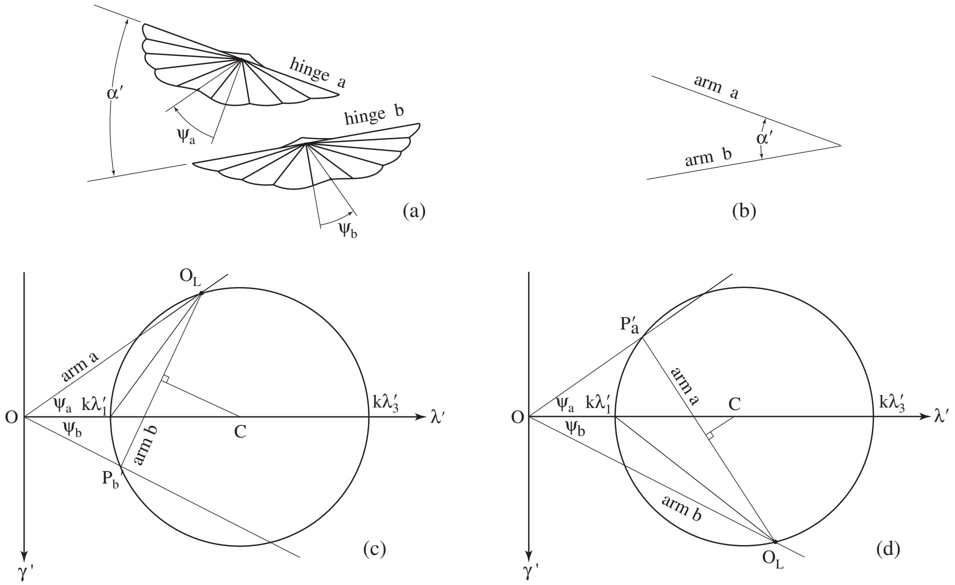

Figure 12.12 Brachiopods: (a) physical plane; (b) rosette; (c) Mohr Circle 1; (d) Mohr Circle 2.

Problem

- The angle between the hinge lines of two deformed brachiopods is $\alpha' = 30°$. The angles of shear associated with these lines are $\psi_a = -35°$ and $\psi_b = +27°$ (Fig. 12.12a). Determine the ratio of the principal stretches and their orientation.

Construction

1. Construct a strain rosette from hinge lines a and b (Figs. 12.12a and 12.12b).
2. On a pair of $\lambda'\gamma'$ axes draw lines making angles of ψ_a and ψ_b with the horizontal λ' axis and passing through the origin (Fig. 12.12c), paying attention to their signs and the clockwise-up convention.
3. Arbitrarily locate O_L on either of these lines.
   (a) If the ψ_a line is used, then position the rosette at that point with arm a along the ψ_a line (Fig. 12.12c). Arm b then intersects the ψ_b line at P_b'.

(b) If the ψ_b line is used, then position the rosette at that point with arm b along the ψ_b line. Arm a then intersects the ψ_b line at P'_a (Fig. 12.12d).

4. The perpendicular bisector of either chords $O_L P'_b$ or $O_L P'_a$ locates the center C on the λ' axis. With radius $O_L C = C P'_b$ complete the circle.
5. Chords $O_L \lambda'_1$ and $O_L \lambda'_3$ fix the orientations of the principal directions relative to arms a and b on the physical plane.

Answer

- Measure the distances to the two intercepts and calculate $R_s = \sqrt{k\lambda'_3/k\lambda'_1} = 2.0$. The S_1 direction makes an angle of $\phi'_a = 11°$ with the arm c.

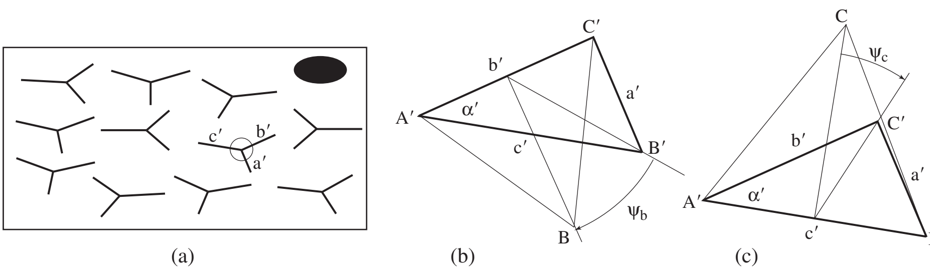

Figure 12.13 Welded tuff: (a) model strain: (b) angle of shear ψ_b; (c) angle of shear ψ_c.

In some situations angles of shear may be constructed from angles that are not initially right angles. The case of the deformed shards in welded tuff is one of these. Typically welded tuff display a strong foliation marked by the planar alignment of flattened pumice and glass shards. Some of these shards have a distinctive Y-shape. These originate between gas bubbles in the original flow and the angles between the three arms of these shards are approximately 120° (Ragan & Sheridan, 1972; Sheridan & Ragan, 1976). After deformation these angles are systematically changed (Fig. 12.13a) and these changes can be converted into angles of shear.

Problem

- For the deformed shard circled in Fig. 12.13a, determine the strain ratio and the orientation of the S_1 direction.

Construction

1. Reassemble the three shard limbs a', b' and c into scalene triangle $A'B'C'$ (Fig. 12.13b,c).
2. On two sides of this triangle construct an equilateral triangle (sides b' and c' are used here). These represent the shape, but not size, of the triangle before deformation.
3. The perpendicular bisector on each of the two base sides is the height of the triangles. As a result of a shear component parallel to these two sides apex point B is transformed to point B' (Fig. 12.13b) and apex point C is transformed to point C' (Fig. 12.13c).

4. As a result of these transformations we have measures of the two shear components: $\psi_b = +36°$ and $\psi_c = -25°$. The angle between sides b' and c' is $\alpha' = 34°$.
5. With these angles we can now construct the Mohr Circle just as before (Fig. 12.14).
   (a) Plot the line with slope ψ_b below and the line with slope ψ_c above the λ' axis.
   (b) Locate the pole O_L on either of these lines (we chose the ψ_b line).
   (c) Plot arm along the ψ_b line. The arm c then intersects the ψ_c at point P_c which is a second point on the circle.
   (d) Line $O_L P_c$ is a chord of the circle and its perpendicular bisection locates the center C of the circle on the λ' axis. The circle can then be completed using as radius $O_L C = P_c C$.
   (e) The line connecting points O_L and λ'_1 gives the orientation of the S_1 direction relative to the arms b and c.

Answer

- $R_s = \sqrt{k\lambda'_3/k\lambda'_1} = 2.0$ and the S_1 direction makes an angle $\phi' = 11°$ with arm c and this is parallel to the foliation. Similar results are obtained in any foliation-normal section. This foliation is essentially horizontal over great distances and R_s increases downward. This implies that the measured strain is due to the compaction of the tuff and that $S_1 = S_2 = 1.0$ and $S_3 = 0.5$.

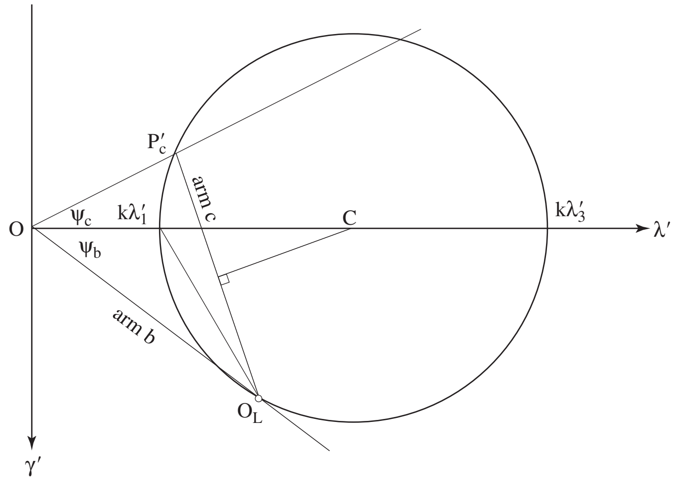

Figure 12.14 Mohr Circle for deformed shard.

An analytical solution for the problem of determining the strain from two angles of shear is also available (Ragan & Groshong, 1993).

If the two measured angles of shear both have the same sign, the accuracy of the Mohr Circle construction can be improved by reversing the sign of one of them. This is easily accomplished by changing the line of reference. In the reference circle (Fig. 12.15a), radius vectors $\mathbf{r}_1 = OP_1$ and $\mathbf{r}_2 = OP_2$ are orthogonal. Then tangent T_1 at point P_1 is parallel to $\mathbf{r}_2$ and T_2 at P_2 is parallel to $\mathbf{r}_1$. These are *conjugate radii* in the circle.

