

Electromagnetismo (2021)

Práctico 5

Ecuación de Laplace con Separación de Variables

Justificado por unicidad de las sol a $\nabla^2\varphi = 0$ (incluyendo C.b.)

simetrías →

- cartesianas: $\varphi = X(x) Y(y) Z(z)$ → sol gral... a ver en cada caso
- cilíndricas: $\varphi = R(r) \Phi(\phi) Z(z)$ ↗ 1. queda para T.E.M.
- esféricos: $\varphi = R(r) \Theta(\theta) \Phi(\phi)$ ↗ 1. queda para T.E.M.

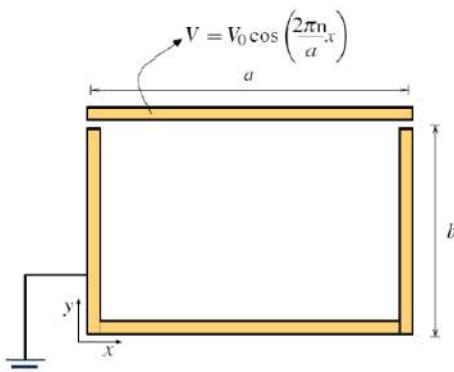
$\varphi(r, \theta) = c_0 + D_0 \ln(r) + \sum_{n=1}^{+\infty} (A_n \sin(n\theta) + B_n \cos(n\theta)) (C_n r^n + D_n r^{-n})$

$\varphi(r, \theta) = \sum_{n=0}^{+\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$

Las funciones $\sin(m\phi), \cos(n\phi) \rightarrow f_n$ cumplen $\int f_n f_m dx = \delta_{nm}$

$P_n(\cos\theta) \rightarrow g_n$ cumple $\int g_n g_m d\theta = \delta_{nm}$
seno dθ

2. Considere una caja rectangular muy larga pero de sección rectangular de ancho a y alto b . A los lados y abajo están conectadas a tierra, es decir, a potencial cero. La superficie superior tiene un potencial periódico $V(x, y = b) = V_0 \cos\left(\frac{2\pi n}{a}x\right)$ (donde n es un número entero). Encuentre el potencial en el interior, $V(x, y)$.



1) cart.

$$\nabla^2 \psi = 0 \rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\psi = X(x)Y(y) \rightarrow \nabla^2 \psi = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$

- Metodología gral. [No aplica del todo]
- 1) Identifico que coord. son cartesianas convenientes.
 - 2) Planteo la sol gral.
 - 3) Escribo las c.b. que debe satisfacer la sol a mi problema
 - 4) Impongo las c.b. (3) en la sol. gral (2) → hallando ⇒ los coef de mi sol gral.
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$$2) \frac{x''}{x} = -k \Rightarrow \frac{y''}{y} = k$$

$$x = A \sin(\sqrt{k}x) + B \cos(\sqrt{k}x)$$

$$\Rightarrow y = C e^{\sqrt{k}y} + D e^{-\sqrt{k}y}$$

se me rompió la metodología

$$X(0)Y(y)$$

$$X(0) = 0 \quad xg' \quad \underline{U(0,y) = 0}$$

$$\Rightarrow X(0) = B = 0$$

$$X(a) = 0 \quad xg' \quad \underline{U(a,y) = 0}$$

$$\Rightarrow X(a) = A \sin(\sqrt{k}a) = 0 \Rightarrow \sqrt{k}a = l\pi \quad \text{con } l=1,2,\dots$$

$$\Rightarrow \varphi(x,y) = \sum_{l=1}^{+\infty} A_l \sin\left(\frac{l\pi}{a}x\right) \cdot \left[C_l e^{\frac{l\pi}{a}y} + D_l e^{-\frac{l\pi}{a}y} \right]$$

$$C'_l \equiv A_l C_l \quad D'_l \equiv A_l D_l$$

$$\Rightarrow \boxed{\varphi(x,y) = \sum_{l=1}^{+\infty} \sin\left(\frac{l\pi}{a}x\right) \left[C'_l e^{\frac{l\pi}{a}y} + D'_l e^{-\frac{l\pi}{a}y} \right]}$$

Sol. gen.

ya fuero impuestas

$$3) \varphi(a,y) = \varphi(0,y) = 0$$

$$-\quad \psi(x,y) = \sum_{k=1}^{\infty} n_k \operatorname{sen}\left(\frac{k\pi}{a}x\right) \cdot [C_k e^{j\frac{k\pi}{a}y} + D_k e^{-j\frac{k\pi}{a}y}]$$

$$\begin{aligned} C'_k &\equiv A_k C_k & D'_k &\equiv A_k D_k \\ \Rightarrow \psi(x,y) &= \sum_{k=1}^{+\infty} \operatorname{sen}\left(\frac{k\pi}{a}x\right) \left[C'_k e^{j\frac{k\pi}{a}y} + D'_k e^{-j\frac{k\pi}{a}y} \right] \end{aligned}$$

