8. Calcular:

a) $\vec{\nabla}\cdot\vec{r}$

b) $\vec{\nabla}\cdot(f(r)\vec{r})$

c) $\vec{\nabla}\cdot(r^n\vec{r})$

¿Cuánto vale la divergencia del campo Coulombiano?

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

a) significa $\nabla\cdot\vec{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(x, y, z)$

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$= 1 + 1 + 1 = 3$$

$\nabla\cdot\vec{r} = 3$

$\boxed{\nabla f(r) = \hat{e}_r f'(r)}$

$$\text{b)}\quad \nabla\cdot(f\vec{A}) = f\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla f = 3f + \overbrace{\vec{r}\cdot\hat{e}_r}^{r} f'(r)$$

$\vec{A} = \vec{r}$

$f = f(r)$

$$\Rightarrow \boxed{\nabla\cdot(f(r)\vec{r}) = 3f + rf'}$$

∇r

$\nabla\cdot\vec{r}$

$\nabla\times\vec{r}$

$$\text{c)}\quad \boxed{\nabla\cdot(\overset{f}{r^n}\,\vec{r}) = 3r^n + nr^n = (3+n)r^n}$$

$r \quad \hat{e}_r$

$\vec{r}$

13. Mostrar que $\boxed{\vec{\nabla}^2\left(\frac{1}{r}\right)} = \boxed{-4\pi\delta^3(\vec{r})}$.

Contextualicemos

$$\left(\frac{1}{r}\right)'$$

$$\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\left[\nabla\left(\frac{1}{r}\right)\right] = \nabla\cdot\left[\hat{e}_r\left\{-\frac{1}{r^2}\right\}\right] = -\nabla\cdot\left[\vec{r}\,r^{-3}\right]$$

xq $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla\cdot\nabla f = \nabla^2 f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

¿qué quiere decir $\delta(\vec{r})$?

La delta de Dirac $D : f \longrightarrow \mathrm{Im}(f)$ es una distribución

$$D[f] = f(0) = \int \delta(\vec{r}\,')\, f(\vec{r}\,')\, d^3\vec{r}\,' = f(0)$$

$$D_{\vec{r}_0}[f] = f(\vec{r}_0) = \int \delta(\vec{r}\,' - \vec{r}_0)\, f(\vec{r}\,')\, d^3\vec{r}\,' = f(\vec{r}_0)$$

Dos distribuciones son iguales $O_1 = O_2$

si $O_1[f] = O_2[f] \quad \forall f$

$\Rightarrow$ el ejercicio refiere a probar que:

$$\int \left[\nabla^2\left(\frac{1}{r}\right)\right] f(\vec{r})\, d^3\vec{r} = \int \left[-4\pi\, \delta(\vec{r})\right] f(\vec{r})\, d^3 r = -4\pi\, D[f]$$

$\forall f$ razonable

La distribución

$$O_F[f] \equiv \int F(\vec{r}) f(\vec{r})\, d^3\vec{r}$$

La derivada de una distribución

$$\underbrace{O'_F[f] = \int \nabla(F(\vec{r})) f(r)\, d^3\vec{r} \quad \equiv -\int F(\vec{r})\, \nabla f(r)\, d^3 r}$$

$$D'[f] \equiv -\int \delta(\vec{r})\, f'(\vec{r})\, d^3\vec{r}$$

1.13

$$\int \left[\nabla^2\left(\frac{1}{r}\right)\right] f(\vec{r})\, d^3\vec{r} \underset{\substack{\text{def}\\ \text{de deriv}\\ \text{de la dist.}}}{\equiv} - \int \nabla\left(\frac{1}{r}\right)\cdot \nabla f(\vec{r})\, d^3\vec{r} \equiv \int \frac{1}{r}\, \nabla^2 f(\vec{r})\, d^3\vec{r}$$

$$= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \frac{1}{r}\, \overbrace{\nabla^2 f(\vec{r})}^{<\infty}\, \overbrace{d^3\vec{r}}^{r^2 \operatorname{sen}\theta\, dr}$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \frac{1}{r}\, \nabla^2 f(\vec{r})\, d^3\vec{r} \underset{\text{partes}}{=} \lim_{\varepsilon\to 0} \left\{ \int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \nabla\cdot\left[\frac{1}{r}\nabla f\right] d^3 r - \underbrace{\int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \nabla\left(\frac{1}{r}\right)\cdot \nabla f\, d^3\vec{r}}_{} \right\}$$

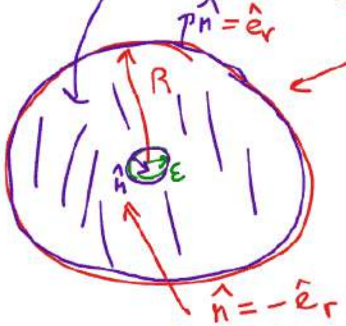
Partes $\rightarrow$ ||

$$\int_{\underline{\mathbb{R}^3 - S_\varepsilon^{(3)}}} \nabla\cdot\left[\nabla\left(\frac{1}{r}\right) f\right] - \cancel{\underbrace{\nabla^2\left(\frac{1}{r}\right)}_{=\,0\ \forall r\neq 0} f}\, d^3\vec{r} \rightarrow 0$$

$$\int \left[\nabla^2\left(\frac{1}{r}\right)\right] f(\vec r)\, d^3\vec r = \lim_{\varepsilon\to 0}\left\{\int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \nabla\cdot\left[\frac{1}{r}\nabla f\right] d^3 r - \int_{\mathbb{R}^3 - S_\varepsilon^{(3)}} \nabla\cdot\left[\nabla\left(\frac{1}{r}\right) f\right]\right\}$$

$$= \lim_{\varepsilon\to 0}\left\{\underbrace{\int_{\partial(\mathbb{R}^3 - S_\varepsilon^{(3)})} \frac{1}{r}\nabla f \cdot \hat n\, d^2\vec r} - \underbrace{\int_{\partial(\mathbb{R}^3 - S_\varepsilon^{(3)})} \nabla\left(\frac{1}{r}\right) f \cdot \hat n\, d^2\vec r}\right\}$$