In the strain ellipse (Fig. 12.15b), the corresponding vectors $\mathbf{r}'_1 = OP'_1$ and $\mathbf{r}'_2 = OP'_2$ are no longer orthogonal but the tangent T'_1 at point P'_1 is still parallel to $\mathbf{r}'_2$ and the tangent T'_2 at P'_2 is parallel to $\mathbf{r}'$. These are conjugate radii in the ellipse; thus any pair of radii derived from conjugate radii are themselves conjugate. The measure of this change is ψ and the magnitude of the angle is the same for each radius vector, but of opposite sign.

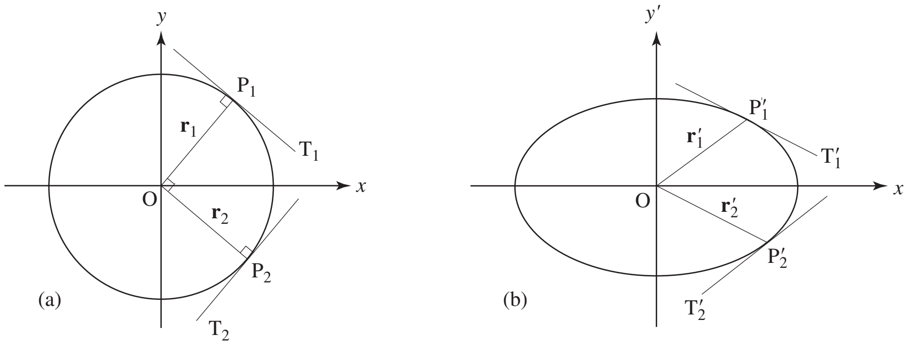

Figure 12.15 Conjugate radii: (a) circle; (b) ellipse.

12.9 Strain from measured stretches

The strain ellipse can also be obtained from measured stretches. The deformed length of a passive line of known initial length would yield an exact value of a stretch. Unfortunately few, if any, such lines exist in nature. There is, however, a class of structures from which the original length can be estimated. These are trains of micro-boudins bounded by fractures. The gaps between the broken fragments may be filled with the ductile material surrounding the boudins or they may be filled with vein material. Examples include broken crystals of tourmaline, rutile and arsenopyrite, amphibole, epidote and kyanite, and some forms of rectangular boudins developed in competent layers embedded in a ductile matrix. Broken fossil parts have also been used.

In the Swiss Alps there are a number of localities where abundant belemnites have been stretched in this manner (Beach, 1979). These have been examined extensively, but the techniques apply to many similar structures. The goal is to estimate the stretch which would have occurred in the absence of the rigid inclusion. Two simple methods have been proposed, each giving different results.

In the conventional method (Ramsay, 1967, p. 248; Ramsay & Huber, 1983, p. 93), the initial length l is taken as the sum of the lengths of the individual fragments and the final length l' as the total sum of the individual gaps G and fragments F (Fig. 12.16a). For N gaps and $N+1$ fragments, we have

$$G_{sum} = \sum_{i=1}^{N} G_i \quad \text{and} \quad F_{sum} = \sum_{i=1}^{N+1} F_i . \tag{12.20}$$

The second method involves a minor but important modification (Hossain, 1979). The final length is taken as the distance between the midpoints of the two end fragments and the initial length is the sum of the fragment lengths between these two points (Fig. 12.16b). These lengths can be written as

$$G_{sum} = \sum_{i=1}^{N} G_i \quad \text{and} \quad F_{sum} = \frac{1}{2}\sum_{i=1}^{N} (F_i + F_{i+1}) . \tag{12.21}$$

This equation for F_{sum} and the corresponding illustration makes clear that Hossain's method is a straightforward extension of the center-to-center technique used for deformed grains of §12.2.

In both Ramsay's and Hossain's methods the total stretch associated with the inclusion train is then calculated from

$$S = l'/l = 1 + (G_{sum}/F_{sum}). \tag{12.22}$$

Problem

- From the following gap and fragment lengths calculate the stretch using the methods of Ramsay and Hossain: $G_i = 7, 5, 9$ mm and $F_i = 11, 5, 7, 9$ mm (Fig. 12.16).

Solution

1. By Ramsay's method (Eq. 12.20) $G_{sum} = 21$ mm, $F_{sum} = 32$ mm and $S = 1 + 21/31 = 1.656\,25$.
2. By Hossain's method (Eq. 12.21) $G_{sum} = 21$ mm, $F_{sum} = 22$ mm and $S = 1 + 21/22 = 1.954\,55$.

As can be seen, the stretch calculated by Hossain's method is significantly greater than that obtained by Ramsay's method.

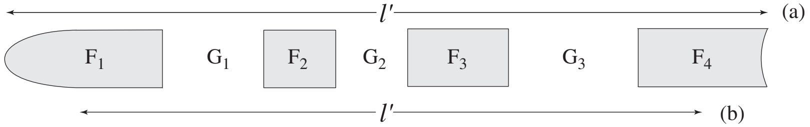

Figure 12.16 Stretch from boudinage: (a) Ramsay's method; (b) Hossain's method.

Both of these approaches belie the complexity of the physical process of boudin formation. In particular, neither method takes into account the evolutionary sequence of the separation of the fragments which must have occurred.

Before the first fracture, the rigid inclusion can not record any strain. As a consequence, the material adjacent to the inclusion must deform inhomogeneously to compensate for the extension which would have occurred in the absence of the inclusion. Once a fracture forms, a part of the extension will be accommodated by the separation of the fragments and a part, as before, by inhomogeneous deformation near the inclusion contact.

Recognizing that the formation of multiple fragments involves a series of such steps, a third method for estimating the stretch involves an iterative strain-reversal technique and gives even better results (Ferguson, 1981, 1987; Ferguson & Lloyd, 1984; Ford & Ferguson, 1985). Lloyd and Condliffe (2003) describe a computer program which automates the process.

Steps

1. With N gaps, N steps are required to reverse the total stretch. Here $N = 3$.
2. The initial and final lengths associated with each gap are given by $l_i = \frac{1}{2}(F_i + F_{1+i})$ and $l'_i = G_i + l_i$. Thus

$$l_1 = 5.5 + 2.5 = 8.0, \quad l'_1 = 7.0 + 8.0 = 15,$$
$$l_2 = 2.5 + 3.5 = 6.0, \quad l'_2 = 5.0 + 6.0 = 11,$$
$$l_3 = 5.5 + 2.5 = 8.0, \quad l'_3 = 9.0 + 8.0 = 17.$$

3. The stretch associated with each gap is calculated from $S_i = l'_i/l_i$ and the gap with the smallest stretch S_{min} is taken as the final increment of stretch (Fig. 12.17a).

$$S_1 = 15/8 = 1.875\,00, \quad S_1^{-1} = 0.533\,33,$$
$$S_2 = 11/6 = 1.833\,33, \quad S_2^{-1} = 0.545\,45,$$
$$S_3 = 17/8 = 2.125\,00, \quad S_3^{-1} = 0.470\,59.$$

4. The gap associated with S_{min} (which is not always the smallest) is now closed by applying the inverse stretch $1/S_{min} = 0.545\,45$ to its length. The other gaps are also reduced by this same factor. There are now $N - 1 = 2$ gaps and $N = 3$ fragments (Fig. 12.17b). Relabeling the gaps and fragments, the lengths are now

$$l_1 = 5.5 + 6.0 = 11.5, \quad l'_1 = 11.5 + 0.181\,82 = 11.681\,82,$$
$$l_2 = 4.5 + 6.0 = 10.5, \quad l'_2 = 10.5 + 1.272\,73 = 11.772\,73,$$

and the stretches associated with the remaining gaps are

$$S_1 = 11.681\,82/11.5 = 1.015\,81, \quad S_1^{-1} = 0.984\,44,$$
$$S_2 = 11.771\,73/10.5 = 1.121\,21, \quad S_2^{-1} = 0.891\,89.$$

5. With this new S_{min} the next gap is closed and the other reduced (Fig. 12.17c). The lengths are now

$$l_1 = 11.5 + 4.5 = 16.0, \quad l_1' = 16.0 + 1.089\,49 = 17.089\,49,$$

and the single remaining stretch is

$$S_1 = 17.089\,49/16 = 1.068\,09, \quad S_1^{-1} = 0.936\,25.$$

With this, the final gap is now closed and the belemnite is whole (Fig. 12.17d).

6. The total stretch is the inverse of the product of the inverse stretches at each stage.[4]

$$S = \left[\prod_{i=1}^{N} S_i^{-1}\right]^{-1} = \frac{1}{0.545\,45 \times 0.984\,44 \times 0.93625} = 1.989\,13. \tag{12.23}$$

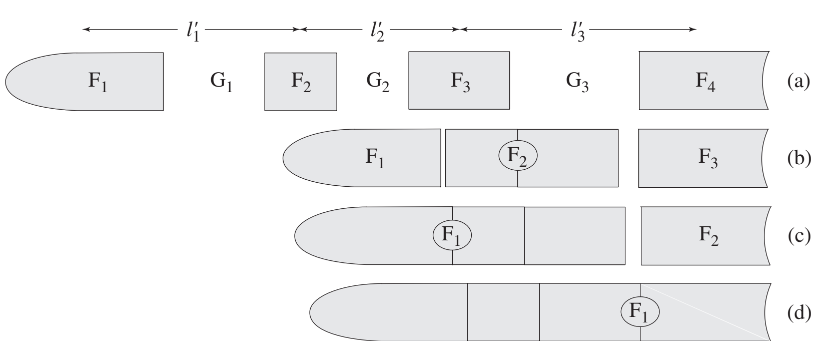

Figure 12.17 Iterative strain reversal technique (Ferguson, 1981).