Sol. gen.

ya fuero impuestas

$$3) \quad \psi(a,y) = \psi(0,y) = 0$$

$$\text{c.b.1)} \quad \psi(x,0) = 0$$

$$\text{c.b.2)} \quad \psi(x,b) = V_0 \cos\left(\frac{2\pi n}{a}x\right)$$

$$4) \quad \text{c.b.1)} \quad \underbrace{\text{sol. gen.}}$$

$$\psi(x,0) = \sum_{k=1}^{+\infty} \operatorname{sen}\left(\frac{k\pi}{a}x\right) \cdot [C'_k + D'_k] = 0 \quad \forall x$$

com. m fijo

4) C. b. 1) sol. gral.

$$Q(x, 0) = \sum_{\lambda=1}^{+\infty} \sin\left(\frac{\lambda\pi}{a}x\right) \cdot [C_\lambda + D_\lambda] = 0 \quad \forall x$$

con m fijo

$$\int_0^a \sin\left(\frac{m\pi}{a}x\right) \cdot \left[\sum_{\lambda=1}^{+\infty} \sin\left(\frac{\lambda\pi}{a}x\right) dx \right] = \sum_{\lambda=1}^{+\infty} \left\{ \int_0^a \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{\lambda\pi}{a}x\right) dx \right\} [C_\lambda + D_\lambda] = \int_0^a 0 \sin\left(\frac{m\pi}{a}x\right) dx = 0$$

\Rightarrow

$$\frac{1}{2} [\cos\left(\frac{\pi x(\lambda-m)}{a}\right) - \cos\left(\frac{\pi x(\lambda+m)}{a}\right)] \Rightarrow \int_0^a \cos(\lambda) - \cos(m) = 0$$

si $\lambda \neq m$

si $\lambda = m$

$$\int_0^a \sin^2\left(\frac{m\pi}{a}x\right) dx = \frac{1}{2} \int_0^a \sin^2 + \cos^2 dx = \frac{a}{2}$$

$$\cos(r+s) = \cos(r)\cos(s) - \sin(r)\sin(s)$$

si $l \neq m$

$$\text{si } l=m$$

$$\int_0^a \sin^2\left(\frac{m\pi}{a}x\right) dx = \frac{1}{2} \int_0^a \sin^2 + \cos^2 dx = \frac{a}{2}$$

$$\Rightarrow \left\{ \int_0^a \sin\left(\frac{l\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx \right\} = \frac{a}{2} S_{lm}$$

$$\Rightarrow \sum_{n=1}^{\infty} (C_n^l + D_n^l) \frac{a}{2} S_{lm} = 0 \Rightarrow (C_m^l + D_m^l) \frac{a}{2} = 0 \\ \Rightarrow D_m^l = -C_m^l$$

$$\boxed{\varphi(x,y) = \sum_{m=1}^{+\infty} C_m^l \cdot 2 \cdot \sin\left(\frac{m\pi}{a}x\right) \sinh\left(\frac{m\pi}{a}y\right)}$$

\uparrow
 $\frac{e^{m\pi y} - e^{-m\pi y}}{2}$

(impongo la c.b.z.) $\varphi(x,b) = v \dots / a \pi n \dots - \sum_{n=1}^{+\infty} 1 \dots \dots (m\pi x) \dots / m\pi b \dots$

$$\varphi(x, y) = \sum_{m=1}^{+\infty} C_m 2 \cdot \sin\left(\frac{m\pi}{a}x\right) \operatorname{senh}\left(\frac{m\pi}{a}y\right)$$

$\frac{e^{m\pi y}}{2} - \frac{e^{-m\pi y}}{2}$

Impongo la c.b. 2.)

para s fixo

$$\int_0^a \sin\left(\frac{s\pi}{a}x\right) \cdot \left[\cos\left(\frac{2\pi n}{a}x\right) \right] dx = \int_0^a \sin^2\left(\frac{s\pi}{a}x\right) dx$$

