Modelos de Crecimiento de una población: Exponencial y Logístico

Miércoles 14:00-17:00

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Objetivos

- Plantear modelos
  - ↳ Integración numérica
  - ↳ Análisis de puntos de equilibrio
- Manejar y graficar datos

Ecuación de conservación para una población

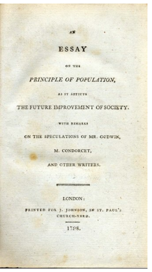
nº de indiv.

$$\frac{dN}{dt} = + \text{Nacimientos} \quad (>0)$$
$$- \text{Muertes} \quad (>0)$$
$$+ \text{Migración} \quad (>0 \text{ o } <0)$$

cómo varía la población en el tiempo

Modelo Malthusiano

Crecimiento exponencial



Thomas Matlhus (1766-1834)

Tasa de natalidad per capita

Nacimientos $= b \cdot N$

Muertes $= d \cdot N$

↳ Tasa de muerte per capita

$$\frac{dN}{dt} = bN - dN$$

↳ $\boxed{\frac{dN}{dt} = \underbrace{(b-d)}_{n \atop r} N} = rN$

$$\frac{dN}{dt} = rN$$

↳ $\int_{t_0}^{t} \frac{dN}{\cancel{dt}} \cdot \frac{1}{N} \cancel{dt} = \int_{t_0}^{t} r \, dt$

$$\int_{N(t_0)}^{N(t)} \frac{dN}{N} = \int_{t_0}^{t} r \, dt = rt \Big|_{t_0=0}^{t} = rt$$

$$= \log(N) \Big|_{N_0}^{N(t)}$$

$$= \log(N(t)) - \log(N_0) = \log\left(\frac{N(t)}{N_0}\right)$$

$$e^{\left(\log\left(\frac{N(t)}{N_0}\right)\right)} = e^{rt}$$

$$\boxed{e^{\log(x)} = x}$$

$$\int_{a}^{b} f(u(x)) \frac{du}{\cancel{dx}} \cancel{dx} = \int_{u(a)}^{u(b)} f(u) \, du$$

$\Rightarrow \frac{N(t)}{N_0} = e^{rt}$

↳ $\boxed{N(t) = N_0 e^{rt}}$

r = tasa de nac. − tasa de muertos ($\gtrless 0$)

Si $N_0 = 0 \Rightarrow$ La población es 0 para siempre

$\Leftrightarrow N^* = 0$ es un estado de equilibrio (del sistema / modelo)

Si $N_0 > 0$: $r < 0 \Rightarrow N \xrightarrow{t \to +\infty} 0$

$r > 0 \Rightarrow N \to +\infty$

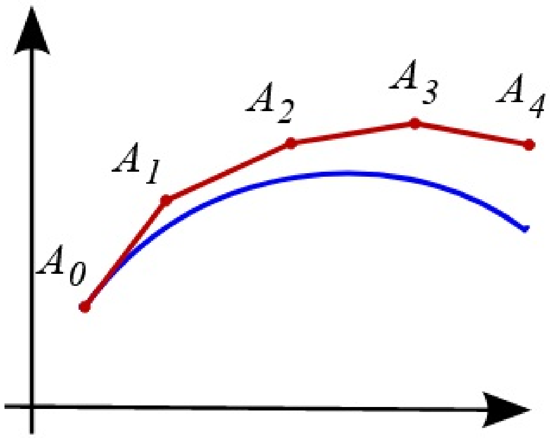
$r = 0 \Rightarrow N = N_0 \ \forall t$

$$\frac{dN}{dt} = rN$$

$$\int_{N_0}^{N} dN \cdot \frac{1}{N} = \int_{0}^{t} r \, dt$$

¿Qué pasa con la población a largo plazo?

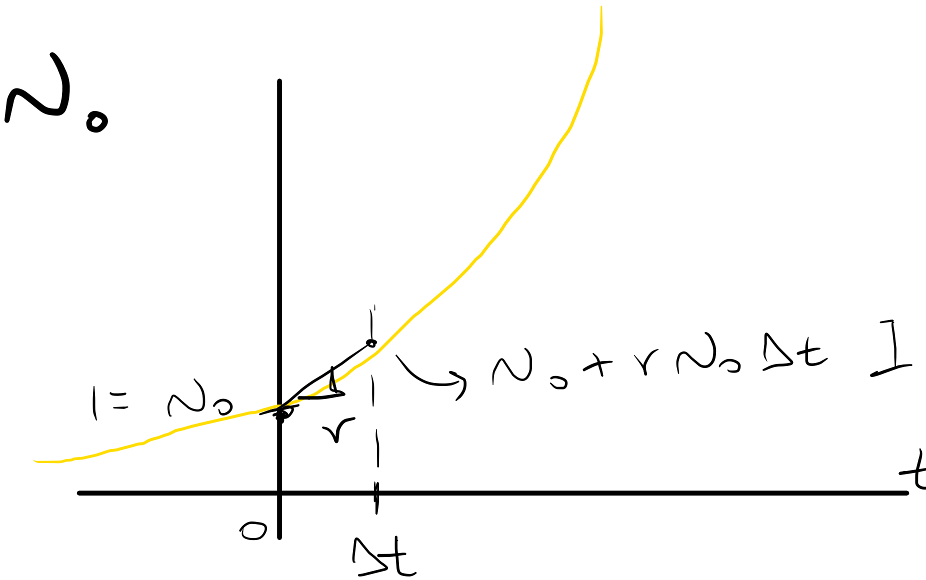
Integración numérica (no explícita): ideas básicas y ode45



