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Dynamical symmetries in classical mechanics

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Abstract

We show how symmetries of a classical dynamical system can be described in terms of operators that act on the state space for the system. We illustrate our results by considering a number of possible symmetries that a classical dynamical system might have, and for each symmetry we give examples of dynamical systems that do and do not possess that symmetry.

1. Introduction

Symmetry principles play a fundamental role in physics [1, 2]. Students usually first encounter symmetry principles in the context of classical mechanics, where Lagrangian or Hamiltonian methods are used to demonstrate the relationship between continuous symmetries and conservation laws [3–5]. In this context, symmetries are often described in terms of invariances of the forms of equations. We will focus our attention on a special class of symmetries, which we call dynamical symmetries, that can be described in terms of the invariances of the forms of equations of motion. For example, the equations of motion for a collection of gravitationally interacting point particles are

$$\ddot{\mathbf{r}}_k = -G \sum_j \frac{m_j(\mathbf{r}_k - \mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|^3}, \tag{1}$$

where $\mathbf{r}_k$ is the position of particle k, m_k is its mass, and G is Newton's constant. The Galilean invariance of this system is demonstrated by the fact that if we apply a Galilean transformation $\mathbf{r}_k \to \mathbf{r}'_k = \mathbf{r}_k - \mathbf{v}t$, where $\mathbf{v}$ is an arbitrary velocity, the form of the equations of motion is unchanged:

$$\ddot{\mathbf{r}}'_k = -G \sum_j \frac{m_j(\mathbf{r}'_k - \mathbf{r}'_j)}{|\mathbf{r}'_k - \mathbf{r}'_j|^3}. \tag{2}$$

In this paper, we consider an alternative formulation of dynamical symmetries, in which dynamical symmetries of a system are described in terms of operators that act on the state space for the system. Our approach has two advantages. First, it can provide more physical intuition, because in the simplest cases it allows dynamical symmetries to be visualized: dynamical symmetries of a one-dimensional system can often be related to geometric symmetries of a flow diagram for that system. Second, it is more closely analogous to the way symmetries are treated in quantum mechanics. We show how the operator formulation of symmetries is related to the formulation in which symmetries are described in terms of invariances of the equations of motion, and we illustrate our results by considering a number of examples. The paper should be accessible to advanced undergraduates and could be used to supplement a discussion of symmetry principles in an undergraduate-level course in classical mechanics.

2. Classical dynamical systems

The state of a classical dynamical system is specified by a set of dynamical variables $\{r_1, \ldots, r_N\}$ that evolve in time according to a set of equations of motion of the form

$$\dot{r}_k = f_k(r_1, \ldots, r_N, t) \tag{3}$$

for some functions f_k. An example of a classical dynamical system is a point particle in one dimension, where the dynamical variables $\{r_1, r_2\}$ are taken to be the particle position $r_1 = z$ and velocity $r_2 = v$. The equations of motion for these quantities are

$$\dot{z} = v, \qquad \dot{v} = (1/m)F(z, v, t), \tag{4}$$

where m is the mass of the particle and $F(z, v, t)$ is the force that acts on the particle at time t.

Let us define an operator $\mathcal{E}(t_f, t_i)$ that evolves the dynamical variables $\{r_1, \ldots, r_N\}$ from time t_i to time t_f by integrating the equations of motion (3):

$$\mathcal{E}(t_f, t_i)\{r_1(t_i), \ldots, r_N(t_i)\} = \{r_1(t_f), \ldots, r_N(t_f)\}. \tag{5}$$

For many dynamical systems, the equations of motion (3) do not explicitly depend on time and can thus be expressed as

$$\dot{r}_k = f_k(r_1, \ldots, r_N). \tag{6}$$

For such systems, the evolution operator $\mathcal{E}(t_f, t_i)$ depends only on the time difference $t_f - t_i$, and for simplicity we will often denote the evolution operator for such systems by $\mathcal{E}(\tau)$, where $\tau \equiv t_f - t_i$ and $\mathcal{E}(\tau) \equiv \mathcal{E}(\tau, 0)$. Such systems are said to be invariant under time translation, because $\mathcal{E}(t_f + \delta t, t_i + \delta t) = \mathcal{E}(t_f, t_i)$ for any time interval δt.