$= C_s 2 \cdot \frac{a}{2} \operatorname{senh}\left(\frac{s\pi b}{a}\right)$

$$V_o \int_0^a \sin\left(\frac{s\pi}{a}x\right) \cos\left(\frac{2\pi n}{a}x\right) dx = \frac{V_o}{2} \int_0^a \left[\sin\left(\frac{\pi x}{a}[s+2n]\right) + \sin\left(\frac{\pi x}{a}[s-2n]\right) \right] dx$$

$$\sin(e \pm f) = \sin(e) \cos(f) \pm \sin(f) \cos(e)$$

$$\sin\left(\frac{s\pi}{a}x\right) \cos\left(\frac{2\pi n}{a}x\right) = \frac{1}{2} [\sin\left(\frac{\pi x}{a}[s+2n]\right) + \sin\left(\frac{\pi x}{a}[s-2n]\right)]$$

$$= -\frac{V_o}{2} \left[\frac{a}{\pi} \left[\begin{array}{l} \cos\left(\frac{\pi x}{a}[s+2n]\right) \\ \cos\left(\frac{\pi x}{a}[s-2n]\right) \end{array} \right] \right]_0^a + \left[\begin{array}{l} \frac{a}{\pi} \cos\left(\frac{\pi x}{a}[s-2n]\right) \\ 0 \end{array} \right]_0^a$$

si $s = 2n$

$\Rightarrow C_s \neq \text{constante}$

$$\Rightarrow c_s \neq \operatorname{senh} \left(\frac{s\pi b}{a} \right) = - \frac{v_0}{2\pi} \left[\underbrace{\frac{1}{s+2n} \left[(-1)^{s+2n} - 1 \right]}_{\begin{array}{l} \text{o si } s \text{ es par} \\ \text{y vale } -2 \text{ si } s \text{ es impar} \end{array}} + \underbrace{\frac{1}{s-2n} \left[(-1)^{s-2n} - 1 \right]}_{\begin{array}{l} \text{o si } s \text{ es par} \\ -2 \text{ si } s \text{ es impar} \end{array}} \right]$$

$$\Rightarrow c_s = \begin{cases} \frac{v_0 2s}{\pi \operatorname{senh} \left(\frac{s\pi b}{a} \right)} & \frac{1}{s^2 - 4n^2} \text{ si } s \text{ impar} \\ 0 & \text{si } s \text{ par} \end{cases}$$

$$\Phi(x, y) = \sum_{s=1}^{+\infty} 2c_s \operatorname{sen} \left(\frac{s\pi x}{a} \right) \operatorname{senh} \left(\frac{s\pi y}{a} \right)$$



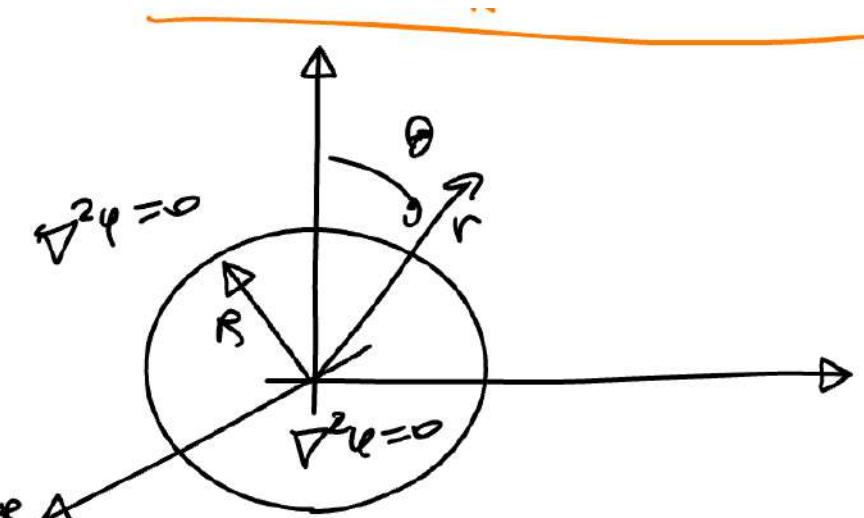
3. Un cascarón esférico de radio R tiene una densidad de carga superficial fija dada por $\sigma(\theta)$. Encuentre el potencial en todo el espacio. Halle la solución completa en el caso en que $\sigma(\theta) = k \cos(\theta)$

1) coord.
estéricas

$$2) \quad \varphi_{\text{gen}}(r, \theta) = \begin{cases} \sum_{n=0}^{+\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) & \text{si } r < R \\ \sum_{n=0}^{+\infty} \left(C_n r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos \theta) & \text{si } r > R \end{cases}$$

