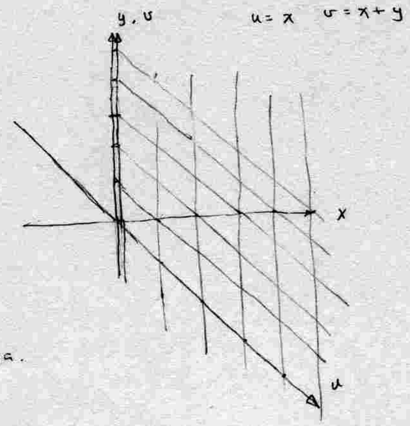
Practico 2 soluciones de algunos ejercicios

3. El vector $\nabla\phi$ con componentes $\partial_x \phi = 0$, $\partial_y \phi = 0$ es el tangente al eje y, $\frac{\partial}{\partial y}$

Entonces en la carta u, v tiene componentes



$$\frac{\partial u}{\partial y} = 0 \quad , \quad \frac{\partial v}{\partial y} = 1$$

- Los mismos como en la carta x, y. Sigue siendo el tangente al eje y, que es igual al eje v

- $\phi = y = v - u \quad \Rightarrow \quad \partial_u \phi = -1 \qquad \partial_v \phi = 1$

Estos componentes se cambiaron al cambiar carta.

De perspectiva levemente distinta:

$$\nabla\phi^u = \frac{\partial u}{\partial x}\nabla\phi^x + \frac{\partial u}{\partial y}\nabla\phi^y = \frac{\partial u}{\partial y}\cdot 1 = 0$$

$$\nabla\phi^v = \frac{\partial v}{\partial x}\nabla\phi^x + \frac{\partial v}{\partial y}\nabla\phi^y = \frac{\partial v}{\partial y}\cdot 1 = 1$$

$$\partial_u \phi = \frac{\partial x}{\partial u}\partial_x \phi + \frac{\partial y}{\partial u}\partial_y \phi = \frac{\partial y}{\partial u}\cdot 1 = -1$$

$$\partial_v \phi = \frac{\partial x}{\partial v}\partial_x \phi + \frac{\partial y}{\partial v}\partial_y \phi = \frac{\partial y}{\partial v}\cdot 1 = 1$$

4. En una carta inercial local z_p^α

$$D_u \omega_\alpha = u^\mu \partial_\mu \omega_\alpha = u^\mu \partial_\mu \left[\frac{\partial x^\nu}{\partial z_p^\alpha} \omega_\nu \right]$$

$$= \frac{\partial x^\nu}{\partial z_p^\alpha} u^\mu \left[\partial_\mu \omega_\nu + \frac{\partial z_p^\beta}{\partial x^\nu} \partial_\mu \left[\frac{\partial x^\sigma}{\partial z_p^\beta} \right] \omega_\sigma \right]$$

$$\frac{\partial z_p^\beta}{\partial x^\nu} \partial_\mu \frac{\partial x^\sigma}{\partial z_p^\beta}\Big|_p = - \frac{\partial x^\sigma}{\partial z_p^\beta} \frac{\partial}{\partial x^\mu} \frac{\partial z_p^\beta}{\partial x^\nu}\Big|_p = - \Gamma^\sigma_{\mu\nu}(p)$$

$$\Rightarrow \quad D_u \omega_\alpha = \frac{\partial x^\nu}{\partial z_p^\alpha} u^\mu \left[\partial_\mu \omega_\nu - \Gamma^\sigma_{\mu\nu} \omega_\sigma \right]$$

Por otro lado, si $D_u \omega$ es un covector entonces

$$D_u \omega_\alpha = \frac{\partial x^\nu}{\partial z_p^\alpha} D_u \omega_\nu$$

Entonces $\quad D_u \omega_\nu = u^\mu \left[\partial_\mu \omega_\nu - \Gamma^\sigma_{\mu\nu} \omega_\sigma \right] \quad$ en cualquier carta x

5.

$$D_\sigma \delta^\mu{}_\nu = \partial_\sigma \delta^\mu{}_\nu + \Gamma^\mu_{\sigma\rho}\, \delta^\rho{}_\nu - \Gamma^\rho_{\sigma\nu}\, \delta^\mu_\rho$$

$$= 0 + \Gamma^\mu_{\sigma\nu} - \Gamma^\mu_{\sigma\nu}$$

$$= 0$$

6. a)

$$D_w u^\sigma = w^\mu \left[\partial_\mu u^\sigma + \Gamma^\sigma_{\mu\nu}\, u^\nu \right]$$

$$= w^\theta \left[\partial_\theta\left[\frac{\partial x^\sigma}{\partial y^\eta} u^\eta \right] + \frac{\partial x^\mu}{\partial y^\theta} \Gamma^\sigma_{\mu\nu} \frac{\partial x^\nu}{\partial y^\iota} u^\iota \right]$$

$$= \frac{\partial x^\sigma}{\partial y^\eta} w^\theta \left\{ \partial_\theta u^\eta + \left(\frac{\partial y^\eta}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial y^\theta \partial y^\iota} + \frac{\partial y^\eta}{\partial x^\rho} \frac{\partial x^\mu}{\partial y^\theta} \frac{\partial x^\nu}{\partial y^\iota} \Gamma^\rho_{\mu\nu} \right) u^\iota \right\}$$