El método de Euler

[t, y] =**ode45** *(fun, trange, init)*

$$\frac{dN}{dt} = rN_0$$



$N_0 + rN_0\Delta t$] $\Delta N = \frac{dN}{dt}\cdot\Delta t$

$\Downarrow$

$$\frac{\Delta N}{\Delta t} \approx \frac{dN}{dt}$$

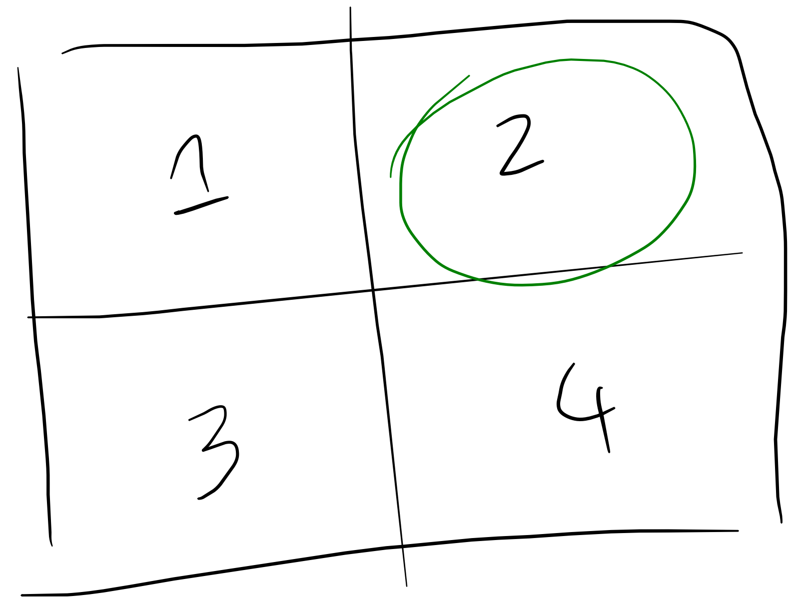
Importar y graficar datos en Octave: csvread, plot y subplot

x =**csvread** *(filename)*

plot *(x, y, fmt)*

subplot *(rows, cols, index)*

2 2



subplot (2,2,2)
plot (x,y)