Although this method still underestimates the total stretch it gives particularly good results and is the recommended approach. By hand, it does, however, require extra work. Hossain's method gives nearly as good results if the gap and the fragment lengths are fairly uniform and is just as quick as Ramsay's approach.

From measured stretches we can determined the state of strain. In two dimensions, there are two cases. If the two stretches and the angles they make with the principal

[4]The symbol $\prod$ means form the product of the series of all N items.

directions are known, the principal stretches can be found, otherwise three stretches are needed. The solutions follow the method of Lisle and Ragan (1988). The problem of three stretches is the simpler one and we start with it.

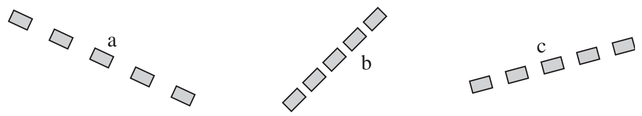

Figure 12.18 Three idealized stretched belemnites.

Problem

- From the three stretched belemnites, determine the principal stretches and their orientations (Fig. 12.18). (Because the gap and fragment lengths are exactly the same in this idealization, the methods of Hossain and Ferguson give identical results.)

Approach

- Before undertaking the full construction it is useful to sketch the strain ellipse as a visual check. This is done by assembling the three scaled stretched lines into a rosette (Fig. 12.19a). Because the ellipse is centro-symmetric, each of these radius vectors has an equal and opposite radius vector, and we then have three complete diameters of the ellipse which can then be sketched with a fair degree of accuracy (Fig. 12.19b).

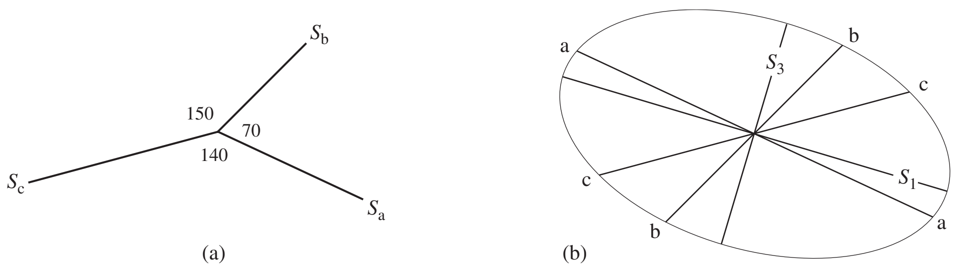

Figure 12.19 Stretch belemnites: (a) scaled rosette; (b) sketched ellipse.

Construction

1. From the three measured stretches, the corresponding reciprocal quadratic elongations are

$$S_a = 2.2\ (\lambda'_a = 0.2066), \quad S_b = 1.4\ (\lambda'_b = 0.5102), \quad S_c = 1.8\ (\lambda'_c = 0.3086).$$

2. Rearrange the three stretch directions into a rosette with arm c between arms a and b (Fig. 12.20a).

3. Draw the vertical γ' axis (but not the horizontal λ' axis) and add three parallel lines at distances equal to the values of λ'_a, λ'_b and λ'_c using a convenient scale (Fig. 12.20b).
4. Arbitrarily locate the pole O_L on the intermediate λ'_c line and at this point draw the rosette so that arm c lies along this same line.
5. Through O_L draw lines parallel to arm a to intersect the λ'_a line at P'_a and parallel to arm b to intersect the λ'_b line at P'_b.
6. Points O_L, P'_a and P'_c lie on the circle, and the perpendicular bisectors of chords $O_L P'_a$ and $O_L P'_c$ intersect to locate its center.
7. Through this center now draw the horizontal λ' axis and complete the circle. Measure the intercepts to determine the values of λ'_1 and λ'_3.
8. Draw the orthogonal lines $\lambda'_1 O_L$ and $\lambda'_3 O_L$ (not shown). These give the orientation of the principal axes relative to the rosette.

Answer

- The principal quadratic elongations and the corresponding principal stretches are

$$\lambda'_1 = 0.20\ (S_1 = 2.24) \quad \text{and} \quad \lambda'_3 = 0.60\ (S_3 = 1.29).$$

Note that the λ' coordinate of $P'a$ is almost the same as λ'_1 (see Fig. 12.20b). The angle between λ'_1 and arm a is 8° measured anticlockwise.

When constructing the rosette at O_L arms a and b may not intersect the two corresponding vertical λ' lines. Then rotate the rosette 180° to reverse the directions of arms and then proceed just as before. In this case the pole O_L will be below the λ' axis rather than above it.

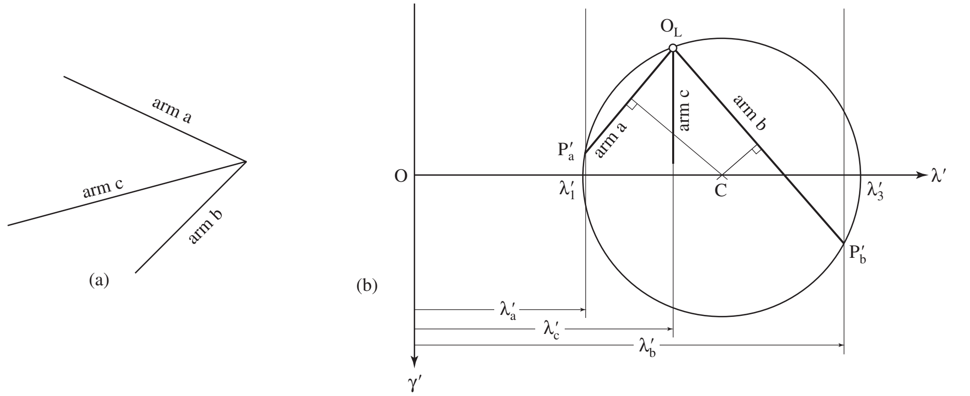

Figure 12.20 Solution of the belemnite problem: (a) rosette; (b) Mohr Circle construction.

The graphical solution of the problem of two stretches in known angular relation with the principal axes of the strain ellipse proceeds in a similar way, except that an extra step is needed to locate a third point on the circle.

Problem

- Two stretched tourmaline crystals are exposed on the plane of schistosity (Fig. 12.21a). A prominent lineation on this plane marks the S_1 direction. Determine the principal stretches.

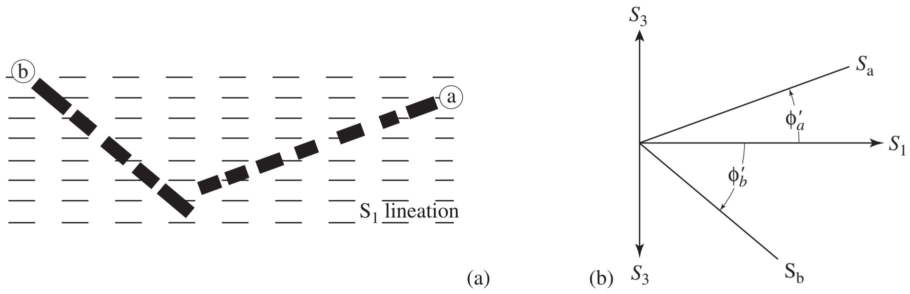

Figure 12.21 Problem of two stretches: (a) tourmaline crystals; (b) rosette.

Construction

1. The two stretches, the corresponding reciprocal quadratic elongations and their orientations relative to the S_1 direction are

$$S_a = 1.7, \quad \lambda'_a = 0.3460, \quad \phi'_a = +20^\circ,$$
$$S_b = 1.4, \quad \lambda'_b = 0.5102, \quad \phi'_b = -40^\circ.$$

2. Construct a rosette representing the two stretches and the principal axes (Fig. 12.21b).
3. Draw the vertical γ' axis and a pair of parallel lines at scale distances equal to the values of λ'_a and λ'_b (Fig. 12.22a).
4. In order to find three points on the circle it is necessary to use the strain rosette twice.
   (a) First, arbitrarily locate pole O_L on the λ'_b line and construct the rosette there with arm b along the λ'_b line. Then draw a line parallel to arm a to intersect the λ'_a line at P'_a. This gives two points on the circle (Fig. 12.22a). The λ'_3 point on the circle lies on the arm representing the S_3 axis.
   (b) Second, if this yet to be located λ'_3 point were the pole, then the S_3 axis would be vertical (see Fig. 12.10b). From this point arm a would intersect the circle at point P'_a. To locate this λ'_3 point we simply reverse this construction by making $O_L = P'_a$ and constructing the rosette there so that the S_3 axis lies along the vertical λ'_a line. Then arm a intersects the line representing the first S_3 direction to give the λ'_3 point. We now add the horizontal λ' axis to the diagram (Fig. 12.22b).
5. The perpendicular bisector of the chord $P'_a P'_b$ intersects the λ' axis at center C and the circle can then be completed with $O_L C = P'_a C$ as radius (Fig. 12.23).