3. Examples

In what follows, it will be useful to have a stock of examples, which we will obtain by considering a point particle in one spatial dimension and making different choices for the force law $F(z, v, t)$ that occurs in equation (4).

For all of our examples, the force law is independent of time, and we will denote the corresponding evolution operator by $\mathcal{E}(\tau)$. Our first example is a free particle, for which the force law is

$$F(z, v) = 0. \tag{7}$$

The corresponding evolution operator is

$$\mathcal{E}(\tau)\{z,\ v\} = \{z + v\tau,\ v\}. \tag{8}$$

Our second example is a harmonically bound particle, for which the force law is

$$F(z, v) = -m\omega^2 z, \tag{9}$$

where ω is the harmonic frequency. The corresponding evolution operator is

$$\mathcal{E}(\tau)\{z,\ v\} = \{z\cos\omega\tau + (v/\omega)\sin\omega\tau,\ v\cos\omega\tau - \omega z\sin\omega\tau\}. \tag{10}$$

Our third example is a damped particle, for which the force law is

$$F(z, v) = -m\gamma v, \tag{11}$$

where γ is the damping constant. The corresponding evolution operator is

$$\mathcal{E}(\tau)\{z,\ v\} = \{z + (v/\gamma)(1 - \mathrm{e}^{-\gamma\tau}),\ v\,\mathrm{e}^{-\gamma\tau}\}. \tag{12}$$

Our fourth example is a uniformly accelerated Newtonian particle, for which the force law is

$$F(z, v) = ma, \tag{13}$$

where a is the acceleration. The corresponding evolution operator is

$$\mathcal{E}(\tau)\{z,\ v\} = \{z + v\tau + a\tau^2/2,\ v + a\tau\}. \tag{14}$$

Our final example is a uniformly accelerated relativistic particle [6]. Let us define $z^\mu \equiv (t, z)$ and $v^\mu \equiv (\gamma, \gamma v)$, where $\gamma \equiv (1 - v^2)^{-1/2}$. The equations of motion for these quantities are

$$\frac{\mathrm{d}z^\mu}{\mathrm{d}s} = v^\mu, \qquad \frac{\mathrm{d}v^\mu}{\mathrm{d}s} = a\epsilon^\mu{}_\nu v^\nu, \tag{15}$$

where s is the proper time, a is the acceleration, and $\epsilon^\mu{}_\nu$ is the Levi-Civita tensor, defined such that $\epsilon^0{}_1 = \epsilon^1{}_0 = 1$ and $\epsilon^0{}_0 = \epsilon^1{}_1 = 0$. Using equation (15), it is straightforward to show that the equations of motion for z and v are given by equation (4), where the force law is

$$F(z, v) = ma(1 - v^2)^{3/2}. \tag{16}$$

The corresponding evolution operator is

$$\begin{aligned}\mathcal{E}(\tau)\{z,\ v\} = \big\{&z + (a^{-2} + (\tau + \tau_v)^2)^{1/2} - \left(a^{-2} + \tau_v^2\right)^{1/2},\\ &(\tau + \tau_v)(a^{-2} + (\tau + \tau_v)^2)^{-1/2}\big\},\end{aligned} \tag{17}$$

where $\tau_v \equiv (v/a)(1 - v^2)^{-1/2}$.

We can visualize the evolution operators for our examples by picking fixed values z_0 and v_0 for the particle position and velocity and plotting $\{z(\tau),\ v(\tau)\} \equiv \mathcal{E}(\tau)\{z_0,\ v_0\}$ as a parametric curve in the z–v plane. By making different choices for z_0 and v_0, we can obtain a family of curves that map out a flow. In figure 1, we plot such flow diagrams for each of our examples.