$D_w u^\sigma = \frac{\partial x^\sigma}{\partial y^\eta} D_w u^\eta$ porque $D_w u$ es vector

$$\Rightarrow D_w u^\eta = w^\theta \left[\partial_\theta u^\eta + \Gamma^\eta_{\theta\iota}\, u^\iota \right]$$

con $\Gamma^\eta_{\theta\iota} = \frac{\partial y^\eta}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial y^\theta \partial y^\iota} + \frac{\partial y^\eta}{\partial x^\rho} \frac{\partial x^\mu}{\partial y^\theta} \frac{\partial x^\nu}{\partial y^\iota} \Gamma^\rho_{\mu\nu}$

7. a)

$$G_y(\omega,u,v) \equiv \Gamma^\eta_{\theta\iota}\, \omega_\eta u^\theta v^\iota$$

$$= \left[\frac{\partial y^\eta}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial y^\theta \partial y^\iota} + \frac{\partial y^\eta}{\partial x^\sigma} \frac{\partial x^\mu}{\partial y^\theta} \frac{\partial x^\nu}{\partial y^\iota} \Gamma^\sigma_{\mu\nu} \right] \omega_\eta u^\theta v^\iota$$

$$= \frac{\partial y^\theta}{\partial x^\mu} \frac{\partial y^\iota}{\partial x^\nu} \frac{\partial^2 x^\sigma}{\partial y^\theta \partial y^\iota} \omega_\sigma u^\mu u^\nu + \Gamma^\sigma_{\mu\nu}\, \omega_\sigma u^\mu u^\nu$$

$$= - \frac{\partial x^\sigma}{\partial y^\iota} \frac{\partial^2 y^\iota}{\partial x^\mu \partial x^\nu} \omega_\sigma u^\mu u^\nu + G_x(\omega,u,v)$$

$G_y \neq G_x$ salvo si y es lineal en x.

b) Para que $D_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\sigma_{\mu\nu}\omega_\sigma$ definan el mismo tensor en cualquier carta es necesario y suficiente que transforma como componentes de tensor:

$$D\omega[u,v] = (D_\mu \omega_\nu)\, u^\mu u^\nu = (D_\mu \omega_\nu) \frac{\partial x^\mu}{\partial y^\theta} u^\theta \frac{\partial x^\nu}{\partial y^\iota} v^\iota$$

$$= D_\theta \omega_\iota\, u^\theta v^\iota \qquad \forall\, u,v$$

$$\Leftrightarrow D_\theta \omega_\iota = \frac{\partial x^\mu}{\partial y^\theta} \frac{\partial x^\nu}{\partial y^\iota} D_\mu \omega_\nu$$

$$D_\theta \omega_\iota = \partial_\theta \omega_\iota - \Gamma^\eta_{\theta\iota}\,\omega_\eta$$

$$= \frac{\partial x^\mu}{\partial y^\theta}\,\partial_\mu\left[\frac{\partial x^\nu}{\partial y^\iota}\,\omega_\nu\right] - \left[\frac{\partial y^\eta}{\partial x^\rho}\frac{\partial x^\mu}{\partial y^\theta}\frac{\partial x^\nu}{\partial y^\iota}\,\Gamma^\rho_{\mu\nu} + \frac{\partial y^\eta}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial y^\theta \partial y^\iota}\right]\frac{\partial x^\sigma}{\partial y^\eta}\,\omega_\sigma$$

$$= \frac{\partial x^\mu}{\partial y^\theta}\frac{\partial x^\nu}{\partial y^\iota}\Big[\partial_\mu \omega_\nu - \Gamma^\sigma_{\mu\nu}\,\omega_\sigma$$

iguales

$$+ \frac{\partial y^\eta}{\partial x^\nu}\,\partial_\mu\left(\frac{\partial x^\sigma}{\partial y^\eta}\right)\omega_\sigma - \frac{\partial y^\eta}{\partial x^\nu}\frac{\partial y^\lambda}{\partial x^\mu}\frac{\partial^2 x^\sigma}{\partial y^\lambda \partial y^\eta}\,\omega_\sigma\Big]$$

$$= \frac{\partial x^\mu}{\partial y^\theta}\frac{\partial x^\nu}{\partial y^\iota}\left[\partial_\mu \omega_\nu - \Gamma^\sigma_{\mu\nu}\,\omega_\sigma\right]$$

8. $$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta}$$

$$= 2\left(\partial_{[\mu}\Gamma^\alpha_{\nu]\beta} + \Gamma^\alpha_{[\mu|\beta|}\Gamma^\gamma_{\nu]\beta}\right)$$

Claramente $R^\alpha{}_{\beta(\mu\nu)} = \frac{1}{2}\left(R^\alpha{}_{\beta\mu\nu} + R^\alpha{}_{\beta\nu\mu}\right) = 0$

$$R^\alpha{}_{[\beta\gamma\delta]} = \sum_{\substack{\text{perm cíclica}\\ \beta\gamma\delta}}\left\{\left(\partial_\mu \Gamma^\alpha_{\nu\beta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta}\right) - \left(\partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\beta\mu}\right)\right\}$$

uno permutación cíclico del otro

$$= 0$$

Nota, siempre se puede ir a coordenadas inerciales locales ($\Gamma^\alpha_{\beta\gamma} = 0$) y calcular solo con las derivadas de Γ. El resultado $R^\alpha{}_{[\beta\gamma\delta]} = 0$ dice que componentes de un tensor son cero en una base. Esto implica que son cero en cualquier base.