Answer

- Measuring the distances of two intercepts gives

$$\lambda_1' = 0.28\ (S_1 = 1.9) \quad \text{and} \quad \lambda_3' = 0.84\ (S_3 = 1.1).$$

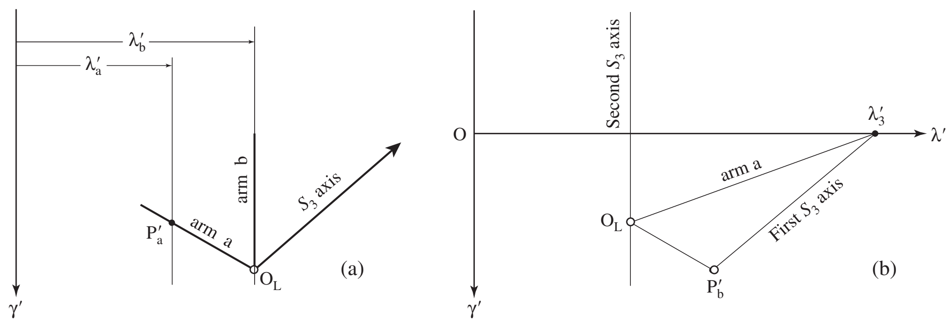

Figure 12.22 Graphical solution for two stretches: (a) Step 1; (b) Step 2.

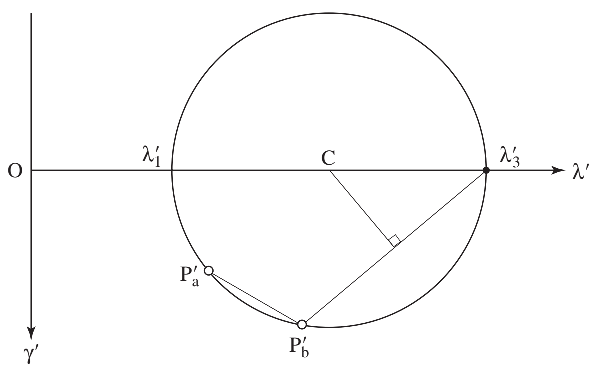

Figure 12.23 Mohr Circle for two stretches.

In this construction the pole may be located on either of the vertical λ_a' or λ_b' lines. If, as here, it is chosen on the b line the pole will be on the lower semi-circle, and if on line a it will be on the upper semi-circle. This method breaks down when the two stretched lines make the same angle with the S_1 direction and lacks sensitivity as this condition is approached.

12.10 Restoration

The importance of determining the state of strain lies in the fact that it describes in the most fundamental way the changes in shape and size which occur as the result of homogeneous deformation. Once we have determined the strain ellipse we immediately know that it was derived from a circle of unit radius. With this information we can then restore any strained object to its initial shape and size.

A simple but important case is the determination of the original thickness of a homogeneously deformed layer. If a material line initially normal to bedding is marked by some physical feature then the associated strain in this single direction could easily be removed. In a few rare situations this may be possible. For example, *Skolithus* is a fossilized worm tube originally normal to bedding surfaces (McLeish, 1971; Wilkinson, *et al.*, 1975). In most cases, however, a more general approach must be used.

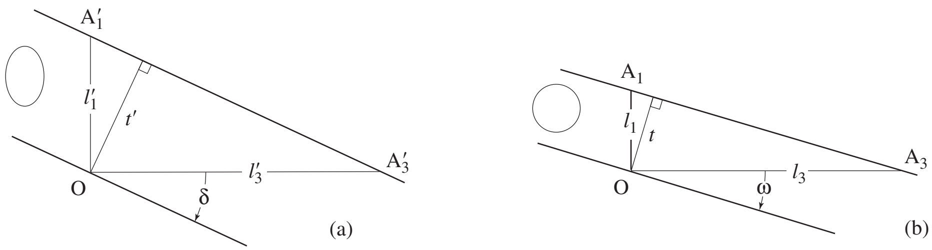

Figure 12.24 Sedimentary bed: (a) deformed thickness t'; (b) restored thickness t.

Problem

- The thickness of a deformed layer is $t' = 1.30$ m. If the S_1 direction is vertical and $S_1 = 1.25$, and $S_3 = 0.80$ what was the original thickness?

Method

1. Arbitrarily locate a point O on the trace of the inclined lower boundary of the layer. Then draw rays parallel to the principal direction to intersect the upper trace at points A'_1 and A'_3 (Fig. 12.24a).
2. Measure the lengths of the vertical and horizontal segments $l'_1 = OA'_1$ and $l'_3 = OA'_3$. Divide these two lengths by the corresponding principal stretches to give original lengths $l_1 = l'_1/S_1$ and $l_3 = l'_3/S_3$.
3. With these restored lengths l_1 and l_2 locate new points A_1 and A_3 on these same rays (Fig. 12.24b). These fix the relative position of the upper boundary of the layer before deformation, and the perpendicular distance between this trace and point O is the original thickness t.

Answer

- The thickness of the layer before deformation was $t = 1.10$ m. It should be especially noted that if the line of measured thickness t' were unstrained directly the result would be in error because t and t' are not generally marked by the same material line. Schwerdtner (1978) described an analytical solution.

Removal of the strain restores not only the initial thickness, but it also removes the inclination due to the strain. Assuming the beds were originally horizontal, this remaining dip is the result of the rotational part of the deformation, hence is equal to the angle of rotation.

For more complicated shapes a more general approach is required. One such application is the restoration of the shape of deformed fossils so that they may be accurately identified (Bambach, 1973; Raup & Stanley, 1978, p. 75). The basic technique, however, applies to any two-dimensional shape.

First, we construct a pair of axes on a drawing or photograph of the deformed object with x' parallel to the long axis and y' parallel to the short axis of the strain ellipse (Fig. 12.25a). In this system, the position vectors $\mathbf{r}'(x', y')$ of points on the outline of the deformed object are determined. This may be done by hand, but it is far easier to record the coordinates with the use a digitizing tablet. Clearly, the more points, the more accurate the description of the deformed object and the more complete the reconstruction can be.

The corresponding position vectors $\mathbf{r}(x, y)$ in the initial state are related to $\mathbf{r}'(x', y')$ by the inverse equations (cf. Eq. 12.5)

$$x = x'/S_1 \quad \text{and} \quad y = y'/S_3.$$

With these we then transform these points back to their initial locations and plot the result (Fig. 12.25b).

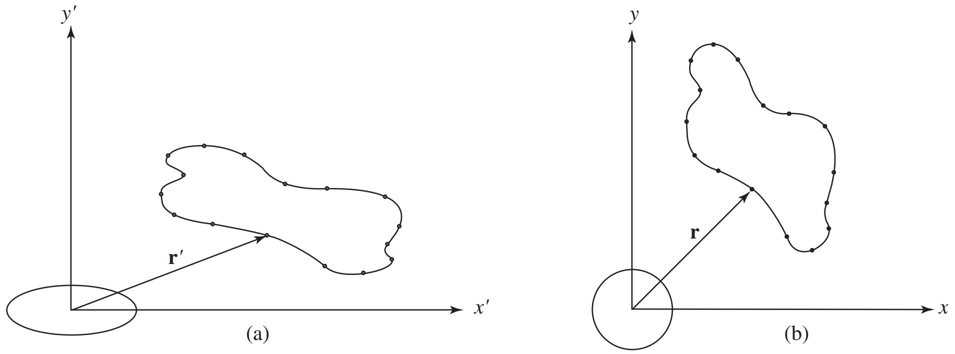

Figure 12.25 General restoration: (a) *x′y′* plane; (b) *xy* plane.

If only R_s is known, the constant-area principal stretches of Eq. 12.19 can be used in this reconstruction of its shape, but if $\Delta > 0$ it will be too small and if $\Delta < 0$ it will be too large.

12.11 Strain and related tensors

To show how strain and deformation are related, we decompose the deformation tensor $\mathbf{D}$ into its stretch and rotational components.[5] We do this by viewing the deformation as having occurred in two steps: first, the rotation of the principal axes from their initial to final state followed by the stretch to produce the strain ellipse. We write this sequence as

$$\mathbf{D} = \mathbf{SR}, \tag{12.24}$$

where the orthogonal *rotation tensor* $\mathbf{R}$ is applied first, followed by the symmetric *left-stretch tensor* $\mathbf{S}$ (called this because it is written on the left of $\mathbf{R}$).[6] For computational purposes we need the matrix form

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \tag{12.25}$$

We can perform this decomposition of $\mathbf{D}$ graphically with the aid of a Mohr Circle construction.