4. Symmetry transformations

Let us return to the case of an arbitrary dynamical system, in which the state of the system is described by a set of dynamical variables $\{r_1, \ldots, r_N\}$ that evolve in time according to the equations of motion (3). We would like to consider a transformation to new dynamical variables, such that if the original dynamical variables at time t are $\{r_1, \ldots, r_N\}$, then the new dynamical variables at time t are $\{\bar{r}_1, \ldots, \bar{r}_N\}$ for some functions $\bar{r}_k(r_1, \ldots, r_N, t)$. Let us define an operator $A(t)$ that implements this transformation:

$$A(t)\{r_1, \ldots, r_N\} = \{\bar{r}_1, \ldots, \bar{r}_N\}. \tag{18}$$

We will assume that the transformation is invertible at all times, so $A(t)$ has a well-defined inverse:

$$A^{-1}(t)\{\bar{r}_1, \ldots, \bar{r}_N\} = \{r_1, \ldots, r_N\}. \tag{19}$$

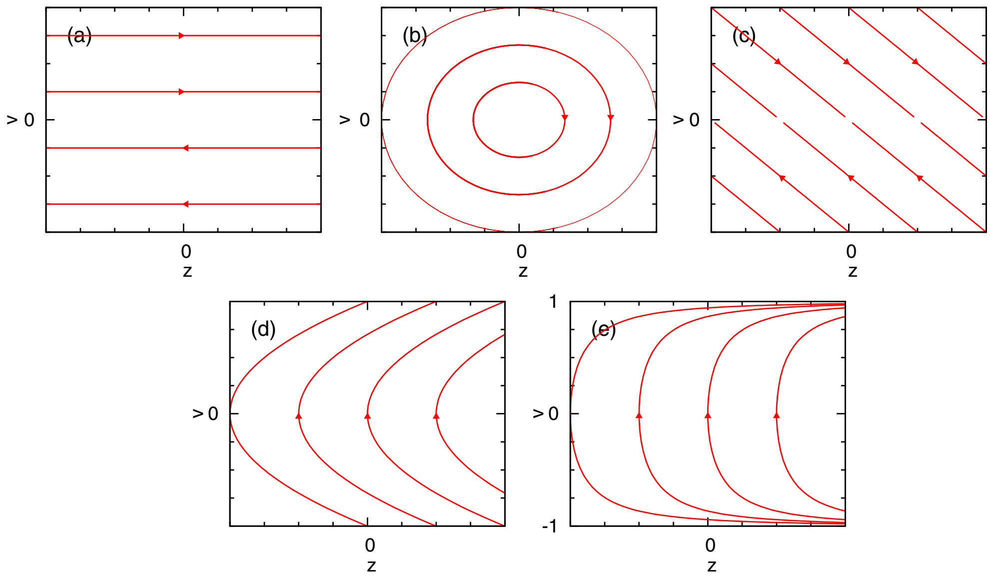

Figure 1. Flow diagrams for example dynamical systems. (a) Free particle; (b) harmonically bound particle; (c) damped particle; (d) uniformly accelerated Newtonian particle; (e) uniformly accelerated relativistic particle. The arrows indicate the direction of increasing τ along each curve.

(This figure is in colour only in the electronic version)

Using the equations of motion (3) for the original dynamical variables, we find that the equations of motion for the new dynamical variables are

$$\dot{\bar{r}}_k = \sum_j \frac{\partial \bar{r}_k}{\partial r_j}\dot{r}_j + \frac{\partial \bar{r}_k}{\partial t} = \sum_j \frac{\partial \bar{r}_k}{\partial r_j} f_j + \frac{\partial \bar{r}_k}{\partial t}. \tag{20}$$

Let us define an operator $\bar{\mathcal{E}}(t_f, t_i)$ that evolves the new dynamical variables $\{\bar{r}_1, \ldots, \bar{r}_N\}$ from time t_i to time t_f by integrating the equations of motion (20). From the definitions of $A(t)$ and $A^{-1}(t)$ given in equations (18) and (19), it follows that

$$\bar{\mathcal{E}}(t_f, t_i) = A(t_f)\mathcal{E}(t_f, t_i)A^{-1}(t_i). \tag{21}$$