3) c.b.1) $\varphi(R^-, \theta) = \varphi(R^+, \theta)$

c.b.2) $\Rightarrow \omega_1 = \omega_2, \quad \sigma(\theta)$



/ -2 -1 \

$$3) \text{ c.b.1)} \quad \varphi(R^-, \theta) = \varphi(R^+, \theta)$$

$$\text{c.b.2)} \quad -\frac{\partial \varphi}{\partial r} \Big|_{R^+} + \frac{\partial \varphi}{\partial r} \Big|_{R^-} = \frac{\sigma(0)}{\epsilon_0} \quad \left(\sim E_\perp^2 - E_\perp^1 = \frac{\sigma}{\epsilon_0} \right)$$

$$\text{c.b.3)} \quad \varphi \xrightarrow[r \rightarrow \infty]{} 0$$

$$\text{c.b.4)} \quad \varphi < \infty \quad r \rightarrow 0$$

$$4) \quad \text{c.b.3} \quad \underbrace{\text{sol gral.}}_{r \rightarrow \infty} \quad \lim_{r \rightarrow \infty} \varphi(r, \theta) = \sum_{n=0}^{+\infty} \left(C_n r^n + D_n \frac{P_n(\cos \theta)}{r^{n+1}} \right) P_n(\cos \theta) = 0$$

$$\Rightarrow C_n = 0 \quad \forall n$$

4) c.b. 3 \curvearrowright sol gral. $\lim_{r \rightarrow \infty} \varphi(r, \theta) = \sum_{n=0}^{+\infty} \left(C_n r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos \theta) = 0$

$$\Rightarrow C_n = 0 \quad \forall n$$

c.b. 4 \curvearrowright sol. gral. $\lim_{r \rightarrow 0} \varphi(r, \theta) = \sum_{n=0}^{+\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) < \infty$

$$\Rightarrow \varphi_{\text{gral}}(r, \theta) = \begin{cases} \sum_{n=0}^{+\infty} A_n r^n P_n(\cos \theta) & \text{si } r < R \\ \sum_{n=0}^{+\infty} \frac{D_n}{r^{n+1}} P_n(\cos \theta) & \text{si } r > R \end{cases} \Rightarrow B_n = 0 \quad \forall n$$

$$+ \infty \quad s_n \quad - \infty \quad f_n$$

Cab. 1 \hookrightarrow sol. gral \Rightarrow
$$\sum_{n=0}^{+\infty} \left\{ A_n R^n \right\} P_n(\cos \theta) = \sum_{n=0}^{+\infty} \left\{ \frac{D_n}{R^{n+1}} \right\} P_n(\cos \theta)$$

$$S_n = F_n$$

$$\hookrightarrow A_n R^n = \frac{D_n}{R^{n+1}} \quad \text{porque } P_n(\cos \theta) \text{ son l.i.}$$

$$\boxed{A_n R^{2n+1} = D_n}$$

Cab. 2 \hookrightarrow sol gral

$$\Rightarrow - \sum_{n=0}^{+\infty} n A_n R^{n-1} P_n(\cos \theta) - \sum_{n=0}^{+\infty} \left\{ A_n R^{2n+1} \right\} \frac{(n+1)}{R^{n+2}} P_n(\cos \theta) = - \frac{\sigma(\theta)}{E_b}$$

C.b.2 sol gral

$$\Rightarrow - \sum_{n=0}^{+\infty} n A_n R^{n-1} P_n(\cos\theta) - \sum_{n=0}^{+\infty} \left\{ A_n R^{2n+1} \right\} (n+1) \frac{P_{n+1}(\cos\theta)}{R^{n+2}} = - \frac{\sigma(\theta)}{\epsilon_0}$$

$\underbrace{- \frac{\partial \phi}{\partial r} \Big|_{R^-}}$ $\underbrace{+ \frac{\partial \phi}{\partial r} \Big|_{R^+}}$

$$\boxed{\sum_{n=0}^{+\infty} \left\{ A_n R^{n-1} (2n+1) \right\} P_n(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0}}$$

los puedo hallar usando que

$$\int_0^\pi P_L(\cos\theta) [\quad] \sin\theta d\theta = \frac{2\pi n}{L+1/2}$$

$$\text{Para el caso } \sigma(\theta) = K \cos\theta = K P_1(\cos\theta)$$

$$\sum_{n=0}^{+\infty} G_n P_n(\cos\theta) = \frac{K}{\epsilon_0} P_1(\cos\theta) \Rightarrow G_n = \delta_{1,n} K \\ \Rightarrow A_R = 0 \text{ si } l \neq 1$$

$$\Rightarrow \psi(r, \theta) = \begin{cases} \frac{K}{3\epsilon_0} \cos\theta & \text{si } r < R \\ \frac{KR^3}{3\epsilon_0 r^2} \cos\theta & \text{si } r > R \end{cases} \quad 3A_1 = \frac{K}{\epsilon_0}$$