Problem

- Determine the stretch component of the simple shear deformation

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Procedure

1. Using the convention of Fig. 12.26a, plot the two points

$$p_1(D_{11}, -D_{21}) = (1, -1) \quad \text{and} \quad p_2(D_{22}, D_{12}) = (1, 0)$$

   on a pair of coordinate axes labeled D_{11}, D_{22} and D_{12}, D_{21} (Fig. 12.27a).
2. Locate the center C at the midpoint of diameter $p_1 p_2$ and complete the *off-axis* circle with radius $p_1C = p_2C$.
3. The sloping line through points O and C intercepts this circle at two points and the lengths of the segments from O represent the principal stretches S_1 and S_3.
4. There are two ways of representing the circle representing $\mathbf{S}$.

[5] As in §11.7 the same notation common to continuum mechanics is used here: majuscules (upper case letters) for the material description and minuscules (lower case letters) for the spatial description.

[6] A second decomposition is $\mathbf{D} = \mathbf{RS}$, that is, a stretch followed by a rotation. In this case $\mathbf{S}$ is the *right-stretch tensor*. The ellipses produced by these two stretch tensors are identical but their orientations differ by the rotation. Because the strain ellipse and its orientation are described in the final state which we observe it is convenient to think of the rotation as having preceded the stretch (Elliott, 1970, p. 2234), so the left-stretch tensor is the one we will use.

(a) A quick way is to draw a second set of axes with the same origin: the S_{11}, S_{22} axis through the center of the circle and the S_{12}, S_{21} axis perpendicular to this (Fig. 12.27b). Having removed the rotation, the circle is now *on axis*.

(b) A more formal way is to rotate the center of the off-axis circle through the angle of rotation ω to the horizontal axis and complete the circle as before (Fig. 12.27c).

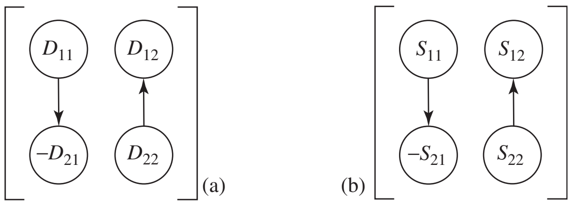

Figure 12.26 Plotting conventions (after Means, 1992, p. 19): (a) Mohr Circle for **D**; (b) Mohr Circle for **S**.

Answer

- Measuring the lengths of the segments OS_1 and OS_3 the principal stretches are $S_1 = 1.62$ and $S_3 = 0.62$. similarly we can determine the coordinates of point p_1 and p_2 which then gives

$$\mathbf{S} = \begin{bmatrix} 1.34 & 0.45 \\ 0.45 & 0.89 \end{bmatrix}.$$

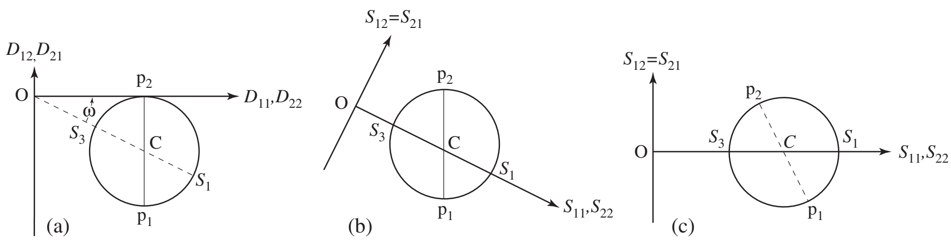

Figure 12.27 Mohr Circles for **D** and **S**.

We can also find **S** analytically but to do so we first need to form the *inverse* of a matrix. In ordinary algebra if a variable is multiplied by its reciprocal or inverse the result is the number 1. We write this as

$$AA^{-1} = 1 \quad \text{or} \quad A^{-1}A = 1,$$

where $A^{-1} = 1/A$. In matrix algebra, the place of 1 is taken by the *unit* matrix **1** (also called the *identity* matrix and represented by the symbol **I**). That is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1}.$$

We then write the product of matrix $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ in two ways

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{1} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}.$$

Note that these two results are the same. This is an exception to the general rule – the product of a matrix and its inverse is *commutative*. With the first of these and denoting the unknown inverse matrix $\mathbf{A}$ by the symbol $\mathbf{B}$ then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{12.26}$$

Performing the multiplication of the two square matrices we obtain four equations containing the four unknown elements of $\mathbf{B}$

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= 1, \\ A_{11}B_{12} + A_{12}B_{22} &= 0, \\ A_{21}B_{11} + A_{22}B_{21} &= 0, \\ A_{21}B_{12} + A_{22}B_{22} &= 1. \end{aligned}$$

Solving for these unknowns gives

$$B_{11} = \frac{A_{22}}{A_{11}A_{22} - A_{12}A_{21}}, \qquad B_{12} = \frac{-A_{12}}{A_{11}A_{22} - A_{12}A_{21}},$$
$$B_{21} = \frac{-A_{21}}{A_{11}A_{22} - A_{12}A_{21}}, \qquad B_{22} = \frac{A_{11}}{A_{11}A_{22} - A_{12}A_{21}},$$

and these B_{ij} are the required elements of the inverse. With these results, we can quickly form the inverse of any 2×2 general matrix $\mathbf{A}$ in three easy steps.

1. Interchange the elements of the main diagonal A_{11} and A_{22}.
2. Change the signs of the off-diagonal elements A_{12} and A_{21}.
3. Divide each element by the determinant of $\mathbf{A}$.

The full result is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}. \tag{12.27}$$

Note that if $\det \mathbf{A} = 0$ then $\mathbf{A}$ is *singular* and the inverse does not exit.

In this context $\mathbf{A}^{-1}$ reverses the effect of $\mathbf{A}$. As in ordinary algebra, doing and then undoing something is the same as not having done anything to begin with, and the matrix operation of doing nothing is the unit matrix.

To solve Eq. 12.24 for $\mathbf{S}$, we *post-multiply* (that is, multiply from the right) both sides by $\mathbf{R}^{-1}$ giving

$$\mathbf{D}\mathbf{R}^{-1} = \mathbf{S}\mathbf{R}\mathbf{R}^{-1}. \tag{12.28}$$

A positive (anticlockwise) rotation is represented by the orthogonal matrix (see Eq. 7.42)

$$\mathbf{R} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}.$$

Applying the three steps, its inverse is

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \tag{12.29}$$

The *transpose* of a matrix is formed by exchanging rows and columns. For $\mathbf{R}$ this gives

$$\mathbf{R}^{T} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \tag{12.30}$$

and we immediately see that the inverse of an orthogonal matrix has a particularly simple form – the inverse and transpose are identical.

$$\mathbf{R}^{T} = \mathbf{R}^{-1},$$

and we can then immediately write down this particular inverse. We can then write Eq. 12.28 as

$$\mathbf{D}\mathbf{R}^{T} = \mathbf{S}\mathbf{R}\mathbf{R}^{T}. \tag{12.31}$$

The product $\mathbf{R}\mathbf{R}^{T} = \mathbf{1}$ and this means that the two rotations cancel. Just as in ordinary algebra, the unit matrix $\mathbf{1}$ is usually not written in such expressions. We then have

$$\mathbf{S} = \mathbf{D}\mathbf{R}^{T}. \tag{12.32}$$

We can obtain the angle of rotation directly from the components of $\mathbf{D}$ from Eq. 11.37,

$$\tan\omega = \frac{D_{21} - D_{12}}{D_{11} + D_{22}} \tag{12.33}$$

and with this angle we can evaluate the elements of both $\mathbf{R}$ and $\mathbf{R}^{T}$.

Problem

- Determine the left-stretch component of the simple shear deformation

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution

1. From Eq. 12.33, $\tan\omega = 0.5$ or $\omega = 26.5651°$ and we can then form the matrix for $\mathbf{R}^T$

$$\mathbf{R}^T = \begin{bmatrix} 0.8944 & 0.4472 \\ -0.4472 & 0.8944 \end{bmatrix}.$$

2. Using this in Eq. 12.32 gives

$$\mathbf{S} = \mathbf{D}\mathbf{R}^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 \\ -0.4472 & 0.8944 \end{bmatrix}.$$

3. Performing the multiplication yields

$$\mathbf{S} = \begin{bmatrix} 0.8944 & 0.4472 \\ 0.4472 & 1.3416 \end{bmatrix}. \tag{12.34}$$

We can construct the Mohr Circle for the left-stretch tensor $\mathbf{S}$ from the components of its matrix representation.

Problem

- Draw the Mohr Circle for $\mathbf{S}$ of Eq. 12.34 and find the principal stretches and their orientation.