We would like to focus our attention on a special class of transformations, called symmetry transformations, that leave the form of the equations of motion invariant. For such transformations, the equations of motion (20) for the new dynamical variables are given by

$$\dot{\bar{r}}_k = f_k(\bar{r}_1, \ldots, \bar{r}_N, t), \tag{22}$$

where the functions f_k are the same functions that occur in the equations of motion (3) for the original dynamical variables. From equations (3) and (22), it follows that the evolution operators for the two sets of dynamical variables are identical:

$$\bar{\mathcal{E}}(t_f, t_i) = \mathcal{E}(t_f, t_i). \tag{23}$$

We substitute equation (23) into equation (21) to obtain

$$\mathcal{E}(t_f, t_i) = A(t_f)\mathcal{E}(t_f, t_i)A^{-1}(t_i). \tag{24}$$

We conclude that for every symmetry transformation, we can define a corresponding time-dependent coordinate transformation operator $A(t)$ such that the evolution operator $\mathcal{E}(t_f, t_i)$

is left invariant under the transformation described by equation (24). For a time-independent symmetry transformation, the operator $A(t) \equiv A$ is constant in time and equation (24) can be expressed as

$$\mathcal{E}(t_f, t_i)A = A\mathcal{E}(t_f, t_i). \tag{25}$$

So for every time-independent symmetry transformation, we can define a corresponding coordinate-transformation operator A that commutes with the evolution operator.

It is instructive to compare the operator formulation of symmetries in classical dynamical systems with symmetries in quantum systems [7]. Consider a quantum system with Hamiltonian $H(t)$. The state of the system is described by a state vector $|\psi\rangle$ that evolves in time according to the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -\mathrm{i}\, H(t)|\psi(t)\rangle, \tag{26}$$

which is the quantum analogue to the equations of motion (3) for a classical dynamical system. We can define a unitary transformation $U(t_f, t_i)$ that evolves the state vector from time t_i to time t_f by integrating equation (26):

$$U(t_f, t_i) = T\left[\exp\left(-\mathrm{i}\int_{t_i}^{t_f} H(t')\,\mathrm{d}t'\right)\right], \tag{27}$$

where T is a time-ordering operator that places operators at later times to the left of operators at earlier times. The unitary transformation $U(t_f, t_i)$ is the quantum analogue to the classical evolution operator $\mathcal{E}(t_f, t_i)$. A symmetry of the quantum system is described by an operator $\mathcal{O}$ that commutes with the Hamiltonian. From equation (27), it follows that $\mathcal{O}$ commutes with the unitary transformation $U(t_f, t_i)$:

$$U(t_f, t_i)\mathcal{O} = \mathcal{O}U(t_f, t_i). \tag{28}$$

We can view equation (28) as the quantum analogue to the invariance condition equation (25) for a classical dynamical system.

5. Parity reversal

Let us now consider some example transformations for the case of a point particle in one dimension. Our first example is parity reversal, which is described by an operator P that flips the signs of the particle position and velocity:

$$P\{z,\ v\} = \{-z,\ -v\}. \tag{29}$$

From equation (25), it follows that if a system is invariant under parity reversal, then the parity reversal operator commutes with the evolution operator:

$$P\mathcal{E}(t_f, t_i) = \mathcal{E}(t_f, t_i)P. \tag{30}$$

A free particle is invariant under parity reversal, as can be verified using the evolution operator given in equation (8):

$$P\mathcal{E}(\tau)\{z,\ v\} = \{-z - v\tau,\ -v\}, \tag{31}$$

$$\mathcal{E}(\tau)P\{z,\ v\} = \{-z - v\tau,\ -v\}. \tag{32}$$

For a uniformly accelerated Newtonian particle, the evolution operator is given by equation (14), and we find that

$$P\mathcal{E}(\tau)\{z,\ v\} = \{-z - v\tau - a\tau^2/2,\ -v - a\tau\}, \tag{33}$$

$$\mathcal{E}(\tau)P\{z,\ v\} = \{-z - v\tau + a\tau^2/2,\ -v + a\tau\}. \tag{34}$$

So this system is not invariant under parity reversal. For the remaining examples, we find that a harmonically bound particle and a free particle are invariant under parity reversal, and a uniformly accelerated relativistic particle is not invariant under parity reversal. Physically, the reason that parity reversal invariance is violated in the two examples involving a uniformly accelerated particle is that for both examples there is a force pushing the particle to the right, and this force breaks the left–right symmetry of the system. In terms of the flow diagrams shown in figure 1, parity reversal corresponds to flipping the diagram across the z and v axes. If a system is invariant under parity reversal, then its flow diagram is symmetric under this operation.