9. Vamos a una carta inercial y elegimos un campo vectorial a tal que $D_\alpha a^\beta = \partial_\alpha a^\beta = 0$ en P. Esto no restringe la elección de $a^\beta(P)$.

$$D_{[\epsilon} D_\gamma D_{\delta]}\, a^\alpha = D_{[\epsilon}\left(R^\alpha{}_{|\beta|\gamma\delta]}\, a^\beta\right) = D_{[\epsilon} R^\alpha{}_{|\beta|\gamma\delta]}\, a^\beta$$

$$+ R^\alpha{}_{\beta[\gamma\delta} D_{\epsilon]}\, a^\beta$$

$$= D_{[\epsilon} R^\alpha{}_{|\beta|\gamma\delta]}\, a^\beta$$

Pero $D_{[\epsilon} D_\gamma D_{\delta]} a^\alpha = R^\alpha{}_{\beta[\epsilon\gamma} D_{\delta]} a^\beta - R^\rho{}_{[\delta\epsilon\gamma]} D_\rho a^\alpha$

$= 0$

Esto establece $D_{[\epsilon} R^\alpha{}_{|\beta|\gamma\delta]}\, a^\beta = 0 \quad \forall\, a(P)$

$\Rightarrow$ Identidad de Bianchi $R^\alpha{}_{\beta[\gamma\delta;\epsilon]} \equiv D_{[\epsilon} R^\alpha{}_{|\beta|\gamma\delta]} = 0$

Hemos usado

$$[D_\mu, D_\nu]\, b_\alpha{}^\beta = - R^\sigma{}_{\alpha\mu\nu}\, b_\sigma{}^\beta + R^\beta{}_{\sigma\mu\nu}\, b_\alpha{}^\sigma$$

¿Porque? Es muy parecida como uno saca los términos con Γs en la expresión explicita para la derivada covariante:

Sea a un campo vectorial, y c un campo covectorial. $a^\sigma c_\sigma$ es un escalar, entonces

$D_{[\mu} D_{\nu]} (a^\sigma c_\sigma) = \partial_{[\mu} \partial_{\nu]} (a^\sigma c_\sigma) = 0$ ya que derivadas parciales conmutan

Pero, también vale la regla de productos:

$D_{[\mu} D_{\nu]}\, a^\sigma c_\sigma = D_{[\mu} \left\{ (D_{\nu]} a)^\sigma c_\sigma + a^\sigma D_{\nu]} c_\sigma \right\}$

$= (D_{[\mu} D_{\nu]} a)^\sigma c_\sigma + a^\sigma D_{[\mu} D_{\nu]} c_\sigma$

$+ \cancel{(D_{[\nu} a)^\sigma D_{\mu]} c_\sigma} + \cancel{(D_{[\mu} a)^\sigma D_{\nu]} c_\sigma}$

Entonces $0 = \frac{1}{2} R^\sigma{}_{\rho\mu\nu}\, a^\rho c_\sigma + a^\rho D_{[\mu} D_{\nu]} c_\rho \quad \forall\, a^\rho$

$\Rightarrow [D_\mu, D_\nu]\, c_\rho = - R^\sigma{}_{\rho\mu\nu}\, c_\sigma$

Ahora considera el escalar $b_\alpha{}^\beta a^\alpha c_\beta$

$0 = D_{[\mu} D_{\nu]} (b_\alpha{}^\beta a^\alpha c_\beta) = (D_{[\mu} D_{\nu]} b)_\alpha{}^\beta a^\alpha c_\beta + b_\alpha{}^\beta (D_{[\mu} D_{\nu]} a)^\alpha c_\beta + b_\alpha{}^\beta a^\alpha D_{[\mu} D_{\nu]} c_\beta$

- las derivadas cruzadas cancelan como en el anterior calculo

$\Rightarrow (D_{[\mu} D_{\nu]} b)_\alpha{}^\beta a^\alpha c_\beta = - b_\alpha{}^\eta a^\alpha \frac{1}{2} R^\beta{}_{\eta\mu\nu} c_\beta - b_\eta{}^\beta \frac{1}{2} R^\eta{}_{\alpha\mu\nu} a^\alpha c_\beta \quad \forall\, a^\alpha, b_\beta$

$\Rightarrow [D_\mu, D_\nu]\, b_\alpha{}^\beta = R^\beta{}_{\eta\mu\nu}\, b_\alpha{}^\eta - R^\eta{}_{\alpha\mu\nu}\, b_\eta{}^\beta$