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The motion of a bead along a path restricted to straight lines (restricted brachistochrone), sliding without friction from rest and accelerated by gravity, is considered. For two shapes of path, the geometry of the route optimized to provide the least travel time from one point to another is obtained. The bead's travel times, path lengths, and average velocities are compared between the two presented models, and with travel along a cycloid path, which (as the solution to the original brachistochrone problem) provides the lowest possible travel time. The calculations are made with and without the use of calculus, and therefore the problems presented are comprehensible for a large variety of students.

Introduction

Problems of optimization in undergraduate physics curricula are rare; and yet, these problems are usually intriguing and of great interest to students. One such fascinating question was formulated by Johann Bernoulli over 300 years ago in 1696:

> **What is the shape of the curve down which a bead sliding from rest and accelerated by gravity will travel, without friction, from one fixed point to another in the least time?**

One can also phrase this question in terms of designing the least-time roller coaster track between two given points. It is called the problem of brachistochrone (from the Greek for "shortest time") curve.[1-4] The solution of the problem was found by five of the great natural philosophers of the day: Bernoulli (Jakob —the brother of Johann), Leibniz, de L'hôpital, von Tschirnhaus, and Newton. The well-known solution is an inverted cycloid. However, the solution of this problem is rather complicated for first-year students and is not offered in introductory physics curricula.

A simplified brachistochrone problem that is suitable for an introductory undergraduate course has been presented elsewhere.[5,6] In this paper two additional cases are presented: a bead moving in a uniform gravitational field along vertical and horizontal wires that form a ⊔-shaped path, and along a V-shaped path. It is assumed that the bead's sliding is purely frictionless translation, and that initial and final gravitational energies of the bead are equal.

Two examples of restricted brachistochrone problems

Imagine a bead with a wire threaded through a hole in it, so that the bead can slide with no friction along the wire. The bead is released from point A and slides down along a vertical frictionless wire of length h. Then it moves along a horizontal line $CD = L$, after which it moves up to point B, which is at the same horizontal level as point A (Fig. 1). Of course, the bead would struggle to turn the corners, so smoothing off the corners would be necessary to prevent a collision between the bead and wire that would cause mechanical energy loss. The question is from what height $AC = h$ a frictionless bead should fall, when traveling between points A and B, in order to minimize its time of travel if gravity alone is the cause of acceleration. This question is not trivial, because there are two competing results of increasing h: the first is that the greater h requires greater distance and time traveled vertically by the bead, but, compensating for that, the greater h also brings about a greater speed and less time for traveling along the horizontal segment of the path. So, to minimize travel time, h is the variable whose optimal value needs to be found.

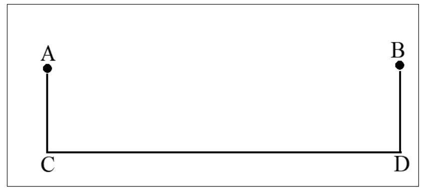

Fig. 1. Restricted frictionless motion of the bead along a ⊔-shaped wire from point A to the given end point B under the influence of a uniform gravitational field (CD = *L*).

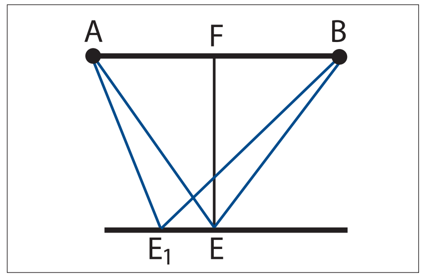

Fig. 2. A bead, released at point A, travels to point B along frictionless paths AEB or AE_1B (AE = EB, AB = *L*).

Let us consider the motion of a bead along path ACDB. On the vertical paths AC and DB, the object moves with acceleration g due to gravity. As initial and final speeds are equal to zero, the travel time t_1 along paths AC and DB is[7]

$$t_1 = \sqrt{\frac{2h}{g}}. \tag{1}$$

Here $h = AC = DB$. The speed at point C is[7]

$$v = \sqrt{2gh}. \tag{2}$$

Along path CD the bead moves with constant speed v and therefore the travel time for path $CD = L$

$$t_2 = \frac{L}{v}. \tag{3}$$

By substitution from Eqs. (1)-(3), the total travel time of $t = 2t_1 + t_2$ becomes