Ajustar datos en Octave: polyfit

$ax + b = [a, b]$

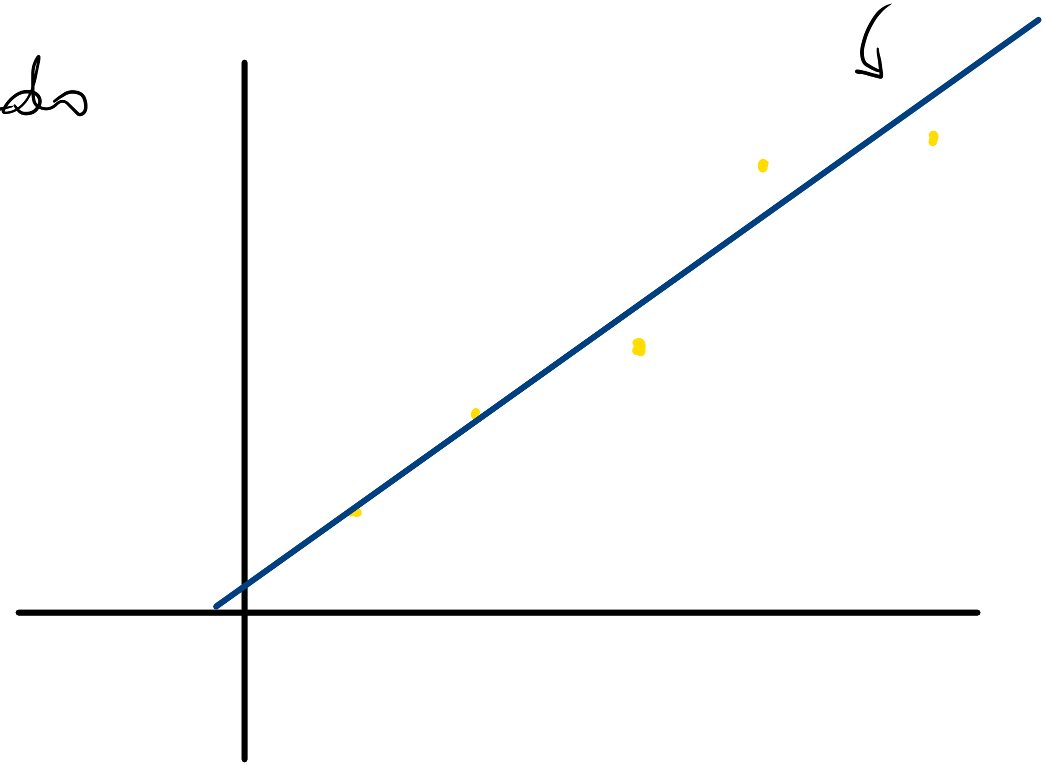
vector

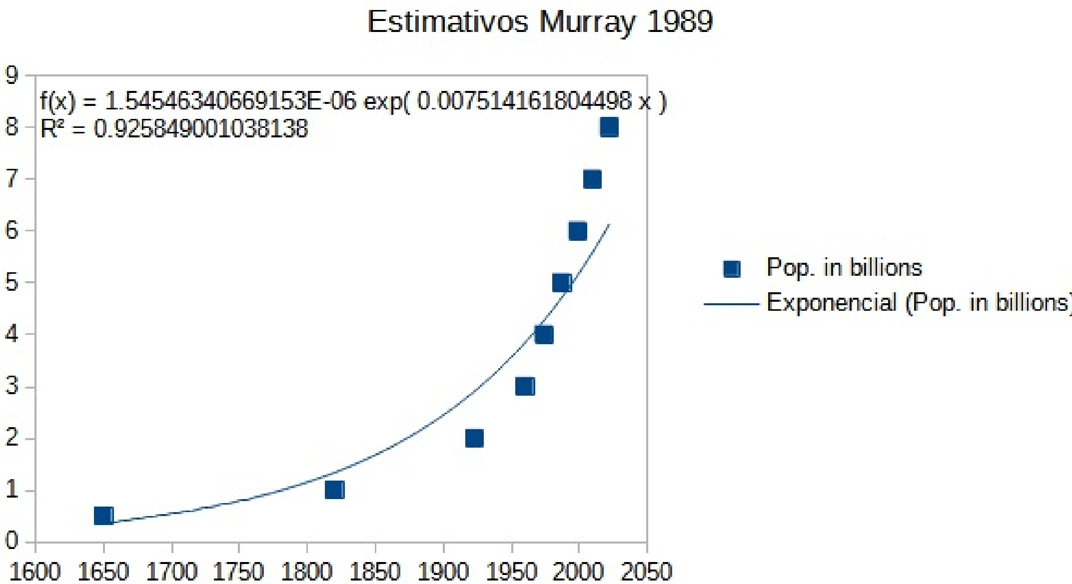
grado

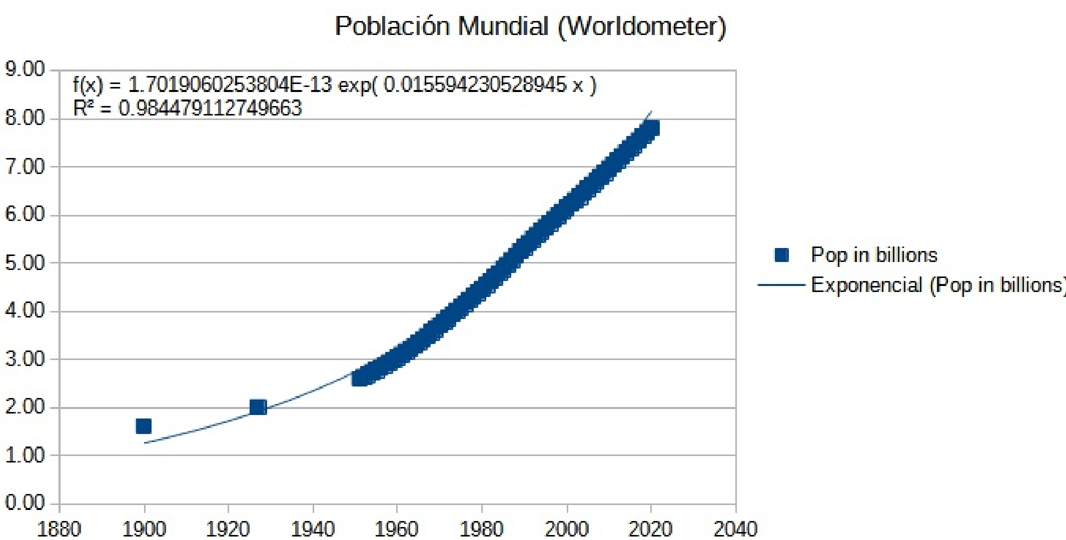