Procedure

1. Draw a pair of coordinate axes and label the horizontal axis S_{11}, S_{22} and the vertical axis $S_{12} = S_{21}$.
2. Using the convention of Fig. 12.26b, plot the two points

$$p_1(S_{11}, -S_{21}) = (0.8944, -0.4472) \quad \text{and} \quad p_2(S_{22}, S_{12}) = (1.3416, 0.4472)$$

   using a convenient scale (Fig. 12.11).
3. The intersection of the diameter p_1p_2 and the horizontal axis locates the center C and the circle can then be completed with $Cp_1 = Cp_2$ as radius.

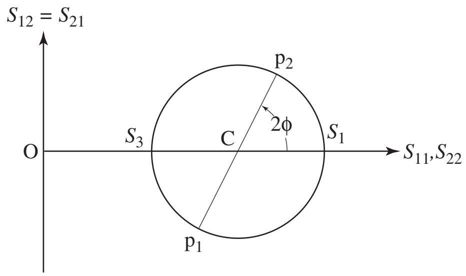

Figure 12.28 Mohr Circle for **S** by direct plot.

Answer

- The angle between the x_2 and the S_1 direction is $2\phi = 64°$ on the Mohr Circle plane or $\phi = 32°$ on the physical plane and this gives the orientation of S_1 in the deformed state.

From this diagram we can also derive expressions for the principal stretches. Distance c along the horizontal axis to the center of the circle and radius r of the circle are given by

$$c = \tfrac{1}{2}(S_{11} + S_{22}) \quad \text{and} \quad r = \tfrac{1}{2}\sqrt{(S_{11} + S_{22})^2 + (S_{12} + S_{21})^2}.$$

Then

$$S_1 = c + r \quad \text{and} \quad S_3 = c - r.$$

In three dimensions we can not so easily form the rotation tensor $\mathbf{R}$ so this method for finding $\mathbf{S}$ from $\mathbf{D}$ by first forming $\mathbf{R}^{-1}$ does not work in three dimensions. There is an alternative approach which also leads to several important insights.

To solve for $\mathbf{S}$ by this more general method, post-multiply each side by its transpose and apply the *reversal rule* whereby the transpose of a product is equal to the product of transposes in reverse order

$$\mathbf{D}\mathbf{D}^T = \mathbf{S}\mathbf{R}(\mathbf{S}\mathbf{R})^T = \mathbf{S}\mathbf{R}\mathbf{R}^T\mathbf{S}^T. \tag{12.35}$$

Because $\mathbf{R}\mathbf{R}^T = \mathbf{1}$ the rotations cancel leaving $\mathbf{S}\mathbf{S}^T$. By symmetry $\mathbf{S} = \mathbf{S}^T$ so

$$\mathbf{S}^2 = \mathbf{D}\mathbf{D}^T, \tag{12.36}$$

where $\mathbf{S}^2$ is the *left Cauchy–Green tensor* (Truesdell, 1991, p. 112). Geometrically $\mathbf{D}^T$ produces the same ellipse as $\mathbf{D}$ but rotates the principal axes from the deformed state back to the undeformed state. *Pre-multiplying* (that is, multiplying from the left) by $\mathbf{D}$ then rotates these principal axes back again to the deformed state and at the same time produces an ellipse whose principal axes are the squares of the principal stretches S_1^2 and S_3^2. This eliminates the rotation and $\mathbf{S}^2$ is symmetric. From $\mathbf{S}^2$ we can then find $\mathbf{S}$ graphically.

Problem 2

- Determine the components of left-stretch tensor $\mathbf{S}$ directly from the simple shear deformation tensor

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Procedure

1. Form the product

$$\mathbf{S}^2 = \mathbf{D}\mathbf{D}^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

2. As before, plot the points $p_1(1, -1)$ and $p_2(2, 1)$ and complete the Mohr Circle for $\mathbf{S}^2$ (Fig. 12.29).
3. The principal values $S_1^2 = 2.62$ and $S_3^2 = 0.38$ are represented by the intercepts on the horizontal axis.
4. Taking the square roots gives $S_1 = 1.62$ and $S_3 = 0.62$. With these construct the Mohr Circle for $\mathbf{S}$.
5. On this smaller $\mathbf{S}$ circle draw a diameter parallel to the diameter p_1p_2 on the larger $\mathbf{S}^2$ circle. The coordinates of these two points give the components of $\mathbf{S}$.

Answer

- The coordinates are $p_1(0.89, -0.45)$ and $p_2(1.34, 0.45)$. Therefore the matrix representation of $\mathbf{S}$ is (compare Eq. 12.34)

$$\mathbf{S} = \begin{bmatrix} 0.89 & 0.45 \\ 0.45 & 1.34 \end{bmatrix}.$$

With $\mathbf{S}$ we can now find $\mathbf{R}$. Pre-multiplying both sides of Eq. 12.24 by $\mathbf{S}^{-1}$ gives

$$\mathbf{S}^{-1}\mathbf{D} = \mathbf{S}^{-1}\mathbf{S}\mathbf{R}.$$

Because $\mathbf{S}^{-1}\mathbf{S} = \mathbf{1}$ we have

$$\mathbf{R} = \mathbf{S}^{-1}\mathbf{D}. \tag{12.37}$$

All the results obtained so far are part of the *material description* of a deformation, that is, the independent variables are the material coordinates. In geological applications we must deal with the deformed state, that is, with the spatial coordinates as independent variables.

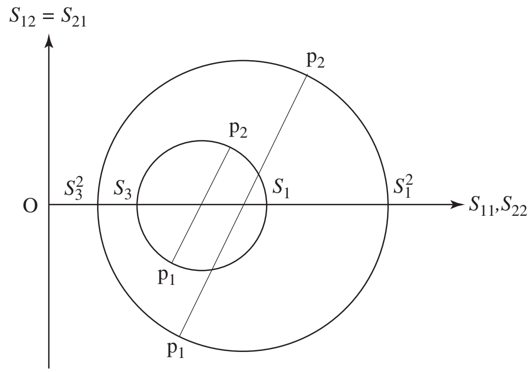

Figure 12.29 Mohr Circle for $\mathbf{S}^2$ and $\mathbf{S}$.

The pair of affine transformation equations which relate the particle at spatial point $p(x_1, x_2)$ back to its initial location at $P(X_1, X_2)$ is

$$X_1 = d_{11}x_1 + d_{12}x_2 - T_1,$$
$$X_2 = d_{21}x_1 + d_{22}x_2 - T_2.$$

The coefficients d_{11}, d_{12}, d_{21} and d_{22} describe the rotation and stretch required to restore the initial configuration, and constants $-T_1$ and $-T_2$ describe the reverse translation. In matrix form these two become

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{12.38}$$

This constitutes the *spatial description* of a homogeneous deformation and the essential part of this description of the reverse transformation is the square matrix representing the *inverse deformation tensor*

$$\mathbf{D}^{-1} = \mathbf{d} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}.$$

Just as before, we can decompose $\mathbf{d}$ into the product of a reverse rotation and an inverse stretch. The first step involves forming the inverse of both sides of Eq. 12.24 to give

$$\mathbf{D}^{-1} = (\mathbf{SR})^{-1}.$$

Applying another version of the *reversal rule* whereby the inverse of the product of two matrices is the product of inverses in reverse order we then have

$$\mathbf{D}^{-1} = \mathbf{R}^{-1}\mathbf{S}^{-1}.$$

This makes sense because if we take two steps forward and then back-up we must reverse the second step first. We may also write this as

$$\mathbf{d} = \mathbf{rs}, \tag{12.39}$$

where $\mathbf{r} = \mathbf{R}^{-1}$ and $\mathbf{s} = \mathbf{S}^{-1}$.

As in the previous example, we first determine the inverse stretch tensor graphically with the aid of a Mohr Circle construction.

Problem

- Using Eq. 12.36, determine the inverse stretch component of the inverse simple shear deformation

$$\mathbf{d} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Procedure

1. With the same method used in Fig. 12.27, plot the points $P_1(1, 1)$ and $P_2(1, 0)$. With C at the midpoint of P_1P_2 complete the off-axis circle (Fig. 12.30a).
2. Just as before there are two ways of drawing the circle for $\mathbf{s}$. Either draw a second set of axes with the s_{11}, s_{22} axis through the center of this circle or rotate the circle to the horizontal axis (Fig. 12.30b).

Answer

- The principal inverse stretches are $s_1 = 1.62$ and $s_3 = 0.62$ and the s_1 direction makes angle $2\phi = 58°$ with the x_1 axis.