$$t = 2\sqrt{\frac{2h}{g}} + \frac{L}{\sqrt{2gh}}, \tag{4}$$

giving t as a function of h. One can see that this function has an extreme point $h = h_{opt}$ for which time is minimum (t_{min}). Taking the derivative of t with respect to h, and equating it to zero,

$$\frac{dt}{dh} = 0 = \frac{1}{\sqrt{gh}}\left(\sqrt{2} - \frac{L}{2h\sqrt{2}}\right)$$

yields optimal height $h_{opt} = \frac{L}{4}$, optimal travel distance $AC + CD + DB = 2h_{opt} + L = 1.5L$, and the minimum travel time:

$$t_{min} = \sqrt{\frac{8L}{g}}.$$

This problem can be modified by considering a V-shaped path. Let a bead be released from point A and move to point B along symmetrical frictionless linear paths AE and EB ($AE = EB$) (Fig. 2). The question is for what ∠AEF (or h = EF) the travel time is minimum. Let $AB = L$, and follow the procedure described above. Since the initial bead speed equals zero, its speed at point E is $v = \sqrt{2gh}$, and its travel time from A to E is $t_1 = \frac{v}{a}$, where $a = g\cos(\angle AEF)$ is the bead's acceleration. Then the total travel time along path AEB is

$$t = 2t_1 = \sqrt{\frac{2(4h^2 + L^2)}{gh}}. \quad (5)$$

Taking the derivative and equating it to zero allows us to find optimal h and ∠AEF:

$$h_{opt} = \frac{L}{2} \quad \text{and} \quad \angle AEF_{opt} = 45^{o},$$

which implies minimum travel time of $t_{min} = \sqrt{\frac{8L}{g}}$.

Now let us compare the travel time along the paths AEB and AE_1B (these two triangles have the same height h). Average speeds along paths AE, EB, AE_1, and E_1B are the same, and are equal to

$$v_{av} = \frac{\sqrt{2gh}}{2},$$

and therefore the travel time along AEB and AE_1B paths are

$$\frac{AE + EB}{v_{av}} \text{and} \frac{AE_1 + E_1B}{v_{av}},$$

respectively. But as $AE + EB < AE_1 + E_1B$,[8] the travel time along path AEB is less than for path AE_1B. That proves that AEB is the optimal path, i.e., the path that requires the least travel time from point A to point B along two straight lines.

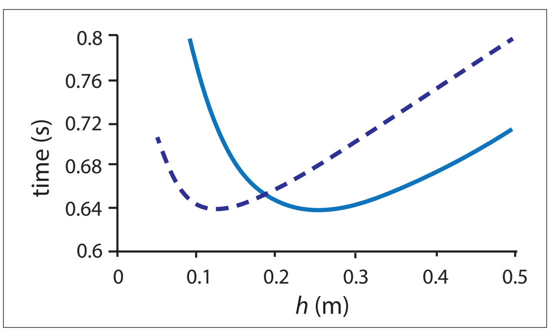

Fig. 3. Travel time as a function of h for a ⊔-shaped (dash line) and V-shaped (solid line) tracks with L = 0.50 m. The minimal time $t_{min} = \sqrt{\frac{8L}{g}} = 0.64$ s **is identical for the two cases.**

As expected, travel time vs. height has a minimum for both basic shapes of path discussed (Fig. 3), while the distances traveled along the two particular optimized shapes are different (Fig. 4). It should also be mentioned that for the two basic shapes, the minimal travel times are identical (Fig. 3).

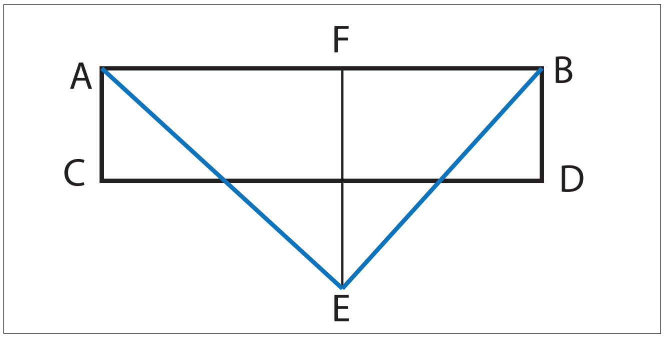

Fig. 4. Optimal profiles for ⊔-shaped (ACDB) and V-shaped (AEB) tracks ($AB = L$, $AC = L/4$, $EF = L/2$, $AE = EB$, $AC + CD + BD = D_1 = 1.5\,L$, $AE + EB = D_2 = \sqrt{2}\,L$).