p = **polyfit** *(x, y, n)*

vector

vector







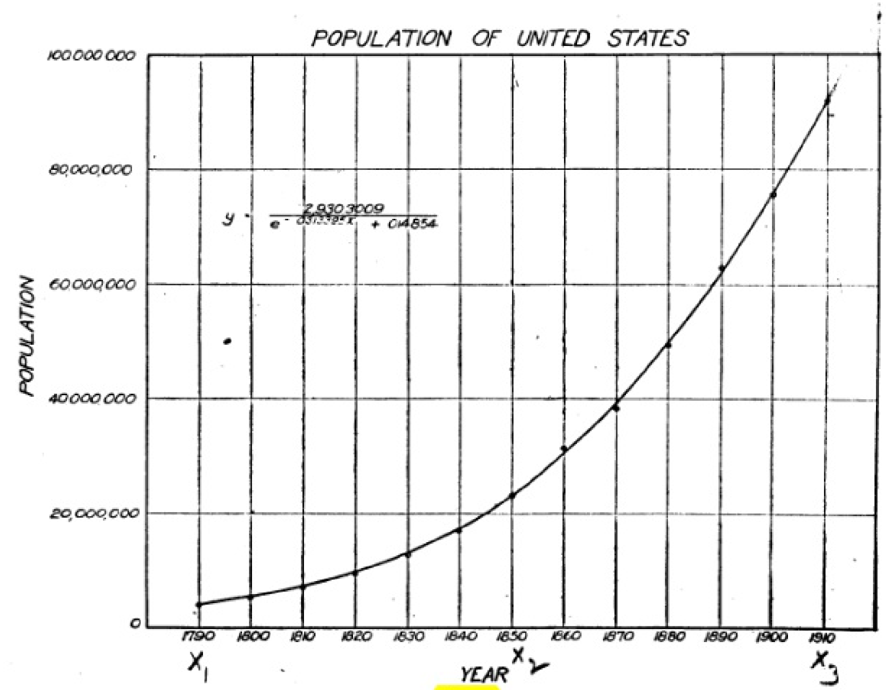
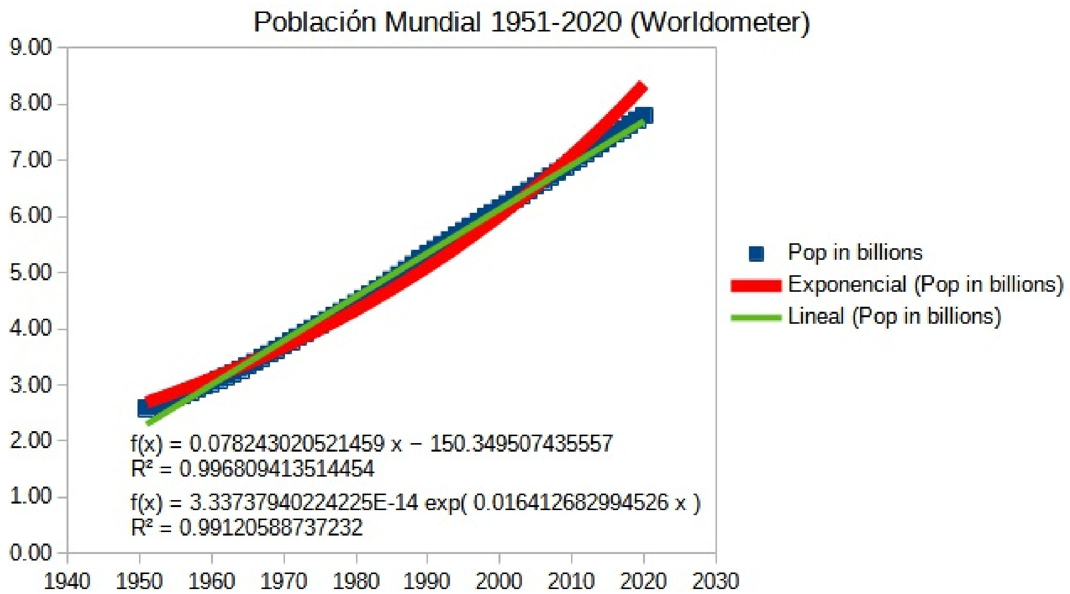
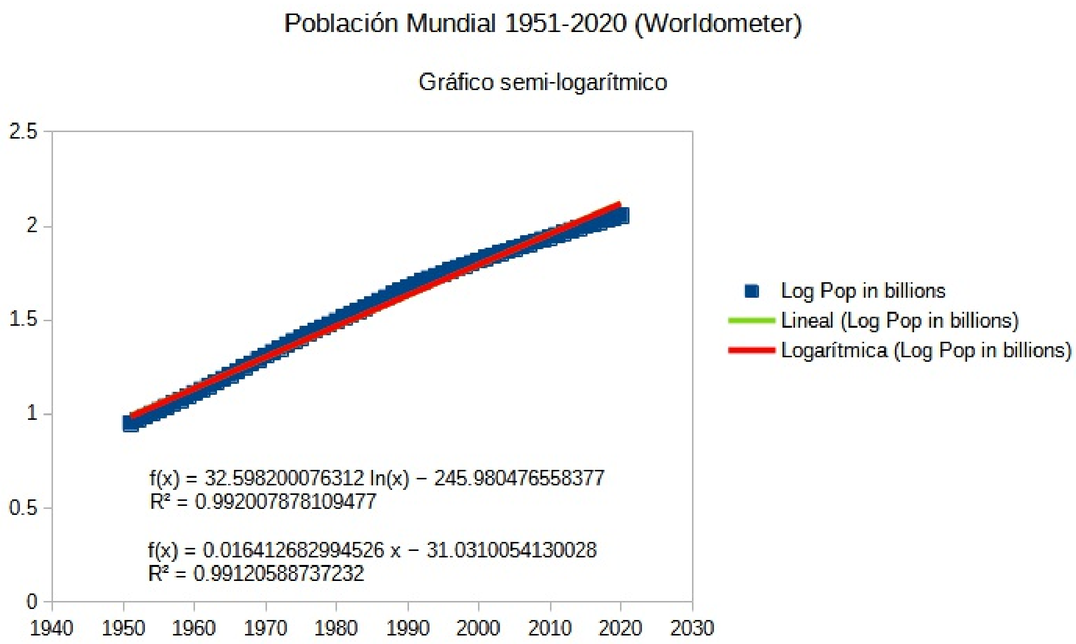