6. Spatial translation

A spatial translation is described by an operator $D(a)$ that displaces the particle position by a constant distance a:

$$D(a)\{z,\ v\} = \{z + a,\ v\}. \tag{35}$$

From equation (25), it follows that if a system is invariant under spatial translation, then the spatial translation operator commutes with the evolution operator:

$$D(a)\mathcal{E}(t_f, t_i) = \mathcal{E}(t_f, t_i)D(a). \tag{36}$$

All of our examples are invariant under spatial translation except for the harmonically bound particle, which is not invariant because the strength of the harmonic binding force depends on the position of the particle. In terms of the flow diagrams shown in figure 1, a spatial translation corresponds to shifting the diagram along the z axis. If a system is invariant under spatial translations, then its flow diagram is symmetric under this operation.

7. Galilean transformation

A Galilean transformation is described by an operator $G(t,w)$ that adds a constant w to the particle velocity and displaces the particle position by wt:

$$G(t, w)\{z,\ v\} = \{z + wt,\ v + w\}. \tag{37}$$

Note that we can express $G(t,w)$ as

$$G(t, w) = D(wt)B_G(w), \tag{38}$$

where $B_G(w)$ is a Galilean boost operator that adds a constant w to the particle velocity:

$$B_G(w)\{z,\ v\} = \{z,\ v + w\}. \tag{39}$$

From equation (24), it follows that if a system is Galilean invariant, then

$$\mathcal{E}(t_f, t_i) = G(t_f, w)\mathcal{E}(t_f, t_i)G^{-1}(t_i, w). \tag{40}$$

In terms of our examples, a free particle and a uniformly accelerated Newtonian particle are Galilean invariant, and the remaining examples are not Galilean invariant.

If a system is time-translation invariant as well as Galilean invariant, then we can express equation (40) in the form

$$B_G^{-1}(w)\mathcal{E}(\tau)B_G(w) = D(w\tau)\mathcal{E}(\tau). \tag{41}$$

Under these circumstances, the system must be invariant under spatial translations, as can be understood from the following argument. First we invert both sides of equation (41) to obtain

$$B_G^{-1}(w)\mathcal{E}(-\tau)B_G(w) = \mathcal{E}(-\tau)D(-w\tau). \tag{42}$$

We then flip the sign of τ:

$$B_G^{-1}(w)\mathcal{E}(\tau)B_G G(w) = \mathcal{E}(\tau)D(w\tau). \tag{43}$$

From equation (40) and (43), it follows that

$$D(a)\mathcal{E}(\tau) = \mathcal{E}(\tau)D(a), \tag{44}$$

where $a \equiv w\tau$. So the system is invariant under spatial translations.

8. Lorentz transformation

It requires some care to describe Lorentz transformations using the operator formalism, since the invariance condition for such transformations is not of the form given in equation (24). We will first derive the appropriate invariance condition for an arbitrary one-dimensional dynamical system and then specialize to the case of a point particle.

Consider a Lorentz transformation from a coordinate system $(t,\ x)$ to a new coordinate system $(t',\ x')$:

$$t' = \gamma(t + \beta x), \qquad x' = \gamma(x + \beta t), \tag{45}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. We will assume that we have defined a set of dynamical variables $\{r_1, \ldots, r_N\}$ relative to the (t, x) coordinate system whose evolution is described by the equations of motion

$$\frac{\mathrm{d}r_k}{\mathrm{d}t} = f_k(r_1, \ldots, r_N). \tag{46}$$

We will define $\{r'_1, \ldots, r'_N\}$ to be the corresponding set of dynamical variables defined relative to the (t', x') coordinate system. If the equations of motion (46) are invariant under Lorentz transformations, it follows that

$$\frac{\mathrm{d}r'_k}{\mathrm{d}t'} = f_k(r'_1, \ldots, r'_N), \tag{47}$$

where the functions f_k that appear in equation (47) are the same functions that appear in equation (46). It is natural to view the dynamical variables $\{r_1, \ldots, r_N\}$ as describing the state of the system along lines of simultaneity $S(t)$ of constant t, and the dynamical variables $\{r'_1, \ldots, r'_N\}$ as describing the state of the system along lines of simultaneity $S'(t')$ of constant t'.