Algebraic solution for minimum travel time

It is possible to find the least travel time and optimal height for the two considered restricted brachistochrone cases without using calculus. To find extreme points and extreme values of the functions defined by Eqs. (4) and (5), an approach similar to that used elsewhere[6, 9-11] is employed.

1. Traveling along a ⊔-shaped path

Equation (4) can be written in the form $t = ax + \frac{b}{x}$, where

$$a = 2\sqrt{\frac{2}{g}}, b = \frac{L}{\sqrt{2g}}, \text{ and } x = \sqrt{h}.$$

The last equation can be written as $ax^2 - tx + b = 0$. Since x is a positive real number, the determinant of this equation must be positive $t_2 - 4ab \geq 0$ and $t_{min} = 2\sqrt{ab} = \sqrt{\frac{8L}{g}}$, and

$$x^2_{opt} = h_{opt} = \frac{t^2_{min}}{4a^2} = \frac{b}{a} = \frac{L}{4}.$$

2. Traveling along a V-shaped path

Squaring both sides of Eq. (5) leads to quadratic equation $8h^2 - gt^2h + 2L^2 = 0$. Since h must be a real positive number, $g^2t^4 \geq 64L^2$. That means that $t_{min} = \sqrt[4]{\frac{64L^2}{g^2}} = \sqrt{\frac{8L}{g}}$,

and

$$h_{opt} = \frac{gt^2_{opt}}{16} = \frac{L}{2}.$$

Comparison of a brachistochrone-cycloid and brachistochrones confined by straight lines

It was found in the 17th century that a cycloid is the curve on which a bead slides under the influence of a uniform gravitational field to a given end point in the shortest time. A cycloid is the curve traced by a point on the rim of a circular wheel of radius r as the wheel rolls along a straight line without slipping (Fig. 5). For one revolution of the wheel[1] $AB = L = 2\pi r$, the length of the ACB arc is

$$D = 8r = \frac{4L}{\pi}, \text{ and the travel time}$$

$$t_{cycl} = 2\pi\sqrt{\frac{r}{g}} = \sqrt{\frac{2\pi L}{g}}.$$

As expected, $t_{cycl} < t_{min}$ for the two restricted cases, and

$$\frac{t_{cycl}}{t_{min}} = \frac{\sqrt{\frac{2\pi L}{g}}}{\sqrt{\frac{8L}{g}}} = \frac{\sqrt{\pi}}{2} = 0.89.$$

One also can see that $D < D_2 < D_1$.

The average speed along a cycloid path

$$\bar{v} = \frac{D}{t_{cycl}} = \frac{4}{\sqrt{2\pi^3}}\sqrt{Lg} \approx 0.51\sqrt{Lg}$$

while along a $\sqcup$-shaped path the average speed

$$\bar{v}_1 = \frac{D_1}{t_{min}} = 0.53\sqrt{Lg}$$ and along a V-shaped path

$$\bar{v}_2 = \frac{D_2}{t_{min}} = 0.50\sqrt{Lg}.$$

Although $\bar{v} \approx \bar{v}_2 \approx \bar{v}_1$, $t_{cycl} < t_{min} = 1.1t_{cycl}$ because $D < D_2 = 1.1D < D_1 = 1.2D$.

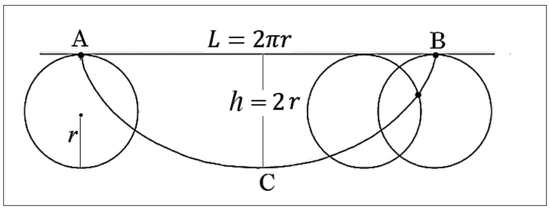

Fig. 5. The inverted cycloid curve.

Conclusion

The brachistochrone problem has historical significance, as its solution contributed to the creation of the calculus of variations,[2,3] on which Lagrangian mechanics is based. Although the problems presented here seek the minimum value for time of motion of an object along a specified simple configuration of linear piecewise continuous lines, it gives some initial idea of the calculus of variations. These extremum problems can also be solved without calculus and therefore are suitable for a large range of students, and, as practically all optimization problems do, these problems usually pique students' curiosity and stimulate their interest in physics.

Acknowledgment

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