FIG. 3

Showing result of fitting equation (xviii) to population data.


Población Mundial 1951-2020 (Worldometer)
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
1940
1950
1960
1970
1980
1990
2000
2010
2020
2030
f(x) = 0.078243020521459 x − 150.349507435557
R² = 0.996809413514454
f(x) = 3.33737940224225E-14 exp(0.016412682994526 x)
R² = 0.99120588737232
Pop in billions
Exponencial (Pop in billions)
Lineal (Pop in billions)


Población Mundial 1951-2020 (Worldometer)
Gráfico semi-logarítmico
2.5
2
1.5
1
0.5
0
1940
1950
1960
1970
1980
1990
2000
2010
2020
2030
f(x) = 32.598200076312 ln(x) − 245.980476558377
R² = 0.992007878109477
f(x) = 0.016412682994526 x − 31.0310054130028
R² = 0.99120588737232
Log Pop in billions
Lineal (Log Pop in billions)
Logarítmica (Log Pop in billions)

Crecimiento logístico

HOW POPULATIONS GROW

It is an observed *fact*, which at this stage of the discussion involves no theoretical implications whatever, or postulates special to it, that the growth of populations of the most diverse organisms follows a regular and characteristic course. In general and everyday terms of common sense this characteristic manner of population growth may be described in the following way. The population at first grows slowly, but gains impetus as it grows, passing gradually into a stage of rapid growth, which finally reaches a maximum of rapidity. After this stage of most rapid growth the population increases ever more and more slowly, until finally there is no more perceptible growth at all. In short, the populations of various forms of life first wax in their speed of growing and then wane.