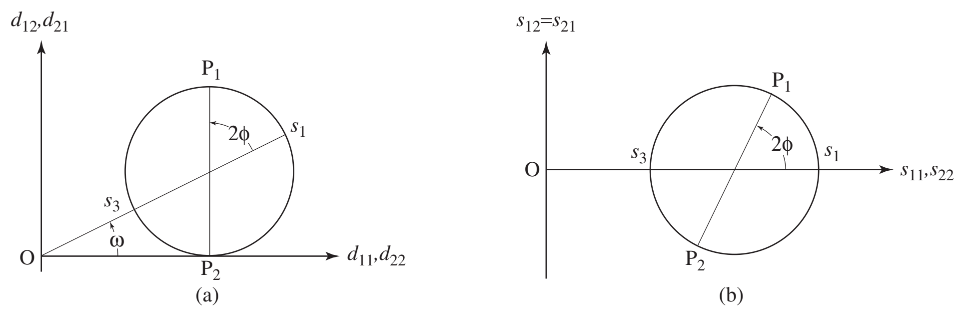

Figure 12.30 Mohr Circles: (a) inverse deformation tensor **d**; (b) inverse stretch tensor **s**.

Note that the circles for $\mathbf{d}$ and $\mathbf{s}$ have the same radii as the circles for $\mathbf{D}$ and $\mathbf{S}$. This special case arises because there is no area change in simple shear. For more general types of deformation these circles will differ in size.

There is one more strain tensor which is of special interest. Just as in the case of **D** we may also determine the inverse stretch tensor **s** directly from **d**. To do this, first pre-multiply both sides of Eq. 12.39 by its transform:

$$\mathbf{d}^T\mathbf{d} = (\mathbf{rs})^T\mathbf{rs}.$$

With the reversal rule

$$\mathbf{d}^T\mathbf{d} = \mathbf{s}^T\mathbf{r}^T\mathbf{rs}.$$

With $\mathbf{r}^T\mathbf{r} = \mathbf{1}$ and $\mathbf{s}^T\mathbf{s} = \mathbf{s}^2$ we have

$$\mathbf{s}^2 = \mathbf{d}^T\mathbf{d}. \tag{12.40}$$

Geometrically, the **d** rotates the axes from the deformed state back again to the initial state. This is followed by $\mathbf{d}^T$ which rotates the axes back to the deformed state.

Although it will not be obvious, $\mathbf{s}^2$ is just the *finite strain tensor* which is the main subject of this chapter. So

$$\mathbf{s}^2 = \begin{bmatrix} \lambda'_{xx} & \gamma'_{xy} \\ \gamma'_{yx} & \lambda'_{yy} \end{bmatrix}.$$

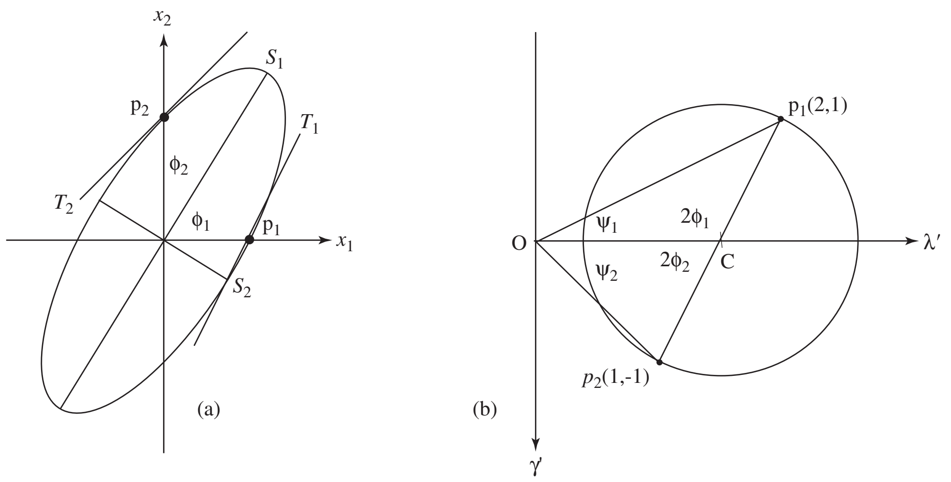

Figure 12.31 Finite strain tensor: (a) simple shear ellipse; (b) corresponding Mohr Circle.

A comparison of a carefully drawn and scaled ellipse and the corresponding Mohr Circle will demonstrate this fact using the simple shear deformation $\psi = 45°$ and the tensor

$$\mathbf{s}^2 = \mathbf{d}^T\mathbf{d} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

1. Strain ellipse (Fig. 12.31a):
   (a) The magnitude of the radius vector in the x_1 direction is 0.72 and in the x_2 direction it is 1.00.
   (b) The principal stretches are $S_1 = 1.62$ and $S_2 = 0.62$.
   (c) The S_1 direction makes angle $\phi_1 = 58°$ with the x_1 axis and $\phi_2 = 32°$ with the x_2 axis.
   (d) The angle tangent T_1 at point p_1 makes with the x_2 axis is the angle of shear $\psi = 27°$ and the angle tangent T_2 at point p_2 makes with the x_1 axis is the angle of shear $\psi = 45°$.
2. Mohr Circle (Fig. 12.31b):
   (a) On a set of $\lambda'\gamma'$ axes plot points $p_1(2, 1)$ and $p_2(1, -1)$ using the convention of Fig. 12.26b.
   (b) Line $p_1 p_2$ is a diameter of the circle with center at C. The intercepts of the circle represent the values of $\lambda'_1 = 0.38$ ($S_1 = 2.62$) and $\lambda'_2 = 2.62$ ($S_2 = 0.62$) (not labeled in the figure).
   (c) From the diagonal elements in the matrix representation of the tensor, the value of λ' associated with the x_1 axis is 2.0000 and with the x_2 axis is 1.0000. The corresponding stretches are $1/\sqrt{2} = 0.7071$ and 1.0000.
   (d) The λ'_1 direction makes angles $2\phi_1 = 116°$ and $2\phi_2 = 64°$ with the x_1 and x_2 axes.
   (e) The slope angles of lines Op_1 and Op_2 are the angles of shear $\psi_1 = 27°$ and $\psi_2 = 45°$ associated with each coordinate axis.

The underlying reason for the closeness of the strain ellipse and the Mohr Circle for finite strain is the fact that the elements of the tensor are simply the coefficients in the equation of the ellipse. The Mohr Circle is just a graphical way of describing the way the coefficients in the equation of an ellipse vary under transforming the axes.

The equation of an ellipse centered at the origin has two forms. For the case where the ellipse axes coincide with the coordinate axes it is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = S_1$ and $b = S_2$ are the lengths of the semi-axes. With the definition of λ' this may also be written as

$$\lambda'_1 x^2 + \lambda'_2 y^2 = 1.$$

This can be written in the form of the matrix equation

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \lambda'_1 & 0 \\ 0 & \lambda'_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

We then see that the square matrix is just the finite strain tensor in diagonal form. Similarly, the equation of the general ellipse centered at the origin is

$$Ax^2 + 2Bxy + Cy^2 = 1.$$

This too can be written as a matrix equation

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

Here, the square matrix of coefficients is just the finite strain tensor in its general form. Hence

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} \lambda'_{xx} & \gamma'_{xy} \\ \gamma'_{yx} & \lambda'_{yy} \end{bmatrix}.$$

Because $\mathbf{s}^2$ is symmetric, $\gamma'_{xy} = \gamma'_{yx}$ and we can write the general equation of the strain ellipse as

$$\lambda'_{xx}x'^2 + 2\gamma'_{xy}x'y' + \lambda'_{yy}y'^2 = 1.$$

With the matrix representation of this tensor, we can also easily find an expression for λ'. We first show how to do this using the diagonal form. In this case, the input is a radius vector of the strain ellipse. The direction cosines of this vector are $(\cos\phi', \sin\phi')$, where ϕ' is measured from the λ' axis. Then

$$\begin{bmatrix} \lambda'_1 & 0 \\ 0 & \lambda'_3 \end{bmatrix} \begin{bmatrix} \cos\phi' \\ \sin\phi' \end{bmatrix} = \begin{bmatrix} \lambda'_1\cos\phi' \\ \lambda'_3\sin\phi' \end{bmatrix}. \tag{12.41}$$

This output vector is normal to the tangent through point P' on the ellipse. The expression for λ' is obtained by forming the dot product of this normal vector $\mathbf{n}$ and the unit vector $(\cos\phi', \sin\phi')$ giving

$$\lambda' = \begin{bmatrix} \cos\phi' & \sin\phi' \end{bmatrix} \begin{bmatrix} \lambda'_1\cos\phi' \\ \lambda'_2\sin\phi' \end{bmatrix} = \lambda'_1\cos^2\phi' + \lambda'_2\sin^2\phi'. \tag{12.42}$$

This is the projection of the normal vector onto the unit vector in the direction of the radius vector. This is identical to the result of Eq. 12.11 obtained algebraically.

