If (2.30) and (2.31) are inserted into (2.29), the potential becomes

$$\Phi(x, \theta, \phi) = \frac{3Va^2}{2x^2}\left(\frac{x^3(x^2 - a^2)}{(x^2 + a^2)^{5/2}}\right)\cos\theta \times \left[1 + \frac{35}{24}\frac{a^2x^2}{(a^2 + x^2)^2}(3 - \cos^2\theta) + \cdots\right] \tag{2.32}$$

We note that only odd powers of $\cos\theta$ appear, as required by the symmetry of the problem. If the expansion parameter is (a^2/x^2), rather than α^2, the series takes on the form:

$$\Phi(x, \theta, \phi) = \frac{3Va^2}{2x^2}\left[\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots\right] \tag{2.33}$$

For large values of x/a this expansion converges rapidly and so is a useful representation for the potential. Even for $x/a = 5$, the second term in the series is only of the order of 2 per cent. It is easily verified that, for $\cos\theta = 1$, expression (2.33) agrees with the expansion of (2.28) for the potential on the axis. [The particular choice of angular factors in (2.33) is dictated by the definitions of the Legendre polynomials. The two factors are, in fact, $P_1(\cos\theta)$ and $P_3(\cos\theta)$, and the expansion of the potential is one in Legendre polynomials of odd order. We shall establish this in a systematic fashion in Section 3.3.]

2.9 Orthogonal Functions and Expansions

The representation of solutions of potential problems (or any mathematical physics problem) by expansions in orthogonal functions forms a powerful technique that can be used in a large class of problems. The particular orthogonal set chosen depends on the symmetries or near symmetries involved. To recall the general properties of orthogonal functions and expansions in terms of them, we consider an interval (a, b) in a variable ξ with a set of real or complex functions $U_n(\xi)$, $n = 1, 2, \ldots$, orthogonal on the interval (a, b). The orthogonality condition on the functions $U_n(\xi)$ is expressed by

$$\int_a^b U_n^*(\xi)U_m(\xi)\,d\xi = 0, \quad m \neq n \tag{2.34}$$

If $n = m$, the integral is finite. We assume that the functions are normalized so that the integral is unity. Then the functions are said to be *orthonormal*, and they satisfy

$$\int_a^b U_n^*(\xi)U_m(\xi)\,d\xi = \delta_{nm} \tag{2.35}$$

An arbitrary function $f(\xi)$, square integrable on the interval (a, b), can be expanded in a series of the orthonormal functions $U_n(\xi)$. If the number of terms in the series is finite (say N),

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n U_n(\xi) \tag{2.36}$$

then we can ask for the "best" choice of coefficients a_n so that we get the "best" representation of the function $f(\xi)$. If "best" is defined as minimizing the mean square error M_N:

$$M_N = \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2 d\xi \tag{2.37}$$

it is easy to show that the coefficients are given by

$$a_n = \int_a^b U_n^*(\xi) f(\xi)\, d\xi \tag{2.38}$$

where the orthonormality condition (2.35) has been used. This is the standard result for the coefficients in an orthonormal function expansion.

If the number of terms N in series (2.36) is taken larger and larger, we intuitively expect that our series representation of $f(\xi)$ is "better" and "better." Our intuition will be correct provided the set of orthonormal functions is *complete*, completeness being defined by the requirement that there exist a finite number N_0 such that for $N > N_0$ the mean square error M_N can be made smaller than any arbitrarily small positive quantity. Then the series representation

$$\sum_{n=1}^{\infty} a_n U_n(\xi) = f(\xi) \tag{2.39}$$

with a_n given by (2.38) is said to *converge in the mean* to $f(\xi)$. Physicists generally leave the difficult job of proving completeness of a given set of functions to the mathematicians. All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete.

Series (2.39) can be rewritten with the explicit form (2.38) for the coefficients a_n:

$$f(\xi) = \int_a^b \left\{ \sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) \right\} f(\xi')\, d\xi' \tag{2.40}$$

Since this represents any function $f(\xi)$ on the interval (a, b), it is clear that the sum of bilinear terms $U_n^*(\xi')U_n(\xi)$ must exist only in the neighborhood of $\xi' = \xi$. In fact, it must be true that

$$\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi) \tag{2.41}$$

This is the so-called *completeness* or *closure relation.* It is analogous to the orthonormality condition (2.35), except that the roles of the continuous variable ξ and the discrete index n have been interchanged.

The most famous orthogonal functions are the sines and cosines, an expansion in terms of them being a Fourier series. If the interval in x is $(-a/2, a/2)$, the orthonormal functions are

$$\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi mx}{a}\right), \quad \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi mx}{a}\right)$$

where m is an integer. The series equivalent to (2.39) is customarily written in the form:

$$f(x) = \tfrac{1}{2}A_0 + \sum_{m=1}^{\infty}\left[A_m \cos\left(\frac{2\pi mx}{a}\right) + B_m \sin\left(\frac{2\pi mx}{a}\right)\right] \tag{2.42}$$

where

$$\begin{aligned} A_m &= \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi mx}{a}\right)dx \\ B_m &= \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi mx}{a}\right)dx \end{aligned} \tag{2.43}$$

If the interval spanned by the orthonormal set has more than one dimension, formulas (2.34)–(2.39) have obvious generalizations. Suppose that the space is two dimensional, and that the variable ξ ranges over the interval (a, b) while the variable η has the interval (c, d). The orthonormal functions in each dimension are $U_n(\xi)$ and $V_m(\eta)$. Then the expansion of an arbitrary function $f(\xi, \eta)$ is

$$f(\xi, \eta) = \sum_n \sum_m a_{nm} U_n(\xi) V_m(\eta) \tag{2.44}$$

where

$$a_{nm} = \int_a^b d\xi \int_c^d d\eta\, U_n^*(\xi) V_m^*(\eta) f(\xi, \eta) \tag{2.45}$$

If the interval (a, b) becomes infinite, the set of orthogonal functions $U_n(\xi)$ may become a continuum of functions, rather than a denumerable set. Then the Kronecker delta symbol in (2.35) becomes a Dirac delta function. An important example is the Fourier integral. Start with the orthonormal set of complex exponentials,

$$U_m(x) = \frac{1}{\sqrt{a}}\, e^{i(2\pi mx/a)} \tag{2.46}$$

$m = 0, \pm 1, \pm 2, \ldots,$ on the interval $(-a/2, a/2)$, with the expansion:

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{\infty} A_m e^{i(2\pi mx/a)} \tag{2.47}$$