Pearl (1927)

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$
$$\Downarrow$$
$$\frac{dN}{N\left(1-\frac{N}{K}\right)} = r\,dt$$

Pierre François Verhulst (1804,1849)

↳ Puede integrarse por fracciones simples! (Ejercicio)

$r>0$

$$\Rightarrow \boxed{N(t) = \frac{N_0 K e^{rt}}{K + N_0(e^{rt}-1)}} \approx \frac{\cancel{N_0} K \cancel{e^{rt}}}{\cancel{N_0}\,\cancel{e^{rt}}} \longrightarrow K$$

$$\boxed{\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)} \qquad \frac{dN}{dt} = rN$$

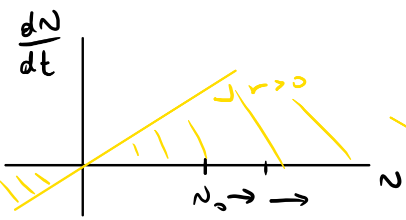
N^* es un estado de equilibrio si $\frac{dN}{dt}(N^*) = 0$

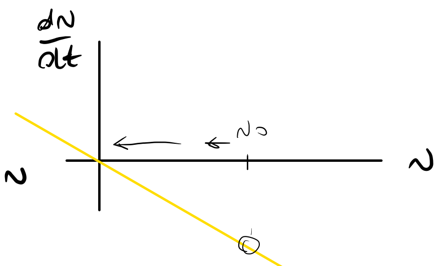
$N^* = K$ es un estado de eq.

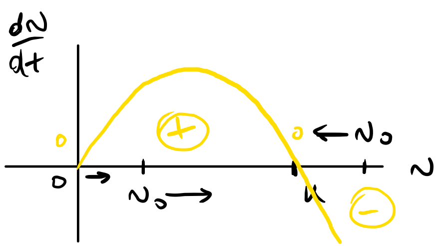
$N^* = 0$ " " " " "

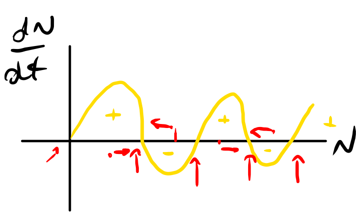
$$rN\left(1-\frac{N}{K}\right) = rN - \frac{r}{K}N^2$$