Let us define a transformation to new dynamical variables, such that if the original dynamical variables at time t are $\{r_1, \ldots, r_N\}$, then the new dynamical variables at time t are $\{\bar{r}_1, \ldots, \bar{r}_N\}$, where $\bar{r}_k(r_1, \ldots, r_N, t)$ is given by the value of r'_k on the line of simultaneity $S'(t')$ that intersects $S(t)$ at $x = 0$. These lines of simultaneity are shown in figure 2(a) for the case $\beta < 0$. Since $S'(t')$ intersects $S(t)$ at $x = 0$, from equation (45) it follows that $t' = \gamma t$, and thus from equation (47) it follows that the equations of motion for the new dynamical variables are

$$\frac{\mathrm{d}\bar{r}_k}{\mathrm{d}t} = \gamma f_k(\bar{r}_1, \ldots, \bar{r}_N). \tag{48}$$

Let us define an operator $\mathcal{E}(t_f - t_i)$ that evolves the original dynamical variables $\{r_1, \ldots, r_N\}$ from time t_i to time t_f by integrating the equations of motion (46), and an

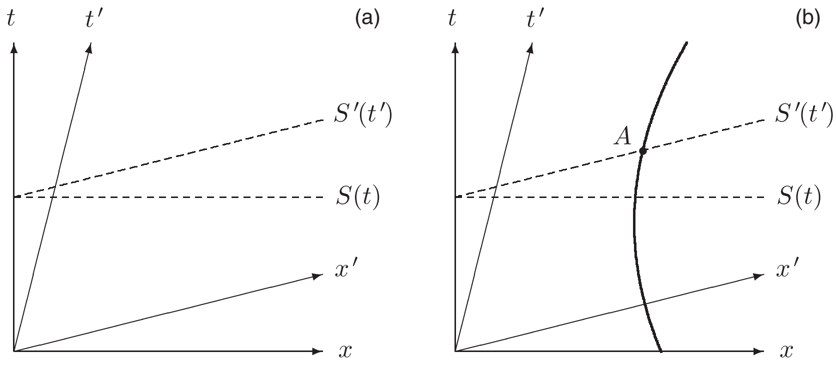

Figure 2. (a) Spacetime diagram for $\beta < 0$. Shown are the x and t axes, the x' and t' axes, the line of simultaneity $S(t)$ for an arbitrary time t, and the corresponding line of simultaneity $S'(t')$ that intersects $S(t)$ at $x = 0$. (b) Particle trajectory (thick line) and the event A at which the particle trajectory intersects the line of simultaneity $S'(t')$ for $t' = \gamma t$.

operator $\bar{\mathcal{E}}(t_f - t_i)$ that evolves the new dynamical variables $\{\bar{r}_1, \ldots, \bar{r}_N\}$ from time t_i to time t_f by integrating the equations of motion (48). From equations (46) and (48), it follows that

$$\bar{\mathcal{E}}(t_f - t_i) = \mathcal{E}(\gamma(t_f - t_i)). \tag{49}$$

Let us define an operator $L(t, \beta)$ that implements the transformation from $\{r_1, \ldots, r_N\}$ to $\{\bar{r}_1, \ldots, \bar{r}_N\}$:

$$L(t, \beta)\{r_1, \ldots, r_N\} = \{\bar{r}_1, \ldots, \bar{r}_N\}. \tag{50}$$

We can then express the evolution operator $\bar{\mathcal{E}}(t_f - t_i)$ as

$$\bar{\mathcal{E}}(t_f - t_i) = L(t_f, \beta)\mathcal{E}(t_f - t_i)L^{-1}(t_i, \beta). \tag{51}$$

We substitute equation (49) into equation (51) to obtain

$$\mathcal{E}(\gamma(t_f - t_i)) = L(t_f, \beta)\mathcal{E}(t_f - t_i)L^{-1}(t_i, \beta). \tag{52}$$

We conclude that if a dynamical system is invariant under Lorentz transformations, then we can define a Lorentz transformation operator $L(t, \beta)$ such that the evolution operator obeys the invariance condition given in equation (52).