In a similar way, we can also obtain an expression for λ' from the full matrix representing the finite strain tensor. Here the direction cosines of the unit vector are $(\cos\theta', \sin\theta')$, where θ' is measured from the x axis.

$$\lambda' = \begin{bmatrix} \cos\theta' & \sin\theta' \end{bmatrix} \begin{bmatrix} \lambda'_{xx} & \gamma'_{xy} \\ \gamma'_{yx} & \lambda'_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta' \\ \sin\theta' \end{bmatrix}.$$

Performing the multiplications and using the equality $\gamma'_{xy} = \gamma'_{yx}$ yields

$$\lambda = \lambda'_{xx}\cos^2\theta' + 2\gamma'_{xy}\cos\theta'\sin\theta' + \lambda'_{yy}\sin^2\theta'. \qquad (12.43)$$

The angle between the unit vector **n** normal to the tangent at P' and the unit radius vector **r** in the direction OP' is the angle of shear ψ, and is obtained by the dot product $\mathbf{n}\cdot\mathbf{r}$. Then the expression for the associated shear strain γ' can be obtained in exactly the same way used to obtain Eq. 12.16.

12.12 Exercises

1. Using the circle and ellipse of Fig. 12.32, graphically determine the stretch S and the angle of shear ψ associated with a radius making an angle of $\phi' = +30°$ with the major axis. Check your result with a Mohr Circle construction.

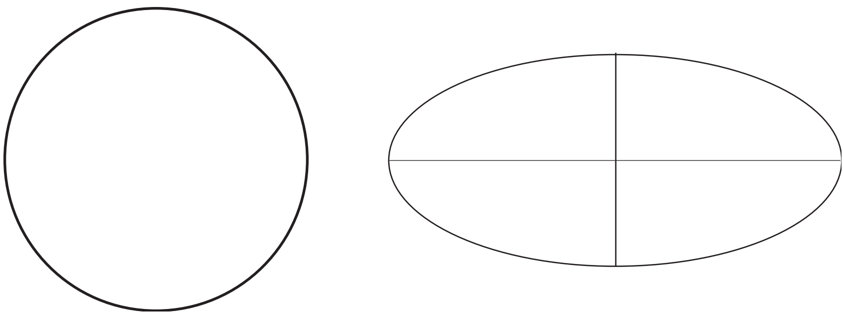

Figure 12.32

2. Using the collection of deformed two-dimensional pebbles of Fig. 2., estimate the orientation and shape of the strain ellipse.

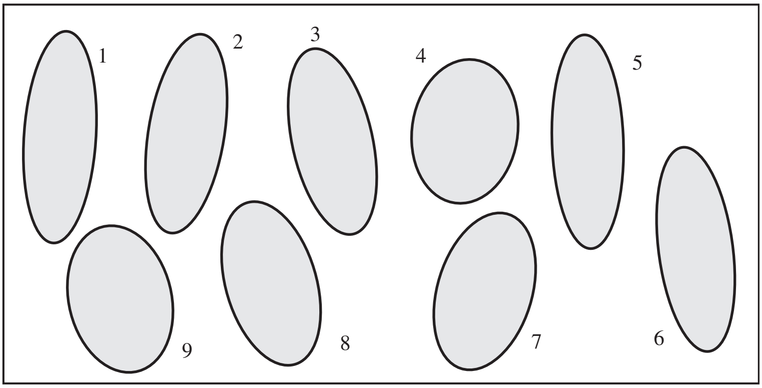

Figure 12.33

3. Using the collection of deformed brachiopods of Fig. 12.34, estimate the orientation and shape of the strain ellipse using Wellman's method.

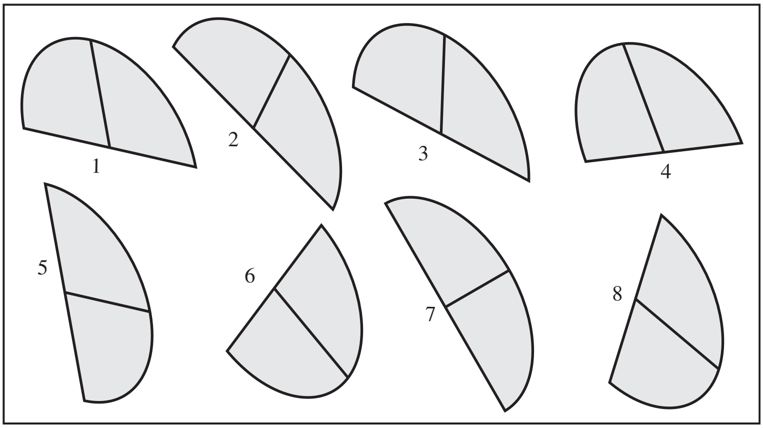

Figure 12.34

4. With a Mohr Circle construction determine the orientation and shape of the strain ellipse from the two deformed brachiopods of Fig. 12.35.

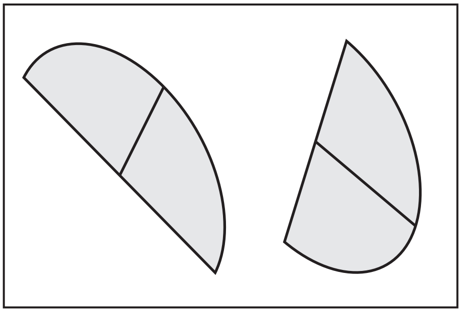

Figure 12.35

5. With a Mohr Circle construction determine the orientation and shape of the strain ellipse from the deformed shard in of Fig. 12.36.

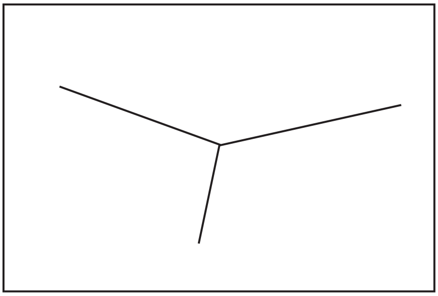

Figure 12.36

6. Determine the stretch of the single boudin shown in Fig. 12.37 and Table 12.1 using the methods of Ramsay, Hossain and Fergusson.

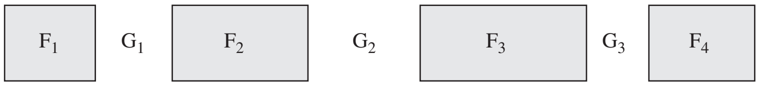

Figure 12.37

Table 12.1

	F	*G*
1	12 mm	10 mm
2	18 mm	16 mm
3	22 mm	8 mm
4	14 mm	

7. Three stakes were placed on the surface of a glacier to form an equilateral triangle 10 m on a side (Fig. 12.38a). After one year the positions of the stakes were resurveyed (Fig. 12.38b). Determine the strain which accumulated over this time span.

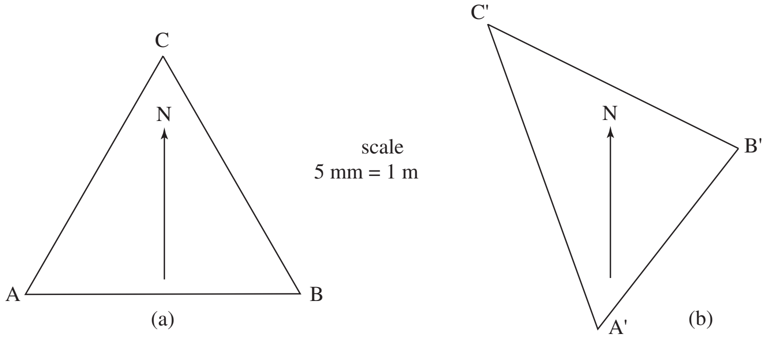

Figure 12.38

13
Flow

13.1 Introduction

As we have seen in Chapter 11, the study of deformation is concerned solely with a comparison of a body of rock in its initial and final configurations: the *translation* compares the initial and final places, the *rotation* compares the initial and final orientations, and the *stretch* compares the initial and final shapes and sizes (see Fig. 11.2). No consideration is given to intermediate configurations or to a particular sequence of configurations (Mase, 1970, p. 77).

However, the *motion* or *flow*[1] by which a particular deformed state is attained is also of considerable interest if we are to understand the processes involved in the formation of geological structures. *Kinematics* is the branch of mechanics concerned with the motion of bodies without regard to any associated forces.

In this chapter we first treat the basic elements of a kinematic analysis by describing the measured velocity field in a tectonically active area and the information that can be derived from it. Second, we consider an approach to the more difficult problem of understanding the flow responsible for old structures. Third, by considering the progressive geometrical evolution of structures we can gain some insight into the geometrical nature of geological flow patterns. Finally, after treating some important theoretical matters, we use these results to consider briefly an alternative approach to estimating the time rates of deformation.

13.2 Active tectonics

The San Andreas Fault zone of California is one of the most heavily instrumented active structures in the world. For about three decades the velocities of many points through-

[1] Flow, like deformation, is a continuum concept. Thus we may speak of fluid flow, ash flow, debris flow, etc. as long as an appropriate scale is used (see §11.2).