Análisis gráfico









$K = 665$

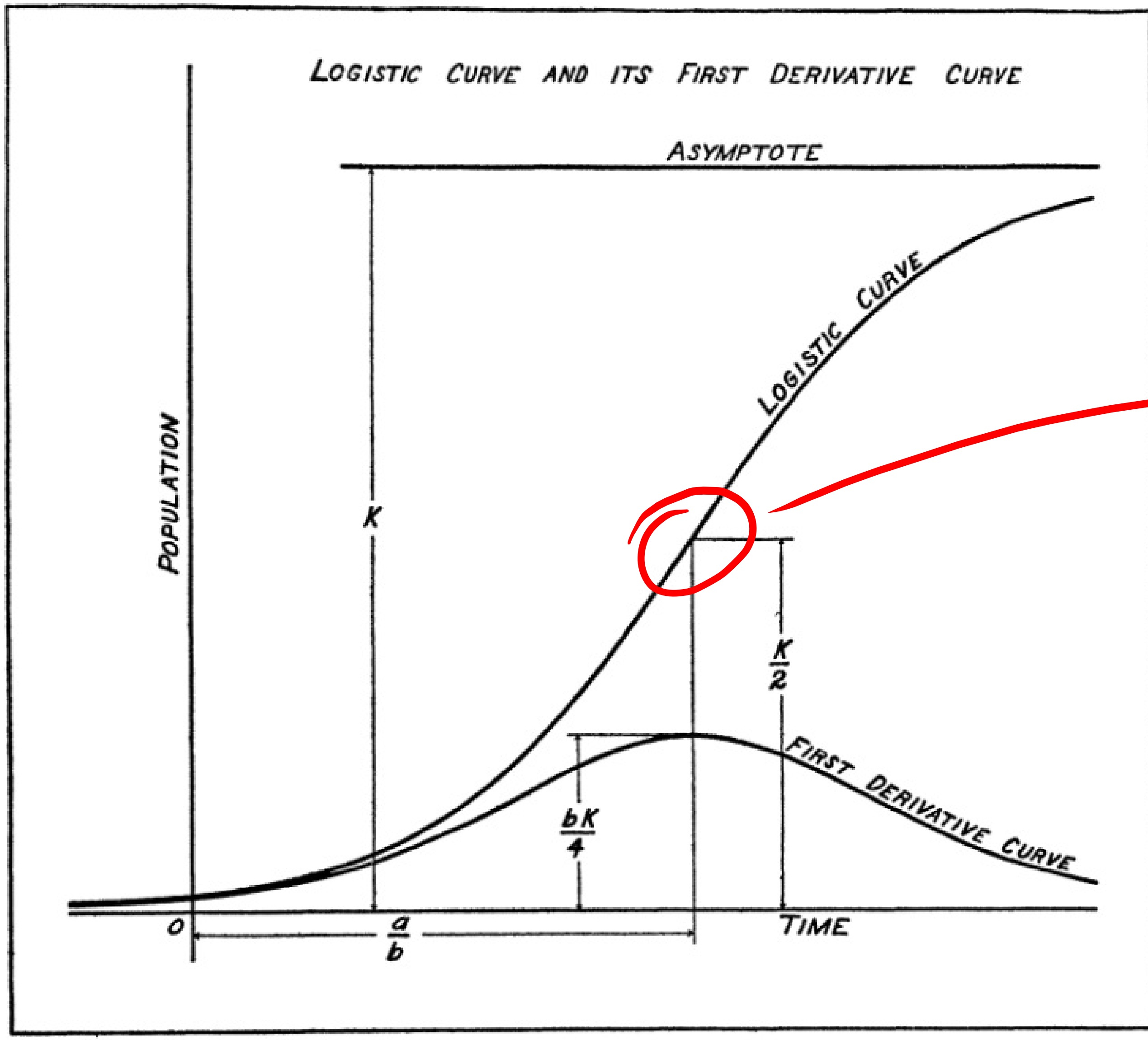
$N_0 = 10$

$r = 0,5355$

I $\quad N = \dfrac{N_0 K e^{rt}}{K + N_0 (e^{rt} - 1)}$

II $\quad \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$

Análisis de estabilidad y
¿Qué pasa con la población a largo plazo?



Ejercicio

FIG. 8. THE LOGISTIC CURVE AND ITS FIRST DERIVATIVE

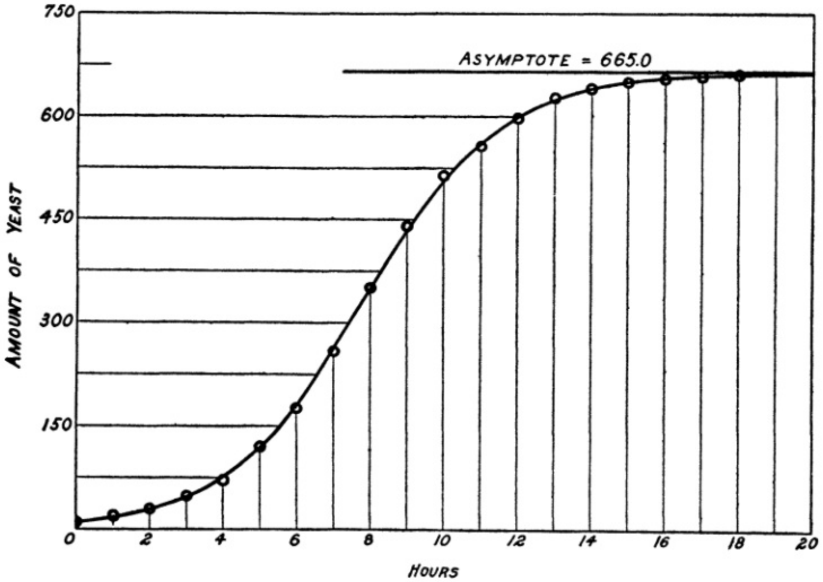


Fig. 1. The Growth of a Population of Yeast Cells

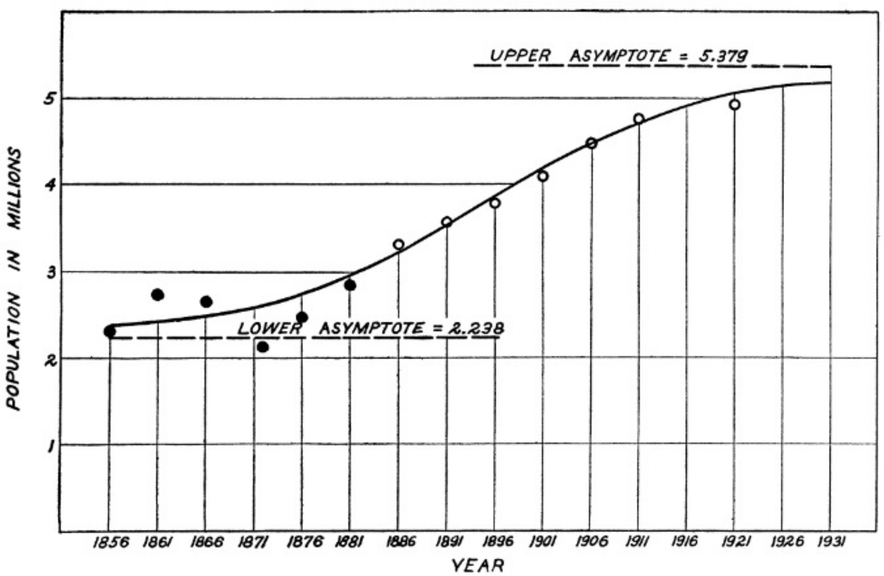


Fig. 7. Observed and Calculated Growth of the Indigenous Native Population of Algeria

