Prueba de que los Polinomios de Legendre son ortogonales.

$P_n(\cos\theta)$

$$\begin{pmatrix}\text{Ec. de}\\ \text{Legendre}\end{pmatrix} \quad \frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{d\Theta(\theta)}{d\theta}\right) + n(n+1)\,\Theta(\theta)\operatorname{sen}\theta = 0 \qquad (*)$$

$$x = \cos\theta \quad \rightarrow \quad \frac{d(\ )}{d\theta} = \frac{d(\ )}{dx}\cdot\frac{dx}{d\theta} = -\operatorname{sen}\theta\frac{d(\ )}{dx}$$

$\Rightarrow$ $(*)$ lo podemos escribir como

$$-\cancel{\operatorname{sen}\theta}\frac{d}{dx}\left[\operatorname{sen}\theta\left\{-\operatorname{sen}\theta\frac{d}{dx}\underbrace{\Theta(\theta(x))}_{P_n(x)}\right\}\right] + n(n+1)\underbrace{\Theta(\theta(x))}_{P_n}\cancel{\operatorname{sen}\theta} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[\underbrace{\mathrm{sen}^2\theta}_{1-\cos^2\theta = 1-x^2} \frac{d\,P_n(x)}{dx}\right] + n(n+1)\,P_n(x) = 0$$

$\Rightarrow$ llegamos a la forma estandar de la ec. de Legendre

$$\boxed{\left[(1-x^2)\,P_n'(x)\right]' + n(n+1)\,P_n(x) = 0} \qquad (1)$$

$$\left[(1-x^2)\,P_m'(x)\right]' + m(m+1)\,P_m(x) = 0 \qquad (2)$$

$(1)\,P_m(x) - (2)\,P_n(x)$

$$\Rightarrow \overbrace{\left[(1-x^2)\,P_m(x)\,P_n'(x)\right]' - \cancel{(1-x^2)\,P_m'(x)\,P_n'(x)}}^{P_m(x)\left[(1-x^2)P_n'(x)\right]'} + n(n+1)\,P_n(x)\,P_m(x)$$

$$- \left[(1-x^2)\,P_n(x)\,P_m'(x)\right]' + \cancel{(1-x^2)\,P_n'(x)\,P_m'(x)} - m(m+1)\,P_m(x)\,P_n(x) = 0$$

$$\Rightarrow \quad \underbrace{\left[(1-x^2)\left\{P_m(x)P_n'(x) - P_n(x)P_n'(x)\right\}\right]' + \overbrace{(n-m)(n+m+1)}^{n^2-m^2+n-m} P_n(x)P_m(x)} = 0$$

$$\Rightarrow \int_{-1}^{1} [\quad\checkmark\quad]\, dx = 0$$

$$\underbrace{\int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\left\{P_m(x)P_n'(x) - P_n(x)P_m'(x)\right\}\right] dx}_{\int_a^b \frac{df}{dx}\,dx \;=\; f\Big|_a^b} + (n-m)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

$$0 \qquad \Rightarrow (1-x^2)\left\{P_m(x)\,P_n'(x) - P_n(x)P_m'(x)\right\}\Big|_{-1}^{1} = 0$$

$$\Rightarrow \quad (n-m)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

$\Rightarrow$ Si $n \neq m \Rightarrow$ es $\displaystyle\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0$.