As an example, let us determine the operator $L(t, \beta)$ for the case of a point particle. We will take the dynamical variables to be the particle position z and velocity v as measured in the (t, x) coordinate system. Suppose that at time t the dynamical variables are $\{z,\ v\}$. As shown in figure 2(b), we will define event A to be the point at which the particle trajectory intersects the line of simultaneity $S'(t')$ for $t' = \gamma t$. The new dynamical variables $\bar{z}$ and $\bar{v}$ are then given by the particle position and velocity at event A as measured in the (t', x') coordinate system.

We will now solve for $\bar{z}$ and $\bar{v}$ as functions of z, v and t. Let us define z_a and v_a to be the particle position and velocity at event A as measured in the (t, x) coordinate system, and t_a to be the time at which event A occurs as measured in the (t, x) coordinate system. From these definitions, it follows that

$$\mathcal{E}(t_a - t)\{z,\ v\} = \{z_a,\ v_a\}. \tag{53}$$

From the Lorentz-transformation equations (45), it follows that the coordinates $(t', \bar{z})$ of event A in the (t', x') coordinate system are related to the coordinates (t_a, z_a) of event A in the (t, x) coordinate system by

$$t' = \gamma(t_a + \beta z_a), \qquad \bar{z} = \gamma(z_a + \beta t_a). \tag{54}$$

We substitute $t' = \gamma t$ into equations (54) and solve for $\bar{z}$ and t_a as functions of z_a and t:

$$\bar{z} = z_a/\gamma + \beta\gamma t, \tag{55}$$

$$t_a = t - \beta z_a. \tag{56}$$

We substitute equation (56) into equation (53) to obtain

$$\mathcal{E}(-\beta z_a)\{z,\ v\} = \{z_a,\ v_a\}. \tag{57}$$

The velocity $\bar{v}$ of the particle at event A as measured in the (t', x') coordinate system is given by the relativistic velocity addition formula:

$$\bar{v} = (\beta + v_a)(1 + \beta v_a)^{-1}. \tag{58}$$

Collecting these results, we find that the operator $L(t, \beta)$ is given by

$$L(t, \beta)\{z,\ v\} = \{\bar{z},\ \bar{v}\}, \tag{59}$$

where $\bar{z}$ is given by equation (55), $\bar{v}$ is given by equation (58), and z_a and v_a are defined implicitly by equation (57). We note that in contrast to the symmetry transformations we have considered so far, the form of the operator $L(t, \beta)$ depends on the evolution operator. We can express $L(t, \beta)$ as

$$L(t, \beta) = D(\beta\gamma t)B_L(\beta), \tag{60}$$

where B_L is a Lorentz boost operator defined such that

$$B_L(\beta)\{z,\ v\} = \{z_a/\gamma,\ (\beta + v_a)(1 + \beta v_a)^{-1}\}, \tag{61}$$

and as before z_a and v_a are defined implicitly by equation (57). Using equation (60), we can express the invariance condition given in equation (52) as

$$B_L^{-1}(\beta)\mathcal{E}(\tau/\gamma)B_L(\beta) = D(\beta\tau)\mathcal{E}(\tau). \tag{62}$$

By the same type of argument we gave in section 7, one can show that Lorentz invariance implies translation invariance, so we could equally well express the invariance condition as

$$B_L^{-1}(\beta)\mathcal{E}(\tau/\gamma)B_L(\beta) = \mathcal{E}(\tau)D(\beta\tau). \tag{63}$$

For the case of a free particle we can obtain an explicit expression for the boost operator $B_L(\beta)$. We substitute the evolution operator for a free particle given in equation (8) into equation (57) and solve for z_a and v_a:

$$z_a = z/(1 + v\beta), \qquad v_a = v. \tag{64}$$

We substitute these expressions into equation (61) to obtain

$$B_L(\beta)\{z,\ v\} = \{(z/\gamma)(1 + v\beta)^{-1},\ (v + \beta)(1 + v\beta)^{-1}\}. \tag{65}$$

It is straightforward to verify that a free particle is Lorentz invariant by checking that the invariance condition (62) is satisfied for the evolution operator (8) and the boost operator (65). For our remaining examples, only the uniformly accelerated relativistic particle is Lorentz invariant. We note that a free point particle is invariant under both Lorentz transformations and Galilean transformations. The full invariance group for a free point particle is derived in [8].