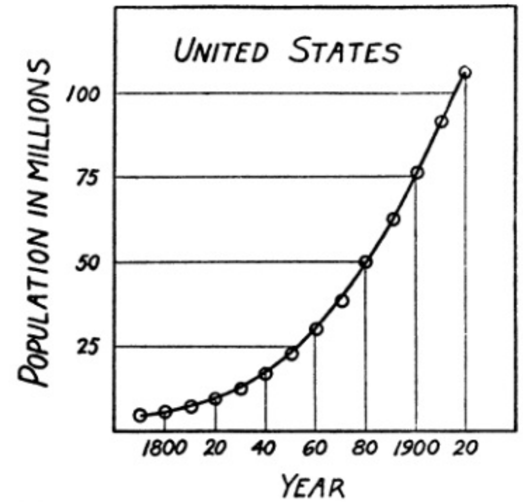


Fig. 3. The Observed and Calculated Growth of the Population of the United States

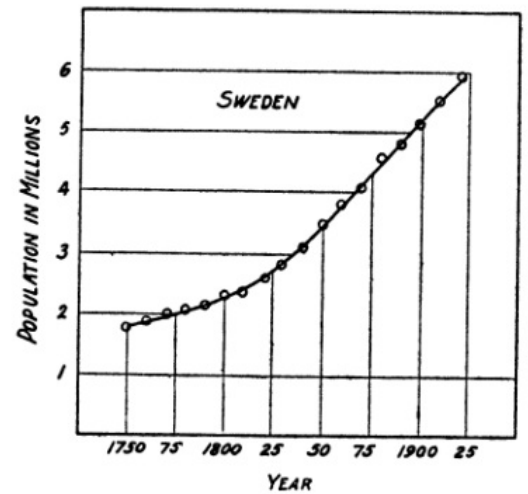


Fig. 4. The Observed and Calculated Growth of the Population of Sweden

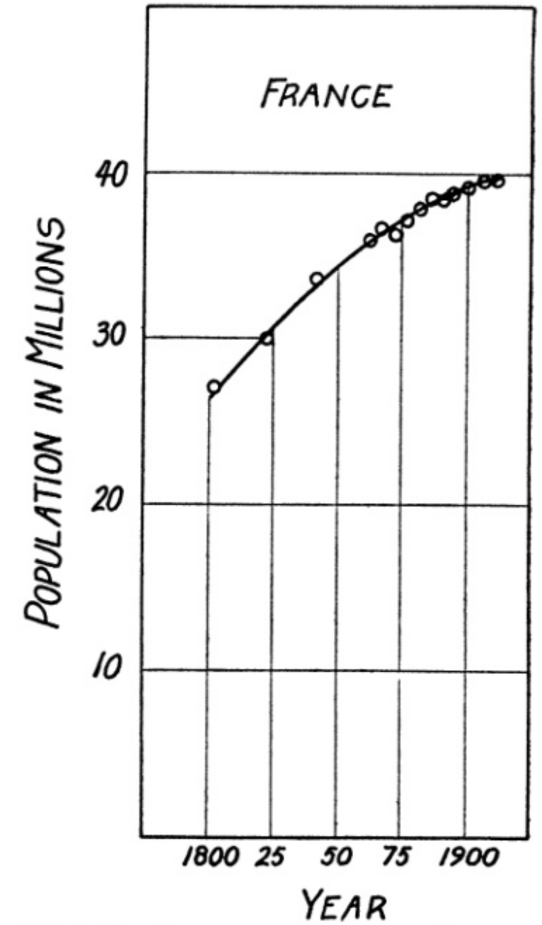


Fig. 5. The Observed and Calculated Growth of the Population of France

Taller de Modelización Matemática y Computacional
Clase 1 – Entrega computacional complementaria

Considere el siguiente modelo poblacional en el que una población que crece según una curva logística en aislamiento es consumida (eg: por humanos) en forma proporcional a la población disponible

$$\frac{dN}{dt} = rN\left(1 - \frac{K}{N}\right) - cN$$

donde $c > 0$ es una constante.
Presente un script de Octave que permita, para diferentes combinaciones de valores iniciales N_0 (por ejemplo, en el intervalo $[0, K]$) y diferentes parámetros ($r > c,\ r = c$ y $r < c$)

(a) Simular el crecimiento poblacional en el tiempo
(b) Graficar usando diferentes opciones
(c) Presentar las gráficas correspondientes a las diferentes combinaciones usando el comando `subplot`.

Obs: La ecuación de este modelo es idéntica a la presente en el ejercicio matemático complementario (2-d), así que si ya realizó ese análisis puede usarlo para comprobar el correcto funcionamiento del script.