Teorema de Gauss



$$= \boxed{\lim_{\varepsilon\to 0} - \int_{\partial S_\varepsilon^{(3)}} \cancel{\frac{1}{\varepsilon}\nabla f\cdot\hat e_r\, \varepsilon^2 \operatorname{sen}\theta\, d\theta\, d\phi} \to 0 \;+\; \lim_{R\to\infty}\int \cancel{\frac{1}{R}\nabla f\cdot\hat e_r\, R^2 \operatorname{sen}\theta\, d\theta\, d\phi} \to 0}$$

$\to$ xq $\lim_{\varepsilon\to 0} \varepsilon\cdot\# = 0$

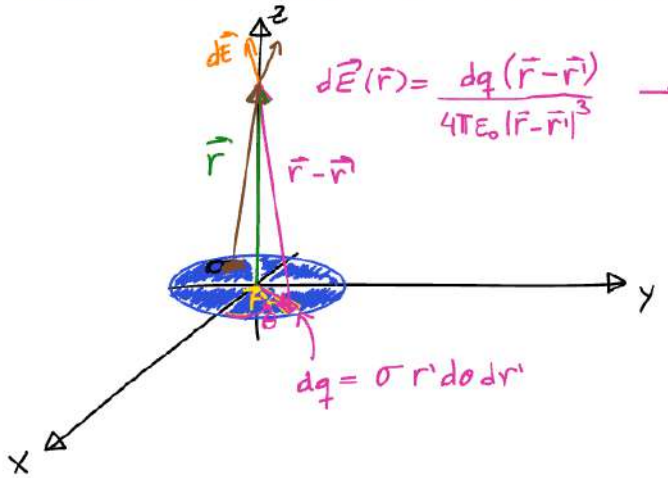
$\to f$ es de soporte compacto

$$\boxed{\lim_{\varepsilon\to 0} - \int_{\partial S_\varepsilon^{(3)}} \underbrace{\frac{1}{\cancel{\varepsilon^2}}\hat e_r}_{\nabla(\frac{1}{r})} f\cdot\hat e_r\, \cancel{\varepsilon^2} \operatorname{sen}\theta\, d\theta\, d\phi \;+\; \lim_{R\to\infty}\int \cancel{\frac{1}{R^2}\hat e_r\, f\cdot\hat e_r\, R^2 \operatorname{sen}\theta\, d\theta\, d\phi} \to 0}$$

$$= \lim_{\varepsilon\to 0} - \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \operatorname{sen}\theta\, d\theta}_{4\pi}\; \underbrace{f(\varepsilon,\theta,\phi)}_{\to f(0,\theta,\phi) = f(\vec 0) = cte} = -4\pi\, f(0)$$

2. Un disco circular de radio a tiene una densidad superficial de carga uniforme σ. El disco se encuentra en el plano xy y su centro coincide con el origen del sistema de coordenadas.

   a) Obtenga una expresión para el campo eléctrico en el eje del disco en función de z.



$$dq = \sigma\, r'\, d\theta\, dr'$$

$$d\vec{E}(\vec{r}) = \frac{dq\,(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3} \longrightarrow \vec{E}(\vec{r}) = \int \frac{\overbrace{dq}^{\sigma r' d\theta dr'}\,(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}$$

$$\Rightarrow \vec{E}(\vec{r}) = \int_0^a dr' \int_0^{2\pi} d\theta \; \frac{\sigma r' \,(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}$$

ojo

$$\vec{r} = z\hat{k}$$

$$\vec{r}' = r'\hat{e}_{r'} = r'\cos\theta\,\hat{\imath} + r'\operatorname{sen}\theta\,\hat{\jmath}$$

$$\vec{r}-\vec{r}' = z\hat{k} - r'\cos\theta\,\hat{\imath} - r'\operatorname{sen}\theta\,\hat{\jmath}$$

$$|\vec{r}-\vec{r}'| = \sqrt{r'^2+z^2}$$

$$\vec{E}(\vec{r}) = -\left\{\int_0^a dr' \int_0^{2\pi} d\theta \, \frac{\sigma r'^2 \cos\theta}{4\pi\varepsilon_0 (r'^2+z^2)^{3/2}}\right\}\hat{\imath} +$$

0 → x la integral en θ

$$-\left\{\int_0^a dr' \int_0^{2\pi} d\theta \, \frac{\sigma r'^2 \operatorname{sen}\theta}{4\pi\varepsilon_0 (r'^2+z^2)^{3/2}}\right\}\hat{\jmath} +$$

0

$$+\left\{\int_0^a dr' \int_0^{2\pi} d\theta \, \frac{\sigma r' z}{4\pi\varepsilon_0 (r'^2+z^2)^{3/2}}\right\}\hat{k} = \frac{\sigma z}{2\varepsilon_0}\left[\int_0^a \frac{\overbrace{r'dr'}^{\frac{1}{2}\frac{d r'^2}{dr'}}}{(r'^2+z^2)^{3/2}}\right]\hat{k}$$