The description of Lorentz symmetry in our formulation is considerably more complicated than the descriptions of the symmetries we have considered so far. We note again that Lorentz symmetry is not of the general form given in equation (24). Also, unlike the symmetry operators we have previously encountered, the form of the boost operator B_L depends on the evolution operator. All of these features are related to the fact that Lorentz symmetry is manifest only if space and time are treated on an equal footing, but in our formulation we have picked out a preferred time parameter t. The symmetry of a Lorentz-invariant system is still present in our formulation, of course, but in a disguised form; for example, it is not at all obvious that the evolution operator (17) satisfies the invariance condition described by equations (62) and (65).

9. Time reversal

Time reversal is another transformation whose invariance condition is not of the form given in equation (24). Let us consider a dynamical system described by dynamical variables $\{r_1, \ldots, r_N\}$ that evolve in time according to a set of equations of motion that do not explicitly depend on time, and thus have the form given in equation (6). Let us consider a transformation to new dynamical variables $\{\bar{r}_1, \ldots, \bar{r}_N\}$ that evolve in time according to the equations of motion (20). We will say that the transformation is a time reversal transformation, and the equations of motion are time-reversal invariant, if the transformed equations of motion (20) have the form

$$\dot{\bar{r}}_k = -f_k(\bar{r}_1, \ldots, \bar{r}_N), \tag{66}$$

where the functions f_k are the same functions that occur in the equations of motion (6) for the original dynamical variables. Let us define an operator $\mathcal{E}(t_f - t_i)$ that evolves the original dynamical variables $\{r_1, \ldots, r_N\}$ from time t_i to time t_f by integrating the equations of motion (6), and an operator $\bar{\mathcal{E}}(t_f - t_i)$ that evolves the new dynamical variables $\{\bar{r}_1, \ldots, \bar{r}_N\}$ from time t_i to time t_f by integrating the equations of motion (66). From equations (6) and (66), it follows that

$$\bar{\mathcal{E}}(\tau) = \mathcal{E}(-\tau). \tag{67}$$

Let us define an operator T that implements the transformation from $\{r_1, \ldots, r_N\}$ to $\{\bar{r}_1, \ldots, \bar{r}_N\}$:

$$T\{r_1, \ldots, r_N\} = \{\bar{r}_1, \ldots, \bar{r}_N\}. \tag{68}$$

We can then express the evolution operator $\bar{\mathcal{E}}(\tau)$ as

$$\bar{\mathcal{E}}(\tau) = T\mathcal{E}(\tau)T^{-1}. \tag{69}$$

We substitute equation (67) into equation (69) to obtain

$$\mathcal{E}(-\tau)T = T\mathcal{E}(\tau). \tag{70}$$

So if a system is time-reversal invariant, we can define a time-reversal operator T such that evolving the system forwards in time and then applying the time-reversal operator is equivalent to applying the time-reversal operator and then evolving the system backwards in time.

All of our examples except the damped particle are time-reversal invariant, where the time-reversal operator is given by

$$T\{z,\ v\} = \{z,\ -v\}. \tag{71}$$

Physically, the reason that the damped particle violates time-reversal invariance is that the damping force causes energy to be irreversibly lost by the particle, and thus picks out a preferred direction in time. In terms of the flow diagrams shown in figure 1, a system is time-reversal invariant if flipping the diagram across the z axis and then reversing the direction of the arrows leaves the diagram unchanged.

10. Conclusion

We have presented a formalism in which symmetries of a classical dynamical system are described in terms of operators that act on the state space for the system, and we have illustrated our formalism by considering a number of specific examples. Symmetry in classical mechanics is often discussed in the context of Noether's theorem, and it is instructive to compare our formalism with this result. We note the following differences. First, Noether's theorem relates symmetries to conservation laws, whereas our formalism provides a description of symmetries in terms of operators acting on the state space for the system. Second, Noether's theorem applies only to systems whose equations of motion can be derived from a Lagrangian, whereas our formalism applies to arbitrary dynamical systems. Finally, Noether's theorem applies only to continuous symmetries, whereas our formalism applies to both continuous and discrete symmetries. Our treatment offers a different perspective on symmetry than is usually given in introductory mechanics classes, and should provide additional insight into this important concept.

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